# VECTOR BUNDLES ON THE UNIVERSAL VECTOR EXTENSION OF AN ELLIPTIC CURVE

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ABSTRACT. We study vector bundles on the universal vector extension $S$ of an elliptic curve $E$ and on its compactification $\overline{S}$. Using the Laumon–Rothstein transform, we relate vector bundles on $S$ having finite-dimensional cohomology with holonomic $\mathcal{D}_E$-modules. We then apply this correspondence to slope stability for polarizations in the chamber adjacent to the boundary divisor. In particular, we give an alternative proof, for this elliptic ruled surface, that the moduli space $\mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}, c_2)$ is nonempty precisely when either $r \geq 2$ and $c_2 \geq 1$, or $r = 1$ and $c_2 = 0$. This result is a special case of results found in [25].

CONTENTS



## 1. INTRODUCTION

We work over the field of complex numbers $\mathbb{C}$. Fix an elliptic curve $(E, x_0)$. The **universal vector extension** $S$ associated to $E$ is a nontrivial $\mathbb{G}_a$-torsor $S \to E^\vee$ over the dual elliptic curve. We fix an isomorphism $E^\vee \cong E$ and simply write $E$ for both the dual elliptic curve and our original elliptic curve. By [14, 17, 18], there is an equivalence of categories induced by a version of the classical Fourier–Mukai transform $\Phi_{LR} : D^b_{coh}(\mathcal{D}_E) \to D^b_{coh}(\mathcal{O}_S)$, called the **Laumon-Rothstein transform**. Our goal is to study vector bundles on $S$ and its compactification through the Laumon-Rothstein transform.

To state the result, we use the following notation. Let $\overline{S}$ denote the compactification of $S$ which is a ruled surface over $E$ whose minimal section $D$ has self-intersection $D^2 = 0$. Let $\mathfrak{M}_H(r, c_1, c_2)$ denote the moduli of $H$-stable vector bundles of Chern invariants $(r, c_1, c_2)$ where $H$ is an ample divisor on $\overline{S}$. It is well known that in this case, there is a chamber structure so that slope stability with respect to

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$H$ depends only on the choice of chamber. Our main result can be roughly phrased as follows.

**Theorem 1.1** (Theorem 5.22)**.** Let $\mathfrak{M}_H(r, \mathcal{O}_{\overline{S}}, c_2)$ denote the moduli of stable bundles where stability is taken with respect to an ample divisor $H$ in the chamber $\mathcal{C}_D$ adjacent to the $[D]$-axis. Then $\mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}, c_2)$ is nonempty if and only if $c_2 \geq 1$ and $r \geq 2$ or $c_2 = 0$ and $r = 1$.

**Remark 1.2.** During preparation of this paper, we found [25]. Its Corollary 0.2 gives the relevant semistable nonemptiness criterion, and Theorem 0.1(3) identifies a general member in positive rank as stable and locally free; together they imply Theorem 5.22. Yoshioka's approach gives a more general result, whereas our argument focuses on bundles arising from holonomic $\mathcal{D}$-modules on the particular elliptic ruled surface with $e = 0$.

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I also acknowledge usage of ChatGPT 5.6 Plus for filling in a missing entry in Table 3 (The entry in Row 3 for $h^1(S, \mathcal{E})$) and providing the baseline TikZ code for Figure 1.

## 2. Geometry of $S$ and $\overline{S}$

Our starting point is the projective compactification of $S$. It is the ruled surface $\overline{S} = \mathbb{P}_E(F_2)$, where we use the quotient convention for projectivization and $F_2$ is the rank-two indecomposable Atiyah bundle, defined by the nonsplit extension [1]

$$0 \to \mathcal{O}_E \to F_2 \to \mathcal{O}_E \to 0.$$

There is a commutative diagram shown Figure 1 where $\sigma$ is the minimal section of $\overline{p} : \overline{S} \to E$ and $j : S \hookrightarrow \overline{S}$ is the inclusion of $S$ as an open subvariety. The complement of $S \subseteq \overline{S}$ is the image of this section $\sigma$ which we denote by $D$. We emphasize the use of $E$ for the abstract elliptic curve and of $D$ for the divisor $\sigma(E) \subset \overline{S}$ given by the section. Furthermore, the normal bundle of $D$ satisfies $\mathcal{N}_{D/\overline{S}} \cong \mathcal{O}_D$.

The Picard group of $\overline{S}$ is

$$\operatorname{Pic}(\overline{S}) \cong \mathbb{Z}\mathcal{O}_{\overline{S}}(D) \oplus \overline{p}^* \operatorname{Pic}(E).$$

We write $\overline{f}$ for the numerical equivalence class of a fibre of $\overline{p}$ over a geometric point of $E$. Then,

$$\overline{f} \cap \overline{f} = D \cap D = 0 \qquad \text{and} \qquad \overline{f} \cap D = 1.$$

For later use, we recall the cohomology of line bundles on $\overline{S}$ and some geometric properties of $S$ and $\overline{S}$.

**Proposition 2.1.** Let $\overline{p}^*\mathcal{L} \otimes \mathcal{O}_{\overline{S}}(nD)$, where $\mathcal{L} \in \operatorname{Pic}(E)$, be a line bundle on $\overline{S}$.

Set

$$h^i := \dim_{\mathbb{C}} H^i(\overline{S}, \overline{p}^*\mathcal{L} \otimes \mathcal{O}_{\overline{S}}(nD)).$$

Then, the cohomology is described as in Table 1.

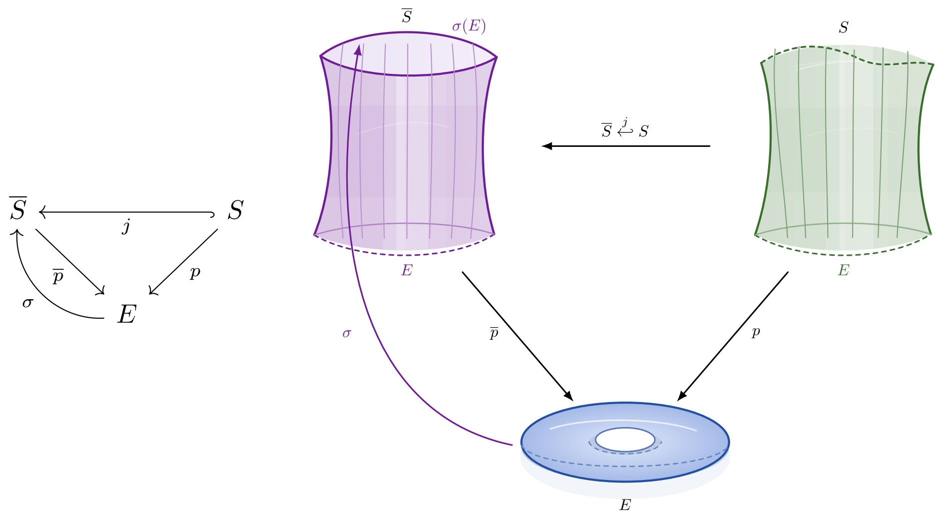


FIGURE 1. The compactification $\overline{S}$, the nontrivial $\mathbb{G}_a$-torsor $S$, and their projections onto $E$.

*Proof.* Computing the cohomology of line bundles on a ruled surface can be done using the results in [9, Section V.2]. The portions of the table for $n \geq 0$ and $n < -1$ are related by Serre duality and Proposition 2.2 (iv) which says $\omega_{\overline{S}} \cong \mathcal{O}_{\overline{S}}(-2D)$. So, we only handle the cases where $n \geq 0$ and $n = -1$.

To verify the table for $n \geq 0$, one can use Lemma V.5.4 of [9], the projection formula, and Exercise III.8.4 of [9]. It is useful to recall from [1] that the symmetric powers of $F_2$ are isomorphic to the rank $n+1$ Atiyah bundle $\mathrm{Sym}^n(F_2) \cong F_{n+1}$. The rank $n+1$ Atiyah bundle is defined inductively as the nontrivial extension of the rank $n$ Atiyah bundle $F_n$ by $\mathcal{O}_E$ given by the image of 1 under $\mathrm{Hom}(\mathcal{O}_E, \mathcal{O}_E) \to \mathrm{Ext}^1(F_n, \mathcal{O}_E)$. As a consequence, these bundles have the property $H^0(E, F_n) \cong H^1(E, F_n) \cong \mathbb{C}$.

For $n = -1$, consider the projection $\overline{p} : \overline{S} \to E$. By Exercise III.8.4 of [9], $R^i\overline{p}_*\mathcal{O}_{\overline{S}}(-D) = 0$ for all $i \geq 0$. By the projection formula,

$$R^i\overline{p}_*(\overline{p}^*\mathcal{L} \otimes \mathcal{O}_{\overline{S}}(-D)) \cong \mathcal{L} \otimes R^i\overline{p}_*\mathcal{O}_{\overline{S}}(-D) = 0.$$

The Leray spectral sequence $H^p(E, R^q\overline{p}_*\mathcal{F}) \Rightarrow H^{p+q}(\overline{S}, \mathcal{F})$ then implies that $H^i(\overline{S}, \overline{p}^*\mathcal{L} \otimes \mathcal{O}_{\overline{S}}(-D)) = 0$ for all $i$. This gives the final row of the table. $\square$

**Proposition 2.2** (Geometric Properties of $\overline{S}$)**.** The following geometric properties of $\overline{S}$ hold:

(i) $\overline{p} : \overline{S} \to E$ is a ruled surface over $E$,
(ii) $\mathrm{Pic}(\overline{S}) \cong \mathbb{Z}\mathcal{O}_{\overline{S}}(D) \oplus \overline{p}^* \mathrm{Pic}(E)$,
(iii) $\mathrm{Num}(\overline{S}) \cong \mathbb{Z}D \oplus \mathbb{Z}\overline{f}$ and $D \cap \overline{f} = 1$, $D \cap D = 0$, and $\overline{f} \cap \overline{f} = 0$,
(iv) $\omega_{\overline{S}} \cong \mathcal{O}_{\overline{S}}(-2D)$,
(v) the normal bundle to $D \subseteq \overline{S}$ is trivial $\mathcal{N}_{D/\overline{S}} \cong \mathcal{O}_D$.

*Proof.* Many of these properties can be found in Chapter V of [9], so we will be terse. Statements (i), (ii), and (iii) can be obtained from Section V.2 of [9]. Statement

| Condition | Case for $\mathcal{L} \in \operatorname{Pic}(E)$ | $h^0$ | $h^1$ | $h^2$ |
|---|---|---|---|---|
| $n \geq 0$ | $d = \deg \mathcal{L} > 0$ | $d(n+1)$ | $0$ | $0$ |
| | $d = \deg \mathcal{L} < 0$ | $0$ | $-d(n+1)$ | $0$ |
| | $\mathcal{L} \in \operatorname{Pic}^0(E) \setminus \{\mathcal{O}_E\}$ | $0$ | $0$ | $0$ |
| | $\mathcal{L} = \mathcal{O}_E$ | $1$ | $1$ | $0$ |
| $n < -1$ | $d = \deg \mathcal{L} > 0$ | $0$ | $-d(n+1)$ | $0$ |
| | $d = \deg \mathcal{L} < 0$ | $0$ | $0$ | $d(n+1)$ |
| | $\mathcal{L} \in \operatorname{Pic}^0(E) \setminus \{\mathcal{O}_E\}$ | $0$ | $0$ | $0$ |
| | $\mathcal{L} = \mathcal{O}_E$ | $0$ | $1$ | $1$ |
| $n = -1$ | All $\mathcal{L} \in \operatorname{Pic}(E)$ | $0$ | $0$ | $0$ |

TABLE 1. Dimensions $h^i$ for $\overline{p}^*\mathcal{L} \otimes \mathcal{O}_{\overline{S}}(nD)$ on the surface $\overline{S}$.

(iv) follows from Lemma V.5.10 of [9]. Statement (v) follows from computing the conormal bundle $\mathcal{I}_D/\mathcal{I}_D^2$ and this can be seen by exhibiting a nowhere vanishing section. Under our conventions, the section $D$ is given by the quotient $F_2 \twoheadrightarrow \mathcal{O}_E$ and $\mathcal{O}_{\overline{S}}(D) \cong \mathcal{O}_{\overline{S}}(1)$. Consequently, $\mathcal{O}(D)|_D \cong \mathcal{O}_D$. $\square$

**Proposition 2.3** (Geometric Properties of $S$)**.** The following geometric properties of $S$ hold:

(i) the morphism $p^* : \operatorname{Pic}(E) \to \operatorname{Pic}(S)$ is an isomorphism,
(ii) $\omega_S \cong \mathcal{O}_S$,
(iii) $S$ does not contain any projective curves as a subscheme,
(iv) $S$ has cohomological dimension 1.

*Proof.* Here, (i), (ii), and (iii)[1] are consequences of the preceding proposition. For (iv), see [10]. $\square$

## 3. Extending Vector Bundles and Finite-Dimensional Cohomology

In this section, we determine when a vector bundle on $S$ will have finite-dimensional cohomology. As $S$ is only quasi-projective, one is not guaranteed to have finite-dimensional cohomology for coherent sheaves. The main result is Theorem 3.8. To do this, we consider extending vector bundles on $S$ to vector bundles on $\overline{S}$.

**Definition 3.1.** Let $\mathcal{E}$ denote a vector bundle on $S$. It admits a vector-bundle extension $\overline{\mathcal{E}}$ to $\overline{S}$. Indeed, first extend $\mathcal{E}$ to a coherent sheaf on $\overline{S}$ and then take its reflexive hull. The restriction to $S$ remains $\mathcal{E}$, and a reflexive coherent sheaf on the smooth surface $\overline{S}$ is locally free [20, Tags 01PD and 0B3N]. We call such a vector bundle $\overline{\mathcal{E}}$ **a locally free extension of $\mathcal{E}$.** Let $\operatorname{LFE}(\mathcal{E})$ denote the set of locally free extensions of $\mathcal{E}$.

**Example 3.2.** If $\mathcal{L} \in \operatorname{Pic}(E)$, then $\operatorname{LFE}(p^*\mathcal{L}) = \{\overline{p}^*\mathcal{L}(nD) \mid n \in \mathbb{Z}\}$.

[1]It is possible to give a quick proof of (iii) by using the Riemann–Hilbert Correspondence.

To state and prove Theorem 3.8, it helps to introduce notation and then compute the cohomology of line bundles on $S$.

**Definition 3.3.** By Proposition 2.3, $\operatorname{Pic}(S)$ can be identified with $\operatorname{Pic}(E)$ by the isomorphism $p^* : \operatorname{Pic}(E) \xrightarrow{\sim} \operatorname{Pic}(S)$. We define the (fibre) **degree of** $\mathcal{E}$ by $\deg(\mathcal{E}) = \deg(\mathcal{L})$ where $\det \mathcal{E} \cong p^*\mathcal{L}$.

**Lemma 3.4.** Let $\overline{\mathcal{E}} \in \operatorname{LFE}(\mathcal{E})$. Then

$$\deg(\mathcal{E}) = c_1(\overline{\mathcal{E}}|_D) = c_1(\overline{\mathcal{E}}) \cap D.$$

In particular, $\deg(\mathcal{E})$ is independent of the choice of $\overline{\mathcal{E}} \in \operatorname{LFE}(\mathcal{E})$.

*Proof.* By using Example 3.2, we have $\det \overline{\mathcal{E}} \cong \overline{p}^*\mathcal{L}(mD)$ where $m \in \mathbb{Z}$ and $\det(\mathcal{E}) \cong p^*\mathcal{L}$. Recalling that $D^2 = 0$,

$$c_1(\overline{\mathcal{E}}) \cap D = c_1(\overline{p}^*\mathcal{L}(mD)) \cap D = c_1(\overline{p}^*\mathcal{L}) \cap D = \deg(\mathcal{L}) = c_1(\overline{\mathcal{E}}|_D).$$

Now use $c_1(\overline{\mathcal{E}}|_D) = c_1(\mathcal{L}) = \deg(\mathcal{L})$. $\square$

**Definition 3.5.** Let $\overline{\mathcal{E}}$ be a vector bundle on $\overline{S}$. Consider the Krull–Schmidt decomposition

$$\overline{\mathcal{E}}\big|_D \cong \bigoplus_{\alpha=1}^{N} V_\alpha$$

into indecomposable vector bundles. Every $V_\alpha$ is semistable, and the Harder–Narasimhan filtration of $\overline{\mathcal{E}}\big|_D$ is obtained by grouping these summands according to their slopes (for proofs, see [21]).

For a vector bundle $\mathcal{E}$ on $S$, define the sets

$$\begin{aligned}
\operatorname{LFE}^+(\mathcal{E}) &:= \left\{\overline{\mathcal{E}} \in \operatorname{LFE}(\mathcal{E}) : \deg V_\alpha \geq 0 \text{ for every } \alpha, \text{ and } \deg V_\alpha > 0 \text{ for at least one } \alpha\right\}, \\
\operatorname{LFE}^-(\mathcal{E}) &:= \left\{\overline{\mathcal{E}} \in \operatorname{LFE}(\mathcal{E}) : \deg V_\alpha \leq 0 \text{ for every } \alpha, \text{ and } \deg V_\alpha < 0 \text{ for at least one } \alpha\right\}, \\
\operatorname{LFE}^{ss,0}(\mathcal{E}) &:= \left\{\overline{\mathcal{E}} \in \operatorname{LFE}(\mathcal{E}) : \deg V_\alpha = 0 \text{ for every } \alpha\right\}.
\end{aligned}$$

Using these sets, define the following sets of vector bundles

$$\begin{aligned}
\operatorname{Bun}^+ &:= \left\{\mathcal{E} \in \operatorname{Bun}(S) : \operatorname{LFE}^+(\mathcal{E}) \neq \varnothing\right\}, \\
\operatorname{Bun}^- &:= \left\{\mathcal{E} \in \operatorname{Bun}(S) : \operatorname{LFE}^-(\mathcal{E}) \neq \varnothing\right\}, \\
\operatorname{Bun}^{ss,0} &:= \left\{\mathcal{E} \in \operatorname{Bun}(S) : \operatorname{LFE}^{ss,0}(\mathcal{E}) \neq \varnothing\right\}, \\
\operatorname{Bun}^{\pm} &:= \left\{\mathcal{E} \in \operatorname{Bun}(S) : \begin{array}{c} \text{for every } \overline{\mathcal{E}} \in \operatorname{LFE}(\mathcal{E}), \text{ there are } \alpha, \beta \\ \text{such that } \deg V_\alpha < 0 < \deg V_\beta \end{array}\right\}.
\end{aligned}$$

In particular, membership in $\operatorname{Bun}^+$ requires at least one positive-degree indecomposable summand, while all remaining summands have nonnegative degree. The analogous statement holds for $\operatorname{Bun}^-$. One can verify easily that $\operatorname{Bun}^+, \operatorname{Bun}^-, \operatorname{Bun}^{ss,0}, \operatorname{Bun}^{\pm}$ are mutually disjoint by considering the determinant bundles and fibre degree. Furthermore, these four subsets of $\operatorname{Bun}(S)$ partition $\operatorname{Bun}(S)$.

**Lemma 3.6.** Let $p^*\mathcal{L} \in \operatorname{Pic}(S)$ where $\mathcal{L} \in \operatorname{Pic}(E)$. Then, its cohomology is described by Table 2.

| $\deg \mathcal{L}$ | Condition | $h^0(S, p^*\mathcal{L})$ | $h^1(S, p^*\mathcal{L})$ |
|---|---|---|---|
| 0 | $\mathcal{L} \cong \mathcal{O}_E$ | 1 | 0 |
| 0 | $\mathcal{L} \ncong \mathcal{O}_E$ | 0 | 0 |
| $> 0$ | | $\infty$ | 0 |
| $< 0$ | | 0 | $\infty$ |

TABLE 2. Cohomology of Line Bundles on $S$

*Proof.* The computations use the same techniques, so we present the first row, where $\mathcal{L} \cong \mathcal{O}_E$. In that case, we are computing the cohomology of $\mathcal{O}_S$. Since $j : S \hookrightarrow \overline{S}$ is an open immersion, $j$ is an affine morphism. Therefore, there are isomorphisms

$$H^i(S, \mathcal{O}_S) \cong H^i(\overline{S}, j_*\mathcal{O}_S) \cong \varinjlim_n H^i(\overline{S}, \mathcal{O}_{\overline{S}}(nD)) \tag{3.1}$$

where the maps in the directed system are induced by the inclusions $\mathcal{O}_{\overline{S}}(nD) \to \mathcal{O}_{\overline{S}}((n+1)D)$. We consider long exact sequences in cohomology associated to

$$0 \to \mathcal{O}_{\overline{S}}(nD) \to \mathcal{O}_{\overline{S}}((n+1)D) \to \sigma_*\mathcal{O}_D \to 0.$$

That the cokernel is $\sigma_*\mathcal{O}_D$ follows from the triviality of the normal bundle. For $n \geq 0$, we get the long exact sequence in cohomology,

$$\begin{aligned} 0 \to H^0(\overline{S}, \mathcal{O}_{\overline{S}}(nD)) &\xrightarrow{\phi_n} H^0(\overline{S}, \mathcal{O}_{\overline{S}}((n+1)D)) \to H^0(\overline{S}, \sigma_*\mathcal{O}_D) \\ \to H^1(\overline{S}, \mathcal{O}_{\overline{S}}(nD)) &\xrightarrow{\psi_n} H^1(\overline{S}, \mathcal{O}_{\overline{S}}((n+1)D)) \to H^1(\overline{S}, \sigma_*\mathcal{O}_D) \to 0. \end{aligned}$$

Every cohomology group in this long exact sequence is one-dimensional for $n \geq 0$ by Proposition 2.1. Examining this exact sequence, for $n \geq 0$, the morphism $\phi_n$ is always an isomorphism, and $\psi_n$ is always zero. Now use (3.1). The remaining rows follow from the same direct-limit calculation after tensoring by $\overline{p}^*\mathcal{L}$ and applying Proposition 2.1. □

**Remark 3.7.** Theorem 2.4.1 in [14] is a stronger version of the statement $H^0(S, \mathcal{O}_S) = \mathbb{C}$ and $H^1(S, \mathcal{O}_S) = 0$ where one considers instead any abelian variety in place of $E$. Laumon's proof utilizes the Koszul complex.

**Theorem 3.8.** A vector bundle $\mathcal{E}$ on $S$ satisfies both

$$h^0(S, \mathcal{E}) < \infty \text{ and } h^1(S, \mathcal{E}) < \infty$$

if and only if $\mathcal{E} \in \mathrm{Bun}^{ss,0}$.

Theorem 3.8 will follow from Lemma 3.9.

**Lemma 3.9.** Let $\mathcal{E}$ be a vector bundle on $S$. Then, we have Table 3. As a consequence of the table, we have another way to see that the sets $\mathrm{Bun}^+, \mathrm{Bun}^-, \mathrm{Bun}^\pm$, and $\mathrm{Bun}^{ss,0}$ are mutually disjoint.

To prove Lemma 3.9, we will break up the proof into a sequence of lemmas. In the process of proving these lemmas, we will consider **elementary modifications** of vector bundles. A detailed exposition on elementary modifications (also known

| | Classification | Condition | $h^0(S,\mathcal{E})$ | $h^1(S,\mathcal{E})$ |
|---|---|---|---|---|
| Row 1 | $\mathcal{E}\in \mathrm{Bun}^+$ | Forced $\deg\mathcal{E}>0$ | $\infty$ | $<\infty$ |
| Row 2 | $\mathcal{E}\in \mathrm{Bun}^-$ | Forced $\deg\mathcal{E}<0$ | $<\infty$ | $\infty$ |
| Row 3 | $\mathcal{E}\in \mathrm{Bun}^\pm$ | $\deg\mathcal{E}\geq 0$ | $\infty$ | $\infty$ |
| Row 4 | | $\deg\mathcal{E}<0$ | $\infty$ | $\infty$ |
| Row 5 | $\mathcal{E}\in \mathrm{Bun}^{ss,0}$ | Forced $\deg\mathcal{E}=0$ | $<\infty$ | $<\infty$ |

TABLE 3. Finiteness of the cohomology groups according to extension type.

as Hecke transforms in the literature) can be found in [5, Appendix A]. To start the process, we deal with the two easiest entries.

**Lemma 3.10.** If $\mathcal{E}\in\mathrm{Bun}^+$, then $h^0(S,\mathcal{E})=\infty$. If $\mathcal{E}\in\mathrm{Bun}^-$, then $h^1(S,\mathcal{E})=\infty$.

*Proof.* Assume $\mathcal{E}\in\mathrm{Bun}^+$. Pick a locally free extension $\overline{\mathcal{E}}$ to $\overline{S}$ such that $\overline{\mathcal{E}}|_D\cong \oplus_{\alpha=1}^h V_\alpha$ with $V_\alpha$ semistable of degree $\geq 0$ and $\deg(V_\alpha)>0$ for at least one $\alpha$. Then, the short exact sequences

$$0\to\overline{\mathcal{E}}(nD)\to\overline{\mathcal{E}}((n+1)D)\to\sigma_*\overline{\mathcal{E}}_D\to 0$$

lead to the fact that $\chi(\overline{\mathcal{E}}(nD))$ is strictly increasing in $n$.

For $n\gg 0$, we know $H^2(\overline{S},\overline{\mathcal{E}}(nD))=0$. Since $\chi(\overline{S},\overline{\mathcal{E}}(nD))\to\infty$ as $n\to\infty$ and $h^0(\overline{S},\overline{\mathcal{E}}(nD))\geq\chi(\overline{S},\overline{\mathcal{E}}(nD))$, it follows $h^0(S,\mathcal{E})=\infty$. The proof of the second statement follows from a similar idea so we omit it. ☐

**Lemma 3.11.** Let $s\in H^0(S,\mathcal{E})$ be a section. Assume $s$ extends to $\overline{s}\in H^0(\overline{S},\overline{\mathcal{E}}(nD))$ and $\overline{s}|_D\in H^0(D,\overline{\mathcal{E}}|_D)$ is a nonzero global section of some semistable summand of degree 0. Then $\overline{s}$ defines a short exact sequence

$$0\to\mathcal{O}_S\overset{\overline{s}|_S}{\to}\mathcal{E}\to\mathcal{F}\to 0. \tag{3.2}$$

Furthermore,

(i) $h^0(S,\mathcal{F})=h^0(S,\mathcal{E})-1$,
(ii) $h^1(S,\mathcal{F})=h^1(S,\mathcal{E})$,
(iii) $h^i(S,\mathcal{F})$ is finite for $i=0,1$ if $h^i(S,\mathcal{F}^{\vee\vee})$ is also finite,
(iv) $\mathrm{rank}(\mathcal{F})=\mathrm{rank}(\mathcal{E})-1$,
(v) one can find an exact sequence $0\to\mathcal{O}_{\overline{S}}\to\overline{\mathcal{E}}(nD)\to\overline{\mathcal{F}}\to 0$ for which

$$0\to\mathcal{O}_D\overset{\overline{s}|_D}{\to}\overline{\mathcal{E}}|_D\to\overline{\mathcal{F}}|_D\to 0$$

is exact, and $\overline{\mathcal{F}}|_D$ is locally free of degree $\deg\overline{\mathcal{E}}|_D$.

*Proof.* For (i) and (ii), take the short exact sequence in (3.2) and consider the long exact sequence in cohomology

$$0\to\mathbb{C}\to H^0(S,\mathcal{E})\to H^0(S,\mathcal{F})\to 0\to H^1(S,\mathcal{E})\to H^1(S,\mathcal{F})\to 0.$$

Here, we use Lemma 3.6 to get $H^0(S,\mathcal{O}_S)=\mathbb{C}$ and $H^1(S,\mathcal{O}_S)=0$.

For (iii), the exact sequence (3.2) shows $\mathcal{F}$ is torsion-free and this exact sequence exists since $s$ cannot vanish on any fibre of $p : S \to E$. Indeed, if $s$ vanished entirely along any fibre $f$, then on $\overline{S}$ we know $s$ vanishes along the closure $\overline{f}$ which then intersects the infinity divisor $D$. This would contradict $\overline{s}|_D$ being a section of a semistable bundle of degree 0. We conclude that the vanishing locus of $s$ must be contained entirely in $S$. Since $S$ has no projective curves, we conclude that $s$ vanishes on a finite set of points. Since $\mathcal{F}$ is torsion-free, there is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}^{\vee\vee} \to \mathcal{T}_Z \to 0$$

where $\mathcal{T}_Z$ is supported on the set of points $Z \subseteq S$ where $\mathcal{F}$ is not locally free. Meanwhile, $\mathcal{F}^{\vee\vee}$ is locally free on all of $S$. Taking cohomology gives

$$0 \to H^0(S, \mathcal{F}) \to H^0(S, \mathcal{F}^{\vee\vee}) \to H^0(S, \mathcal{T}_Z) \to H^1(S, \mathcal{F}) \to H^1(S, \mathcal{F}^{\vee\vee}) \to 0.$$

From here, it is immediate that finiteness of $h^i(\mathcal{F}^{\vee\vee})$ implies finiteness of $h^i(\mathcal{F})$ for $i = 0, 1$ since $h^0(\mathcal{T}_Z)$ is finite due to the coherence and zero-dimensional support of $\mathcal{T}_Z$.

Part (iv) follows immediately by taking ranks in (3.2).

Finally, (v) follows from picking a coherent extension $\overline{\mathcal{F}}$ of $\mathcal{F}$ and using the fact that $\overline{s}|_D$ is a nonzero section of a semistable degree-zero summand. Let $\overline{s} \in H^0(\overline{S}, \overline{\mathcal{E}}(nD))$ denote the chosen extension of $s$.

In particular, one can take $\overline{\mathcal{F}} := \operatorname{coker}(\mathcal{O}_{\overline{S}} \xrightarrow{\overline{s}} \overline{\mathcal{E}}(nD))$. Indeed, the latter condition ensures that $\overline{s}|_D$ is nowhere vanishing on $D$ (since any section with zeros would generate a positive degree subsheaf). This is sufficient to imply $\overline{\mathcal{F}}|_D$ is locally free. The claim about degree is obvious. Since the degree along $E$ is independent of choice of extension, and $\mathcal{F} \hookrightarrow \mathcal{F}^{\vee\vee}$, we see that the degree along $E$ of any extension of $\mathcal{F}^{\vee\vee}$ is the same as $\deg \overline{\mathcal{E}}|_D$. $\square$

**Lemma 3.12.** If $\mathcal{E} \in \mathrm{Bun}^{\pm}$, then $h^0(S, \mathcal{E}) = \infty$.

*Proof.* Choose a locally free extension $\overline{\mathcal{E}}^0$ of $\mathcal{E}$. Since $\mathcal{E} \in \mathrm{Bun}^{\pm}$, there is an indecomposable summand $\mathcal{M}^0 \subseteq \overline{\mathcal{E}}^0|_D$ of positive degree. After replacing $\overline{\mathcal{E}}^0$ by $\overline{\mathcal{E}}^0(nD)$ for $n \gg 0$, we may assume that $H^2(\overline{S}, \overline{\mathcal{E}}^0) = 0$. Starting from $\overline{\mathcal{E}}^0$, repeatedly perform the upper elementary modification associated to the quotient onto the positive-degree summand. Thus we obtain locally free extensions $\overline{\mathcal{E}}^m$ of $\mathcal{E}$ and exact sequences

$$0 \longrightarrow \overline{\mathcal{E}}^m \longrightarrow \overline{\mathcal{E}}^{m+1} \longrightarrow \sigma_* \mathcal{M}^m \longrightarrow 0,$$

where $\mathcal{M}^m$ is chosen to be indecomposable and $\deg \mathcal{M}^m > 0$. We are permitted to make such a choice of $\mathcal{M}^m$ since $\mathcal{E} \in \mathrm{Bun}^{\pm}$. Since a sheaf supported on $D$ has no second cohomology, $H^2(\overline{S}, \overline{\mathcal{E}}^m) = 0$ for every $m$. Moreover,

$$\chi(\overline{\mathcal{E}}^{m+1}) = \chi(\overline{\mathcal{E}}^m) + \chi(\mathcal{M}^m) > \chi(\overline{\mathcal{E}}^m).$$

Consequently, $\chi(\overline{\mathcal{E}}^m) \to \infty$ as $m \to \infty$. Because $H^2(\overline{S}, \overline{\mathcal{E}}^m) = 0$ for $m \gg 0$,

$$h^0(\overline{S}, \overline{\mathcal{E}}^m) = \chi(\overline{\mathcal{E}}^m) + h^1(\overline{S}, \overline{\mathcal{E}}^m) \geq \chi(\overline{\mathcal{E}}^m).$$

Every $\overline{\mathcal{E}}^m$ restricts to $\mathcal{E}$ on $S$, so $H^0(\overline{S}, \overline{\mathcal{E}}^m) \hookrightarrow H^0(S, \mathcal{E})$. The left hand side diverges as $m \to \infty$ and so $h^0(S, \mathcal{E}) = \infty$. $\square$

**Lemma 3.13.** Suppose that $\mathcal{E} \in \mathrm{Bun}^{\pm}$ and $\deg \mathcal{E} < 0$. Then $h^1(S, \mathcal{E}) = \infty$.

*Proof.* Proceed by induction on $\operatorname{rank}(\mathcal{E})$. The rank one case is vacuous, since a line bundle cannot belong to $\mathrm{Bun}^{\pm}$. Dualizing gives $\mathcal{E}^{\vee} \in \mathrm{Bun}^{\pm}$. By Lemma 3.12, $h^0(S, \mathcal{E}^{\vee}) = \infty$. Choose a nonzero section of $\mathcal{E}^{\vee}$ and let $\mathcal{L} \subseteq \mathcal{E}^{\vee}$ be the saturation of the line subsheaf that it generates. Then $\mathcal{L}$ is a line bundle, $\mathcal{E}^{\vee}/\mathcal{L}$ is torsion-free, and $\deg \mathcal{L} \geq 0$, because $\mathcal{L}$ has a nonzero global section.

Dualizing the inclusion $\mathcal{L} \hookrightarrow \mathcal{E}^{\vee}$ gives an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{J} \longrightarrow 0.$$

Here, $\mathcal{K} = (\mathcal{E}^{\vee}/\mathcal{L})^{\vee}$ is locally free of rank one less than $\mathcal{E}$, and $\mathcal{J} \cong \mathcal{I}_Z \otimes \mathcal{L}^{\vee}$ for some zero-dimensional subscheme $Z \subseteq S$, possibly empty.

Suppose first that $\deg \mathcal{L} > 0$. Then $\deg \mathcal{L}^{\vee} < 0$ and so $h^1(S, \mathcal{L}^{\vee}) = \infty$. The exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{L}^{\vee} \longrightarrow \mathcal{T}_Z \longrightarrow 0$$

shows that $H^1(S, \mathcal{J})$ surjects onto $H^1(S, \mathcal{L}^{\vee})$ since $\mathcal{T}_Z$ is torsion with finite support. Hence $h^1(S, \mathcal{J}) = \infty$. Since $S$ has cohomological dimension one, $H^2(S, \mathcal{K}) = 0$, and therefore $H^1(S, \mathcal{E}) \longrightarrow H^1(S, \mathcal{J})$ is surjective. Thus $h^1(S, \mathcal{E}) = \infty$.

Now suppose $\deg \mathcal{L} = 0$. In this case $\deg \mathcal{K} = \deg \mathcal{E} < 0$. A negative-degree bundle cannot belong to $\mathrm{Bun}^{+}$ or $\mathrm{Bun}^{ss,0}$. Hence either $\mathcal{K} \in \mathrm{Bun}^{-}$ or $\mathcal{K} \in \mathrm{Bun}^{\pm}$. In the first case, Lemma 3.10 gives $h^1(S, \mathcal{K}) = \infty$; in the second case, the induction hypothesis gives the same conclusion. On the other hand, $h^0(S, \mathcal{J}) < \infty$, since $\mathcal{J} \subseteq \mathcal{L}^{\vee}$ and $\deg \mathcal{L}^{\vee} = 0$. The cohomology sequence contains

$$H^0(S, \mathcal{J}) \longrightarrow H^1(S, \mathcal{K}) \longrightarrow H^1(S, \mathcal{E}),$$

so $H^1(S, \mathcal{E})$ is an infinite-dimensional space that is being quotiented by a finite-dimensional subspace. Thus, $h^1(S, \mathcal{E}) = \infty$. ∎

**Lemma 3.14.** Let $\mathcal{G}$ be a vector bundle on $S$ such that $h^0(S, \mathcal{G}) = \infty$. Let $\mathcal{F} \subseteq \mathcal{G}$ be the saturation of the image of the evaluation morphism $H^0(S, \mathcal{G}) \otimes_{\mathbb{C}} \mathcal{O}_S \longrightarrow \mathcal{G}$. Then $\mathcal{F}$ is locally free, $H^0(S, \mathcal{F}) = H^0(S, \mathcal{G})$, and $\deg \mathcal{F} > 0$.

*Proof.* The quotient $\mathcal{G}/\mathcal{F}$ is torsion-free by construction. Note that $\mathcal{F}$ is coherent. Since $\mathcal{F}$ is the saturation and $\mathcal{G}$ is a vector bundle, it follows $\mathcal{F}$ is reflexive. As $S$ is a smooth surface, we conclude $\mathcal{F}$ is also locally free.

Every global section of $\mathcal{G}$ factors through $\mathcal{F}$. Indeed, if $s\colon \mathcal{O}_S \to \mathcal{G}$ is a global section, then $s$ is obtained by restricting the evaluation morphism to the copy of $\mathcal{O}_S$ corresponding to $s$. Hence $\operatorname{Im}(s) \subseteq \operatorname{Im}(\mathrm{ev}) \subseteq \mathcal{F}$. Since $\mathcal{F} \hookrightarrow \mathcal{G}$ is injective, the inclusion induces an isomorphism $H^0(S, \mathcal{F}) \xrightarrow{\sim} H^0(S, \mathcal{G})$. In particular, $h^0(S, \mathcal{F}) = \infty$.

Let $q = \operatorname{rank}(\mathcal{F})$. Since taking saturation does not affect the generic fibre, one can choose sections

$$s_1, \ldots, s_q \in H^0(S, \mathcal{G}) = H^0(S, \mathcal{F})$$

that are linearly independent at the generic point. Their exterior product gives a nonzero section $s_1 \wedge \cdots \wedge s_q \in \det \mathcal{F}$. By Lemma 3.6, $\deg \mathcal{F} \geq 0$.

Suppose $\deg \mathcal{F} = 0$. A nonzero section of the degree-zero line bundle $\det \mathcal{F}$ is nowhere vanishing. Therefore the chosen $q$ sections give an isomorphism $\mathcal{O}_S^{\oplus q} \cong \mathcal{F}$. Since $H^0(S, \mathcal{O}_S) = \mathbb{C}$, this would imply $h^0(S, \mathcal{F}) = q < \infty$. Now the triviality of $\mathcal{F}$ contradicts the fact that $H^0(S, \mathcal{G})$ is infinite-dimensional. We conclude that $\deg \mathcal{F} > 0$. ∎

**Lemma 3.15.** Suppose that $\mathcal{E} \in \mathrm{Bun}^{\pm}$ and $\deg \mathcal{E} \geq 0$. Then $h^1(S, \mathcal{E}) = \infty$.

*Proof.* Set $\mathcal{G} = \mathcal{E}^\vee$. By Lemma 3.12, $h^0(S, \mathcal{G}) = \infty$. Let $\mathcal{F} \subsetneq \mathcal{G}$ be the positive-degree subbundle supplied by Lemma 3.14. Here, $\mathcal{F} \neq \mathcal{G}$ as $\deg \mathcal{F} > 0$ and $\deg \mathcal{G} \leq 0$. Set $\mathcal{Q} := \mathcal{G}/\mathcal{F}$. The sheaf $\mathcal{Q}$ is torsion-free. Since $\deg \mathcal{F} > 0$, either $\mathcal{F} \in \mathrm{Bun}^+$ or $\mathcal{F} \in \mathrm{Bun}^\pm$.

If $\mathcal{F} \in \mathrm{Bun}^+$, then $\mathcal{F}^\vee \in \mathrm{Bun}^-$, and Lemma 3.10 gives $h^1(S, \mathcal{F}^\vee) = \infty$. If $\mathcal{F} \in \mathrm{Bun}^\pm$, then $\deg \mathcal{F}^\vee < 0$. So, Lemma 3.13 again gives $h^1(S, \mathcal{F}^\vee) = \infty$. Dualizing

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{Q} \longrightarrow 0$$

gives

$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow \mathcal{E} \longrightarrow \mathcal{J} \longrightarrow 0$$

and $\mathcal{J} = \mathrm{Im}(\mathcal{E} \to \mathcal{F}^\vee)$ sits in an exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{T}_Z \longrightarrow 0,$$

where $\mathcal{T}_Z$ has zero-dimensional support $Z$. One has that $H^1(S, \mathcal{J}) \longrightarrow H^1(S, \mathcal{F}^\vee) \longrightarrow 0$ is surjective, so $H^1(S, \mathcal{J})$ is infinite-dimensional. Since $H^2(S, \mathcal{Q}^\vee) = 0$, we know $H^1(S, \mathcal{E}) \longrightarrow H^1(S, \mathcal{J}) \longrightarrow 0$ is surjective. Thus, $h^1(S, \mathcal{E}) = \infty$. □

We return to proving Lemma 3.9.

*Proof of Lemma 3.9.* To verify the validity of the table in Lemma 3.9, we verify each row using the preceding lemmas.

**Row 1.** Assume $\mathcal{E} \in \mathrm{Bun}^+$ and pick $\overline{\mathcal{E}} \in \mathrm{LFE}(\mathcal{E})$ satisfying the condition for $\mathcal{E}$ to be in $\mathrm{Bun}^+$. For $n \gg 0$, the short exact sequence

$$0 \to \overline{\mathcal{E}}(nD) \to \overline{\mathcal{E}}((n+1)D) \to \sigma_* \overline{\mathcal{E}}_D \to 0,$$

produces the long exact sequence in cohomology

$$\begin{aligned} 0 \to H^0(\overline{S}, \overline{\mathcal{E}}(nD)) &\to H^0(\overline{S}, \overline{\mathcal{E}}((n+1)D)) \overset{a_{n+1}}{\to} H^0(\overline{S}, \sigma_* \overline{\mathcal{E}}_D) \\ &\to H^1(\overline{S}, \overline{\mathcal{E}}(nD)) \to H^1(\overline{S}, \overline{\mathcal{E}}((n+1)D)) \to H^1(\overline{S}, \sigma_* \overline{\mathcal{E}}_D) \to 0. \end{aligned}$$

From this, we build another exact sequence

$$0 \to \frac{H^0(\sigma_* \overline{\mathcal{E}}_D)}{\mathrm{Im}(a_{n+1})} \to H^1(\overline{\mathcal{E}}(nD)) \to H^1(\overline{\mathcal{E}}((n+1)D)) \to H^1(\sigma_* \overline{\mathcal{E}}_D) \to 0. \tag{3.3}$$

Let

$$d_+ := \sum_{\deg V_\alpha > 0} \deg V_\alpha = \deg(\overline{\mathcal{E}}_D).$$

For $n \gg 0$, exactness gives us the equation

$$h^1(\overline{S}, \overline{\mathcal{E}}((n+1)D)) - h^1(\overline{S}, \overline{\mathcal{E}}(nD)) = \dim \mathrm{Im}(a_{n+1}) - d_+.$$

If $\dim \mathrm{Im}(a_{n+1}) \leq d_+$ for every $n \gg 0$, then the dimension of the first cohomology groups is nonincreasing and are therefore bounded. Otherwise, the image $\mathrm{Im}(a_{n+1})$ contains a nonzero section of a degree zero indecomposable summand of $\overline{\mathcal{E}}|_D$. Lemma 3.11 then produces a torsion-free quotient $\mathcal{F}$ whose reflexive hull $\mathcal{F}^{\vee\vee}$ belongs to $\mathrm{Bun}^+$. Moreover, $H^1(S, \mathcal{E}) \cong H^1(S, \mathcal{F})$ and finiteness follows by induction on rank.

**Row 2.** The proof is exactly the same as the proof of Row 1. To show $h^0(S, \mathcal{E}) < \infty$, we consider (3.3) and whether or not $\dim \mathrm{Im}(a_{n+1}) > 0$ for some $n$. If it never occurs, then the dimensions of $H^0(\overline{S}, \overline{\mathcal{E}}(nD))$ stay uniformly bounded for $n \gg 0$. If $\dim \mathrm{Im}(a_{n+1}) > 0$ occurs, then the image contains a lifted section of a degree-zero summand, so Lemma 3.11 reduces the rank and preserves the negative-degree part.

Now, like before, use induction on rank and Lemma 3.6. For $h^1(S, \mathcal{E}) = \infty$, use Lemma 3.10.

**Rows 3 and 4.** These rows follow from combining Lemmas 3.12, 3.13, 3.14, and 3.15.

**Row 5.** In this case every indecomposable summand of $\overline{\mathcal{E}}_D$ has degree zero. Whenever a nonzero section of a degree-zero summand lifts from $D$, using Lemma 3.11 reduces the rank without changing the finiteness of either cohomology group. Now apply induction on rank and use Lemma 3.6 for the base case.

Finally, the four classes in the table exhaust all vector bundles on $S$. The table therefore also proves Theorem 3.8. $\square$

Before ending this section, we point out a convenient corollary of Theorem 3.8. It suggests that we may possibly extend the scope of this paper to torsion-free sheaves instead, but we do not pursue these ideas here.

**Corollary 3.16.** Let $\mathcal{F}$ be a torsion-free sheaf on $S$. Then $h^0(S, \mathcal{F})$ and $h^1(S, \mathcal{F})$ are both finite if and only if $h^0(S, \mathcal{F}^{\vee\vee})$ and $h^1(S, \mathcal{F}^{\vee\vee})$ are both finite.

*Proof.* Consider the long exact sequence in cohomology arising from $0 \to \mathcal{F} \to \mathcal{F}^{\vee\vee} \to \mathcal{T} \to 0$ where $\mathcal{T}$ is a finite length coherent sheaf. $\square$

## 4. Relationship to $\mathcal{D}$-modules

This section uses the relationship between $\mathcal{O}$-modules on $S$ and $\mathcal{D}$-modules on $E$ to study vector bundles on $S$. What we call the Laumon–Rothstein transform was constructed independently by Laumon [14] and Rothstein [18] in 1996 and was subsequently extended to $D$-algebras by Polishchuk and Rothstein [17]. Schnell later described the perverse coherent $t$-structure corresponding to holonomic $\mathcal{D}$-modules on abelian varieties [19] and used it to study cohomology support loci. We specialize Schnell's results in Subsection 4.1, use the transform to describe low-rank vector bundles in Subsection 4.2, and compare invariants in Subsection 4.3.

### 4.1. **The Laumon–Rothstein Transform.**

Let $\mathrm{Hol}(E)$ denote the abelian category of holonomic $\mathcal{D}_E$-modules. Let $\mathcal{H}$ denote the subcategory of $\mathrm{Coh}(\mathcal{O}_S)$ consisting of coherent $\mathcal{O}_S$-modules $\mathcal{F}$ with finite-dimensional cohomology $H^i(S, \mathcal{F} \otimes p^*\mathcal{L})$ for all $\mathcal{L} \in \mathrm{Pic}^0(E)$ and all $i$. The pair of torsion sheaves and torsion-free sheaves satisfying this finiteness condition provides a **torsion pair** on $\mathcal{H}$. The data of a torsion pair and $\mathcal{H}$ is called a **torsion theory** for $\mathcal{H}$. Using this torsion pair, we can define a new abelian category

$$\mathcal{H}^\natural := \left\{ C^\bullet \in D^b(\mathcal{H}) \mid \begin{array}{cc} H^1(C^\bullet) & \text{is a torsion sheaf,} \\ H^0(C^\bullet) & \text{is a torsion-free sheaf,} \\ H^i(C^\bullet) = 0 & \text{for all other } i \end{array} \right\}.$$

This construction is typically referred to as taking the **tilt** of the abelian category $\mathcal{H}$ with respect to the given torsion theory. The following is a restatement of some of the results in [19] to our situation.

**Theorem 4.1** (Laumon–Rothstein; Schnell)**.** The **Laumon–Rothstein transform**[2] is a functor inducing an equivalence of derived categories

$$\Phi_{LR} : D^b_{\text{coh}}(\mathcal{D}_E) \longrightarrow D^b_{\text{coh}}(\mathcal{O}_S).$$

Furthermore, $\Phi_{LR}$ restricts to an equivalence of abelian categories $\Phi_{LR} : \mathrm{Hol}(E) \to \mathcal{H}^\natural$. If $\mathcal{M} \in \mathrm{Hol}(E)$ is a simple holonomic $\mathcal{D}_E$-module, then one of the following is true

(i) $\mathcal{M} \cong \delta_x$ is a $\mathcal{D}_E$-module supported at a point in which case $\Phi_{LR}(\mathcal{M})$ is the line bundle $p^*\mathcal{O}_E(x - x_0)$,
(ii) $\mathcal{M} \cong (\mathcal{L}, \nabla)$ is an integrable connection and so $\Phi_{LR}(\mathcal{M}) \cong \mathbb{C}_{(\mathcal{L},\nabla)}[-1]$ is a skyscraper sheaf supported at the point of $S$ corresponding to $(\mathcal{L}, \nabla)$,
(iii) $\mathcal{M}$ falls into neither of the above cases; thus, $\Phi_{LR}(\mathcal{M})$ is a vector bundle concentrated in cohomological degree zero and is an element of $\mathcal{H}$.

The next two lemmas track when a holonomic $\mathcal{D}_E$-module transforms into a torsion-free $\mathcal{O}_S$-module.

**Lemma 4.2.** Suppose $\mathcal{E}$ is a coherent sheaf on $S$ which is the Laumon–Rothstein transform of some holonomic $\mathcal{D}$-module $\mathcal{M}$. Then the following are equivalent

(i) $\mathcal{E}$ is a vector bundle,
(ii) for every rank-one integrable connection $(\mathcal{L}, \nabla)$ and for $i = 0, 1$, one has
$$\mathrm{Ext}^{i-1}_{\mathcal{D}}((\mathcal{L}, \nabla), \mathcal{M}) = \mathrm{Ext}^i_{\mathcal{D}}((\mathcal{L}, \nabla)[1], \mathcal{M}) = 0,$$
(iii) $\mathcal{M}$ has no $\mathcal{D}$-submodule which is an integrable connection.

*Proof.* The condition that $\mathcal{E}$ is locally free is equivalent to the vanishing of the Ext-sheaves $\mathcal{E}xt^i_{\mathcal{O}_S}(\mathbb{C}_x, \mathcal{E})$ for $i = 0, 1$ where $\mathbb{C}_x$ denotes the skyscraper sheaf at $x \in S$. Indeed, for $x$ a closed point, these Ext-sheaves are supported at $x$ and after taking stalks, this is equivalent to the statement that $\mathrm{Ext}^i_{\mathcal{O}_{S,x}}(\mathcal{O}_{S,x}/\mathfrak{m}_x, \mathcal{E}_x)$ vanishes for $i = 0, 1$. This is equivalent to saying that the depth of $\mathcal{E}_x$ is at least 2 and since $S$ is smooth, this would imply $\mathcal{E}_x$ is a free $\mathcal{O}_{S,x}$-module.

Furthermore, we have the local-to-global spectral sequence

$$H^p(S, \mathcal{E}xt^q_{\mathcal{O}_S}(\mathbb{C}_x, \mathcal{E})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_S}(\mathbb{C}_x, \mathcal{E}).$$

Since these Ext-sheaves have zero-dimensional support, we know

$$H^p(S, \mathcal{E}xt^q_{\mathcal{O}_S}(\mathbb{C}_x, \mathcal{E})) = 0$$

for $p > 0$. So, the spectral sequence degenerates at the $E_2$-page and we get an isomorphism

$$H^0(S, \mathcal{E}xt^q_{\mathcal{O}_S}(\mathbb{C}_x, \mathcal{E})) \cong \mathrm{Ext}^q_{\mathcal{O}_S}(\mathbb{C}_x, \mathcal{E}).$$

Under the Laumon–Rothstein transform, this is equivalent to having

$$\mathrm{Ext}^i_{\mathcal{D}}((\mathcal{L}, \nabla)[1], \mathcal{M}) \cong \mathrm{Ext}^i_{\mathcal{O}_S}(\mathbb{C}_{(\mathcal{L},\nabla)}, \mathcal{E}) = 0$$

for $i = 0, 1$ and all $(\mathcal{L}, \nabla) \in S$. So, we have the equivalence of the first two statements.

For equivalence with (iii), the shift implies

$$\mathrm{Ext}^i_{\mathcal{O}_S}(\mathbb{C}_x, \mathcal{E}) \cong \mathrm{Ext}^{i-1}_{\mathcal{D}}((\mathcal{L}, \nabla), \mathcal{M}).$$

For $i = 0$, the vanishing of $\mathrm{Ext}^{-1}_{\mathcal{D}}((\mathcal{L}, \nabla), \mathcal{M})$ is automatic since both objects are in the heart of the standard t-structure. For $i = 1$, the vanishing requires

[2]For more on the Laumon–Rothstein transform including its definition, see Section B of [19].

$\mathrm{Ext}^0_{\mathcal{D}}((\mathcal{L},\nabla),\mathcal{M}) = \mathrm{Hom}_{\mathcal{D}}((\mathcal{L},\nabla),\mathcal{M}) = 0$, meaning $\mathcal{M}$ has no $\mathcal{D}$-submodule that is an integrable connection.

□

**Lemma 4.3.** Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module. Then the following are equivalent

(i) $\Phi_{LR}(\mathcal{M})$ is a torsion-free sheaf on $S$,
(ii) $\mathcal{M}$ does not have a nontrivial quotient which is an integrable connection.

*Proof.* Set $\mathcal{E} := \Phi_{LR}(\mathcal{M})$. It is quasi-isomorphic to a two-term complex $0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to 0$. For $\mathcal{E}$ to be a torsion-free sheaf on $S$, it is equivalent to require $\mathcal{H}^1(\mathcal{E}) = 0$. This condition is equivalent to requiring $\mathrm{Hom}(\mathcal{E}, \mathbb{C}_x[-1]) = 0$ for all $x \in S$. Under the Laumon–Rothstein transform, this is then equivalent to the condition $\mathrm{Hom}(\mathcal{M},(\mathcal{L},\nabla)) = 0$ for all $(\mathcal{L},\nabla) \in S$. □

Combined with results of Section 3, we have a description of vector bundles arising from holonomic $\mathcal{D}_E$-modules.

**Proposition 4.4.** Let $\mathcal{M}$ be a $\mathcal{D}_E$-module on the elliptic curve and let $\Phi_{LR}(\mathcal{M})$ denote its Laumon–Rothstein transform. The following are equivalent:

(1) $\mathcal{M}$ is a holonomic $\mathcal{D}$-module which does not have a submodule or quotient module that is isomorphic to an integrable connection,
(2) $\mathcal{E} := \Phi_{LR}(\mathcal{M})$ is a vector bundle in cohomological degree zero and $\mathcal{E} \in \mathrm{Bun}^{ss,0}$.

**Remark 4.5.** Specializing [19, Proposition 24.1] to the setting of Proposition 4.4, we see that holonomicity of the $\mathcal{D}_E$-module $\mathcal{M}$ is equivalent to the finiteness of the cohomology groups $H^i(S, \mathcal{E} \otimes p^*\mathcal{L})$ for all $\mathcal{L} \in \mathrm{Pic}^0(E)$ and for all $i$ where $\mathcal{E} = \Phi_{LR}(\mathcal{M})$ is a vector bundle. However, (2) of Proposition 4.4 implies one only needs to check finiteness for when $\mathcal{L} \cong \mathcal{O}_E$.

4.2. **Low Rank Vector Bundles on $S$.** In this subsection, we describe $\mathcal{D}_E$-modules corresponding to rank 1 and 2 vector bundles on $S$. For rank at least three, we were unable to find an interesting description. Further study for rank $\geq 3$ is left for future work. We define the notation

$$h^0_{\mathrm{Pic}}(\mathcal{E}) := \sum_{\mathcal{L} \in \mathrm{Pic}^0(E)} h^0(S, \mathcal{E} \otimes p^*\mathcal{L}).$$

To verify $h^0_{\mathrm{Pic}}(\mathcal{E})$ is finite, one can pick a locally free extension $\overline{\mathcal{E}}$ in which $\overline{\mathcal{E}}_D$ is semistable of degree zero, and consider what happens if one tries to compute cohomology of its twists by degree zero line bundles by using the isomorphism $j_*\mathcal{E} \cong \varinjlim_n \overline{\mathcal{E}}(nD)$.

**Proposition 4.6.** Let $\mathcal{M}$ be a holonomic $\mathcal{D}_E$-module and assume $\Phi_{LR}(\mathcal{M}) = \mathcal{E}$ is a vector bundle. Then the following are equivalent

(i) $\mathcal{M} \cong \bigoplus_{i=1}^r \delta_{x_i}$ is a direct sum of $\mathcal{D}$-modules supported at points,
(ii) $h^0_{\mathrm{Pic}}(\mathcal{E}) = \mathrm{rank}\, \mathcal{E}$ or in other words $h^0_{\mathrm{Pic}}(\mathcal{E})$ is maximal,
(iii) for every extension $\overline{\mathcal{E}}$ whose restriction to $D$ is semistable of degree zero, $\overline{p}_*\overline{\mathcal{E}}(nD)$ is semistable of degree zero for $n \gg 0$,
(iv) $h^1(S, \mathcal{E} \otimes p^*\mathcal{L}) = 0$ for all $\mathcal{L} \in \mathrm{Pic}^0(E)$.

*Proof.* First we prove (i)$\iff$(ii), and then prove (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii). Applying the Laumon–Rothstein transform produces a direct sum of line bundles $p^*\mathcal{L}_{x_i}$ where $\mathcal{L}_{x_i} \in \mathrm{Pic}^0(E)$. By our cohomology computations, (ii) follows.

(ii) $\Longrightarrow$ (i). We argue by induction on $\mathrm{rank}(\mathcal{E})$. A nonzero section after a degree-zero twist gives

$$0 \longrightarrow p^*\mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

with $\mathcal{F}$ torsion-free. Summing the resulting inequalities in $h^0$ over $\mathrm{Pic}^0(E)$ gives

$$h^0_{\mathrm{Pic}}(\mathcal{E}) \leq 1 + h^0_{\mathrm{Pic}}(\mathcal{F}^{\vee\vee}) \leq \mathrm{rank}(\mathcal{E}).$$

Equality forces equality at each step and so induction gives a filtration of $\mathcal{E}$ with quotients in $p^* \mathrm{Pic}^0(E)$. Under the Laumon–Rothstein transform this is a filtration of $\mathcal{M}$ by $\delta$ $\mathcal{D}_E$-modules. Kashiwara's equivalence makes the category of modules supported at a point equivalent to finite-dimensional vector spaces, so the filtration splits and $\mathcal{M}$ is a direct sum of delta $\mathcal{D}_E$-modules.

(i) $\Longrightarrow$ (iii). For the standard extension of $\bigoplus_i p^*\mathcal{L}_i$, one has

$$\overline{p}_*\overline{\mathcal{E}}(nD) \cong \bigoplus_i \mathcal{L}_i \otimes F_{n+1}$$

for $n \gg 0$, where $F_{n+1}$ is Atiyah's indecomposable bundle of rank $n+1$ and degree zero. Any other extension with semistable degree-zero restriction is related to a common refinement by elementary modifications along degree-zero semistable bundles on $D$; after pushing forward for $n \gg 0$, these give extensions inside the abelian category of semistable bundles of slope zero on $E$. Thus the pushforward remains semistable of degree zero.

(iii) $\Longrightarrow$ (iv). This is a consequence of the Leray spectral sequence.

(iv) $\Longrightarrow$ (i). The rank of $\mathcal{M}$ is equal to $-\chi(S, \mathcal{E} \otimes p^*\mathcal{L})$ for generic $\mathcal{L} \in \mathrm{Pic}^0(E)$. For generic $\mathcal{L}$, we know that $h^0(S, \mathcal{E} \otimes p^*\mathcal{L}) = 0$ and so our assumption implies the rank of $\mathcal{M}$ is zero. So on some Zariski open subset $U \subseteq E$, $\mathcal{M}|_U = 0$. Therefore, $\mathcal{M}$ is supported at a finite set of points. By Kashiwara's equivalence [11, Theorem 1.6.1], $\mathcal{M}$ is a direct sum of $\mathcal{D}$-modules of $\delta$-functions. □

**Proposition 4.7.** Let $\mathcal{E} \in \mathrm{Bun}^{ss,0}$ and assume $\mathcal{E}$ has rank 2. Then $\mathcal{E}$ falls into one of the following classes:

(i) If $h^0_{\mathrm{Pic}}(\mathcal{E}) = 2$, then $\mathcal{E} \cong p^*\mathcal{L} \oplus p^*\mathcal{L}'$ is a direct sum of two line bundles where $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}^0(E)$,

(ii) If $h^0_{\mathrm{Pic}}(\mathcal{E}) = 1$, then $\mathcal{E}$ sits in an exact sequence

$$0 \to p^*\mathcal{L} \to \mathcal{E} \to p^*\mathcal{L}' \otimes \mathcal{I}_Z \to 0$$

where $Z \subseteq S$ is some finite-length subscheme of $S$,

(iii) If $h^0_{\mathrm{Pic}}(\mathcal{E}) = 0$, then $\mathcal{E}$ is the Laumon–Rothstein transform of a simple holonomic $\mathcal{D}$-module.

*Proof.* Case (i) follows from the previous proposition. For Case (ii), since $h^0_{\mathrm{Pic}}(\mathcal{E}) = 1$, there must exist a nontrivial morphism $\phi : p^*\mathcal{L} \hookrightarrow \mathcal{E}$ where $\mathcal{L} \in \mathrm{Pic}^0(E)$. This is also a morphism in the abelian category $\mathcal{H}^\natural$ and the cokernel in this category is given by $\mathcal{H}^0(\mathrm{Cone}(\phi)) \cong \mathcal{Q}$. Since the cokernel must be an object of $\mathcal{H}^\natural$, it is a torsion-free sheaf of rank one. Let us denote it by $\mathcal{Q}$. It must have finite-dimensional cohomology, which holds if and only if $\mathcal{Q} \cong p^*\mathcal{L}' \otimes \mathcal{I}_Z$ where $\mathcal{L}' \in \mathrm{Pic}^0(E)$. Finally, Case (iii) follows from a combination of Cases (i) and (ii), and the description of simple objects in the tilt of a Jordan-Hölder category in Lemma 5.5. ☐

4.3. **Correspondence between Invariants.** Let $\mathcal{E} := \Phi_{LR}(\mathcal{M})$ be a vector bundle in cohomological degree zero where $\mathcal{M}$ is a holonomic $\mathcal{D}_E$-module. From before, there exists an extension $\overline{\mathcal{E}}$ to $\overline{S}$ such that $\overline{\mathcal{E}}|_D$ is semistable of degree zero. Replacing $\overline{\mathcal{E}}$ by $\overline{\mathcal{E}}(nD)$ for $n \gg 0$, we may assume $R^1\overline{p}_*\overline{\mathcal{E}} = 0$. So, we have a filtration of $p_*\mathcal{E}$ given by

$$\overline{p}_*\overline{\mathcal{E}} \subseteq \overline{p}_*\overline{\mathcal{E}}(D) \subseteq \overline{p}_*\overline{\mathcal{E}}(2D) \subseteq \cdots \subseteq p_*\mathcal{E}.$$

By $\Phi_{LR}$'s compatibility with the classical Fourier–Mukai transform $\Phi_{FM}$ (see e.g. [17]), we obtain on the $\mathcal{D}$-module side a **good filtration**

$$F_0\mathcal{M} \subseteq F_1\mathcal{M} \subseteq F_2\mathcal{M} \subseteq \cdots \subseteq \mathcal{M}$$

where $\Phi_{FM}(F_k\mathcal{M}) = \overline{p}_*\overline{\mathcal{E}}(kD)$. We can form the associated graded module

$$\operatorname{gr}_F \mathcal{M} := \bigoplus_{i=0}^{\infty} F_i\mathcal{M}/F_{i-1}\mathcal{M}$$

where $F_{-1}\mathcal{M} = 0$.

To check that $\overline{p}_*\overline{\mathcal{E}}(kD)$ should correspond to a good filtration on $\mathcal{M}$, it suffices to check that there exists an $i_0$ such that for all $j \geq 0$ and $i \geq i_0$, the morphism

$$\overline{p}_*\mathcal{O}_{\overline{S}}(jD) \otimes \overline{p}_*\overline{\mathcal{E}}(iD) \to \overline{p}_*\overline{\mathcal{E}}((j+i)D)$$

is surjective. Indeed, such an $i_0$ does exist as the morphism is surjective when $i_0$ is chosen large enough so that $R^1\overline{p}_*\overline{\mathcal{E}}(i_0D) = 0$. The fact that such an $i_0$ exists is due to the relative ampleness of $D$ with respect to $\overline{p} : \overline{S} \to E$.

By our assumptions, $F_i\mathcal{M}/F_{i-1}\mathcal{M}$ is the torsion sheaf corresponding to $\overline{\mathcal{E}}|_D$ under the Fourier–Mukai transform for $i \geq 1$. Therefore, the characteristic cycle of $\mathcal{M}$ is of the form

$$\operatorname{Cyc}(\mathcal{M}) = d[T_E^*E] + \sum_{i=1}^{s} r_i[T_{x_i}^*E]$$

where the $S$-equivalence type of $\overline{\mathcal{E}}|_D$ is $\bigoplus_{i=1}^{s} \mathcal{L}_i^{\oplus r_i}$ with $x_i$ corresponding to $\mathcal{L}_i$. We also know that the degree of $\overline{p}_*\overline{\mathcal{E}}$ is fixed as $-d$ and under the Laumon–Rothstein transform this means $F_0\mathcal{M}$ has rank $d$. This provides the coefficient for $[T_E^*E]$.

This description is independent of the chosen extension. Any two locally free extensions admit a common refinement by elementary modifications along $D$. For one such modification

$$0 \longrightarrow \overline{\mathcal{E}}' \longrightarrow \overline{\mathcal{E}} \longrightarrow \sigma_*\mathcal{Q} \longrightarrow 0,$$

the triviality of the normal bundle of $D$ gives exact sequences

$$0 \longrightarrow \mathcal{Q} \longrightarrow \overline{\mathcal{E}}'|_D \longrightarrow \mathcal{K} \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{K} \longrightarrow \overline{\mathcal{E}}|_D \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Thus the two restrictions have the same class in the Grothendieck group $K_0(D)$. Since both are semistable of degree zero, have the same class in $K_0(D)$, and are related by the above short exact sequences, they have the same Jordan–Hölder factors and hence the same $S$-equivalence type. Under the Fourier–Mukai transform, the factors $\mathcal{L}_i$ correspond to the skyscraper factors at $x_i$, so their multiplicities give the coefficients $r_i$ of $[T_{x_i}^*E]$. For generic $\mathcal{L} \in \operatorname{Pic}^0(E)$, the Riemann–Roch Theorem and the Laumon–Rothstein transform identify $h^1(S, \mathcal{E} \otimes p^*\mathcal{L})$ with $d = \operatorname{rank} \mathcal{M}$, which gives the zero-section coefficient.

Kashiwara's index theorem (see, for example, [11]) says

$$\begin{aligned}\chi_{dR}(\mathcal{M}) &= \sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{C}} H^i(E, \mathrm{DR}(\mathcal{M})[-1]) \\ &= \mathrm{Cyc}(\mathcal{M}) \cap [T^*_E E] = \sum_{i=1}^{s} r_i > 0.\end{aligned}$$

Since we are in the case of a curve, we also have

$$\chi_{dR}(\mathcal{M}) = \sum_{x\in E}(\operatorname{rank}\Phi_x(\mathcal{M}) + \operatorname{irreg}\Phi_x(\mathcal{M}))$$

where $\Phi_x$ is the formal vanishing cycles functor on the formal disk composed with restriction to the formal neighborhood around $x$ and the sum is over closed points $x \in E$. Consequently, the rank of the vector bundle $\Phi_{LR}(\mathcal{M})$ can be understood as being related to the irregularity of the $\mathcal{D}_E$-module $\mathcal{M}$. We package everything above into a single proposition.

**Proposition 4.8.** Let $\mathcal{E} \in \mathrm{Bun}^{ss,0}$ and assume $\mathcal{M}$ satisfies $\mathcal{E} = \Phi_{LR}(\mathcal{M})$.

(i) If $\overline{\mathcal{E}}$ is a locally free extension of $\mathcal{E}$ to $\overline{S}$ which is semistable of degree zero along $D$, then the $S$-equivalence type of $\overline{\mathcal{E}}|_D$ is independent of the choice of locally free extension whose restriction to $D$ is semistable of degree zero and we call it **the $S$-equivalence type at infinity of $\mathcal{E}$**.

(ii) The $S$-equivalence type at infinity of $\mathcal{E}$, together with $h^1(S, \mathcal{E} \otimes p^*\mathcal{L})$ for generic $\mathcal{L}$, determines the characteristic cycle of $\mathcal{M}$.

(iii) $\mathrm{Cyc}(\mathcal{M}) = d[T^*_E E] + \sum_{i=1}^{s} r_i[T^*_{x_i} E]$ where $\operatorname{rank}\mathcal{E} = \sum_{i=1}^{s} r_i$, $c_2(\overline{\mathcal{E}}) = d$ for any locally free extension $\overline{\mathcal{E}}$ with $\overline{\mathcal{E}}_D$ semistable of degree zero, and $h^1(S, \mathcal{E} \otimes p^*\mathcal{L}) = d$ for generic $\mathcal{L} \in \mathrm{Pic}^0(E)$.

| $\mathcal{D}_E$-modules | $\mathcal{O}_S$-modules |
|---|---|
| No IC (integrable connection) quotient module | Torsion-free sheaf |
| No IC submodule nor IC quotient module | Vector bundle |
| Holonomic | Finite-dimensional cohomology for all degree-zero twists |
| Rank | $h^1(S, \mathcal{E} \otimes p^*\mathcal{L})$ for generic degree-zero $\mathcal{L}$ |
| $\mathcal{D}_E$-modules of $\delta$-functions $\delta_x$ | $p^*\mathcal{O}_E(x - x_0)$ for $\mathcal{O}_E(x - x_0) \in \mathrm{Pic}^0(E)$ |
| ICs $(\mathcal{L}, \nabla)$ | Skyscraper sheaves on $S$ |
| $\mathcal{D}_E$-submodules that are $\delta$-functions | $h^0_{\mathrm{Pic}}(\mathcal{E})$ |
| Characteristic cycle | Rank and $S$-equivalence type at infinity |
| Good filtrations | Extensions to $\overline{S}$ |

Table 4. Informal dictionary relating the two sides of the Laumon–Rothstein transform.

## 5. Applications to Stable Vector Bundles

In this section, we apply our previous results to study slope stability on the ruled surface $\overline{S}$. To do so, we must define stability conditions using the minimal section $D$. We achieve this through two interrelated methods. The first relies on the observation that $[D]$ is a movable curve class. This allows us to use the results of [8]. The second approach evaluates stability asymptotically with respect to the ample divisors $H_n = nD + \overline{f}$ as $n \to \infty$. By synthesizing these two techniques, we prove Theorem 5.22.

### 5.1. Stability Conditions.

The numerical class $[D]$ is a **movable curve class** in the sense that $D \cap C \geq 0$ for every effective divisor $C \subseteq \overline{S}$. Thus the slope stability formalism for movable curve classes applies [8, Definitions 2.2 and 2.10].

**Definition 5.1.** Let $\overline{\mathcal{E}}$ be a torsion-free sheaf on $\overline{S}$. Its $\mu_D$**-slope** is denoted

$$\mu_D(\overline{\mathcal{E}}) := \frac{c_1(\overline{\mathcal{E}}) \cap D}{\operatorname{rank} \overline{\mathcal{E}}}.$$

Note that if $\overline{\mathcal{E}}$ is torsion-free, then $c_1(\overline{\mathcal{E}}) \cap D = c_1(\overline{\mathcal{E}}^{\vee\vee}) \cap D$. We say $\overline{\mathcal{E}}$ is $\mu_D$-**semistable** (resp. $\mu_D$-stable) if for all coherent subsheaves $0 \subsetneq \overline{\mathcal{F}} \subseteq \overline{\mathcal{E}}$ with $\operatorname{rank}(\overline{\mathcal{F}}) < \operatorname{rank}(\overline{\mathcal{E}})$ one has $\mu_D(\overline{\mathcal{F}}) \leq \mu_D(\overline{\mathcal{E}})$ (resp. the inequality is strict).

**Lemma 5.2.** Assume $\overline{\mathcal{F}}$ is a $\mu_D$-stable torsion-free coherent sheaf on $\overline{S}$. Then $\overline{\mathcal{F}}^{\vee\vee}$ is a $\mu_D$-stable vector bundle on $\overline{S}$ with $\mu_D(\overline{\mathcal{F}}^{\vee\vee}) = \mu_D(\overline{\mathcal{F}})$.

*Proof.* This follows by consideration of the exact sequence $0 \to \overline{\mathcal{F}} \to \overline{\mathcal{F}}^{\vee\vee} \to \mathcal{G} \to 0$ where $\mathcal{G}$ is some finite-length coherent $\mathcal{O}_{\overline{S}}$-module. Note $c_1(\overline{\mathcal{F}}) = c_1(\overline{\mathcal{F}}^{\vee\vee})$.

To check $\mu_D$-stability, let $\overline{\mathcal{E}} \subseteq \overline{\mathcal{F}}^{\vee\vee}$ be a subbundle. We can consider the intersection $\overline{\mathcal{F}} \cap \overline{\mathcal{E}}$ inside of $\overline{\mathcal{F}}^{\vee\vee}$. The intersection $\overline{\mathcal{F}} \cap \overline{\mathcal{E}}$ differs from $\overline{\mathcal{E}}$ along a finite-length subscheme of $\overline{S}$ and so they have the same slope. Since $\overline{\mathcal{F}} \cap \overline{\mathcal{E}} \subseteq \overline{\mathcal{F}}$, the claim follows from stability of $\overline{\mathcal{F}}$. $\square$

**Proposition 5.3.** Pulling back along the inclusion $j : S \hookrightarrow \overline{S}$ induces a bijection between $\mu_D$-semistable bundles of slope zero up to modification along $D$ and $\mathrm{Bun}^{ss,0}$ and it preserves rank

$$j^* : \frac{\left\{\overline{\mathcal{F}} \in \mathrm{Bun}(\overline{S}) \mid \mu_D(\overline{\mathcal{F}}) = 0 \text{ and } \overline{\mathcal{F}} \text{ is } \mu_D\text{-semistable}\right\}}{\overline{\mathcal{F}} \sim \overline{\mathcal{F}}' \text{ iff } j^*\overline{\mathcal{F}} \cong j^*\overline{\mathcal{F}}'.} \xrightarrow{\sim} \mathrm{Bun}^{ss,0}.$$

*Proof.* First, we show that the map is surjective. Suppose $\mathcal{E} \in \mathrm{Bun}^{ss,0}$. Pick a locally free extension $\overline{\mathcal{E}}$ to $\overline{S}$ such that $\overline{\mathcal{E}}|_D$ is semistable of degree 0. To prove that $\overline{\mathcal{E}}$ is $\mu_D$-semistable, we proceed by contradiction. Suppose not and there exists a nonzero coherent subsheaf $\overline{\mathcal{F}} \subseteq \overline{\mathcal{E}}$ such that $\overline{\mathcal{F}}$ is torsion-free and $\mu_D(\overline{\mathcal{F}}) > 0$. Since $\mu_D(\overline{\mathcal{F}}) = \mu_D(\overline{\mathcal{F}}^{\vee\vee})$ and $\overline{\mathcal{F}}^{\vee\vee} \subseteq \overline{\mathcal{E}}$, we can replace $\overline{\mathcal{F}}$ by its reflexive hull. Since $j : S \to \overline{S}$ is flat, we have an inclusion $j^*\overline{\mathcal{F}} \subseteq j^*\overline{\mathcal{E}}$. Since $\mu_D(\overline{\mathcal{F}}) > 0$, there are two possibilities: either (a) $\mathrm{LFE}^+(j^*\overline{\mathcal{F}}) \neq \emptyset$ or (b) $j^*\overline{\mathcal{F}} \in \mathrm{Bun}^{\pm}$. Applying Theorem 3.8, both situations imply that $h^0(S, j^*\overline{\mathcal{F}}) = \infty$ and therefore $j^0(S, \mathcal{E}) = \infty$. This is a contradicts $h^0(S, \mathcal{E}) < \infty$. So, $\overline{\mathcal{E}}$ is $\mu_D$-semistable. Everything above did not depend on the choice of locally free extension $\overline{\mathcal{E}}$. So, the map is surjective.

Next, we show that the map is well-defined. Suppose we are given a $\mu_D$-semistable vector bundle $\overline{\mathcal{E}}$ on $\overline{S}$ with $\mu_D(\overline{\mathcal{E}}) = 0$. It suffices to show that $h^i(S, j^*\overline{\mathcal{E}}) <$

$\infty$ for $i = 0, 1$. Since Jordan-Hölder filtrations exist, we denote it by

$$0 \subsetneq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_k = \overline{\mathcal{E}}$$

and each quotient in this filtration is a $\mu_D$-stable torsion-free sheaf of slope zero. Pulling back this filtration along $j : S \to \overline{S}$, we see that finiteness of $h^i(S, j^*\overline{\mathcal{E}}) < \infty$ is implied by the statement: any torsion-free $\mu_D$-stable sheaf $\mathcal{F}$ of slope zero has $h^i(S, j^*\mathcal{F}) < \infty$.

Given such an $\mathcal{F}$, consider

$$0 \to \mathcal{F} \to \mathcal{F}^{\vee\vee} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is a finite-length $\mathcal{O}_{\overline{S}}$-module. Then, $h^i(S, j^*\mathcal{F}) < \infty$ is equivalent to $h^i(S, j^*\mathcal{F}^{\vee\vee}) < \infty$. By Lemma 5.2, $\mu_D$-stability of $\mathcal{F}$ implies $\mu_D$-stability of $\mathcal{F}^{\vee\vee}$ and also $\mu_D(\mathcal{F}) = \mu_D(\mathcal{F}^{\vee\vee})$. The statement we want to prove is obvious in the case $\operatorname{rank}\mathcal{F} = 1$ so assume $\operatorname{rank}\mathcal{F} > 1$. Then $\mu_D$-stability implies $\overline{p}^*\mathcal{L}(mD) \nsubseteq \mathcal{F}^{\vee\vee}$ for all $\mathcal{L} \in \operatorname{Pic}^0(E)$ and $m \in \mathbb{Z}$. In particular, there can be no nontrivial inclusion $\mathcal{O}_S \hookrightarrow j^*\mathcal{F}^{\vee\vee}$. Thus, $\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, j^*\mathcal{F}^{\vee\vee}) = H^0(S, j^*\mathcal{F}^{\vee\vee}) < \infty$. By Theorem 3.8, we can rule out $\operatorname{LFE}^+(j^*\mathcal{F}^{\vee\vee}) \neq \emptyset$ and $j^*\mathcal{F}^{\vee\vee} \in \operatorname{Bun}^{\pm}$. We can never have $\operatorname{LFE}^-(j^*\mathcal{F}^{\vee\vee}) \neq \emptyset$ since $\mu_D(\mathcal{F}^{\vee\vee}) = 0$. This rules out all possibilities besides the one where $j^*\mathcal{F}^{\vee\vee} \in \operatorname{Bun}^{ss,0}$. □

**Corollary 5.4.** If $\overline{\mathcal{F}} \in \operatorname{Bun}(\overline{S})$ such that $\mu_D(\overline{\mathcal{F}}) = 0$ and $\overline{\mathcal{F}}$ is $\mu_D$-semistable, then $\mathcal{F} := j^*\overline{\mathcal{F}} \in \operatorname{Bun}^{ss,0}$. In particular, there is some sequence of elementary modifications (allowing any quotients, not just semistable degree zero ones) which produces an $\overline{\mathcal{F}}' \in \operatorname{Bun}(\overline{S})$ such that $\overline{\mathcal{F}}'|_D$ is semistable of degree zero. Additionally, there is a one-to-one correspondence between the following sets

(i) $\operatorname{Bun}^{ss,0}$,

(ii) $\dfrac{\{\overline{\mathcal{F}} \in \operatorname{Bun}(\overline{S}) \mid \mu_D(\overline{\mathcal{F}}) = 0, \overline{\mathcal{F}} \text{ is } \mu_D\text{-semistable}\}}{\overline{\mathcal{F}} \sim \overline{\mathcal{F}}' \text{ iff } j^*\overline{\mathcal{F}} \cong j^*\overline{\mathcal{F}}'.}$,

(iii) $\dfrac{\{\overline{\mathcal{F}} \in \operatorname{Bun}(\overline{S}) \mid \overline{\mathcal{F}}|_D \text{ semistable of degree } 0\}}{\overline{\mathcal{F}} \sim \overline{\mathcal{F}}' \text{ iff } j^*\overline{\mathcal{F}} \cong j^*\overline{\mathcal{F}}'.}$.

The following lemmas help us to describe vector bundles on $S$ corresponding to simple holonomic $\mathcal{D}$-modules.

**Lemma 5.5.** [22, Lemma 2.4] Let $(\mathfrak{T}, \mathfrak{F})$ be a torsion theory for the heart $\mathcal{A}$ of a bounded $t$-structure. Let $\mathcal{A}^\natural$ denote the tilt with respect to this torsion theory. Then any simple object of $\mathcal{A}^\natural$ lies either in $\mathfrak{F} = \mathfrak{T}^\perp$ or in $\mathfrak{T}[-1]$ and

(i) $\mathcal{F} \in \mathfrak{F}$ is simple iff there are no distinguished triangles

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to \mathcal{F}'[1] \qquad \text{or} \qquad \mathcal{T}'[-1] \to \mathcal{F} \to \mathcal{F}' \to \mathcal{T}'$$

where $\mathcal{F}', \mathcal{F}'' \in \mathfrak{F}$ and $\mathcal{T}' \in \mathfrak{T}$ are nonzero.

(ii) $\mathcal{T}[-1] \in \mathfrak{T}[-1]$ is simple iff there are no distinguished triangles

$$\mathcal{T}'[-1] \to \mathcal{T}[-1] \to \mathcal{T}''[-1] \to \mathcal{T}' \qquad \text{or} \qquad \mathcal{T}'[-1] \to \mathcal{T}[-1] \to \mathcal{F}' \to \mathcal{T}'$$

where $\mathcal{T}', \mathcal{T}'' \in \mathfrak{T}$ and $\mathcal{F}' \in \mathfrak{F}$ are nonzero.

The following are now easy consequences of what we have done so far.

**Lemma 5.6.** If $\mathcal{E} \in \operatorname{Bun}^{ss,0}$ such that there does not exist a nonzero proper subbundle $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ such that $\mathcal{F} \in \mathcal{H}^\natural$, then $\mathcal{E}$ is a simple object of $\mathcal{H}^\natural$.

*Proof.* According to Lemma 5.5, failure of simplicity occurs in one of two ways. We leave this verification as an easy exercise to the reader. We remind the reader that if $\mathcal{T}$ is a torsion sheaf with finite support on $S$ and $\mathcal{E}$ is a vector bundle on $S$, then $\mathrm{Ext}^1_S(\mathcal{T}, \mathcal{E}) = 0$. □

**Lemma 5.7.** Suppose that $\mathcal{E} \in \mathrm{Bun}^{ss,0}$. Then $\mathcal{E} \in \mathcal{H}^\natural$ is simple if and only if every locally free extension $\overline{\mathcal{E}}$ to $\overline{S}$ with $\overline{\mathcal{E}}|_D$ semistable of degree zero is actually $\mu_D$-stable of slope zero.

*Proof.* Since any two extensions whose restrictions to $D$ are semistable of degree zero are related by a sequence of modifications, and elementary modifications do not affect $\mu_D$-stability (see Lemma 5.11 below), the statement can be strengthened to the existence of one such extension $\overline{\mathcal{E}}$. The result then follows from existence of a Jordan-Hölder filtration and Corollary 5.4. □

Our description also applies to semisimple holonomic $\mathcal{D}_E$-modules and the associated locally free sheaf.

**Proposition 5.8.** A locally free sheaf $\mathcal{E}$ on $S$ is the Laumon–Rothstein transform of a simple holonomic $\mathcal{D}_E$-module if and only if

(i) there is a locally free extension $\overline{\mathcal{E}}$ to $\overline{S}$ such that $\overline{\mathcal{E}}|_D$ is semistable of degree zero,

(ii) and $\overline{\mathcal{E}}$ is $\mu_D$-stable on $\overline{S}$ for every locally free extension with $\overline{\mathcal{E}}|_D$ semistable of degree zero.

*Proof.* When viewed as a complex $\mathcal{E} \in D^b_{\mathrm{coh}}(\mathcal{O}_S)$ concentrated in cohomological degree zero, condition (i) implies $\mathcal{E}$ is the transform of some holonomic $\mathcal{D}_E$-module $\mathcal{M}$. Now use condition (ii) together with Lemma 5.7. □

**Proposition 5.9.** A locally free sheaf $\mathcal{E}$ on $S$ is the Laumon–Rothstein transform of a semisimple holonomic $\mathcal{D}_E$-module if and only if

(i) $\mathcal{E} \in \mathrm{Bun}^{ss,0}$ and

(ii) there exists a locally free extension $\overline{\mathcal{E}}$ such that $\overline{\mathcal{E}}|_D$ is semistable of degree zero and $\overline{\mathcal{E}}$ is $\mu_D$-polystable of slope zero.

*Proof.* Since ($\Leftarrow$) is obvious, we prove ($\Rightarrow$) instead. If $\mathcal{E} = \Phi_{LR}(\mathcal{M})$ is the Laumon–Rothstein transform and $\mathcal{M} = \bigoplus_{i=1}^s \mathcal{M}_i$ is its direct sum decomposition into simple holonomic modules, then $\mathcal{E} = \bigoplus_{i=1}^s \Phi_{LR}(\mathcal{M}_i)$. Clearly, $\mathcal{E}$ satisfies Condition (i).

To prove Condition (ii) holds, we note that since $\Phi_{LR}(\mathcal{M}_i)$ is a summand of a vector bundle, $\Phi_{LR}(\mathcal{M}_i)$ is itself a vector bundle. Since $\mathcal{M}_i$ is simple and holonomic, we can apply Proposition 5.8 to deduce that every extension of $\Phi_{LR}(\mathcal{M}_i)$ whose restriction to $D$ is semistable of degree zero is $\mu_D$-stable of slope zero. Therefore $\Phi_{LR}(\mathcal{M})$ has a $\mu_D$-polystable locally free extension to $\overline{S}$ of slope zero $\overline{\Phi_{LR}(\mathcal{M})}$ which restricts to a semistable of degree zero vector bundle along $D$. □

Let $H_n$ be the ample divisor $nD + \overline{f}$ on $\overline{S}$ for $n \geq 1$. Let $\mathfrak{M}_{H_n}(r, L, c_2)$ denote the moduli scheme of $\mu_{H_n}$-stable bundles of rank $r$ with fixed determinant $L$ and second Chern class $c_2$. If instead of fixing the determinant to be $L$, we choose to fix the numerical equivalence class of the first Chern class $c_1$, then we will be clear in writing $\mathfrak{M}_{H_n}(r, c_1, c_2)$ instead.

**Lemma 5.10.** Let $\overline{\mathcal{E}}$ be a torsion-free sheaf of rank $r \geq 2$ on $\overline{S}$ with $c_1(\overline{\mathcal{E}}) \equiv aD$. If $\overline{\mathcal{E}}$ is $\mu_D$-stable, then there exists an integer $N = N(\overline{\mathcal{E}})$ such that $\overline{\mathcal{E}}$ is $\mu_{H_n}$-stable for every $n \geq N$.

*Proof.* Set $M := \mu_{\overline{f}}^{\max}(\overline{\mathcal{E}}) = \sup_{0 \neq \overline{\mathcal{F}} \subseteq \overline{\mathcal{E}}} \mu_{\overline{f}}(\overline{\mathcal{F}})$. Since $[\overline{f}]$ is movable, $M < \infty$ by [8, Proposition 2.22]. It suffices to consider saturated subsheaves $0 \neq \overline{\mathcal{F}} \subsetneq \overline{\mathcal{E}}$. Set $s = \operatorname{rank}(\overline{\mathcal{F}})$, and $c_1(\overline{\mathcal{F}}) \equiv a'D + b'\overline{f}$. Then,

$$\frac{a'}{s} = \mu_{\overline{f}}(\overline{\mathcal{F}}) \leq M.$$

Moreover, $\mu_D$-stability and $\mu_D(\overline{\mathcal{E}}) = 0$ give $b'/s < 0$. Since $b' \in \mathbb{Z}$, we have $b' \leq -1$. Consequently,

$$\mu_{H_n}(\overline{\mathcal{F}}) = \frac{nb' + a'}{s} \leq -\frac{n}{s} + M \leq M - \frac{n}{r-1}.$$

On the other hand, $\mu_{H_n}(\overline{\mathcal{E}}) = \frac{a}{r}$. Choosing $N$ so that $M - \frac{N}{r-1} < \frac{a}{r}$ proves the claim uniformly for every proper subsheaf. $\square$

**Lemma 5.11.** If $\overline{\mathcal{E}}$ is a $\mu_D$-stable bundle of degree zero, then any elementary modification $\overline{\mathcal{F}}$ along $D$ is also $\mu_D$-stable.

*Proof.* We start with lower elementary modifications. Consider a lower elementary modification exact sequence

$$0 \to \overline{\mathcal{F}} \to \overline{\mathcal{E}} \to \sigma_* \mathcal{Q} \to 0.$$

Since $c_1(\overline{\mathcal{F}}) = c_1(\overline{\mathcal{E}}) - \operatorname{rank}(\mathcal{Q})D$ and $D^2 = 0$, the two bundles have the same $\mu_D$-slope. Every proper subsheaf of $\overline{\mathcal{F}}$ is also a proper subsheaf of $\overline{\mathcal{E}}$, so the $\mu_D$-stability of $\overline{\mathcal{E}}$ implies that of $\overline{\mathcal{F}}$. This proves the claim for lower elementary modifications. On the other hand, twisting by $D$ preserves $\mu_D$-stability and any upper elementary modification can be realized as a lower elementary modification of a single positive twist by $D$. So, the claim also follows for upper elementary modifications. $\square$

5.2. **Moduli of Stable Bundles.** In this subsection, we study stable bundles on the compactification $\overline{S}$. As the study of stable vector bundles on algebraic surfaces is rich, we begin by reviewing notation and well-known results specialized to $\overline{S}$.

**Definition 5.12.** Let $H$ be an ample divisor on $\overline{S}$. There is[3] a (coarse) moduli space of $\mu_H$-slope-stable vector bundles with Chern invariants $(r, c_1, c_2)$ denoted by $\mathfrak{M}_H(r, c_1, c_2)$.

**Proposition 5.13** (Additional Properties of $\mathfrak{M}_H(r, c_1, c_2)$)**.** Using the notation above,

(i) $\mathfrak{M}_H(r, c_1, c_2)$ is a quasi-projective variety of finite type over $\mathbb{C}$ [12, Chapter 4].
(ii) If $[\overline{\mathcal{E}}] \in \mathfrak{M}_H(r, c_1, c_2)$ represents a point and $\operatorname{Ext}^2(\overline{\mathcal{E}}, \overline{\mathcal{E}})^0 = 0$, then $[\overline{\mathcal{E}}]$ represents a smooth point [12, Section 4.5].
(iii) There is a locally finite **chamber structure** on the ample cone so that if two ample divisors $H$ and $H'$ lie in the same chamber $\mathcal{C}$, then $\overline{\mathcal{E}}$ is $\mu_H$-stable if and only if it is $\mu_{H'}$-stable [23].

[3]Use [12, Theorem 4.3.4] and take the locus corresponding to locally free $\mu_H$-stable sheaves which forms an open subscheme.

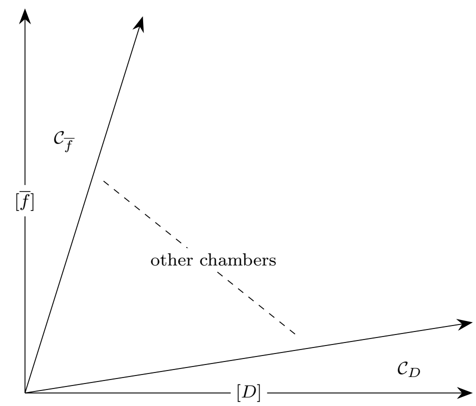


FIGURE 2. Chamber structure on $\mathrm{Amp}(\overline{S})$.

By Proposition 5.13 (iii), one can make sense of the chamber $\mathcal{C}_D$ adjacent to the $[D]$-axis in the ample cone $\mathrm{Amp}(\overline{S}) = \mathbb{R}_{>0}[\overline{f}] \oplus \mathbb{R}_{>0}[D]$. We write $\mu_{\mathcal{C}_D}$ if we are considering slope stability with respect to an ample divisor $H \in \mathcal{C}_D$.

**Corollary 5.14.** Let $\overline{\mathcal{E}}$ be a torsion-free sheaf of rank $r$ with $c_1(\overline{\mathcal{E}}) \equiv aD$. The following conditions are equivalent:

(i) $\overline{\mathcal{E}}$ is $\mu_{\mathcal{C}_D}$-stable;
(ii) $\overline{\mathcal{E}}$ is $\mu_{H_n}$-stable for every sufficiently large $n$;
(iii) for every proper saturated subsheaf $0 \neq \overline{\mathcal{F}} \subsetneq \overline{\mathcal{E}}$, one has either

$$\mu_D(\overline{\mathcal{F}}) < \mu_D(\overline{\mathcal{E}}) \qquad \text{or} \qquad \mu_D(\overline{\mathcal{F}}) = \mu_D(\overline{\mathcal{E}}) \quad \text{and} \quad \mu_{\overline{f}}(\overline{\mathcal{F}}) < \mu_{\overline{f}}(\overline{\mathcal{E}}).$$

In particular,

$$\mu_D\text{-stable} \quad \Longrightarrow \quad \mu_{\mathcal{C}_D}\text{-stable} \quad \Longrightarrow \quad \mu_D\text{-semistable}.$$

*Proof.* The $H_n$ lie in $\mathcal{C}_D$ for all sufficiently large $n$, so the first two conditions are equivalent. For every proper saturated subsheaf $\overline{\mathcal{F}} \subsetneq \overline{\mathcal{E}}$, one has

$$\mu_{H_n}(\overline{\mathcal{F}}) - \mu_{H_n}(\overline{\mathcal{E}}) = n\big(\mu_D(\overline{\mathcal{F}}) - \mu_D(\overline{\mathcal{E}})\big) + \big(\mu_{\overline{f}}(\overline{\mathcal{F}}) - \mu_{\overline{f}}(\overline{\mathcal{E}})\big).$$

Letting $n \to \infty$ shows that asymptotic stability requires $\mu_D(\overline{\mathcal{F}}) \leq \mu_D(\overline{\mathcal{E}})$. When equality holds, stability is determined by the $\overline{f}$-slopes. Conversely, the boundedness used in Lemma 5.10 makes these inequalities uniform over all subsheaves and therefore implies $\mu_{H_n}$-stability for all sufficiently large $n$. □

**Lemma 5.15.** Fix a line bundle $\overline{\mathcal{L}}$ on $\overline{S}$ such that its numerical equivalence class satisfies $c_1(\overline{\mathcal{L}}) \equiv aD$. The family of $\mu_D$-stable torsion-free sheaves $\overline{\mathcal{G}}$ satisfying $\mathrm{rank}(\overline{\mathcal{G}}) = t$, $\det(\overline{\mathcal{G}}) \cong \overline{\mathcal{L}}$, $c_2(\overline{\mathcal{G}}) = e$ has dimension at most $2te$.

*Proof.* Set $\delta := \mathrm{length}\big(\overline{\mathcal{G}}^{\vee\vee}/\overline{\mathcal{G}}\big)$. There is an exact sequence

$$0 \longrightarrow \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}^{\vee\vee} \longrightarrow Q \longrightarrow 0,$$

where $\mathrm{length}(Q) = \delta$. Since $\overline{S}$ is smooth, $\overline{\mathcal{G}}^{\vee\vee}$ is locally free. Moreover, $c_2(\overline{\mathcal{G}}^{\vee\vee}) = e - \delta$. By the Bogomolov–Gieseker inequality, $\delta \leq e$ and so only finitely many values of $\delta$ occur. The $\mu_D$-stability of $\overline{\mathcal{G}}$ implies that $\overline{\mathcal{G}}^{\vee\vee}$ is $\mu_D$-stable. Consequently, $\overline{\mathcal{G}}^{\vee\vee}$ is $\mu_{\mathcal{C}_D}$-stable and the family of possible reflexive hulls with determinant $\overline{\mathcal{L}}$ has dimension at most $2t(e-\delta)$. For a fixed reflexive hull, the possible kernels are parameterized by the Quot scheme $\mathrm{Quot}_{\overline{S}}(\overline{\mathcal{G}}^{\vee\vee}, \delta)$, whose dimension is $(t+1)\delta$ by [6]. Therefore, the locus with fixed $\delta$ has dimension at most

$$2t(e-\delta) + (t+1)\delta = 2te - (t-1)\delta \leq 2te.$$

Taking the union over $\delta$ proves the claim. □

5.3. **Brill-Noether Loci.**

**Definition 5.16.** [4] Let $k > 0$ be an integer. The Brill–Noether locus is the subscheme $W^k_H(r, c_1, c_2) \subseteq \mathfrak{M}_H(r, c_1, c_2)$ whose support is

$$\mathrm{Supp}(W^k_H(r, c_1, c_2)) = \{\overline{\mathcal{E}} \in \mathfrak{M}_H(r, c_1, c_2) \mid h^0(\overline{S}, \overline{\mathcal{E}}) \geq k\}.$$

Using the relation to holonomic $\mathcal{D}$-modules, we can easily prove the following.

**Proposition 5.17.** Assume $c_1$ has numerical equivalence class $c_1 \equiv mD$. If $H \in \mathcal{C}_D$, then the Brill–Noether locus $W^k_H(r, c_1, c_2) \subseteq \mathfrak{M}_H(r, c_1, c_2)$ is empty if $k \geq r+1$. For $r = 2$, we have the stronger statement that $W^k_H(2, c_1, c_2) = \emptyset$ if $k \geq 2$.

*Proof.* Suppose $\overline{\mathcal{E}} \in W^k_H(r, c_1, c_2)$ and assume $H \in \mathcal{C}_D$. We can assume $H$ is of the form $H = nD + \overline{f}$ with $n \gg 0$. Then $\overline{\mathcal{E}}$ is $\mu_{H_n}$-stable for $H_n \equiv nD + \overline{f}$ for all $n \gg 0$. This implies $\overline{\mathcal{E}}$ is $\mu_D$-semistable and so $j^*\overline{\mathcal{E}} \in \mathrm{Bun}^{ss,0}$ is the Laumon–Rothstein transform of a holonomic $\mathcal{D}$-module. In particular, $h^0(S, j^*\overline{\mathcal{E}}) \leq r$ and therefore

$$h^0(\overline{S}, \overline{\mathcal{E}}) \leq h^0(S, j^*\overline{\mathcal{E}}) \leq r.$$

Hence, $\mathrm{Supp}\, W^k_H(r, c_1, c_2) = \emptyset$. The strengthened statement for $r = 2$ can be deduced from Proposition 4.7. □

5.4. **Proving Theorem 5.22.** In this subsection, we prove our main theorem. We start by establishing nonemptiness of $\mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}, 1)$ and then use extensions to increase rank and $c_2$.

**Lemma 5.18.** $\mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}, 1)$ is nonempty for all $r \geq 2$.

*Proof.* Using the dictionary developed above, it suffices to show there exists a meromorphic connection $(\mathcal{O}_E, d+\omega)$ which has an irregularity index $r-1 \geq 1$ at the base point $x_0 \in E$. It is enough to construct a meromorphic 1-form $\omega$ which has a pole of order $r \geq 2$ at $x_0 \in E$ and no poles elsewhere.

The exact sequence

$$0 \to \Omega_E \to \Omega_E(r[x_0]) \to \mathbb{C}_{r[x_0]} \to 0$$

gives us the long exact sequence

$$\begin{aligned} 0 \to H^0(E, \Omega_E) \to H^0(E, \Omega_E(r[x_0])) \to H^0(E, \mathbb{C}_{r[x_0]}) \\ \xrightarrow{\mathrm{Res}} H^1(E, \Omega_E) = \mathbb{C} \to 0. \end{aligned}$$

Choose a local parameter $z$ at $x_0$. The principal part $z^{-r}\,dz$ has residue zero and exactly a pole of order $r$. By the exact sequence, it lifts to a global meromorphic differential $\omega \in H^0(E, \Omega_E(rx_0))$ with no pole away from $x_0$.

Let $\mathcal{M}$ be the minimal extension of this rank-one meromorphic connection. The dictionary of Subsection 4.3 gives $\mathrm{rank}\,\Phi_{LR}(\mathcal{M}) = 1 + \mathrm{irreg}_{x_0}(\mathcal{M}) = r$, boundary type $\mathcal{O}_D^{\oplus r}$, and generic first cohomology equal to $\mathrm{rank}\,\mathcal{M} = 1$. Applying the Riemann–Roch Theorem therefore gives $c_2 = 1$.

Tensoring by a degree-zero line bundle and applying elementary modifications along $D$ normalizes the determinant to $\mathcal{O}_{\overline{S}}$ without changing $c_2$ or $\mu_D$-stability. Since $\mathcal{M}$ is simple, Proposition 5.8 gives $\mu_D$-stability and hence $\mu_{\mathcal{C}_D}$-stability. $\square$

**Lemma 5.19.** Let $r \geq 2$. Suppose there exists $\overline{\mathcal{F}} \in \mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}(\ell D), c_2)$. Assume that $\overline{\mathcal{F}}$ is $\mu_H$-stable for $H := nD + \overline{f}$ with $n \gg 0$. Then there exists a $\mu_H$-stable bundle $\overline{\mathcal{G}} \in \mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}(\ell D), c_2 + 1)$.

*Proof.* The idea is to use elementary modifications at a point to increase $c_2$. Let $z \in \overline{S} \setminus D$ be a closed point. Consider the kernel

$$0 \to \overline{\mathcal{E}} \to \overline{\mathcal{F}} \to \mathbb{C}_z \to 0.$$

Then $\overline{\mathcal{E}}$ is torsion-free of rank $r$ with $\det(\overline{\mathcal{E}}) \cong \mathcal{O}_{\overline{S}}(\ell D)$ and $c_2(\overline{\mathcal{E}}) = c_2 + 1$. It is easily verified that $\overline{\mathcal{E}}$ is $\mu_H$-stable and determines a point in the moduli space of $H$-stable torsion-free sheaves with invariants $(r, \mathcal{O}_{\overline{S}}(\ell D), c_2 + 1)$. Denote this moduli by $\mathfrak{M}^{tf}$.

By Serre duality,

$$\mathrm{Ext}^2(\overline{\mathcal{E}}, \overline{\mathcal{E}}) \cong \mathrm{Hom}(\overline{\mathcal{E}}, \overline{\mathcal{E}} \otimes \omega_{\overline{S}})^\vee = 0$$

and this is zero due to $H \cap K_{\overline{S}} < 0$ and $\mu_H$-stability of $\overline{\mathcal{E}}$. An application of the Grothendieck-Riemann-Roch Theorem gives

$$\begin{aligned}\chi(\overline{\mathcal{E}}, \overline{\mathcal{E}}) &= r^2\chi(\mathcal{O}_{\overline{S}}) + (r-1)c_1(\overline{\mathcal{E}})^2 - 2rc_2(\overline{\mathcal{E}}) \\ &= -2rc_2(\overline{\mathcal{E}}) \\ &= -2r(c_2+1).\end{aligned}$$

Since $\overline{\mathcal{E}}$ is $\mu_H$-stable, $\mathrm{Hom}(\overline{\mathcal{E}}, \overline{\mathcal{E}}) = \mathbb{C}$and from above, $\mathrm{Ext}^2(\overline{\mathcal{E}}, \overline{\mathcal{E}}) = 0$. Therefore,

$$\dim \mathrm{Ext}^1(\overline{\mathcal{E}}, \overline{\mathcal{E}}) = 1 + 2r(c_2+1).$$

The tangent space of the moduli is the kernel of the trace map from $\mathrm{Ext}^1(\overline{\mathcal{E}}, \overline{\mathcal{E}}) \to H^1(\overline{S}, \mathcal{O}_{\overline{S}})$ denoted by $\mathrm{Ext}^1(\overline{\mathcal{E}}, \overline{\mathcal{E}})^0$. Altogether, $\dim \mathrm{Ext}^1(\overline{\mathcal{E}}, \overline{\mathcal{E}})^0 = 2r(c_2+1)$, $\mathfrak{M}^{tf}$ is smooth at $[\overline{\mathcal{E}}]$, and the component passing through $[\overline{\mathcal{E}}]$ has dimension $2r(c_2+1)$.

Now let $\mathfrak{N}_\delta \subset \mathfrak{M}^{tf}$ denote the locus of torsion-free sheaves $\overline{\mathcal{G}}$ such that

$$\mathrm{length}(\overline{\mathcal{G}}^{\vee\vee}/\overline{\mathcal{G}}) = \delta.$$

Setting $\overline{\mathcal{V}} = \overline{\mathcal{G}}^{\vee\vee}$, we get a $\mu_H$-stable vector bundle with determinant $\mathcal{O}_{\overline{S}}(\ell D)$ and second Chern class $c_2(\overline{\mathcal{V}}) = c_2 + 1 - \delta$. By the same deformation-theoretic calculations as before, the family of possible such $\overline{\mathcal{V}}$ has dimension at most $2r(c_2 + 1 - \delta)$.

Now suppose we fixed $\overline{\mathcal{V}}$ and considered possible exact sequences

$$0 \to \overline{\mathcal{G}} \to \overline{\mathcal{V}} \to \mathcal{Q} \to 0$$

where $\mathrm{length}(\mathcal{Q}) = \delta$. Such quotients are parameterized by the Quot scheme $\mathrm{Quot}_{\overline{S}}(\overline{\mathcal{V}}, \delta)$ and it has dimension $(r+1)\delta$ by [6]. Combining everything thus far,

$$\begin{aligned}\dim \mathfrak{N}_\delta &\leq 2r(c_2 + 1 - \delta) + (r+1)\delta \\ &= 2r(c_2+1) - (r-1)\delta < 2r(c_2+1)\end{aligned}$$

and the last inequality uses $r \geq 2$ and $\delta \geq 1$.

By the Bogomolov-Gieseker inequality and $c_1(\overline{\mathcal{V}})^2 = 0$, we have $c_2(\overline{\mathcal{V}}) \geq 0$. The possible $\delta$ which can arise is bounded by $1 \leq \delta \leq c_2 + 1$. Therefore, the non-locally free locus $\bigcup_{\delta \geq 1} \mathfrak{N}_\delta$ in $\mathfrak{M}^{tf}$ has dimension strictly smaller than the component passing through $[\overline{\mathcal{E}}]$. As the property of being locally free is open in flat families, the component passing through $[\overline{\mathcal{E}}]$ contains a locally free point outside of $\bigcup_{\delta \geq 1} \mathfrak{N}_\delta$. So, there must exist a locally free $\mu_H$-stable bundle $\overline{\mathcal{G}} \in \mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}(\ell D), c_2 + 1)$. $\square$

The following proposition gives an interesting construction of stable vector bundles.

**Proposition 5.20.** Let $\overline{\mathcal{E}} \in \mathfrak{M}_{\mathcal{C}_D}\big(2, \mathcal{O}_{\overline{S}}, c_2\big)$, and assume $c_2 > 0$. Then $\mathcal{F} := \operatorname{End}^0(\overline{\mathcal{E}})$, the bundle of trace-free endomorphisms of $\overline{\mathcal{E}}$, is $\mu_{\mathcal{C}_D}$-stable and $\mathcal{F} \in \mathfrak{M}_{\mathcal{C}_D}\big(3, \mathcal{O}_{\overline{S}}, 4c_2\big)$.

*Proof.* Choose an ample divisor $H = nD + \overline{f}$ in the chamber $\mathcal{C}_D$. Since $\operatorname{End}(\overline{\mathcal{E}}) \cong \mathcal{O}_{\overline{S}} \oplus \operatorname{End}^0(\overline{\mathcal{E}})$, the bundle $\mathcal{F}$ has rank 3. Moreover,

$$\det \operatorname{End}(\overline{\mathcal{E}}) \cong \det(\overline{\mathcal{E}}^\vee \otimes \overline{\mathcal{E}}) \cong \mathcal{O}_{\overline{S}},$$

and therefore $\det \mathcal{F} \cong \mathcal{O}_{\overline{S}}$.

We first compute the second Chern class of $\mathcal{F}$. Since $c_1(\overline{\mathcal{E}}) = 0$, we have $\operatorname{ch}(\overline{\mathcal{E}}) = 2 - c_2(\overline{\mathcal{E}})$. Hence

$$\operatorname{ch}\big(\mathcal{F}\big) = \operatorname{ch}(\overline{\mathcal{E}}^\vee)\operatorname{ch}(\overline{\mathcal{E}}) - 1 = \big(2 - c_2(\overline{\mathcal{E}})\big)^2 - 1 = 3 - 4c_2(\overline{\mathcal{E}}).$$

Since $\operatorname{End}(\overline{\mathcal{E}}) \cong \mathcal{O}_{\overline{S}} \oplus \mathcal{F}$, it follows that $c_2(\mathcal{F}) = 4c_2(\overline{\mathcal{E}}) = 4c_2$.

It remains to prove that $\mathcal{F}$ is $\mu_H$-stable. The trace pairing gives an isomorphism $\mathcal{F} \cong \mathcal{F}^\vee$. It is therefore enough to rule out rank-one subsheaves of nonnegative $H$-slope. Indeed, suppose first that $\mathcal{A} \subseteq \mathcal{F}$ is a saturated rank-one subsheaf with $\mu_H(\mathcal{A}) \geq 0$. Since $\overline{S}$ is smooth, the inclusion extends across the finite set where $\mathcal{A}$ is not locally free to an inclusion $L := \mathcal{A}^{\vee\vee} \hookrightarrow \mathcal{F}$, where $L$ is a line bundle and $\mu_H(L) = \mu_H(\mathcal{A}) \geq 0$. On the other hand, if $\mathcal{A} \subseteq \mathcal{F}$ is a saturated rank-two subsheaf with $\mu_H(\mathcal{A}) \geq 0$, then the quotient $\mathcal{Q} := \mathcal{F}/\mathcal{A}$ is torsion-free of rank one. Dualizing the quotient map gives an injection $\mathcal{Q}^\vee \hookrightarrow \mathcal{F}^\vee \cong \mathcal{F}$, and

$$\mu_H(\mathcal{Q}^\vee) = c_1(\mathcal{A}) \cap H = 2\mu_H(\mathcal{A}) \geq 0.$$

Thus it suffices to rule out line subbundles $L \hookrightarrow \mathcal{F}$ with $\mu_H(L) \geq 0$.

Such an inclusion induces a nonzero morphism $\phi : \overline{\mathcal{E}} \otimes L \longrightarrow \overline{\mathcal{E}}$, because $\operatorname{Hom}(L, \operatorname{End}(\overline{\mathcal{E}})) \cong \operatorname{Hom}(\overline{\mathcal{E}} \otimes L, \overline{\mathcal{E}})$.

Suppose first that $\mu_H(L) > 0$. Twisting by a line bundle preserves $H$-stability, and

$$\mu_H(\overline{\mathcal{E}} \otimes L) = \mu_H(\overline{\mathcal{E}}) + \mu_H(L) > \mu_H(\overline{\mathcal{E}}).$$

Therefore, $\operatorname{Hom}(\overline{\mathcal{E}} \otimes L, \overline{\mathcal{E}}) = 0$, contradicting the existence of $\phi$.

It remains to consider the case $\mu_H(L) = 0$. If $L \cong \mathcal{O}_{\overline{S}}$, then the inclusion $\mathcal{O}_{\overline{S}} \hookrightarrow \operatorname{End}^0(\overline{\mathcal{E}})$ would define a nonzero trace-free endomorphism of $\overline{\mathcal{E}}$. This is impossible because the stability of $\overline{\mathcal{E}}$ implies $\operatorname{End}(\overline{\mathcal{E}}) \cong \mathbb{C}$.

We may therefore assume that $L$ is nontrivial. The bundles $\overline{\mathcal{E}} \otimes L$ and $\overline{\mathcal{E}}$ are $H$-stable of the same rank and slope. Consequently, the nonzero morphism $\phi$ is an isomorphism: $\overline{\mathcal{E}} \otimes L \cong \overline{\mathcal{E}}$. Taking determinants gives $L^{\otimes 2} \cong \mathcal{O}_{\overline{S}}$. Since $\operatorname{Pic}(\overline{S}) \cong \mathbb{Z}\mathcal{O}_{\overline{S}}(D) \oplus \overline{p}^* \operatorname{Pic}(E)$, we have $L \cong \overline{p}^*\eta$ for some nontrivial two-torsion line bundle $\eta \in \operatorname{Pic}^0(E)$.

Let

$$\widetilde{\pi} : \widetilde{\overline{S}} := \mathrm{Spec}_{\overline{S}}\big(\mathcal{O}_{\overline{S}} \oplus L\big) \longrightarrow \overline{S}$$

be the associated connected étale double cover, and let $\tau$ denote its deck involution. The composition

$$\overline{\mathcal{E}} \cong \overline{\mathcal{E}} \otimes L^{\otimes 2} \xrightarrow{\phi \otimes \mathrm{id}_L} \overline{\mathcal{E}} \otimes L \xrightarrow{\phi} \overline{\mathcal{E}}$$

is an automorphism of the stable bundle $\overline{\mathcal{E}}$ and is therefore multiplication by a nonzero scalar. After rescaling $\phi$, we may assume that this composition is the identity.

Consequently, there exists a line bundle $M \in \mathrm{Pic}(\widetilde{\overline{S}})$ such that $\overline{\mathcal{E}} \cong \widetilde{\pi}_* M$. Pulling back to $\widetilde{\overline{S}}$ gives $\widetilde{\pi}^*\overline{\mathcal{E}} \cong M \oplus \tau^* M$. Since $\det(\overline{\mathcal{E}}) \cong \mathcal{O}_{\overline{S}}$, we obtain $M \otimes \tau^* M \cong \det\big(\widetilde{\pi}^*\overline{\mathcal{E}}\big) \cong \mathcal{O}_{\widetilde{\overline{S}}}$.

The involution $\tau$ acts trivially on $\mathrm{Num}(\widetilde{\overline{S}})$, since it preserves the numerical classes of the minimal section and of a fibre. Hence $c_1(\tau^* M) \equiv c_1(M)$. The relation $M \otimes \tau^* M \cong \mathcal{O}_{\widetilde{\overline{S}}}$ therefore gives $2c_1(M) \equiv 0$. Since $\mathrm{Num}(\widetilde{\overline{S}})$ is torsion-free, we conclude that $c_1(M) \equiv 0$. It follows that $c_2\big(\widetilde{\pi}^*\overline{\mathcal{E}}\big) = c_1(M) \cap c_1(\tau^* M) = 0$. On the other hand, functoriality of Chern classes gives $c_2\big(\widetilde{\pi}^*\overline{\mathcal{E}}\big) = \widetilde{\pi}^* c_2(\overline{\mathcal{E}})$, and hence $c_2\big(\widetilde{\pi}^*\overline{\mathcal{E}}\big) = 2c_2(\overline{\mathcal{E}}) = 2c_2 > 0$, which is a contradiction. Therefore, $\mathcal{F}$ has no rank-one subsheaf of nonnegative $H$-slope. By the self-duality of $\mathcal{F}$, it also has no rank-two destabilizing subsheaf. Hence $\mathcal{F}$ is $H$-stable, and consequently $\mathcal{F} \in \mathfrak{M}_{\mathcal{C}_D}\big(3, \mathcal{O}_{\overline{S}}, 4c_2\big)$. $\square$

The following lemma provides a way to construct $\mu_{\mathcal{C}_D}$-stable bundles from $\mu_D$-stable bundles. We do not end up using it to prove the main theorem. See Remark 5.23.

**Lemma 5.21.** Let $\overline{\mathcal{F}} \in \mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}, c_2)$ where $c_2 > 0$. Assume further that $\overline{\mathcal{F}}$ is $\mu_D$-stable. Then there is a vector bundle $\overline{\mathcal{E}}$ fitting into an exact sequence

$$0 \to \overline{\mathcal{F}}(-D) \to \overline{\mathcal{E}} \to \mathcal{I}_z(rD) \to 0 \tag{5.1}$$

where $z \in \overline{S} \setminus D$ is a closed point. Furthermore, $\overline{\mathcal{E}} \in \mathfrak{M}_{\mathcal{C}_D}(r+1, \mathcal{O}_{\overline{S}}, c_2+1)$.

*Proof.* From the short exact sequence

$$0 \to \mathcal{I}_z(rD) \to \mathcal{O}_{\overline{S}}(rD) \to \mathbb{C}_z \to 0$$

we get an exact sequence

$$\mathrm{Ext}^1(\mathcal{I}_z(rD), \overline{\mathcal{F}}(-D)) \to \mathrm{Ext}^2(\mathbb{C}_z, \overline{\mathcal{F}}(-D)) \to \mathrm{Ext}^2(\mathcal{O}_{\overline{S}}(rD), \overline{\mathcal{F}}(-D)).$$

By Serre duality,

$$\mathrm{Ext}^2\big(\mathcal{O}_{\overline{S}}(rD), \overline{\mathcal{F}}(-D)\big) \cong H^0\big(\overline{S}, \overline{\mathcal{F}}^\vee((r-1)D)\big)^\vee,$$

and this is zero because $\overline{\mathcal{F}}$ is assumed to be $\mu_D$-stable. Indeed, $\overline{\mathcal{F}}^\vee((r-1)D)$ is also $\mu_D$-stable of slope zero and any injection from $\mathcal{O}_{\overline{S}}$ would contradict this. So, we can find a nontrivial extension class in $\mathrm{Ext}^1(\mathcal{I}_z(rD), \overline{\mathcal{F}}(-D))$ mapping to a nonzero element of $\mathrm{Ext}^2(\mathbb{C}_z, \overline{\mathcal{F}}(-D))$. Let $\overline{\mathcal{E}}$ denote the corresponding extension.

By the local-to-global Ext sequence, the middle term $\overline{\mathcal{E}}$ is locally free at $z$ exactly when the extension class has nonzero image in $\mathrm{Ext}^2(\mathbb{C}_z, \overline{\mathcal{F}}(-D))$. Hence, our choice makes $\overline{\mathcal{E}}$ locally free. It remains to show $\overline{\mathcal{E}}$ is $\mu_{\mathcal{C}_D}$-stable of slope zero. Let $\mathcal{H} \subseteq \overline{\mathcal{E}}$ be a subsheaf. Then we get an exact sequence

$$0 \to \mathcal{H}_0 \to \mathcal{H} \to \mathcal{H}_1 \to 0$$

where $\mathcal{H}_0 \subseteq \overline{\mathcal{F}}(-D)$ and $\mathcal{H}_1 \subseteq \mathcal{I}_z(rD)$. We will use Corollary 5.14 repeatedly without comment.

**Case 1.** If $0 < \operatorname{rank} \mathcal{H}_0 < r$, then $D \cap c_1(\mathcal{H}_0) < 0$ and $D \cap c_1(\mathcal{H}_1) \leq 0$. Therefore, $\mu_D(\mathcal{H}) < 0$.

**Case 2.** If $\operatorname{rank} \mathcal{H}_0 = r$, then $\mathcal{H}_1 = 0$. Furthermore, $\mu_{\overline{f}}(\mathcal{H}) \leq \mu_{\overline{f}}(\overline{\mathcal{F}}(-D)) = -1 < 0$.

**Case 3.** If $\mathcal{H}_0 = 0$, then the saturation of $\mathcal{H}$ in $\mathcal{I}_z(rD)$ is isomorphic to a line bundle $\mathcal{O}_{\overline{S}}(rD - D')$ where $D'$ is an effective divisor. If $D \cap D' > 0$, then $\mu_D(\mathcal{H}) < 0$. On the other hand, if $D \cap D' = 0$, then considering the cohomology of line bundles implies $\mathcal{O}_{\overline{S}}(D') \cong \mathcal{O}_{\overline{S}}(a'D)$. Apply $\operatorname{Hom}(\mathcal{O}_{\overline{S}}((r-a')D), -)$ to (5.1) gives

$$\begin{aligned}0 &\to \operatorname{Hom}(\mathcal{O}_{\overline{S}}(rD - a'D), \overline{\mathcal{F}}(-D)) \to \operatorname{Hom}(\mathcal{O}_{\overline{S}}(rD - a'D), \overline{\mathcal{E}}) \\ &\to \operatorname{Hom}(\mathcal{O}_{\overline{S}}(rD - a'D), \mathcal{I}_z(rD)).\end{aligned}$$

The first Hom vanishes since $\overline{\mathcal{F}}(-D)$ is $\mu_D$-stable of $\mu_D$-slope zero and $\mathcal{O}_{\overline{S}}(rD - a'D)$ has $\mu_D$-slope zero. The third Hom is isomorphic to $H^0(\overline{S}, \mathcal{I}_z(a'D))$. Since $h^0(\mathcal{O}_{\overline{S}}(a'D)) = 1$, the only effective divisor in this linear system is $a'D$ itself. Because we explicitly chose $z \in \overline{S} \setminus D$, the global section of $\mathcal{O}_{\overline{S}}(a'D)$ does not vanish at $z$. Therefore, $H^0(\overline{S}, \mathcal{I}_z(a'D)) = 0$. With the first and third terms vanishing, the middle Hom vanishes as well, completing the proof.

□

**Theorem 5.22.** $\mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}, c_2)$ is nonempty if and only if $c_2 \geq 1$ and $r \geq 2$ or $c_2 = 0$ and $r = 1$.

*Proof.* ($\Leftarrow$) Let $r \geq 2$ and $c_2 \geq 1$. By Lemma 5.18, there is a $\mu_D$-stable bundle $\overline{\mathcal{F}}$ with invariants $(r, \mathcal{O}_{\overline{S}}, 1)$. Choose $n$ sufficiently large so that $H_n = nD + \overline{f}$ lies in $\mathcal{C}_D$ and that $\overline{\mathcal{F}}$ is $H_n$-stable. Applying Lemma 5.19 repeatedly with this fixed polarization produces an $H_n$-stable bundle of invariants $(r, \mathcal{O}_{\overline{S}}, c_2)$.

($\Rightarrow$) Let $\overline{\mathcal{E}}$ be an $H$-stable vector bundle with trivial determinant. If $r = 1$, then $\overline{\mathcal{E}}$ is a line bundle and therefore $c_2(\overline{\mathcal{E}}) = 0$. If $r \geq 2$, then the Bogomolov–Gieseker inequality gives $c_2(\overline{\mathcal{E}}) \geq 0$.

Now assume $c_2(\overline{\mathcal{E}}) = 0$. Then equality holds in the Bogomolov–Gieseker inequality. By the Kobayashi–Hitchin correspondence, $\overline{\mathcal{E}}$ is projectively unitary flat. Since $c_1(\overline{\mathcal{E}}) = 0$, it is unitary flat. Moreover, $\pi_1(\overline{S}) \cong \pi_1(E)$ is abelian, so every irreducible unitary representation of $\pi_1(\overline{S})$ is one-dimensional. Slope stability forces $r = 1$, a contradiction. □

**Remark 5.23.** We prove Theorem 5.22 by showing that every nonempty moduli space $\mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}, c_2)$ contains a $\mu_{\mathcal{C}_D}$-stable bundle, and then applying Lemma 5.19. Except for the initial case supplied by Lemma 5.18, the proof uses only $\mu_D$-stable bundles.

A dimension estimate for the strictly $\mu_D$-semistable locus proves the required existence in rank two (see Proposition 5.24) but the argument does not work for higher ranks.

**Proposition 5.24.** $\mathfrak{M}_{\mathcal{C}_D}(2, \mathcal{O}_{\overline{S}}, c_2)$ for $c_2 > 0$ has a point corresponding to a $\mu_D$-stable bundle.

*Proof.* The locus of strictly $\mu_D$-semistable bundles $\overline{\mathcal{E}}$ can be described by short exact sequences

$$0 \to \overline{p}^*\mathcal{L}(-aD) \to \overline{\mathcal{E}} \to \overline{p}^*\mathcal{M}(aD) \otimes \mathcal{I}_Z \to 0$$

where $Z$ is a subscheme of $\overline{S}$ of length $c_2$, $\mathcal{L}, \mathcal{M} \in \operatorname{Pic}^0(D)$, and $\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O}_E$. The short exact sequence above provides the $\mu_D$-Jordan-Hölder filtration of $\overline{\mathcal{E}}$. Since $\mathbb{P}\operatorname{Ext}^1(\overline{p}^*\mathcal{M}(aD) \otimes \mathcal{I}_Z, \overline{p}^*\mathcal{L}(-aD))$ parameterizes nontrivial extensions and a cohomology computation shows $\dim \operatorname{Ext}^1(\overline{p}^*\mathcal{M}(aD) \otimes \mathcal{I}_Z, \overline{p}^*\mathcal{L}(-aD)) \leq c_2 + 1$, we conclude that the space of such $\overline{\mathcal{E}}$ has dimension at most $3c_2 + 1$. This is strictly less than $\dim \mathfrak{M}_{\mathcal{C}_D}(2, \mathcal{O}_{\overline{S}}, c_2) = 4c_2$ provided $c_2 > 1$ and so a $\mu_D$-stable bundle exists if $c_2 > 1$. On the other hand, the construction in Lemma 5.18 provides a $\mu_D$-stable bundle in $\mathfrak{M}_{\mathcal{C}_D}(2, \mathcal{O}_{\overline{S}}, 1)$. □

5.5. **Existence of $\mu_D$-stable Bundles.** The main purpose of this subsection is to prove the following theorem.

**Theorem 5.25.** For every $r \geq 2$ and $c_2 > 0$, there exists a $\mu_D$-stable vector bundle $\overline{\mathcal{E}}$ on $\overline{S}$ such that $\operatorname{rank} \overline{\mathcal{E}} = r$, $\det \overline{\mathcal{E}} \cong \mathcal{O}_{\overline{S}}$, and $c_2(\overline{\mathcal{E}}) = c_2$. In particular, $[\overline{\mathcal{E}}] \in \mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}, c_2)$.

From the work above, the existence of a vector bundle $\overline{\mathcal{E}}$ as in Theorem 5.25 implies there exists a simple holonomic $\mathcal{M}$ of generic rank $c_2$ with $\operatorname{Cyc}(\mathcal{M}) \cap [T^*_E E] = r$ for any $r \geq 2$ and $c_2 > 0$.

First, we recall a standard avoidance statement. We only actually use the statement for schemes.

**Lemma 5.26.** Let $\mathfrak{X}$ be an irreducible algebraic stack of finite type over $\mathbb{C}$. A nonempty open substack of $\mathfrak{X}$ cannot be written as a countable union of proper closed substacks.

*Proof.* It is enough to prove the corresponding statement for an irreducible scheme of finite type over $\mathbb{C}$. Indeed, after replacing the open substack by itself, choose a smooth surjective atlas $U \to \mathfrak{X}$ and an irreducible component $U_0 \subset U$ whose image is dense in $\mathfrak{X}$. The inverse image in $U_0$ of every proper closed substack of $\mathfrak{X}$ is a proper closed subset of $U_0$. The assertion for schemes over an algebraically closed field is standard. □

The next lemma adapts the proof of openness of slope stability in [12, Proposition 2.3.1], except we do not impose boundedness on the possible $\mu_D$-destabilizing subsheaves. So, we obtain a countable union of closed subsets rather than a finite union.

**Lemma 5.27.** Let $V$ be an irreducible scheme of finite type over $\mathbb{C}$, and let $\mathcal{E}$ be a $V$-flat coherent sheaf on $\overline{S} \times V$. Assume that every geometric fiber $\mathcal{E}_v$ is torsion-free and satisfies $\operatorname{rank}(\mathcal{E}_v) = r$ and $c_1(\mathcal{E}_v) \cap D = 0$. Then,

$$B_D(\mathcal{E}) := \{v \in V \mid \mathcal{E}_v \text{ is not } \mu_D\text{-stable}\} \subseteq V$$

is a countable union of closed subsets of $V$. If one fiber $\mathcal{E}_{v_0}$ is $\mu_D$-stable, then these closed subsets are all proper subsets.

*Proof.* Fix an ample divisor on $\overline{S}$. For each Hilbert polynomial $P$, let

$$q_P \colon Q_P := \mathrm{Quot}^P_{\overline{S}\times V/V}(\mathcal{E}) \longrightarrow V$$

be the relative Quot scheme of quotients with Hilbert polynomial $P$ with respect to the chosen ample divisor. By Grothendieck's Representability Theorem [12, Theorem 2.2.4], $q_P$ is projective.

On $\overline{S} \times Q_P$ there is a universal exact sequence

$$0 \longrightarrow \mathcal{K}_P \longrightarrow \mathcal{E}_{Q_P} \longrightarrow \mathcal{Q}_P \longrightarrow 0. \tag{5.2}$$

Both $\mathcal{E}_{Q_P}$ and $\mathcal{Q}_P$ are flat over $Q_P$, and hence so is $\mathcal{K}_P$. It follows that $\mathrm{rank}(\mathcal{K}_{P,q})$ is locally constant on $Q_P$. The integer $c_1(\mathcal{K}_{P,q}) \cap D$ is also locally constant. Indeed, $\mathcal{K}_P(D)$ is flat over $Q_P$, the Riemann–Roch Theorem gives

$$\chi(\mathcal{F}(D)) - \chi(\mathcal{F}) = c_1(\mathcal{F}) \cap D + \frac{\mathrm{rank}(\mathcal{F})}{2}\left(D^2 - D \cap K_{\overline{S}}\right) \tag{5.3}$$

$$= c_1(\mathcal{F}) \cap D, \tag{5.4}$$

because $D^2 = D \cap K_{\overline{S}} = 0$. Now recall that Euler characteristics are locally constant in proper flat families.

Let $Q_P^{dest} \subseteq Q_P$ be the union of the connected components where $0 < \mathrm{rank}(\mathcal{K}_{P,q}) < r$, and $c_1(\mathcal{K}_{P,q}) \cap D \geq 0$. Since $Q_P$ is of finite type, it has only finitely many connected components. So, $Q_P^{dest}$ is both open and closed in $Q_P$. Then, the restricted morphism $Q_P^{dest} \longrightarrow V$ is projective. Its image $Z_P := q_P(Q_P^{dest})$ is closed in $V$. We claim

$$B_D(\mathcal{E}) = \bigcup_P Z_P \subseteq V. \tag{5.5}$$

Every fiber $\mathcal{E}_v$ has $\mu_D(\mathcal{E}_v) = 0$. If $\mathcal{E}_v$ is not $\mu_D$-stable, then it contains a destabilizing subsheaf $\mathcal{F} \subsetneq \mathcal{E}_v$ with $0 < \mathrm{rank}(\mathcal{F}) < r$ and $c_1(\mathcal{F}) \cap D \geq 0$. The quotient $\mathcal{E}_v \twoheadrightarrow \mathcal{E}_v/\mathcal{F}$ has some Hilbert polynomial $P$, and the corresponding point of $Q_P$ belongs to $Q_P^{dest}$. Hence $v \in Z_P$. Conversely, every point of $Q_P^{dest}$ over $v$ produces such a destabilizing subsheaf of $\mathcal{E}_v$ and so $\mathcal{E}_v$ is not $\mu_D$-stable. This proves (5.5).

There are only countably many Hilbert polynomials given our conditions. So (5.5) is a countable union. If $\mathcal{E}_{v_0}$ is $\mu_D$-stable, then $v_0 \notin Z_P$ for every $P$, so every $Z_P$ is a proper subset. $\square$

**Lemma 5.28.** Let $\overline{\mathcal{G}}$ be a $\mu_D$-stable torsion-free sheaf on $\overline{S}$ with $\mathrm{rank}(\overline{\mathcal{G}}) = r \geq 2$, $c_1(\overline{\mathcal{G}}) \equiv 0$, $c_2(\overline{\mathcal{G}}) = c > 0$.

Then there exists a $\mu_D$-stable vector bundle $\overline{\mathcal{E}}$ with the same numerical invariants. By twisting by an appropriate line bundle in $\overline{p}^* \, \mathrm{Pic}^0(E)$, the $\overline{\mathcal{E}}$ may be chosen so that $\det(\overline{\mathcal{E}}) \cong \mathcal{O}_{\overline{S}}$.

*Proof.* By Corollary 5.14, choose an ample divisor $H \in \mathcal{C}_D$ for which $\overline{\mathcal{G}}$ is $\mu_H$-stable. Let $\mathfrak{S}$ be the stack of $\mu_H$-semistable torsion-free sheaves $\overline{\mathcal{H}}$ satisfying $\mathrm{rank}(\overline{\mathcal{H}}) = r, c_1(\overline{\mathcal{H}}) \equiv 0, c_2(\overline{\mathcal{H}}) = c$. The Riemann–Roch Theorem gives $\chi(\overline{\mathcal{H}}) = -c$ and so Theorem 0.1(1) of [25] applies to get that $\mathfrak{S}$ is irreducible. Let $\mathfrak{S}^{\mu s} \subseteq \mathfrak{S}$ be the open substack of $\mu_H$-stable sheaves. It is nonempty because it contains $[\overline{\mathcal{G}}]$, and hence it is irreducible. Moreover, $\mathfrak{S}^{\mu s}$ is smooth: this follows from [25, Proposition 1.5(2)], since $K_{\overline{S}} \cap H < 0$. By [25, Proposition 2.7], the open substack $\mathfrak{V} \subseteq \mathfrak{S}^{\mu s}$ parameterizing vector bundles is dense and nonempty.

Choose a smooth atlas $U \to \mathfrak{S}^{\mu s}$ and a point $u \in U$ lying above $[\overline{\mathcal{G}}]$. Since $\mathfrak{S}^{\mu s}$ is smooth over $\mathbb{C}$, the scheme $U$ is smooth. Its irreducible components are therefore disjoint open-and-closed subschemes. Replacing $U$ by the component containing $u$, we obtain an irreducible scheme $V$ whose image in $\mathfrak{S}^{\mu s}$ is open and therefore dense. Let $\mathcal{U}$ be the induced flat family on $\overline{S} \times V$.

By Lemma 5.27, the locus of points $v \in V$ for which $\mathcal{U}_v$ is not $\mu_D$-stable is a countable union $\bigcup_{i\geq 1} Z_i$ of proper closed subsets of $V$. Set $V^{lf} = V \times_{\mathfrak{S}^{\mu s}} \mathfrak{V}$. Since the image of $V$ and $\mathfrak{V}$ are both dense open substacks of $\mathfrak{S}^{\mu s}$, the scheme $V^{lf}$ is a nonempty open subset of $V$. Each $Z_i \cap V^{lf}$ is proper: otherwise the closed subset $Z_i$ would contain the dense open subset $V^{lf}$ of $V$, and hence would equal $V$. By Lemma 5.26, there exists a point $v \in V^{lf} \setminus \bigcup_{i\geq 1} Z_i$. Then $\overline{\mathcal{E}} := \mathcal{U}_v$ is our desired $\mu_D$-stable vector bundle with the same numerical invariants as $\overline{\mathcal{G}}$.

$$\begin{tikzcd} V \arrow[r, hook] & U \arrow[r] & \mathfrak{S}^{\mu s} \arrow[r, hook] & \mathfrak{S} \\ V \times_{\mathfrak{V}} \mathfrak{S}^{\mu s} \arrow[u] \arrow[rr] & & \mathfrak{V} \arrow[u, hook] & \end{tikzcd}$$

Since $c_1(\overline{\mathcal{E}}) \equiv 0$, there is an $L \in \mathrm{Pic}^0(E)$ such that $\det(\overline{\mathcal{E}}) \cong \overline{p}^* L$. Multiplication by $r$ on $\mathrm{Pic}^0(E)$ is surjective (as we work over $\mathbb{C}$), so there exists an $N \in \mathrm{Pic}^0(E)$ with $N^{\otimes r} \cong L^{-1}$. Replacing $\overline{\mathcal{E}}$ by $\overline{\mathcal{E}} \otimes \overline{p}^* N$ makes its determinant trivial. Since $\overline{p}^* N$ is numerically trivial, this twist preserves $\mu_D$-stability and does not affect the Chern classes. □

*Proof of Theorem 5.25.* Fix $r \geq 2$. The $c_2 = 1$ construction established in Lemma 5.18 gives a $\mu_D$-stable vector bundle of rank $r$ and determinant $\mathcal{O}_{\overline{S}}$.

Assume inductively that such a vector bundle exists with second Chern class $m \geq 1$. Elementary modification at a point like in Lemma 5.19 produces a $\mu_D$-stable torsion-free sheaf of rank $r$, trivial determinant, and second Chern class $m+1$. Indeed, if $z \in \overline{S} \setminus D$ then consider a surjection $\overline{\mathcal{E}} \twoheadrightarrow \mathbb{C}_z$, and let $\overline{\mathcal{G}}$ be its kernel. Every positive-rank proper subsheaf of $\overline{\mathcal{G}}$ is a subsheaf of $\overline{\mathcal{E}}$, $\mu_D(\overline{\mathcal{G}}) = \mu_D(\overline{\mathcal{E}})$, and hence $\overline{\mathcal{G}}$ is $\mu_D$-stable. Lemma 5.28 then replaces it by a $\mu_D$-stable vector bundle with the same invariants. Induction gives a $\mu_D$-stable vector bundle of rank $r$, trivial determinant, and second Chern class $c_2$ for every $c_2 > 0$. In particular, the conclusion holds whenever $\mathfrak{M}_{\mathcal{C}_D}(r, \mathcal{O}_{\overline{S}}, c_2)$ is nonempty. □

**Example 5.29.** Choose distinct points $x_1, x_2 \in E$ and an $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. There exists a meromorphic differential $\omega$ on $E$ with simple poles at $x_1, x_2$ with residues $\alpha$, $-\alpha$ respectively. Consider the rank one meromorphic connection $(\mathcal{O}_{E\setminus\{x_1,x_2\}}, d + \omega)$ and let $\mathcal{M}$ be the minimal extension of this meromorphic connection. Then, $\mathcal{M}$ is a simple regular holonomic $\mathcal{D}_E$-module of generic rank 1 and $\mathrm{Cyc}(\mathcal{M}) = [T^*_E E] + [T^*_{x_1} E] + [T^*_{x_2} E]$. Furthermore, $\Phi_{LR}(\mathcal{M})$ is a rank 2 vector bundle whose $S$-equivalence type at $\infty$ is $\mathcal{L}_{x_1} \oplus \mathcal{L}_{x_2}$. By considering elementary modifications of a locally free extension $\overline{\mathcal{E}}$ of $\mathcal{E}$ to $\overline{S}$ with $\overline{\mathcal{E}}_D$ semistable of degree zero, one sees that the locally free $\mu_{\mathcal{C}_D}$-stable extensions of $\mathcal{E}$ need not be unique even if one fixes numerical invariants.

**Remark 5.30.** It is possible to use Lemma 5.18 to explicitly construct an example of a $\mu_D$-stable vector bundle $\overline{\mathcal{E}}$ satisfying $\mathrm{rank}(\overline{\mathcal{E}}) = r$, $\det(\overline{\mathcal{E}}) \cong \mathcal{O}_{\overline{S}}$, and $c_2(\overline{\mathcal{E}}) = m$ for any $m \geq 1$ if $r \geq 2$. One can do this by considering pushforwards along finite isogenies. We leave a write-up to the future.

## Further Directions and Questions

We conclude with questions motivated by the above work. We expect to address some of these topics in future work (either in a follow-up paper or in an update to this one).

(i) Can the Laumon–Rothstein transform provide an alternative computation of the Picard group? Such computations of Picard groups of moduli of stable sheaves have already been performed in [23, 24] so the main interest would be in giving an alternative argument.

(ii) Can one define an appropriate moduli space $\mathcal{X}(r, c_1, c_2)$ of filtered holonomic $\mathcal{D}$-modules? Does the Laumon–Rothstein transform identify it with the corresponding stable-bundle moduli space? For moduli of $\mathcal{D}$-modules, see [16].

(iii) The Laumon–Rothstein transform satisfies $\Phi_{LR}(\mathcal{M} * \mathcal{N}) \cong \Phi_{LR}(\mathcal{M}) \otimes \Phi_{LR}(\mathcal{N})$ where $\mathcal{M} * \mathcal{N}$ denotes convolution of $\mathcal{D}$-modules on the elliptic curve. The work above can be used to compute the characteristic cycle of $\mathcal{M} * \mathcal{N}$. Is it possible to find a formula for the formal types of $\mathcal{M} * \mathcal{N}$ as well? We are only aware of the related works in [7] for the local formal setting and [3] for the regular holonomic setting. We are not quite sure the extent to which [7] answers our question.

(iv) Can one generalize the above work to study moduli spaces of stable torsion-free sheaves on $\overline{S}$? What happens if we consider other stability conditions such as Gieseker stability? The works [25] and [15] have already addressed the study of such sheaves so we are simply asking about an interpretation using the Laumon–Rothstein transform.

(v) Can one generalize the above work to replace the elliptic curve $E$ with any abelian variety $A$? For an abelian variety of dimension $g > 1$, the boundary of the projective compactification of its universal vector extension is a $\mathbb{P}^{g-1}$-bundle over the abelian variety, rather than a section. This introduces many new difficulties.

(vi) The results of [2] are applicable given the cohomology of $\omega_{\overline{S}}^{\vee} \cong \mathcal{O}_{\overline{S}}(2D)$ and provide a Poisson structure on $\mathfrak{M}_{\mathcal{C}_D}(r, c_1, c_2)$. How does this Poisson structure translate to the $\mathcal{D}_E$-module side under the Laumon-Rothstein transform?

(vii) Can one use the results above to construct examples with interesting Tannakian Galois groups as described in [13]?

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