Solutions to Hartshorne's Algebraic Geometry

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Preface

These are my algebraic geometry notes. The first five chapters consist of solutions to exercises in [6]. The citation for material in Hartshorne's text are labeled as follows: I write Hartshorne II.2.1 to refer to exercise 1 in chapter II, section 2 while I write Hartshorne Lemma II.3.2 to mean the lemma 2 in chapter II, section 3. Obviously, any of definition, theorem, proposition, remark, or corollary could be used in place of lemma.

Included among these notes are solutions to exercises from other algebraic geometry textbooks that I have looked at as well.

For a lot of these exercises, I have referred to various online sources, other textbooks, and talked with others. Since there is no reason to leave the list of online references to a bibliography, I shall list them here.

(1) James McKernan's "Model" Answers.
(2) Another of James Mckernan's "Model" Answers.
(3) Third page of James Mckernan's "Model" Answers.
(4) Algebraic Geometry By: Robin Hartshorne Solutions by Joe Cutrone and Nick Marshburn
(5) Andrew Egbert's Solutions to Hartshorne Problems

Part 1

Algebraic Geometry by Robin Hartshorne

CHAPTER 1

Chapter I Exercises

1. Affine Varieties

Hartshorne I.1.1.

(a) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

(c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

PROOF. (a) We have $A(Y) \cong k[x, y]/(y - x^2) \cong k[x, x^2] \cong k[x]$. Hence, $A(Y)$ is isomorphic to a polynomial ring in one variable over k. The isomorphisms here are easy to see:

$$k[x, y]/(y - x^2) \to k[x, x^2], \qquad x \mapsto x,\ y \mapsto x^2 \qquad \& \qquad k[x, x^2] \to k[x], \qquad x \mapsto x.$$

(b) We have[1] $A(Z) \cong k[x, y]/(xy - 1) \cong k[x, \frac{1}{x}]$. Suppose not and $\phi : A(Z) \xrightarrow{\cong} k[z]$ with $k[z]$ a single variable polynomial ring over k. Observe that in $A(Z)$, the element x has an inverse, namely $\frac{1}{x}$. However, $\phi(x) \in k[z]$ cannot have an inverse since $(k[z])^* = k$ and isomorphisms must preserve units. Contradiction.

(c) Let $f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ be an irreducible quadratic in $k[x, y]$. Let $W = Z(f)$. Throughout this proof, the constants will change by the line and we abuse notation by using the same letters for constants and scalars. First, we factor the first three terms to get

$$f(x, y) = (Ax + By)(Cx + Dy) + Ex + Fy + G.$$

Now, we should observe that linear change of coordinates does not affect the coordinate ring of a variety.

Case 1: Suppose $Ax + By$ and $Cx + Dy$ are linearly dependent. Then

$$f(x, y) = G(Ax + By)^2 + (Dx + Ey) + F.$$

Applying a linear change of coordinates gives

$$f(x, y) = x^2 + Ax + By + C.$$

Replacing $Ax + By + C$ by $-w$, since it depends only on the original y, we get

$$f(x, y) = x^2 - w.$$

This gives $A(W) \cong A(Y)$.

[1] Note that $k[x, \frac{1}{x}] \ncong k(x)$ so we cannot show that $A(W) \ncong k[z]$ for dimension reasons.

Case 2: Suppose $Ax+By$ and $Cx+Dy$ are linearly independent. Let $w = Ax+By$ and $z = Cx + Dy$ be the new variables to write and then factor:

$$f(w,z) = wz + Aw + Bz + C \implies f(w,z) = (w+A)(z+B) + C.$$

Another linear change of coordinates gives

$$f(w,z) = wz - 1.$$

Therefore, $A(W) \cong A(Z)$.

□

Hartshorne I.1.2. *The Twisted Cubic Curve.* Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \{(t,t^2,t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1 . Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t, y = t^2, z = t^3$.

PROOF. The generators for the ideal $I(Y)$ are $y-x^2, z-x^3$. That is, $I(Y) = (y-x^2, z-x^3)$. To see this, note that

$$Y = Z(y-x^2, z-x^3) = Z(\mathfrak{a}) \qquad \text{where} \qquad \mathfrak{a} = (y-x^2, z-x^3).$$

Indeed, elements of Y are of form $(x,y,z) = (t,t^2,t^3)$ which satisfy the equations inside the RHS. So $Y \subseteq Z(y-x^2, z-x^3)$. Meanwhile, if (x,y,z) is in the RHS, then $x^2 = y$, $x^3 = z$ and so $(x,y,z) = (x,x^2,x^3) \in Y$. We will show below $k[x,y,z]/\mathfrak{a} \cong k[x]$, but for now, this implies $\mathfrak{a}$ is a prime ideal. Then,

$$I(Y) = I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}.$$

Since $A(Y) \cong k[x,y,z]/(x^2-y, x^3-z) \cong k[x,x^2,x^3] \cong k[x]$, we get $A(Y) \cong k[x]$. It is clear Y has dimension 1 since $\dim A(Y) = 1$ and by Hartshorne Proposition I.1.7. □

Hartshorne I.1.3 Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

PROOF. We know $Y = Z(I(x^2-yz, xz-x))$. We wish to understand when $x^2 - yz = xz - x = 0$. If $x(z-1) = xz - x = 0$, then we have $x = 0$ or $z = 1$. We consider these two cases.

Case $x = 0$: If $x = 0$, we get $-yz = 0$. In this case, we need $y = 0$ or $z = 0$. Thus, when $x = 0$, we get $Z(x,y)$ or $Z(x,z)$ which are the lines $x = y = 0$ and $x = z = 0$ respectively.

Case $z = 1$: The equation becomes $x^2 - y = x - x = 0$. The equation $x - x$ is vacuous so if $z = 1$, then $x^2 - y = 0$. So, when $z = 1$, we get $Z(z-1, x^2-y)$ which is a surface in $\mathbb{A}^3$.

We claim $Y = Z(x,y) \cup Z(x,z) \cup Z(z-1, x^2-y)$ is the decomposition into irreducible components. It follows form our work above that $Y \subseteq Z(x,y) \cup Z(x,z) \cup Z(z-1, x^2-y)$. On the other hand, if P is in the RHS, then P fulfills both polynomials defining Y. To see that these are irreducible components, let $A := k[x,y,z]$ and observe that

$$A/I(x,y) = A/(x,y) \cong k[z]$$
$$A/I(x,z) = A/(x,z) \cong k[y]$$
$$A/I(z-1, x^2-y) = A/(z-1, x^2-y) \cong k[x, x^2] \cong k[x]$$

which are all integral domains. Application of the Nullstellensatz completes the proof. □

Hartshorne I.1.4. If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

PROOF. **Method 1:** It is known that X is Hausdorff iff the diagonal of $X \times X$ with the product topology is closed. The diagonal of $\mathbb{A}^2$ is $Z(x-y)$. However, $\mathbb{A}$ is not Hausdorff since the open sets are complements of sets of form $Z(x-a)$ for some $a \in k$. So, the diagonal of $\mathbb{A}^2$ with the product topology is not closed. However, $Z(x-y)$ is closed in the Zariski topology so the two topologies cannot coincide. □

PROOF. The proper algebraic sets of $\mathbb{A}$ are precisely the finite collection of points. So, the closed sets of $\mathbb{A} \times \mathbb{A}$ with the product topology are precisely the finite collection of pairs of points or $\mathbb{A} \times \{a_1, \ldots, a_n\}$. If we consider when the pairs are always equal i.e. (x, x) for $x \in k$, then this determines the algebraic subset $Z(x-y) \in \mathbb{A}^2$ in the product topology. However, the set of solutions to this is infinite. This leaves only $Z(x-y) = \mathbb{A} \times \{a_1, \ldots, a_n\}$. Picking $b \in k \setminus \{a_1, \ldots, a_n\}$, we find $(b, b) \in Z(x-y)$, but it is not in the latter set. So, the Zariski product topology does not coincide with the Zariski topology. □

Hartshorne I.1.5. Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

PROOF. ($\implies$): Since $k[x_1, \ldots, x_n]$ is finitely generated and Y algebraic, $A(Y)$ is finitely generated. We show that $A(Y)$ contains no nilpotent elements. For if $f \in A(Y)$ s.t. $f^r = 0$, we see that $f^r \in I(Y)$. Hence, by the Nullstellensatz $f \in I(Y)$ and we conclude that $f = 0$. ($\impliedby$) Since B is a finitely generated k-algebra, we can take B to be the quotient of $k[x_1, \ldots, x_n]$ by an ideal $\mathfrak{a}$. Then, $Y = Z(\mathfrak{a})$ is an algebraic set because B is reduced. Indeed, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ is a radical ideal and since our ring B is reduced, $\mathfrak{a}$ itself is radical and we apply Hartshorne Corollary I.1.4. □

Hartshorne I.1.6. Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\bar{Y}$ is also irreducible.

PROOF. Let $U \subseteq X$ be a nonempty open subset of an irreducible space. Since $X = \overline{U} \cup (X \setminus U)$, irreducibility and $U \neq \emptyset$ implies $\overline{U} = X$. So U is dense. If $U = Z_1 \cup Z_2$ were reducible, then $X = (Z_1 \cup Z_2) \cup (X \setminus U)$ is also reducible which is a contradiction.

Let $Y \subseteq X$ s.t. Y is irreducible in its induced topology. Suppose not and $\overline{Y} = Z_1 \cup Z_2$ is reducible. Then $Y = (Y \cap Z_1) \cup (Y \cap Z_2)$ which means WLOG that $Y = Y \cap Z_1$. Then $Y \subseteq Z_1$ and taking closures says $\overline{Y} \subseteq Z_1$ and hence, $\overline{Y} = Z_1$. Contradiction. □

Hartshorne I.1.7.

(a) Show that the following conditions are equivalent for a topological space X : (i) X is noetherian; (ii) every nonempty family of closed subsets has a minimal element; (iii) X satisfies the ascending chain condition for open subsets; (iv) every nonempty family of open subsets has a maximal element.

(b) A noetherian topological space is quasi-compact, i.e., every open cover has a finite subcover.

(c) Any subset of a noetherian topological space is noetherian in its induced topology.

(d) A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

PROOF. (a) **(i) $\implies$ (ii)**: Consider a family of closed subsets $\mathfrak{A} := \{Z_i : i \in I\}$. Suppose this family has no minimal element. Choose a $Z_{i_0} \in \mathfrak{A}$. There is no minimal element so there is a Z_{i_1} s.t. $Z_{i_0} \supsetneq Z_{i_1}$. Continuing inductively, we have a infinite descending chain of closed subsets

$$Z_{i_0} \supsetneq Z_{i_1} \supsetneq Z_{i_2} \supsetneq \dots$$

which contradicts X Noetherian.

(ii) $\implies$ (iii): An ascending chain of open subsets gives a descending chain of closed subsets by taking complements which has a minimal element by (ii) and so, every chain terminates. More precisely, let

$$U_1 \subsetneq U_2 \subsetneq U_3 \subsetneq$$

be an ascending chain of open subsets. Consider the family $\{X \setminus U_i : i \in \mathbb{N}\}$ of closed subsets. This has a minimal element $X \setminus U_j$ which implies

$$X \setminus U_j \subseteq X \setminus U_{j+1} \subseteq X \setminus U_{j+2} \subseteq \dots$$

and taking complements, $U_j \supseteq U_{j+1} \supseteq \dots$. But we already know $U_j \subseteq U_{j+1}$ so $U_j = U_{j+1} = U_{j+2} = \dots$ which means the original chain terminates.

(iii) $\implies$ (iv): Suppose not and there was no maximal element. Then we can obtain a sequence $U_1 \subsetneq U_2 \subsetneq U_3 \subsetneq \dots$ which never terminates thereby contradicting the ascending chain condition for open sets.

(iv) $\implies$ (i): If every nonempty family of open subsets has a maximal element, then certainly a descending chain of closed subsets has a minimal element by taking complements. So the chain terminates.

(b) To show that a noetherian topological space is quasicompact, fix a topological space X. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Let

$$\mathfrak{S} := \left\{ \bigcup_\alpha U_\alpha : U_\alpha \in \{U_\alpha\}_{\alpha \in A} \text{ and the union is finite} \right\}$$

and $\mathfrak{S}$ has a maximal element due to X noetherian and (a). The maximal element $\mathfrak{M}$ must cover X for if not, there is a U_β s.t. $U_\beta \cap \mathfrak{M} \neq \emptyset$ and $\mathfrak{M} \subsetneq \mathfrak{M} \cup U_\beta \in \mathfrak{S}$ contradicting maximality.

Alternately, we can index the cover and define a chain $U_1 \subseteq U_1 \cup U_2 \subseteq \dots \subseteq \bigcup_{i=1}^n U_i$. By the ascending chain condition on open sets (since we have the descending chain condition on closed sets), we can choose n such that $\bigcup_1^n U_i = \bigcup_1^{n+1} U_i$. So, $\bigcup_1^n U_i = \bigcup_\alpha U_\alpha \supseteq X$ as desired.

(c) Use the equivalences of (a). Let $\{U_i\}_{i \in \mathbb{N}} = \{V_i \cap U\}_{i \in \mathbb{N}}$ be an ascending chain of open subsets of U i.e. $V_i \cap U \subsetneq V_{i+1} \cap U$. Then $V_i \subsetneq V_{i+1}$ and then $\{V_i\}_{i \in I}$ is an ascending of open subsets of X which must terminate for some n. Then $U_n = U_m$ for all $m \geq n$ and (a) says that U must be noetherian.

(d) We can take open subsets U_x containing an element of $x \in X$ which is disjoint from other U_y's if $x \neq y$. Then, the chain $U_{x_1} \subseteq U_{x_1} \cup U_{x_2} \subseteq \dots$ must eventually terminate at say $\cup_{i=1}^n U_{x_i}$. Since each $x \in X$ is in a U_{x_i}, X must be finite.

Suppose not and there are infinitely many points. Choose $x_1, x_2 \in X$. Separate them $x_1 \in U_{x_1}$ and $x_2 \in U_{x_2}$. WLOG assume U_{x_2} has infinitely many points. Then choose $x_3 \in U_{x_2}$. Separate x_2 from x_3 using open sets V_{x_2} and V_{x_3} lying inside U_{x_2}. WLOG assume V_{x_3} has infinitely many points (otherwise switch the index for

x_2 and x_3). Replace U_{x_2} by V_{x_2}. Then choose a point $x_4 \in V_{x_3}$ and choose open sets W_{x_3} and W_{x_4} in V_{x_3} separating x_3 and x_4. WLOG assume W_{x_4} has infinitely many elements. Set $U_{x_3} = W_{x_3}$. Repeat this process inductively. Then, we have an ascending chain of open subsets

$$U_1 \subsetneq U_1 \cup U_2 \subsetneq U_1 \cup U_2 \cup U_3 \subsetneq$$

which does not terminate. Contradiction. So $|X| < \infty$. Since X is finite and Hausdorff, the topology is automatically discrete.

□

Hartshorne I.1.8. Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \nsubseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$. (See (7.1) for a generalization.)

PROOF. Fix $I(H) = (f)$ since it is a hypersurface and let $Y \cap H = \bigcup_1^m G_i$ be its decomposition into irreducible components. Let $\overline{f}$ be the image of f in $A(Y)$.

First, $\overline{f}$ is not a zerodivisor. This is because $Y \nsubseteq H$ and hence, $I(Y) \not\supseteq (f)$ and so, $f \notin I(Y)$. So, $\overline{f}$ is nonzero and we are done by $A(Y)$ be an integral domain. Now, the image $I(G_i)/I(Y)$ of $I(G_i)$, a prime ideal, is a prime ideal in $A(Y)$. Also, $\overline{f} \in I(G_i)$ and $I(G_i)$ is minimal over $\overline{f}$ by irreducibility of G_i.

Second, $\overline{f}$ is not a unit. For if it were a unit, then a minimal prime $I(G_i)$ containing $\overline{f}$ would be the unit ideal which is not prime. By Theorem 1.11A, height $I(G_i) = 1$. So, by Theorem 1.8A,

$$\begin{aligned}\dim A/I(G_i) &= \dim A/I(Y \cap G_i) = \dim(A/I(Y))/I(G_i) \\ &= \dim A(Y)/I(G_i) \\ &= \dim A(Y) - \text{height } I(G_i) = r-1.\end{aligned}$$

□

Remark 1. It is crucial that we require the base field to be algebraically closed. For instance, if we had worked in $\mathbb{R}^3$, two spheres touching at a point would be a counter example. See Hartshorne Remark 1.4.5.

Hartshorne I.1.9. Let $\mathfrak{a} \subseteq A = k[x_1, \ldots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geqslant n-r$.

PROOF 1. Let $\mathfrak{a} = (f_1, \ldots, f_r)$. We will do induction on r. Exercise 1.8 already gives us $\dim G \geq n-1$ for any irreducible component G of $Z(f_1)$.

Suppose the result is true for $r-1$ i.e. every irreducible component of $Z((f_1, \ldots, f_{r-1}))$ has dimension $\geq n-r+1$. Now, $Z((f_1, \ldots, f_{r-1})) \supseteq Z((f_1, \ldots, f_r))$.

An irreducible component G of $Z((f_1, \ldots, f_r))$ is contained in an irreducible component G' of $Z((f_1, \ldots, f_{r-1}))$. In this case, G is an irreducible component of $G' \cap Z(f_r)$ since G is contained in G' and in $Z(f_r)$. So, $\dim G \geq \dim G' \cap Z(f_r)$.

The image of f_r in $A(G')$ is neither a zero divisor nor a unit. It isn't a zero divisor since the image does not vanish on G'. It isn't a unit since $G' \cap Z(f_r) \neq \emptyset$. By Theorem 1.11A, a minimal prime $\mathfrak{p} \subseteq A(G')$ containing the image of f_r has height 1. Next,

$$\begin{aligned}&\dim A(G')/\mathfrak{p} + \text{height}\,\mathfrak{p} = \dim A(G') \\ \implies\quad &\dim A(G')/\mathfrak{p} = \dim A(G') - 1 \geq n-r+1-1 = n-r.\end{aligned}$$

Now we show $G' \cap Z(\mathfrak{p}) \subseteq G$. Suppose not and $G \subsetneq G' \cap Z(\mathfrak{p}) = Z(\mathfrak{p})$. Then $I(G) \supsetneq I(Z(\mathfrak{p}))$ in which Nullstellensatz implies $I(G) = \mathfrak{q} \supsetneq \mathfrak{p}$ where $G = Z(\mathfrak{q})$ and $I(G) = \mathfrak{q}$. Since G was an irreducible component, that means $\mathfrak{q}$ is a minimal prime ideal. Contradiction.

Hence, $G' \cap Z(\mathfrak{p}) \subseteq G$ which means $\dim A(G')/\mathfrak{p}) \leq \dim A(G)$. So,

$$n - r \leq \dim A(G')/\mathfrak{p} \leq \dim A(G).$$

□

PROOF 2. Let $\mathfrak{a} = (f_1, \ldots, f_r)$. Consider $A(Z(\mathfrak{a}))$ and $Z(\mathfrak{a}) = \bigcup_1^m G_i$ the decomposition into irreducible components. By Theorem 1.8A, we have

$$\dim(A/I(G_i)) = \dim A - \text{height } I(G_i).$$

It suffices to show that height $I(G_i) \leq r$. Since $G_i \subseteq Z(\mathfrak{a})$ and $\mathfrak{a}$ is generated by r elements, the height of $I(G_i)$ is $\leq r$ by the generalized Krull's Hauptidealsatz (see Theorem 10.2[2] of Eisenbud). Indeed, $I(G_i)$ is minimal among the primes that contain the elements $f_1, \ldots, f_r$ and so height $I(G_i) \leq r$. Therefore,

$$\dim G_i = \dim(A/I(G_i)) \geq \dim A - r = n - r.$$

□

Hartshorne I.1.10.

(a) If Y is any subset of a topological space X, then $\dim Y \leqslant \dim X$.
(b) If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$
(c) Give an example of a topological space X and a dense open subset U with $\dim U < \dim X$
(d) If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.
(e) Give an example of a noetherian topological space of infinite dimension.

PROOF. **(a)** Consider a chain of irreducible closed subsets of Y with $m := \dim Y$

$$Z_0 \subsetneq Z_1 \subsetneq Z_2 \subseteq \cdots \subsetneq Z_m.$$

Taking the closure of these sets gives $\overline{Z_0} \subsetneq \overline{Z_1} \subsetneq \cdots \subsetneq \overline{Z_m}$ which is a chain of distinct irreducible subsets of X by Hartshorne I.1.6. Thus, $\dim X \geq \dim Y$ as desired.
(b)[3] Let X be a topological space X covered by a family of open subsets $\{U_i\}$. By (a), it is clear $\dim X \geq \sup \dim U_i$. So, we show the other inequality.

Let $Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \ldots$ be longest chain of irreducible closed subsets obtainable in X (and it may potentially be infinite). Choose U in the cover s.t. $U \cap Z_0 \neq \emptyset$. We claim that $U \cap Z_n \subsetneq U \cap Z_{n+1}$ for all n. Suppose not. Then $Z_{n+1} \setminus Z_n$ is nonempty and open in Z_{n+1}. Furthermore, $Z_{n+1} \cap U = Z_n \cap U$ means that $Z_n \cap U$ is open in Z_{n+1}. From Hartshorne II.1.6, $Z_n \cap U$ and $Z_{n+1} \setminus Z_n$ are open and dense in Z_{n+1} which means

$$(Z_n \cap U) \cap (Z_{n+1} \setminus Z_n) \neq \emptyset \iff (Z_n \cap Z_{n+1} \setminus Z_n) \cap U \neq \emptyset.$$

[2] "Theorem 10.2 (Final version of the Principal Ideal Theorem). If $x_1, \ldots, x_c \in R$, and P is minimal among primes of R containing $x_1, \ldots, x_c$ then $\operatorname{codim} P \leq c$." Note that codim is Eisenbud's notation for the height.
[3] Wenbo's comment: "1.1.10 (b) will also need to show that $Z_i \cap U$ is irreducible in U: if $Z_i \cap U = A \cup B$, then $Z_i = \bar{A} \cup \bar{B} \cup (Z_i \setminus U)$, where closure is taken in X. then say $Z_i = \bar{A}$, $A = \bar{A} \cap U = Z_i \cap U$. Thus $Z_i \cap U$ is irreducible." which I am unsure of.

But this is absurd since $Z_n \cap Z_{n+1} \setminus Z_n = \emptyset$. Contradiction. Therefore $\dim U \geq \dim X$ and that means $\sup_i \dim U_i \geq \dim X$.
(c) We give an example of a topological space X and a dense open subset U with $\dim U < \dim X$. Let $X = \{u, v\}$ and take the open sets as $\emptyset, \{u\}, X$. This clearly defines a topology. Set $U := \{u\}$. Then the smallest closed set that contains U is going to be X and so, $\overline{U} = X$. On the other hand, $\dim U = -\infty$ since U contains no irreducible closed subsets[4]. Meanwhile, $\dim X = 1$ since the longest chain of of irreducible closed subsets is $\{v\}$.
(d) Suppose $Y_0 \subset \cdots \subset Y_n$ is a chain of closed irreducible subsets of Y such that $Y \neq X$. Add Y to the end of this chain to deduce $\dim(X) \geq \dim(Y) + 1$. So, either $\dim(X) = \infty$ or $\dim(X) > \dim(Y)$. Since neither are possible, we have a contradiction,
(e) Take $\mathbb{N}$ and define the closed subsets to be sets of the form $\{1, \ldots, n\}$. This defines a topology on $\mathbb{N}$ and the descending chain condition. However, $\dim \mathbb{N} = \infty$ because we have the chain $\{1\} \subseteq \{1, 2\} \subseteq \{1, 2, 3\} \ldots$ of irreducible closed and nonempty subsets of $\mathbb{N}$. □

Hartshorne I.1.11 Let $Y \subseteq \mathbb{A}^3$ be the curve given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say Y is not a local complete intersection-cf. (Ex. 2.17).

Proof. The fact that $I(Y)$ is a prime ideal of height 2 can be understood geometrically. By definition, Y is the image of the map $t \to (t^3, t^4, t^5)$ from $\mathbb{A}^1$ and hence, $\dim(Y) = 1$. But then $\dim(A(Y) = 1$ which means that $\text{height}(I(Y)) = 2$ if we use Hartshorne Theorem I.1.8A (noting that Y is an irreducible algebraic set).

We show that $I(Y)$ cannot be generated by 2 elements. First off, any element $f \in I(Y)$ cannot have a term with x, x^2, y, or z by considering the generators of $I(Y)$. The ideal $I(Y)$ can be generated by $y^2 - xz, x^3 - yz, z^2 - xy$.

Now if $\langle f, g \rangle = I(Y)$, then we can write

$$af + bg = x^3 - yz, \qquad cf + dg = y^2 - xz, \qquad ef + hg = z^2 - xy.$$

Then, we can write

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} x_f & y_f & z_f \\ x_g & y_g & z_g \end{pmatrix} \begin{pmatrix} x^3 \\ y^2 \\ z^2 \end{pmatrix} + \text{other terms.}$$

Set $a_0 := a(0, 0, 0)$ and do the same for b, c, d, e, h. Here, the x_f, y_f, z_f are constants (and the same with subscripts g). Then

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \\ e_0 & h_0 \end{pmatrix} \begin{pmatrix} x_f & y_f & z_f \\ x_g & y_g & z_g \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

The which is absurd (view this as a composition of maps from a 3-dimensional vector space to a 2-dimensional vector space and then to a 3-dimensional vector space which cannot be the identity). The computation of the above matrices is due to the fact that f and g have no terms purely in x, x^2, y, z. □

Hartshorne I.1.12 Give an example of an irreducible polynomial $f \in \mathbb{R}[x, y]$, whose zero set $Z(f)$ in $\mathbb{A}^2_{\mathbb{R}}$ is not irreducible (cf. 1.4.2).

[4]Note that the supremum of the emptyset is $-\infty$.

PROOF. Take $f(x,y) = (x^2-1)^2 + (y^2-1)^2$. This is irreducible, but its vanishing set has more than one component. In fact, its zero set has four points $(\pm 1, \pm 1)$. □

2. Projective Varieties

Hartshorne I.2.1 Prove the "homogeneous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$. [Hint: Interpret the problem in terms of the affine $(n+1)$-space whose affine coordinate ring is S, and use the usual Nullstellensatz, (1.3A).]

PROOF. Write $W(\mathfrak{a})$ when we are referring to the zero set of $\mathfrak{a}$ in $\mathbb{P}^n$. Then the zero set in affine space is

$$Z(\mathfrak{a}) = \{a = (a_0, \ldots, a_{n+1}) : \forall g \in \mathfrak{a}, g(a) = 0\}.$$

This set in particular contains the origin since $\deg f > 0$ and f is homogeneous. Also, this set contains all linear multiples of elements $[a_0 : a_1 : \cdots : a_n] \in W(\mathfrak{a})$. The condition that $f(P) = 0$ for all $P \in W(\mathfrak{a})$ is equivalent to saying $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ provided f is homogeneous. Indeed, if f is homogeneous of degree d,

$$f([a_0 : \cdots : a_n]) = 0 \iff \forall \lambda \in k^\times, f((\lambda a_0, \ldots, \lambda a_n)) = 0$$
$$\iff \lambda^d (f(a_0, \ldots, a_n)) = 0 \iff f((a_0, \ldots, a_n)) = 0.$$

By the usual Nullstellensatz, there is a $q > 0$ s.t. $f^q \in \mathfrak{a}$. □

Hartshorne I.2.2 For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(i) $Z(\mathfrak{a}) = \varnothing$ (the empty set);
(ii) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;
(iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

PROOF. (i) $\implies$ (ii): We know $Z(\mathfrak{a}) = \emptyset$ iff for all $x \in \mathbb{P}^n$, there is an $f \in \mathfrak{a}$ s.t. $f(x) \neq 0$. There are two cases.

Suppose $\mathfrak{a}$ has a unit scalar. Then $\mathfrak{a} = S$ and we can always take this unit scalar as our f.

Suppose $\mathfrak{a}$ has no units. Then $\sqrt{\mathfrak{a}} \subsetneq S$. Consider the polynomial x_i which has degree 1. Then $x_i = 0$ on all of $Z(\mathfrak{a})$ by properties of the empty set. By Hartshorne I.2.1, $x_i^{q_i} \in \mathfrak{a}$ for some q_i. This implies $x_i \in \sqrt{\mathfrak{a}}$. As i was arbitrary, $\sqrt{\mathfrak{a}} \supseteq S_+$. Now show $S_+ \subseteq \sqrt{\mathfrak{a}}$. Suppose not and there is an $a \in \sqrt{\mathfrak{a}}$ and $a \in S \setminus S_+$. This means a is a scalar. But then $a^d \in \mathfrak{a}$ which is also a scalar. Contradiction. This means the other containment is true and $S_+ = \sqrt{\mathfrak{a}}$.
(ii) $\implies$ (iii): If $\sqrt{\mathfrak{a}} = S$, we know $\sqrt{\mathfrak{a}} \supseteq S_+$ so it suffices to prove the result in the latter case. If $\sqrt{\mathfrak{a}} = S_+$, each of x_i has a power $d_i > 0$ in which $x_i^{d_i} \in \mathfrak{a}$. Take $d > \sum_{i=0}^n d_i$. Then any monomial $x_0^{l_0} \ldots x_n^{l_n}$ in S_d must have $\sum_{i=0}^n l_n = d > \sum_{i=0}^n d_i$. So at least some e_j satisfies $l_j \geq d_j$. Therefore, the monomial is in $\mathfrak{a}$.
(iii) $\implies$ (ii): It is clear $Z(\mathfrak{a}) \subseteq Z(S_d)$. But we can consider the polynomials $x_0^d, x_1^d, \ldots, x_n^d$ so any point x vanishing at each of these polynomials will have 0 in each of its coordinates. Thus, $x = 0$ and $x \notin \mathbb{P}^n$. Therefore, $Z(S_d) = \emptyset$. So, $Z(\mathfrak{a}) = \emptyset$. □

Hartshorne I.2.3 (a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.
(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.
(c) For any two subsets Y_1, Y_2 of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \varnothing$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
(e) For any subset $Y \subseteq \mathbb{P}^n$, $Z(I(Y)) = \bar{Y}$.

PROOF. (a) Assume $T_1 \subseteq T_2$ are subsets of S^h.[5] If $P \in Z(T_2)$, then certainly $P \in Z(T_1)$.
(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $f \in S$ homogeneous and zero on Y_2 implies that it is zero on Y_1. So, $I(Y_2) \subseteq I(Y_1)$.
(c) Omitted. Proving the two inclusions is straightforward.
(d) The proof follows the same approach as Hartshorne Proposition I.1.2.
(e) The proof follows the same apparoch as Hartshorne Proposition I.1.2.

□

Hartshorne I.2.4 (a) There is a 1-1 inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$, and homogeneous radical ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Note: Since S_+does not occur in this correspondence, it is sometimes called the irrelevant maximal ideal of S.
(b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.
(c) Show that $\mathbb{P}^n$ itself is irreducible.

PROOF. (a) This is immediate by combining Hartshorne I.2.3(d) and Hartshorne I.2.2.
(b) Suppose $I(Y)$ prime. Then, suppose $Z(I(Y)) = Y_1 \cup Y_2$. So, $I(Z(I(Y))) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2) = I(Y)$. However, $I(Y)$ is prime so either $I(Y) = I(Y_1)$ or $I(Y) = I(Y_2)$. So, $Z(I(Y)) = Y_1$ or $Z(I(Y)) = Y_2$. So, Y is irreducible.
(c) This is immediate from (b) as $I(\mathbb{P}^n) = (0)$.

□

Hartshorne I.2.5 (a) $\mathbb{P}^n$ is a noetherian topological space.
(b) Every algebraic set in $\mathbb{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its irreducible components.

PROOF. **(a)** We know $\mathbb{P}^n$ is noetherian since it is a union of finitely many Noetherian topological spaces. In particular, the open sets $U_i \cong \mathbb{A}^n_k$ from $i = 0, 1, \ldots, n$ cover $\mathbb{P}^n$ and are all noetherian.
(b) Since (a) holds true, apply Hartshorne Proposition I.1.5. □

Hartshorne I.2.6 If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [Hint: Let $\varphi_i : U_i \to \mathbb{A}^n$ be the homeomorphism of (2.2), let Y_i be the affine variety $\varphi_i(Y \cap U_i)$, and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)\left[x_i, x_i^{-1}\right]$. Now use (1.7), (1.8A), and (Ex 1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is nonempty.]

PROOF. The proof makes use of Hartshorne's hint. Choose Y_i nonempty. First, we show that $A(Y_i)$ can be identified with $(S(Y)_{x_i})_0$ where we write $(\cdot)_0$ to indicate the degree zero terms of the graded ring inside. For this proof, we write $A = k[y_1, \ldots, y_n]$ and $S = k[x_0, \ldots, x_n]$.

Observe $A(Y_i) \cong A/I(Y_i)$ consists of polynomials f that vanish on Y but not where $x_i = 0$. From the homeomorphism φ_i, we know that $f \in I(Y_i)$ iff

$$f\left(\frac{a_0}{a_i}, \ldots, \frac{\widehat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i}\right) = 0$$

[5] S^h denotes the set of homogeneous polynomials of S.

where $\left(\frac{a_0}{a_i}, \ldots, \widehat{\frac{a_i}{a_i}}, \ldots, \frac{a_n}{a_i}\right) \in Y_i$. Here, the hat indicates that entry is omitted. But $f = f(y_1, \ldots, y_n) \in A$ can be rewritten to

$$f(y_1, \ldots, y_n) = f\left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right) = \frac{P(x)}{x_i^{\deg(P(x))}}$$

for $P(x) \in k[x_0, \ldots, x_n]$. In particular, an arbitrary element $g(y_1, \ldots, y_n) \in A$ corresponds formally to an element $g\left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right) \in (S_{x_i})_0$. Furthermore, $(S(Y)_{x_i})_0 = (S_{x_i})_0/(I(Y))_0$ since taking the degree zero elements commutes with quotients[6]. So, A corresponds to $(S_{x_i})_0$ and quotienting by $I(Y_i)$ is the same as quotienting out by $(I(Y))_0$. Then $A(Y_i) \cong (S(Y)_{x_i}))_0 = (S_{x_i})_0/(I(Y))_0 = (S(Y)_{x_i})_0$.

Next, we show $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Define a map $S_{x_i} \to A(Y_i)[x_i, x_i^{-1}]$ by

$$x_0 \mapsto y_1 \cdot x_i \qquad x_1 \mapsto y_2 \cdot x_i \qquad \ldots \qquad \frac{1}{x_i} \mapsto x_i^{-1} \qquad \ldots \qquad x_n \mapsto y_n \cdot x_n.$$

This map is clearly surjective so we determine the kernel. The kernel consists of $\frac{f(x_0,\ldots,x_n)}{x_i^d}$ s.t. $\frac{f(y_1x_i,\ldots,y_nx_i}{x_i^d} = 0$. But this is zero iff $f(y_1x_i, \ldots, y_nx_i) = 0$ and this is iff $f(y_1x_i, \ldots, y_nx_i)$ vanishes on Y_i. The points of Y_i have form $\left(\frac{a_0}{a_i}, \ldots, \frac{a_n}{a_i}\right)$ where $[a_0 : a_1 : \cdots : a_n] \in Y$ so[7] this is iff $f(a_0, a_1, \ldots, a_n) = 0$. This last statement is iff $f \in I(Y)$ and so iff $\frac{f(x_0,\ldots,x_n)}{x_i^d} \in I(Y)_{x_i}$. So when $Y_i \neq \emptyset$,

$$(S(Y)_{x_i}) \cong S_{x_i}/I(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}].$$

Now, use 1.7, 1.8A, and Ex 1.10 as hinted by Hartshorne. We get

$$\begin{aligned}
\dim S(Y) &= \text{tr. deg } K(S(Y)) && \text{(Hartshorne Proposition I.1.8A)}\\
&= \text{tr. deg } K(S(Y)_{x_i}) \text{(Obvious)} &&\\
&= \text{tr. deg } K(A(Y_i)[x_i, x_i^{-1}]) && \text{(From above)}\\
&= \text{tr. deg } K(S(Y)_{x_i})_0[x_i, x_i^{-1}] && \text{(From above)}\\
&= 1 + \text{tr. deg } K(S(Y)_{x_i})_0[x_i^{-1}] && (x_i \text{ is independent})\\
&= 1 + \text{tr. deg } K(S(Y)_{x_i})_0) &&\\
&= 1 + \dim A(Y_i) && \text{(Hartshorne Proposition I.1.8A)}
\end{aligned}$$

Since $\dim S(Y) = \dim S - \text{height } I(Y)$, we know

$$\begin{aligned}
&\dim S - \text{height } I(Y) = 1 + \dim A(Y_i)\\
\iff\ &n + 1 - \text{height } I(Y) = 1 + \dim A(Y_i)\\
\iff\ &n - \text{height } I(Y) = \dim(Y_i).
\end{aligned}$$

The final LHS equals $\dim A(Y)$ by Theorem 1.7. So, by Hartshorne I.1.10,

$$\dim Y = \sup_{j=1,\ldots,n} \dim Y_j = \dim Y_i.$$

Putting this together with our work above, $\dim S(Y) = 1 + \dim Y$. □

[6]Follows from the theory of graded rings.

[7]We may assume WLOG that $a_i = 1$ via the equivalence class in projective space.

Hartshorne I.2.7

(a) $\dim \mathbb{P}^n = n$

(b) If $Y \subseteq \mathbb{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \bar{Y}$. [Hint: Use (Ex. 2.6) to reduce to (1.10).]

PROOF. (a) We have

$$\dim \mathbb{P}^n = \dim S(\mathbb{P}^n) - 1 = \dim k[x_0, \ldots, x_n] - 1 = n + 1 - 1 = n.$$

This is assuming $I(\mathbb{P}^n) = 0$ and this is not hard to see. Indeed, for f to vanish on $\mathbb{P}^n$ means f must vanish on points $[0 : \cdots : 1 : \cdots : 0]$ that are zero except in the ith coordinate. This implies f must be homogeneous with each variable appearing at least once in each of the term in f. However, such a polynomial would not vanish at $[1 : \cdots : 1]$. Thus, the only polynomial in $I(\mathbb{P}^n)$ the zero polynomial.

(b) Let Y be a quasi-projective variety. We may cover Y by a family of open sets $Y \cap U_i$. Let $Y_i := Y \cap U_i$ using $\varphi_i^{-1}(\mathbb{A}^n)$. We know that this is a family of open subsets since Y is the intersection of a projective variety and an open set.

Next, by Hartshorne I.1.10c, we know that $\dim Y = \sup \dim Y_i$. Also by Hartshorne I.1.10c, we have $\overline{Y} = \sup \dim(\overline{Y} \cap U_i)$

Reduce to the affine case since one can view Y_i as a quasi-affine variety by Hartshorne Corollary I.2.3. We have that $\dim Y_i = \dim \overline{Y_i}$ where the closure is taken in affine n-space.

Next, we claim that $\dim \overline{Y_i} = \dim \overline{Y} \cap U_i$. It suffices to show $\dim \overline{Y \cap U_i} = \dim \overline{Y} \cap U_i$. Certainly, we have $\overline{Y} \cap U_i \subseteq \overline{Y \cap U_i}$ because U_i is open. Since $\overline{Y \cap U_i} \subseteq \overline{Y} \cap \overline{U_i}$, Hartshorne I.1.10a gives

$$\dim \overline{Y} \cap U_i \leq \dim \overline{Y \cap U_i} \leq \dim \overline{Y} \cap \overline{U_i} \leq \dim \overline{Y}. \tag{1}$$

Now $\overline{Y} \cap U_i$ is an open subset of $\overline{Y}$ when the intersection is nonempty. Therefore, $\dim \overline{Y} \cap U_i = \dim \overline{Y}$ by our work in Hartshorne I.2.6. So we have equalities in 1. In particular, $\dim \overline{Y} \cap U_i = \dim \overline{Y \cap U_i} = \dim \overline{Y_i}$. At last, we have a chain of equalities from what we have done

$$\dim Y = \sup \dim Y_i = \sup \dim \overline{Y_i} = \sup \dim \overline{Y \cap U_i} = \sup \dim \overline{Y} \cap U_i = \dim \overline{Y}.$$

□

Hartshorne I.2.8 A projective variety $Y \subseteq \mathbb{P}^n$ has dimension $n - 1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a hypersurface in $\mathbb{P}^n$

PROOF. ($\implies$): Assume $\dim Y = n - 1$. We have

$$\dim Y = \dim S(Y) - 1 = \dim S - \text{height } I(Y) - 1 = n - \text{height } I(Y).$$

So, height $I(Y) = 1$. From Hartshorne Proposition I.12A, we deduce it is principal $I(Y) = (f)$. Primality ensures f is irreducible and $f \notin k^*$. Also $f \neq 0$ because $\dim Y = n - 1$. Homogeneity of $I(Y)$ ensures f is a linear combination of homogeneous elements.

($\impliedby$): Given f an irreducible homogeneous polynomial. By (1.11A), we get

$$\dim Y = \dim S(Y) - 1 = \dim S - \text{height } I(Y) - 1 = n + 1 - 1 - 1 = n - 1$$

as desired. We note that the hypothesis of Hartshorne Theorem 1.11A are fulfilled since f irreducible and homogeneous with S an integral domain. The ideal generated by f is certainly minimal over f and prime. □

Hartshorne I.2.9 *Projective Closure of an Affine Variety.* If $Y \subseteq \mathbb{A}^n$ is an affine variety, we identify $\mathbb{A}^n$ with an open set $U_0 \subseteq \mathbb{P}^n$ by the homeomorphism φ_0. Then we can speak of $\bar{Y}$, the closure of Y in $\mathbb{P}^n$, which is called the projective closure of Y. (a) Show that $I(\bar{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2). (b) Let $Y \subseteq \mathbb{A}^3$ be the twisted cubic of (Ex. 1.2). Its projective closure $\bar{Y} \subseteq \mathbb{P}^3$ is called the twisted cubic curve in $\mathbb{P}^3$. Find generators for $I(Y)$ and $I(\bar{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do not necessarily generate $I(\bar{Y})$.

PROOF. (a) We prove the two inclusions.

We show $I(\overline{Y}) \subseteq \beta(I(Y))$ first. An element of $I(\overline{Y})$ is a homogeneous polynomial f which vanishes on $\overline{Y}$. Since it vanishes on the projective closure of Y, we know that f has the to be the homogenization of some polynomial vanishing on Y. This is because we can take $f = g(1, x_1, \ldots, x_n)$. Hence, $f = \beta(g)$. This implies $f \in \beta(I(Y))$.

Suppose we have $g \in \beta(I(Y))$. Then, g is the homogenization of some polynomial $f \in I(Y)$. So, g certainly vanishes in Y. Also, g vanishes on $\overline{Y}$ since g is homogeneous. So, $g \in I(\overline{Y})$ as desired.

(b) Let Y be the twisted cubic as in (Ex. 1.2). The generators for $I(Y)$ are $y - x^2$ and $z - y^3$. So, the generators of $I(\overline{Y})$ are

$$\beta(y - x^2) = w^2\left(\frac{y}{w} - \left(\frac{x}{w}\right)^2\right) = yw - x^2$$

and

$$\beta(z - x^3) = w^3\left(\frac{z}{w} - \left(\frac{x}{w}\right)^3\right) = zw^2 - x^3.$$

It can be seen that $xz - y^2 \in I(\overline{Y})$ since it is in $I(Y)$ and vanishes at zero, homogeneous, and hence, it is in $I(\overline{Y})$. However, $xz - y^2$ is not in $(yw - x^2, zw^2 - x^3)$. □

Hartshorne I.2.10 *The Cone Over a Projective Variety* (Figure 1). Let $Y \subseteq \mathbb{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbb{A}^{n+1} - \{(0, \ldots, 0)\} \to \mathbb{P}^n$ be the map which sends the point with affine coordinates $(a_0, \ldots, a_n)$ to the point with homogeneous coordinates $(a_0, \ldots, a_n)$. We define the affine cone over Y to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \ldots, 0)\}.$$

(a) Show that $C(Y)$ is an algebraic set in $\mathbb{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \ldots, x_n]$

(b) $C(Y)$ is irreducible if and only if Y is.

(c) $\dim C(Y) = \dim Y + 1$ Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbb{P}^{n+1}$. This is called the projective cone over Y.

PROOF. (a) One knows $C(Y)$ is an algebraic set because $C(Y) = Z(T) \subseteq \mathbb{A}^{n+1}$ where $Y = Z(T) \subseteq \mathbb{P}^n$ (we assume that T does not contain a constant polynomial, otherwise this is trivial). The proof of $C(Y) = Z(T)$ is straightforward.

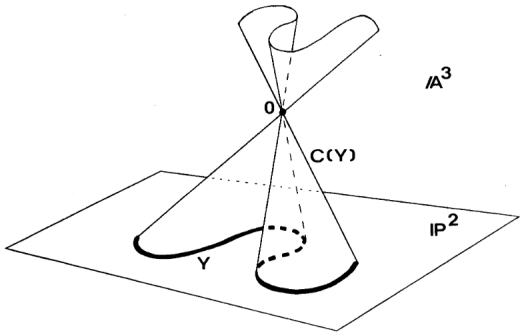


FIGURE 1. The cone over a curve Y in $\mathbb{P}^2$. The cone is "embedded" inside $\mathbb{A}^3$.

A homogeneous polynomial $f \in I(Y)$ will also vanish on $C(Y)$ because $f(\not\vdash) = 0$ and f vanishes on all linear multiples of $(a_0, \ldots, a_n) \in Y$ viewed as affine coordinates. As elements of $I(Y)$ are linear combinations of such homogeneous polynomials, $I(Y) \subseteq I(C(Y))$. On the other hand, an element $f \in I(C(Y))$ must vanish at $\not\vdash$ which means f has no constant term. Furthermore, if $a = (a_0, \ldots, a_n) \in C(Y)$, then any linear multiple of a is in $C(Y)$. But $f(\lambda a) = 0$ for all $\lambda \in k^*$ iff f is homogeneous. Therefore, $f \in I(Y)$.

One has $I(Y) \subseteq I(C(Y))$ by the above discussion. For the other inclusion, let $f \in I(C(Y))$. Then, f is a homogeneous polynomial and it vanishes on Y when viewed as an algebraic set in $\mathbb{P}^n$. But that means $f \in I(Y)$ viewed as an ideal of S. Thus, $f \in I(Y)$ viewed as an ideal in $k[x_0, \ldots, x_n]$.

(b) By (a), we know that the ideal of $C(Y)$ is the same as the ideal $I(Y)$ in $k[x_0, \ldots, x_n]$. So $C(Y)$ is irreducible iff $I(C(Y))$ is prime iff $I(Y)$ is prime iff Y is irreducible.

(c) Note $\dim C(Y) = \dim A(C(Y)) = \dim A - \text{height}\, I(C(Y))$. Then $\dim C(Y) = n + 1 - \text{height}\, I(C(Y))$ since $\dim A = n + 1$. So,

$$\dim C(Y) = \dim A - \text{height}\, I(C(Y)) = \dim A - \text{height}\, I(Y) = n+1-\text{height}\, I(Y) = \dim Y + 1$$

because the ideal $I(C(Y))$ is equal to the ideal $I(Y)$ viewed as an ideal in A.

□

Hartshorne I.2.11 *Linear Varieties in* $\mathbb{P}^n$. A hypersurface defined by a linear polynomial is called a hyperplane.
(a) Show that the following two conditions are equivalent for a variety Y in $\mathbb{P}^n$:
(i) $I(Y)$ can be generated by linear polynomials.
(ii) Y can be written as an intersection of hyperplanes.
In this case we say that Y is a *linear variety* in $\mathbb{P}^n$.
(b) If Y is a linear variety of dimension r in $\mathbb{P}^n$, show that $I(Y)$ is minimally generated by $n - r$ linear polynomials.
(c) Let Y, Z be linear varieties in $\mathbb{P}^n$, with $\dim Y = r, \dim Z = s$. If $r + s - n \geqslant 0$, then $Y \cap Z \neq \varnothing$. Furthermore, if $Y \cap Z \neq \varnothing$, then $Y \cap Z$ is a linear variety of dimension $\geqslant r+s-n$. (Think of $\mathbb{A}^{n+1}$ as a vector space over k, and work with its subspaces.)

PROOF. (a) The two directions are obvious since Y is a variety in $\mathbb{P}^n$ and hyperplanes are cut out by equations of the form

$$a_0X_0 + a_1X_1 + \cdots + a_nX_n = 0$$

for $a_i \in k$ and $X_0, \ldots, X_n$ the coordinates of $\mathbb{P}^n$.

(b) We do reverse induction on r starting with $r = n$. In this case, $I(Y)$ is minimally generated by zero linear polynomials i.e. $I(Y) = (1)$.

Now assume the result for $r+1 \leq n$ and we prove the result for r. Let Y be a linear variety of dimension r and $Y = \bigcap_{i=1}^{l} H_i$ is an intersection of hyperplanes. We may assume that this intersection is irredundant so that removing any hyperplane yields a linear variety Y' which is of dimension $r+1$. In that case, $I(Y')$ is minimally generated by $n-r-1$ linear polynomials which we use to cut out $n-r-1$ hyperplanes. Then the intersection of these $n-r-1$ hyperplanes with the hyperplane we removed gives a set of $n-r$ linear polynomials which generate $I(Y)$.

Essentially, the argument is that we can remove a hyperplane, use induction, and then augment our choices of hyperplanes.

We have then shown that $I(Y)$ is can be generated by $n-r$ linear polynomials. This has to be the minimum number of generate because a single linear polynomial cuts out a codimension 1 surface. Then intersecting with another hyperplane increases the codimension by ≤ 1 by considering the projective cone in Hartshorne I.2.10 and using Hartshorne I.1.8. Therefore, $n-r$ linear polynomials gives a variety whose codimension is $\geq n-r$. So we cannot use less than $n-r$ linear polynomials to cut out Y.

(c) By the assumptions, $I(Y)$ can be generated by $n-r$ polynomials and $I(Z)$ by $n-s$ polynomials. Look at $Y \cap Z$ and consider their affine cone sitting $\mathbb{A}^{n+1}$. Their affine cone is cut out by $(n-r)+(n-s) = 2n-r-s \leq n$ linear equations. This gives a homogeneous underdetermined system of equations which always has non-trivial solutions. Hence, $Y \cap Z \neq \emptyset$.

Assume $Y \cap Z \neq \emptyset$. Clearly $Y \cap Z$ is a linear variety since it is cut out by the linear polynomials cutting out Y and Z. So we can at least say that $Y \cap Z$ is cut out by $2n-r-s$ equations. That means

$$\dim(Y \cap Z) \geq n - (2n-r-s) = -n+r+s = r+s-n$$

since adding a new equation can at worst drop the dimension by 1.

□

Remark 2. For a scheme theoretic proof of a more general statement implying (c), see [10, Tag 0B2N]. [**Reread**]

Hartshorne I.2.12 *The d-Uple Embedding.* For given $n, d > 0$, let $M_0, M_1, \ldots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ by sending the point $P = (a_0, \ldots, a_n)$ to the point $\rho_d(P) = (M_0(a), \ldots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the *d-uple embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n = 1, d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic.

(a) Let $\theta : k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then a is a homogeneous prime ideal, and so $Z(a)$ is a projective variety in $\mathbb{P}^N$.

(b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

(c) Now show that ρ_d is a homeomorphism of $\mathbb{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

(d) Show that the twisted cubic curve in $\mathbb{P}^3$ (Ex. 2.9) is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

PROOF. (a) If $f \in k[y_0, \dots, y_n]$, let $f = f_1 + \cdots + f_m$ be the degree i components. If $\theta(f) = 0$, then $\theta(f_i) = 0$ for each i. So, $\mathfrak{a}$ is a homogeneous ideal. It is prime since $k[x_0, \dots, x_n]$ is an integral domain and the pullback of a prime ideal is always a prime ideal. Thus, $Z(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

(b) The inclusion $\mathrm{im}(\rho_d) \subseteq Z(\mathfrak{a})$ is obvious. See Bercherd's solution for the nontrivial direction.

(c) See Borcherd's solution.

(d) This is clear; an element of $\mathbb{P}^1$ is of the form $(x_0 : x_1)$. The 3-uple embedding maps such an element to $(x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$.

□

Remark 3. The d-uple embedding in this exercise is more commonly known as the Veronese map. See [8] or [1].

The idea here is to consider the space of all monomials of degree d in $x_0, \dots, x_n$ variables. This vector space has $\binom{n}{d}$. many elements. In projective space, we remove the possibility of 0 so we get $\binom{n}{d} - 1$.

For another example, consider the 2-uple embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$. This is just defined by

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2).$$

Some interesting situations occur. If $n = 1$ and $d = 1$, we have the identity map on $\mathbb{P}^1$. If $n = 1$ and $d = 2$, we get $(x_0 : x_1) \mapsto (x_0^2 : x_0x_1 : x_1^2)$ which is a quadratic. If $n = 1$ and $d = 3$, we get the twisted cubic.

Question: What occurs if $n = 2$ and $d = 3$? Hartshorne I.2.13 will explore the case of $n = 2$ and $d = 3$ which is known as the Veronese surface.

Hartshorne I.2.13 Let Y be the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. This is the *Veronese surface*. If $Z \subseteq Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbb{P}^5$ such that $V \cap Y = Z$.

PROOF 1. The know that the map is defined by

$$[x : y : z] \to [x^2 : y^2 : z^2 : xy : xz : yz].$$

Any curve Z in $\mathbb{P}^2$ is defined by a homogeneous form $F(x, y, z) = 0$ and so

$$F(x, y, z)^2 = G(x^2, y^2, z^2, xy, xz, yz).$$

Then some factor H of G defines a hypersurface in $\mathbb{P}^5$ and its vanishing corresponds to the vanishing of $F(x, y, z)$. So $Z(H) \cap Y \supseteq Z$. Assuming we knew the 2-uple embedding gives an isomorphism onto its image, we get equality. □

PROOF 2. Let $Z \subseteq Y$ be the curve. Since it is a closed curve of Y, Z corresponds to a prime ideal $\mathfrak{p}$ of the homogeneous coordinate ring $S(Y)$. Since it's a curve, $S(Y)/\mathfrak{p}$ is dimension 1. So, choose an irreducible element of $\mathfrak{p}$ (which is also a nonunit) and call it f. Lift f to $k[x_0, \dots, x_5]$. Here, f is still irreducible. So f cuts out a hypersurface V in $\mathbb{P}^5$. But then $V \cap Y$ is cut out by the equations of f and the equations of Y. Since f was contained in $\mathfrak{p}$, we deduce that $V \cap Y \supseteq Z$. But $V \cap Y$ and Z have the same dimensions and since $V \cap Y$ is irreducible, we are done. □

Hartshorne I.2.14 *The Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_i b_j, \ldots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a *subvariety* of $\mathbb{P}^N$.
[Hint: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} \mid i = 0, \ldots, r, j = 0, \ldots, s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $k[\{z_{ij}\}] \to k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ which sends z_{ij} to $x_i y_j$. Then show that $\operatorname{Im} \psi = Z(\mathfrak{a})$.]

Proof. We show $\operatorname{im} \psi = Z(\mathfrak{a})$. The fact that $\operatorname{im} \psi \subseteq Z(\mathfrak{a})$ is clear, so we prove $\supseteq$. Suppose a point $P = (x_{00} \ldots, x_{ij}, \ldots, x_{rs})$ is in $Z(\mathfrak{a})$. Since this is a point in projective space, assume WLOG that $x_{00} \neq 0$. Then $f(P) = 0$ for all $f \in \mathfrak{a}$. One can observe that $\mathfrak{a}$ contains the elements

$$z_{ij} z_{kl} - z_{kj} z_{il} \;\Rightarrow\; z_{ij} z_{kl} = z_{il} z_{kj}$$

give the relations. It follows that

$$x_{00} x_{ij} = x_{0j} x_{k0} \;\Rightarrow\; x_{ij} = \frac{x_{0j} x_{k0}}{x_{00}}.$$

Now consider the point

$$(x_{00}, \ldots, x_{r0}) \times \left(\frac{x_{00}}{x_{00}}, \frac{x_{01}}{x_{00}}, \ldots, \frac{x_{0s}}{x_{00}}\right)$$

and observe that this maps to P under the Segre embedding. So $P \in \operatorname{im}(\psi)$. □

Hartshorne I.2.15 *The Quadric Surface in* $\mathbb{P}^3$ (Fig. 2). Consider the surface Q (a surface is a variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw = 0$.
(a) Show that Q is equal to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.
(b) Show that Q contains two families of lines (a line is a linear variety of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbb{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \varnothing$; if $M_t \neq M_u$, $M_t \cap M_u = \varnothing$, and for all t, u, $L_t \cap M_u =$ one point.
(c) Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ (where each $\mathbb{P}^1$ has its Zariski topology).

Proof.

(a) I claim that Q is the same as the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ for a suitable choice of coordinates.

Indeed, if we have $(a_0, a_1) \times (b_0, b_1) \in \mathbb{P}^1 \times \mathbb{P}^1$, the image under ψ is $(a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1)$ which clearly satisfies the equation $xy - zw = 0$. The converse is straightforward (and tedious) algebra.

(b) Consider when we fix a point $p = (p_0, p_1) \in \mathbb{P}^1$ and look at the Segre embedding of $\{p\} \times \mathbb{P}^1$ into $\mathbb{P}^3$. We would have, $(p_0 x_0, p_0 x_1, p_1 x_0, p_1 x_1)$ for $(x_0, x_1) \in \mathbb{P}^1$. So, the image of $\{p\} \times \mathbb{P}^1$ is a line. Do the same thing with $\mathbb{P}^1 \times \{q\}$, I have $(q_0 x_0, q_1 x_0, q_0 x_1, q_1 x_1)$. Letting p and q vary, we get families of lines L_p and M_q. It is easy to check the first two conditions we want in the problem.

Now, I claim that $L_p \cap M_q$ is a single point for each p, q. Indeed, the two lines intersect when $x_0 = q_0$ and $x_1 = q_1$ for the line L_p and when $x_0 = p_0$ and $x_1 = p_1$ for the line M_q.

(c) Q contains the curve defined by $x = y$. This curve is closed in Q, but it is not either of the two family of lines above. So, the Zariski topology on Q is not homeomorphic to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ via ψ.

□

Hartshorne I.2.16 (a) The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbb{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.
(b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbb{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

PROOF. (a) Notice that the relations give us the following equations to help cut out $Q_1 \cap Q_2$;

$$x(x^2 - yw) = x^3 - xyw = x^3 - (zw)w = x^3 - zw^2$$

and

$$y(x^2 - yw) = yx^2 - y^2w = x(zw) - y^2w = (xz - y^2)w = 0.$$

So the equations $x^2 - yw = xy - zw = x^3 - zw^2 = 0$ help cut out the twisted cubic portion (see our solution to Hartshorne I.2.9). Meanwhile, if $w = 0$, then we get $x^2 - yw = x^2 = 0$ implying $x = 0$. There are no constraints on y or z that are imposed by this. So the line is cut by $x = w = 0$. Hence, $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.
(b) Where $C \cap L = 0$ is givenby $x^2 - yz = y = 0$. But this condition forces $x^2 = 0$ and so $x = 0$. Therefore, we have a single point $P = [0 : 0 : 1]$. Meanwhile,

$$I(C) = (x^2 - yz), \qquad I(L) = (y), \qquad I(P) = (x, y)$$

but

$$I(C) + I(L) \neq I(P)$$

since the LHS does not contain x. □

Hartshorne I.2.17 *Complete intersections.* A variety Y of dimension r in $\mathbb{P}^n$ is a (strict) complete intersection if $I(Y)$ can be generated by $n - r$ elements. Y is a set-theoretic complete intersection if Y can be written as the intersection of $n - r$ hypersurfaces.
(a) Let Y be a variety in $\mathbb{P}^n$, let $Y = Z(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated by q elements. Then show that $\dim Y \geqslant n - q$.
(b) Show that a strict complete intersection is a set-theoretic complete intersection.
*(c) The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbb{P}^3$ (Ex. 2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1, H_2 of degrees 2,3 respectively, such that $Y = H_1 \cap H_2$.
**(d) It is an unsolved problem whether every closed irreducible curve in $\mathbb{P}^3$ is a set-theoretic intersection of two surfaces. See Hartshorne [1] and Hartshorne [5, III, §5] for commentary.

PROOF. (a) Consider the affine cone and use Hartshorne I.1.9.
(b) Trivial. A strict complete intersection has $I(Y)$ generated by $n - r$ elements. These elements then cut out $n - r$ hypersurfaces and Y is the intersection of these.
(c) Let $I(Y) = \langle XZ - Y^2, YW - Z^2, XW - YZ \rangle$. Take $XZ - Y^2 = 0$ as the degree 2 hypersurface. Now take $XW^2 - 2YWZ + X^3 = 0$ as the degree 3 hypersurface. Then their intersection is where these two equations vanish. But they imply the vanishing of the equations of the twisted cubic as follows.

We have

$$(YW-Z^2)^2 = (YW)^2-2YWZ^2+Z^4 = XZW^2-2YWZ^2+Z^4 = Z(XW^2-2YWZ+Z^3) = 0$$

and

$$(XW - YZ)^2 = (XW)^2 - 2XWYZ + (YZ)^2 = X(XW^2 - 2WYZ + Z^3) = 0.$$

(d) This is an open problem. □

3. Morphisms

Hartshorne I.3.1 (a) Show that any conic in $\mathbb{A}^2$ is isomorphic either to $\mathbb{A}^1$ or $\mathbb{A}^1 - \{0\}$ (cf. Ex. 1.1).
(b) Show that $\mathbb{A}^1$ is not isomorphic to any proper open subset of itself. (This result is generalized by (Ex. 6.7) below.)
(c) Any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.
(d) We will see later (Ex. 4.8) that any two curves are homeomorphic. But show now that $\mathbb{A}^2$ is not even homeomorphic to $\mathbb{P}^2$.
(e) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

Proof. (a) This is immediate from Hartshorne I.1.1. Use the fact that isomorphic coordinate rings yield isomorphic varieties.

(b) Suppose not and that there is a $U \subseteq \mathbb{A}^1$ be open. Without loss of generality, we can take $U = Z(k[x]_f)$ where $f \neq 1$. Then, $\mathcal{O}(U) \cong A_f$. So, $k[x] \cong A_f$ which is obviously false unless $f = 1$.

(c) Let V be a conic in $\mathbb{P}^2$. Then V is cut out by an equation of form

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0.$$

Translate the conic so that it misses the point $[0 : 0 : 1]$. Plugging this in, the conic has form cZ^2 and so the requirement is that $c \neq 0$. Dehomogenize w.r.t. X to get the equation

$$a + b(Y/X)^2 + c(Z/X)^2 + d(Y/X) + e(Y/X) + f(Y/X)(Z/X) = 0$$

in the chart where $X \neq 0$. This is a conic in $\mathbb{A}^2$ so it is isomorphic to either $\mathbb{A}^1$ or $\mathbb{A}^1 - \{0\}$. One can check that this is isomorphic to $\mathbb{A}^1$. Meanwhile, if we dehomogenize w.r.t Y we get

$$a(X/Y)^2 + b + c(Z/Y)^2 + d(X/Y) + e(X/Y)(Z/Y) + f(Z/Y) = 0.$$

These two equations are related by taking the first equation and multiplying by $\frac{X^2}{Y^2}$. But this precisely is the gluing between two copies of $\mathbb{A}^1$ which yield projective space $\mathbb{P}^1$.

(d) Two curves in $\mathbb{P}^2$ intersect but there are pairs of curves in $\mathbb{A}^2$ which do not intersect.

(e) If the affine variety is isomorphic to a projective variety, then it has no globally defined regular functions except the constant functions. But that means that the affine coordinate ring of the affine variety is just k. Hence, it is isomorphic to just a point.

□

Hartshorne I.3.2 A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.
(a) For example, let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.
(b) For another example, let the characteristic of the base field k be $p > 0$, and define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism.*

Proof. (a) The fact that φ defines a bijective bicontinuous morphism is easy to check. To see that φ is not an isomorphism, use the coordinate rings and note that the coordinate ring of $y^2 = x^3$ is not integrally closed.
(b) The map is clearly bijective and bicontinuous. However, it is given by the map of rings

$$k[t] \to k[t]$$

defined by $t \to t^p$. This is certainly not an isomorphism. □

Hartshorne I.3.3 (a) Let $\varphi : X \to Y$ be a morphism. Then for each $P \in X, \varphi$ induces a homomorphism of local rings $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P.X}$.
(b) Show that a morphism φ is an isomorphism if and only if φ is a homeomorphism, and the induced map φ_P^* on local rings is an isomorphism, for all $P \in X$.
(c) Show that if $\varphi(X)$ is dense in Y, then the map φ_P^* is *injective* for all $P \in X$.

Proof.
(a) Suppose $\varphi : X \to Y$ is a morphism and $P \in X$. We wish to show that $\varphi_p^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$. We know that $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular for $f : V \to k$ and hence, we get an induced morphism $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_X$. So, taking colimits, we get $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$ which is the induced morphism we wanted.
(b) The converse is obvious (read up on sheaves). Suppose φ is an isomorphism. Clearly, it is a homeomorphism from X to Y. All that is left to do is to verify that φ_P^* is an isomorphism for each $P \in X$. We can take φ^{-1} to be the continuous inverse of φ. Then, $\varphi{-}1$ induces (by part (a))

$$\varphi_Q^{-1*} : \mathcal{O}_{\varphi^{-1}(Q),X} \to \mathcal{O}_{Q,Y}$$

for all $Q \in Y$. So, $\varphi_{\varphi(P)}^{-1*}$ is an inverse local ring homomorphism of φ_P^*.
(c) This is straightforward from definition. Essentially, φ is dominant iff the ring map φ is injective and this follows from the fact that the localization maps φ_P^* are injective for all $P \in X$.

□

Hartshorne I.3.4 Show that the d-uple embedding of $\mathbb{P}^n$(Ex. 2.12) is an isomorphism onto its image.

Proof. Relatively straightforward exercise (I did this as homework for Dima's Algebraic Geometry I course). □

Hartshorne I.3.5 By abuse of language, we will say that a variety "is affine" if it is isomorphic to an affine variety. If $H \subseteq \mathbb{P}^n$ is any hypersurface, show that $\mathbb{P}^n - H$ is affine.
[Hint: Let H have degree d. Then consider the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ and use the fact that $\mathbb{P}^N$ minus a hyperplane is affine.

PROOF. I did this as homework for Dima's Algebraic Geometry I course. Here's how it goes. Let H be a degree d hypersurface. Then take the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$. Then the equation cutting out $H \subseteq \mathbb{P}^n$ is a linear equation in the coordinates of $\mathbb{P}^N$. Then $\mathbb{P}^N$ subtracted by the vanishing set of this linear equation is affine. But the d-uple embedding is an isomorphism onto its image. So $\mathbb{P}^n - H$ is affine. □

Hartshorne I.3.6 There are quasi-affine varieties which are not affine. For example, show that $X = \mathbb{A}^2 - \{(0,0)\}$ is not affine.
[Hint: Show that $\mathcal{C}(X) \cong k[x,y]$ and use (3.5). See (III, Ex. 4.3) for another proof.]

PROOF. Suppose not and that $X = \mathbb{A}^2 - \{(0,0)\}$ is affine. Then $\mathcal{O}(X) \cong k[x,y]$ since $\mathcal{O}(X) \cong A(X)$ and $I(X) = (0)$. Indeed, $I(X) = 0$ because any regular function cannot contain the variables x and y. Proposition II.3.5 implies $\mathbb{A}^2 \cong X$ which is absurd. □

Hartshorne I.3.7 (a) Show that any two curves in $\mathbb{P}^2$ have a nonempty intersection.
(b) More generally, show that if $Y \subseteq \mathbb{P}^n$ is a projective variety of dimension $\geqslant 1$, and if H is a hypersurface, then $Y \cap H \neq \varnothing$.
[Hint: Use (Ex. 3.5) and (Ex. 3.1e). See (7.2) for a generalization.]

PROOF. This follows from Bezout's Theorem. Alternatively, this follows from the Projective Dimension Theorem in Hartshorne I.7.

A simple proof is as follows.

Suppose $C \cap C' = \emptyset$. We have $C' \subset \mathbb{P}^2 \setminus C$ and $\mathbb{P}^2 \setminus C$ is affine by Hartshorne I.3.5. Now, $\mathbb{P}^2 \setminus C$ is both affine *and* projective so it is just a single point by Hartshorne I.3.1e. However, the dimension of C' is at least 1. So, $\mathbb{P}^2 \setminus C$ cannot be a point. Contradiction.

□

Hartshorne I.3.8 Let H_i and H_j be the hyperplanes in $\mathbb{P}^n$ defined by $x_i = 0$ and $x_j = 0$, with $i \neq j$. Show that any regular function on $\mathbb{P}^n - (H_i \cap H_j)$ is constant. (This gives an alternate proof of (3.4a) in the case $Y = \mathbb{P}^n$.)

PROOF. Let H_i and H_j be hyperplanes given by $x_i = 0$ and $x_j = 0$ in $\mathbb{P}^n$. Suppose f is a regular function on $\mathbb{P}^n - (H_i \cap H_j)$. Then, $f = g/h$ for $h \neq 0$ on $\mathbb{P}^n - (H_i \cap H_j)$. Thus, h must contain homogeneous terms in x_i and x_j only. However, on $H_i \setminus H_j$, x_i can't be zero. So, h must contain no terms in x_i. Similarly, h must contain no terms in x_j. Thus, h must be constant. Since f must be of degree zero, f is constant. □

Hartshorne I.3.9 The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let $X = \mathbb{P}^1$, and let Y be the 2 -uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. Then $X \cong Y$ (Ex. 3.4). But show that $S(X) \ncong S(Y)$.

PROOF. The isomorphism $X \cong Y$ is clear from Hartshorne I.3.4. Now $S(X) = k[x_0, x_1]$ while $S(Y) = k[t_0, t_1, t_2]/(t_1^2 - t_0 t_2)$ and so $S(X) \ncong S(Y)$. □

Hartshorne I.3.10 *Subvarieties.* A subset of a topological space is locally closed if it is an open subset of its closure, or, equivalently, if it is the intersection of an open set with a closed set.

If X is a quasi-affine or quasi-projective variety and Y is an irreducible locally closed subset, then Y is also a quasi-affine (respectively, quasi-projective) variety, by virtue of being a locally closed subset of the same affine or projective space. We call this the *induced structure* on Y, and we call Y a subvariety of X.

Now let $\varphi : X \to Y$ be a morphism, let $X' \subseteq X$ and $Y' \subseteq Y$ be irreducible locally closed subsets such that $\varphi(X') \subseteq Y'$. Show that $\varphi|_{X'} : X' \to Y'$ is a morphism.

PROOF. [Incomplete] □

Hartshorne I.3.11 Let X be any variety and let $P \in X$. Show there is a $1-1$ correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P.

PROOF. [Incomplete] □

Hartshorne I.3.12 If P is a point on a variety X, then $\dim \mathcal{O}_P = \dim X$.

[Hint: Reduce to the affine case and use (3.2c).]

PROOF. Suppose X is a variety. Because there is a natural affine open neighborhood that covers projective space and we have homeomorphisms from the discussion on projective varieties and quasi-projective varieties, we can reduce to X affine. Then, Hartshorne Theorem I.3.2c completes the proof.

□

Hartshorne I.3.13 *The Local Ring of a Subvariety.* Let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes $\langle U, f\rangle$ where $U \subseteq X$ is open, $U \cap Y \neq \varnothing$, and f is a regular function on U. We say $\langle U, f\rangle$ is equivalent to $\langle V, g\rangle$, if $f = g$ on $U \cap V$. Show that $\mathcal{O}_{Y,X}$ is a local ring, with residue field $K(Y)$ and dimension $= \dim X - \dim Y$. It is the *local ring* of Y on X. Note if $Y = P$ is a point we get $\mathcal{O}_P$, and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.

PROOF. [Incomplete] □

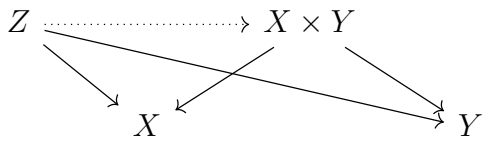
Hartshorne I.3.14 *Projection from a Point.* Let $\mathbb{P}^n$ be a hyperplane in $\mathbb{P}^{n+1}$ and let $P \in \mathbb{P}^{n+1} - \mathbb{P}^n$. Define a mapping $\varphi : \mathbb{P}^{n+1} - \{P\} \to \mathbb{P}^n$ by $\varphi(Q) =$ the intersection of the unique line containing P and Q with $\mathbb{P}^n$.
(a) Show that φ is a morphism.
(b) Let $Y \subseteq \mathbb{P}^3$ be the twisted cubic curve which is the image of the 3 -uple embedding of $\mathbb{P}^1$ (Ex. 2.12). If t, u are the homogeneous coordinates on $\mathbb{P}^1$, we say that Y is the curve given *parametrically* by $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$. Let $P = (0, 0, 1, 0)$, and let $\mathbb{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is a cuspidal cubic curve in the plane, and find its equation.

PROOF. (a) WLOG assume $P := [1 : 0 : \cdots : 0]$. The φ sends $[x_0 : x_1 : \cdots : x_n]$ to $[x_1 : x_2 : \cdots : x_n]$. This is clearly a morphism away from P.

(b) [Incomplete] □

Hartshorne I.3.15 Products of Affine Varieties. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.
(a) Show that $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology is irreducible. [Hint: Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}, i = 1, 2$. Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.] The affine variety $X \times Y$ is called the product of X and Y. Note that its topology is in general not equal to the product topology (Ex. 1.4).
(b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) Show that $X \times Y$ is a product in the category of varieties, i.e., show (i) the projections $X \times Y \to X$ and $X \times Y \to Y$ are morphisms, and (ii) given a variety Z, and the morphisms $Z \to X, Z \to Y$, there is a unique morphism $Z \to X \times Y$ making a commutative diagram



(d) Show that $\dim X \times Y = \dim X + \dim Y$.

PROOF. (a) Clear from the hint.
(b) See Shafarevich's Basic Algebraic Geometry I.
(c) This is an easy check of the universal property.
(d) Use (b) and some facts of commutative algebra. □

Hartshorne I.3.16 *Products of Quasi-Projective Varieties.* Use the Segre embedding (Ex. 2.14) to identify $\mathbb{P}^n \times \mathbb{P}^m$ with its image and hence give it a structure of projective variety. Now for any two quasi-projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, consider $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$.
(a) Show that $X \times Y$ is a quasi-projective variety.
(b) If X, Y are both projective, show that $X \times Y$ is projective.
*(c) Show that $X \times Y$ is a product in the category of varieties.

PROOF. All of this is relatively straightforward. The only difficulty is *(c) since one needs to check the universal property and this can be highly nontrivial. □

Hartshorne I.3.17 *Normal Varieties.* A variety Y is *normal at a point* $P \in Y$ if $\mathcal{O}_P$ is an integrally closed ring. Y is *normal* if it is normal at every point.
(a) Show that every conic in $\mathbb{P}^2$ is normal.
(b) Show that the quadric surfaces Q_1, Q_2 in $\mathbb{P}^3$ given by equations $Q_1 : xy = zw$; $Q_2 : xy = z^2$ are normal (cf. (II. Ex. 6.4) for the latter.)
(c) Show that the cuspidal cubic $y^2 = x^3$ in $\mathbb{A}^2$ is not normal.
(d) If Y is affine, then Y is normal $\Leftrightarrow A(Y)$ is integrally closed.
(e) Let Y be an affine variety. Show that there is a normal affine variety $\tilde{Y}$, and a morphism $\pi : \tilde{Y} \to Y$, with the property that whenever Z is a normal variety, and $\varphi : Z \to Y$ is a *dominant* morphism (i.e., $\varphi(Z)$ is dense in Y), then there is a unique morphism $\theta : Z \to \tilde{Y}$ such that $\varphi = \pi \circ \theta$. $\tilde{Y}$ is called the *normalization* of Y. You will need (3.9A) above.

Definition 3.1. A variety Y is **normal at a point** $P \in Y$ if $\mathcal{O}_P$ is an integrally closed ring. Y is **normal** if it is normal at every point.

PROOF. (a) By a linear change of coordaintes, we can assume the conic C is cut out by an equation of the form $x_0^2+x_1^2+x_2^2$. The singular locus is where $2x_0 = 2x_1 = 2x_2 = 0$. But this singular locus is empty. Normality and nonsingularity correspond for curves and so we deduce that the conic is normal in $\mathbb{P}^2$.
(b) Omitted. Just look at Hartshorne II.6.4 for the proof.
(c) Clear since it is not nonsingular at the origin.

(d) This eqvuialence is obvious. Indeed, this follows from the fact that the ring $k[x_1, \ldots, x_n]/I$ is integrally closed iff each of its localization at maximal ideals is locally closed as well. See this post.[8]

(e)

[**Incomplete**] □

Hartshorne I.3.18 *Projectively Normal Varieties.* A projective variety $Y \subseteq \mathbb{P}^n$ is *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.

(a) If Y is projectively normal, then Y is normal.

(b) There are normal varieties in projective space which are not projectively normal. For example, let Y be the twisted quartic curve in $\mathbb{P}^3$ given parametrically by $(x, y, z, w) = (t^4, t^3u, tu^3, u^4)$. Then Y is normal but not projectively normal. See (III, Ex. 5.6) for more examples.

(c) Show that the twisted quartic curve Y above is isomorphic to $\mathbb{P}^1$, which is projectively normal. Thus projective normality depends on the embedding.

Proof. [**Incomplete**] □

Hartshorne I.3.19 *Automorphisms of* $\mathbb{A}^n$. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ be a morphism of $\mathbb{A}^n$ to $\mathbb{A}^n$ given by n polynomials $f_1, \ldots, f_n$ of n variables $x_1, \ldots, x_n$. Let $J = \det |\partial f_i/\partial x_j|$ be the Jacobian polynomial of φ.

(a) If φ is an isomorphism (in which case we call φ an *automorphism* of $\mathbb{A}^n$) show that J is a nonzero constant polynomial.

**(b) The converse of (a) is an unsolved problem, even for $n = 2$. See, for example. Vitushkin [1].

Proof. (a) [**Incomplete**]

(b) Take $Y := \mathbb{A}^1_k$. Then define the map $f : Y \to k$ by $f(t) = \frac{1}{t}$. This cannot be extended. □

Hartshorne I.3.20 Let Y be a variety of dimension $\geqslant 2$, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

(a) Show that f extends to a regular function on Y.

(b) Show this would be false for $\dim Y = 1$.

See (III, Ex. 3.5) for generalization.

Proof. [**Incomplete**] □

Hartshorne I.3.21 *Group Varieties.* A group variety consists of a variety Y together with a morphism $\mu : Y \times Y \to Y$, such that the set of points of Y with the operation given by μ is a group, and such that the inverse map $y \to y^{-1}$ is also a morphism of $Y \to Y$.

(a) The additive group $\mathbb{G}_a$ is given by the variety $\mathbb{A}^1$ and the morphism $\mu : \mathbb{A}^2 \to \mathbb{A}^1$ defined by $\mu(a, b) = a + b$. Show it is a group variety.

(b) The multiplicative group $\mathbb{G}_m$ is given by the variety $\mathbb{A}^1 - \{(0)\}$ and the morphism $\mu(a, h) = ab$. Show it is a group variety.

(c) If G is a group variety, and X is any variety, show that the set $\mathrm{Hom}(X, G)$ has a natural group structure.

[8] I could not find the reference in Eisenbud. I will need to update to and find the reference in AM. I think it is in Chapter 5.

(d) For any variety X, show that $\mathrm{Hom}\,(X, \mathbb{G}_a)$ is isomorphic to $\mathcal{O}(X)$ as a group under addition.
(e) For any variety X, show that $\mathrm{Hom}\,(X, \mathbb{G}_m)$ is isomorphic to the group of units in $\mathcal{O}(X)$, under multiplication.

PROOF. (a) The **additive group** $\mathbb{G}_a$ is given by $\mathbb{A}^1$ and $\mu : \mathbb{A}^2 \to \mathbb{A}^1$ defined by $\mu(a, b) = a + b$. The operation given by μ defines G_a as a group. All that is left is to verify that $y \mapsto y^{-1}$ is a morphism of $\mathbb{A}^1 \to \mathbb{A}^1$. Indeed, this defines a k-algebra morphism $k[y] \to k[y]$.

(b) The verification is the same. However, note that $\mathbb{A}^1 - \{(0)\}$ has affine coordinate ring $k[y]_y$.

(c) $\mathrm{Hom}(X, G)$ has a natural group structure.
[**Incomplete**]

(d) $\mathrm{Hom}(X, G)$ is isomorphic to $\mathcal{O}(X)$ as a group under addition. Indeed,
[**Incomplete**]

(e) [**Incomplete**]

□

4. Rational Maps

Hartshorne I.4.1 If f and g are regular functions on open subsets U and V of a variety X, and if $f = g$ on $U \cap V$, show that the function which is f on U and g on V is a regular function on $U \cup V$. Conclude that if f is a *rational* function on X, then there is a largest open subset U of X on which f is represented by a regular function. We say that f is *defined* at the points of U.

PROOF. Define $h(p)$ to be $f(p)$ if $p \in U$ and $g(p)$ if $p \in V$. This defines a regular function. Checking that this is so is a tad bit tedious.

An easier way to see why this is so is to note that we have a sheaf of regular functions on the variety and the sheaf axioms allow us to glue to our desired h.

For the second statement. Suppose f is a rational function X. Then it is an equivalence class of pairs of regular functions and open subsets that where the equivalence relation is when the regular functions agree on the intersection. By the result we just proved, we can always find a regular function on the union of the open subsets. So, we can find a largest open subset.[9] □

Hartshorne I.4.2 Same problem for rational maps. If φ is a rational map of X to Y, show there is a largest open set on which φ is represented by a morphism. We say the rational map is defined at the points of that open set.

PROOF. Take the union over all open sets on which φ is defined. □

Hartshorne I.4.3 (a) Let f be the rational function on $\mathbb{P}^2$ given by $f = x_1/x_0$. Find the set of points where f is defined and describe the corresponding regular function.
(b) Now think of this function as a rational map from $\mathbb{P}^2$ to $\mathbb{A}^1$. Embed $\mathbb{A}^1$ in $\mathbb{P}^1$, and let $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$ be the resulting rational map. Find the set of points where φ is defined, and describe the corresponding morphism.

[9] I am pretty sure the argument is straight from Zorn's Lemma.

PROOF. (a) Notice that f is defined away from $V(x_0)$. The corresponding regular function $V(x_0)^c$ is just x_1/x_0.

(b) The map is actually defined everywhere because $\mathbb{P}^2 \to \mathbb{P}^1$ maps $[a_0 : a_1]$ to $[a_1/a_0 : 1]$ when $a_0 \neq 0$ and $[a_0 : a_1]$ to $[1 : 0]$ when $a_0 = 0$. □

Hartshorne I.4.4 A variety Y is *rational* if it is birationally equivalent to $\mathbb{P}^n$ for some n (or, equivalently by (4.5), if $K(Y)$ is a pure transcendental extension of k).
(a) Any conic in $\mathbb{P}^2$ is a rational curve.
(b) The cuspidal cubic $y^2 = x^3$ is a rational curve.
(c) Let Y be the nodal cubic curve $y^2z = x^2(x + z)$ in $\mathbb{P}^2$. Show that the projection φ from the point $P = (0, 0, 1)$ to the line $z = 0$(Ex. 3.14) induces a birational map from Y to $\mathbb{P}^1$. Thus Y is a rational curve.

PROOF.
(a) By Harthsorne I.3.1, every conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$ and isomorphisms are birational. Hence, every conic is rational.
(b) Define a map from Y the cuspidal curve to $\mathbb{P}^1$ by sending (x, y) to $\frac{1}{t}$. This map has an inverse sending t to (t^2, t^3).
Alternatively, the blow-up of the cupsidal cubic is the twisted cubic in $\mathbb{P}^2$ which is known to be a rational curve.
(c) Let Y be the nodal cubic curve $y^2z = x^2(x + z)$ in $\mathbb{P}^2$. I claim that the projection φ given induces a birational map $Y \to \mathbb{P}^1$ and hence, Y is rational.
Consider the open set of $\mathbb{P}^2$ defined when $z = 1$. On this open set, the nodal cubic curve is $y^2 = x^2(x+1)$. Since the projection map is defined only on $\mathbb{P}^2-(0, 0, 1)$ by Hartshorne I.3.14, I have a rational map.
This projection sends (x, y, z) to (x, y). The inverse map will be defined by $(x, y) \mapsto ((y^2 - x^2)x, (y^2 - x^2)y, x^3)$ for $(x, y) \neq (1, 1)$ and $(x, y) \neq (1, -1)$.
[**Incomplete**]

□

Remark 4. This is an important characterization of varieties and will be used later in the text. Varieties birationally equivalent to $\mathbb{P}^n$ are called rational.

Hartshorne I.4.5 Show that the quadric surface $Q : xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$ (cf. Ex. 2.15).

PROOF. It is birational since I can define a map $(x, y, z, w) \mapsto (\frac{x}{z}, \frac{y}{z})$ for $z \neq 0$. This map has an inverse defined by sending (x, y) to $(x, y, 1, xy)$.

One can find this inverse by noting that it sends $(\frac{x}{z}, \frac{y}{z})$ to $(\frac{x}{z}, \frac{y}{z}, 1, \frac{xy}{z^2})$. Note that $\frac{xy}{z^2} = \frac{zw}{z^2} = \frac{w}{z}$ and that we can then scale $(\frac{x}{z}, \frac{y}{z}, 1, \frac{w}{z})$ to (x, y, z, w). This shows that it is indeed birational because these are rational maps.

To see that the quadric is not isomorphic to $\mathbb{P}^2$, consider a pair of skew lines given in Hartshorne I.2.15 say L_u and L_t. Then if $u \neq t$, the two lines do not intersect. However, any two lines in $\mathbb{P}^2$ intersect by Hartshorne I.3.7a. □

Remark 5. This shows that birational equivalence and isomorphisms between varieties are different.

Hartshorne I.4.6 *Plane Cremona Transformations.* A birational map of $\mathbb{P}^2$ into itself is called a *plane Cremona transformation.* We give an example, called a *quadratic transformation.* It is the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $(a_0, a_1, a_2) \to (a_1a_2, a_0a_2, a_0a_1)$ when no two of a_0, a_1, a_2 are 0 .

(a) Show that φ is birational, and is its own inverse.
(b) Find open sets $U, V \subseteq \mathbb{P}^2$ such that $\varphi : U \to V$ is an isomorphism.
(c) Find the open sets where φ and φ^{-1} are defined, and describe the corresponding morphisms. See also (V, 4.2.3).

PROOF. (a) [**Incomplete**]

We consider the composition i.e. so we check

$$(a_1a_2, a_0a_2, a_0a_1) \mapsto (a_0a_2a_0a_1, a_1a_2a_0a_1, a_1a_2a_0a_2) = (a_0a_1a_2a_0, a_0a_1a_2a_1, a_0a_1a_2a_2)$$

and so we divide out by $a_0a_1a_2$ to get the claim. This requires that none of the a_0, a_1, a_2 are zero.
(b) It is easy to check that the open sets are where U, V correspond to $\mathbb{P}^2 \setminus V(a_0) \cup V(a_1) \cup V(a_2)$.
(c) [**Incomplete**] □

Hartshorne I.4.7 Let X and Y be two varieties. Suppose there are points $P \in X$ and $Q \in Y$ such that the local rings $\mathcal{O}_{P.X}$ and $\mathcal{O}_{Q.Y}$ are isomorphic as k-algebras. Then show that there are open sets $P \in U \subseteq X$ and $Q \in V \subseteq Y$ and an isomorphism of U to V which sends P to Q.

PROOF. The proof itself is actually quite tedious and difficult to write out. I wrote out a proof for Math 763 at UW-Madison when it was assigned for homework. A solution can be found online here. □

Hartshorne I.4.8 (a) Show that any variety of positive dimension over k has the same cardinality as k.
[Hints: Do $\mathbb{A}^n$ and $\mathbb{P}^n$ first. Then for any X, use induction on the dimension n. Use (4.9) to make X birational to a hypersurface $H \subseteq \mathbb{P}^{n+1}$. Use (Ex. 3.7) to show that the projection of H to $\mathbb{P}^n$ from a point not on H is finite-to-one and surjective.]
(b) Deduce that any two curves over k are homeomorphic (cf. Ex. 3.1).

PROOF. (a) It is clear for $\mathbb{A}^n$ and $\mathbb{P}^n$. Fix X birational to a hypersurface $H \subseteq \mathbb{P}^{n+1}$. Then by Hartshorne I.3.7, the intersection of H with another variety is always nonempty. Take the projection of H to $\mathbb{P}^n$ from a point not on H and then Hartshorne I.3.7 implies that this map is finite-to-one and surjective (literally by definition of the projection). So the cardinality of points on H corresponds to the cardinality of points on the image of the projection and then the inductive hypothesis completes the proof.

There are some comments regarding the usefulness of this hint online.

(b) It suffices to show that any curve over k is homeomorphic the affine line. By the Axiom of Choice, pick a bijection between the set of points of the curve and the set of points of k. Then this induces a homeomorphism since closed sets map to closed sets. The closed subsets are precisely $C, \emptyset$, and finite collection of points of C, and these map to k, $\emptyset$, and finite collection of points of k. So the topologies are the same. □

Hartshorne I.4.9 Let X be a projective variety of dimension r in $\mathbb{P}^n$. with $n \geqslant r+2$. Show that for suitable choice of $P \notin X$, and a linear $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$, the projection from P to $\mathbb{P}^{n-1}$ (Ex. 3.14) induces a *birational* morphism of X onto its image $X' \subseteq \mathbb{P}^{n-1}$. You will need to use (4.6 A).(4.7 A), and (4.8 A). This shows in particular that the birational map of (4.9) can be obtained by a finite number of such projections.

PROOF. [**Incomplete**] □

Hartshorne I.4.10 Let Y be the cuspidal cubic curve $y^2 = x^3$ in $\mathbb{A}^2$. Blow up the point $O = (0,0)$, let E be the exceptional curve, and let $\tilde{Y}$ be the strict transform of Y. Show that E meets $\tilde{Y}$ in one point, and that $\tilde{Y} \cong \mathbb{A}^1$. In this case the morphism $\varphi : \tilde{Y} \to Y$ is bijective and bicontinuous, but it is not an isomorphism.

PROOF. To compute that E meets $\widetilde{Y}$ in one point, compute $\widetilde{Y}$ over the two charts $t_0 = 1$ and $t_1 = 1$ as a subset of $\mathbb{A}^2 \times \mathbb{P}^1$ where t_0, t_1 are the coordinates of $\mathbb{P}^1$.

Now $\widetilde{Y}$ is the twisted cubic i.e. it is cut out in $\mathbb{A}^3$ by (t, t^2, t^3). This is isomorphic to $\mathbb{A}^1$.

The morphism is clearly bijective and bicontinuous, but it is not an isomorphism since the blow-up is $\mathbb{A}^1$ while Y is the cuspidal cubic (which has a singularity at the origin). □

5. Nonsingular Varieties

Hartshorne I.5.1 Locate the singular points and sketch the following curves in $\mathbb{A}^2$ (assume char $k \neq 2$). Which is which in Figure 4 ?

(a) $x^2 = x^4 + y^4$;
(b) $xy = x^6 + y^6$;
(c) $x^3 = y^2 + x^4 + y^4$;
(d) $x^2y + xy^2 = x^4 + y^4$.

PROOF. For each of (a)-(d), sketching the curve entails testing points on the x-axis and y-axis.

(a) Tacnode; it has a singular point at the origin. To see that it has a singular point at the origin, notice that the Jacobian matrix is

$$\begin{bmatrix} 3x^4 - 2x \\ 4y^3 \end{bmatrix}$$

and at $P = (0,0)$, the rank of the Jacobian matrix is 0. However, for our affine variety to be singular at a point P, the rank should be $2 - 1$ since the dimension of our variety is 1.

(b) Node; it has a singular point at the origin. The Jacobian matrix is

$$\begin{bmatrix} y - 5x^6 \\ x - 5y^6 \end{bmatrix}$$

which has rank 0 at $P = (0,0)$.

(c) Cusp; it has a singular point at the origin. As before, $\begin{bmatrix} 3x^2 - 4x^3 \\ -2y - 4y^3 \end{bmatrix}$ has only zero entries at $P = (0,0)$.

(d) Triple point; it has a singular point at the origin. The Jacobian matrix is $\begin{bmatrix} 2x + y^2 - 3x^4 \\ x^2 + 2xy - 4y^3 \end{bmatrix}$.

□

Hartshorne I.5.2 Locate the singular points and describe the singularities of the following surfaces in A^3 (assume char $k \neq 2$). Which is which in Figure 5 ?

(a) $xy^2 = z^2$;
(b) $x^2 + y^2 = z^2$;
(c) $xy + x^3 + y^3 = 0$.

PROOF. (a) Pinch point; there are singularities along $(x, 0, 0)$. The Jacobian matrix is $\begin{bmatrix} y^2 \\ 2xy \\ -2z \end{bmatrix}$. Since we are considering these in $\mathbb{A}^3$, the equation cuts out a hypersurface. Therefore, a point is nonsingular iff the Jacobian matrix there has rank 1. The matrix above indicates there is a singular point at $(0, 0, 0)$ and in particular, the points $(x, 0, 0)$ are all singular.

(b) Conical double point; there is a singularity at the origin. The Jacobian matrix is $\begin{bmatrix} 2x \\ 2y \\ -2z \end{bmatrix}$ which singular only at the origin.

(c) Double line; the z-value can freely variable and the only graph that has this is the double line. The Jacobian matrix is $\begin{bmatrix} y + 3x^2 \\ x + 3y^2 \\ 0 \end{bmatrix}$ which is singular along $(0, 0, z)$. Again, only the double line has this.

□

Hartshorne I.5.3 *Multiplicities.* Let $Y \subseteq \mathbb{A}^2$ be a curve defined by the equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + f_1 + \ldots + f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Then we define the *multiplicity* of P on Y, denoted $\mu_P(Y)$ to be the least r such that $f_r \neq 0$. (Note that $P \in Y \Leftrightarrow \mu_P(Y) > 0$.) The linear factors of f_r are called the *tangent directions* at P.

(a) Show that $\mu_P(Y) = 1 \Leftrightarrow P$ is a nonsingular point of Y.

(b) Find the multiplicity of each of the singular points in (Ex. 5.1) above.

PROOF. **(a)** If $\mu_P(Y) = 1$, I know that $f_1 \neq 0$ and 1 is the least such where $f = f_0 + \ldots f_r$. In this case, the Jacobian at P is $\begin{bmatrix} a \\ b \end{bmatrix}$ where a is the coefficient for the term ay and b is the coefficient for the term bx. The rank is 1 which implies that Y is nonsingular at P.

For the converse, we prove the contrapositive. Clearly, if $\mu_P(Y) = 0$, then it is singular at P. Similarly, if $\mu_P(Y) > 1$, I know that the linear terms $ax + by + c + d$ is zero. So, the Jacobian at P is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which is of rank 2. Hence, it is singular at that point.

(b) All letter references are to Hartshorne I.5.1. All of the curves there are singular at the origin. The multiplicities are are all 2. □

Remark 6. The exercise here introduces some definitions that we should keep in mind.

Given a curve Y defined by f in $\mathbb{A}^2$ and P a point in $\mathbb{A}^2$, we will assume $P = (0, 0)$ by a change of coordinates. Write $f = f_0 + \cdots + f_d$ as sum of homogeneous polynomials f_s of degree s. The **multiplicity of P on Y** is $\mu_P(Y)$ which is the least r s.t. $f_r \neq 0$. The linear factors f_r are called the **tangent directions at P**.

Hartshorne I.5.4 *Intersection Multiplicity.* If $Y, Z \subseteq \mathbb{A}^2$ are two distinct curves, given by equations $f = 0, g = 0$, and if $P \in Y \cap Z$, we define the intersection multiplicity $(Y \cdot Z)_P$ of Y and Z at P to be the length of the $\mathcal{O}_P$-module $\mathcal{O}_P/(f, g)$.

(a) Show that $(Y \cdot Z)_P$ is finite, and $(Y \cdot Z)_P \geq \mu_P(Y) \cdot \mu_P(Z)$.

(b) If $P \in Y$, show that for almost all lines L through P (i.e., all but a finite number), $(L \cdot Y)_P = \mu_P(Y)$.

(c) If Y is a curve of degree d in $\mathbf{P}^2$, and if L is a line in $\mathbf{P}^2, L \neq Y$, show that $(L \cdot Y) = d$. Here we define $(L \cdot Y) = \sum (L \cdot Y)_P$ taken over all points $P \in L \cap Y$, where $(L \cdot Y)_P$ is defined using a suitable affine cover of P^2.

PROOF. **(a)** [Incomplete]
(b) [Incomplete]
(c) [Incomplete] □

Remark 7. The exercise defines the **intersection multiplicity** of two curves Y and Z in affine space at a point P. It is defined as the length of the $\mathcal{O}_P$-module $\mathcal{O}_P/(f, g)$ where f and g give Y and Z.

Hartshorne I.5.5 For every degree $d > 0$, and every $p = 0$ or a prime number, give the equation of a nonsingular curve of degree d in $\mathbb{P}^2$ over a field k of characteristic p.

PROOF. Fix $p = 0$ and $d > 0$. Let x, y, z be the coordaintes of $\mathbb{P}^2$. Then the equation $x^d + y^d + z^d = 0$ works.

Fix $p \neq 0$ and $d > 0$. When $d \neq p$, the equation $x^d + y^d + z^d = 0$ works as before. When $d = p$, the equation $x^d + y^d + z^d = (x + y + z)^p = 0$ does not work since the partial derivatives vanish. In this case, we must modify the equation. The equation $x^{d-1}y + z^d + y^{d-1}z = 0$ works because the partial derivatives w.r.t. x, y, z are

$$(d-1)x^{d-2}y, \qquad x^{d-1} + (d-1)y^{d-2}z, \qquad y^{d-1}.$$

For all of these to be zero, we first need $y = 0$ by the last partial dervative. Then $y = 0$ in the original equation says that $z = 0$. Then the second partial derivative forces $x^{d-1} = 0$ if we want it to equal zero so $x = 0$. This does not give a point in projective space. □

Hartshorne I.5.6 *Blowing Up Curve Singularities.*

(a) Let Y be the cusp or node of (Ex. 5.1). Show that the curve $\tilde{Y}$. obtained by blowing up Y at $O = (0, 0)$ is nonsingular (cf. (4.9.1) and (Ex. 4.10)).

(b) We define a *node* (also called *ordinary double point*) to be a double point (i.e., a point of multiplicity 2) of a plane curve with distinct tangent directions (Ex. 5.3). If P is a node on a plane curve Y, show that $\varphi^{-1}(P)$ consists of two distinct nonsingular points on the blown-up curve $\tilde{Y}$. We say that "blowing up P resolves the singularity at $P \gg$.

(c) Let $P \in Y$ be the tacnode of (Ex. 5.1). If $\varphi : \tilde{Y} \to Y$ is the blowing-up at P, show that $\varphi^{-1}(P)$ is a node. Using (b) we see that the tacnode can be resolved by two successive blowings-up.

(d) Let Y be the plane curve $y^3 = x^5$, which has a "higher order cusp" at O. Show that O is a triple point; that blowing up O gives rise to a double point (what kind?) and that one further blowing up resolves the singularity.

Note: We will see later (V, 3.8) that any singular point of a plane curve can be resolved by a finite sequence of successive blowings-up.

PROOF. [Incomplete] □

Hartshorne I.5.7 Let $Y \subseteq \mathbb{P}^2$ be a nonsingular plane curve of degree > 1, defined by the equation $f(x, y, z) = 0$. Let $X \subseteq \mathbb{A}^3$ be the affine variety defined by f (this is the cone over Y; see (Ex. 2.10)). Let P be the point $(0, 0, 0)$, which is the *vertex* of the cone. Let $\varphi : \tilde{X} \to X$ be the blowing-up of X at P.

(a) Show that X has just one singular point, namely P.

(b) Show that $\tilde{X}$ is nonsingular (cover it with open affines).

(c) Show that $\varphi^{-1}(P)$ is isomorphic to Y.

PROOF. (a) This is easy since $f(x, y, z)$ is a homogeneous polynomial of degree $\deg(Y)$. This requires $\deg(Y) > 1$ because if $\deg(Y) = 1$, then Y is a line and variety defined by $f(x, y, z)$ in the dgree 1 case is literally a plane and hence, it is nonsingular.
(b) The blow-up resolves the singularity at P and so $\widetilde{X}$ is nonsingular.
(c) $\varphi^{-1}(P) \cong Y$ is also clear. □

Hartshorne I.5.8 Let $Y \subseteq \mathbb{P}^n$ be a projective variety of dimension r. Let $f_1, \ldots, f_t \in S = k[x_0, \ldots, x_n]$ be homogeneous polynomials which generate the ideal of Y. Let $P \in Y$ be a point, with homogeneous coordinates $P = (a_0, \ldots, a_n)$. Show that P is nonsingular on Y if and only if the rank of the matrix $\|(\partial f_i/\partial x_j)(a_0, \ldots, a_n)\|$ is $n - r$.
[Hint: (a) Show that this rank is independent of the homogeneous coordinates chosen for P; (b) pass to an open affine $U_i \subseteq \mathbb{P}^n$ containing P and use the affine Jacobian matrix; (c) you will need Euler's lemma, which says that if f is a homogeneous polynomial of degree d, then $\sum x_i (\partial f/\partial x_i) = d \cdot f$.]

PROOF. [Incomplete] □

Hartshorne I.5.9 Let $f \in k[x, y, z]$ be a homogeneous polynomial, let $Y = Z(f) \subseteq \mathbb{P}^2$ be the algebraic set defined by f, and suppose that for every $P \in Y$, at least one of $(\partial f/\partial x)(P), (\partial f/\partial y)(P), (\partial f/\partial z)(P)$ is nonzero. Show that f is irreducible (and hence that Y is a nonsingular variety). [Hint: Use (Ex. 3.7).]

PROOF. For f to be irreducible, we need Y irreducible. Suppose not and Y were reducible. Then Y is a union of at least two curves and each pair of curves has nonempty intersection. Pick a point of intersection Q.

It suffices to show that all of the partial derivatives vanish at Q. Indeed, reducibility of f gives $f = g_1 g_2 h$ where we can assume g_1, g_2 are irreducible and cut out the two curves (and h here may be reducible). The fact that the curves intersect at Q means $g_1(Q) = g_2(Q) = 0$. Now,

$$\frac{\partial f}{\partial x}(Q) = \left(\frac{\partial g_1}{\partial x} g_2 h + \frac{\partial g_2}{\partial x} g_1 h + \frac{\partial h}{\partial x} g_1 g_2\right)(Q) = 0.$$

The same is true for y, z in place of x. Contradiction. □

Hartshorne I.5.10 For a point P on a variety X, let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_P$. We define the *Zariski tangent space* $T_P(X)$ of X at P to be the dual k-vector space of $\mathfrak{m}/\mathfrak{m}^2$.
(a) For any point $P \in X$, $\dim T_P(X) \geqslant \dim X$, with equality if and only if P is nonsingular.

(b) For any morphism $\varphi : X \to Y$, there is a natural induced k-linear map $T_P(\varphi) : T_P(X) \to T_{\varphi(P)}(Y)$.
(c) If φ is the vertical projection of the parabola $x = y^2$ onto the x-axis, show that the induced map $T_0(\varphi)$ of tangent spaces at the origin is the zero map.

PROOF. (a) □

Hartshorne I.5.11 *The Elliptic Quartic Curve in* $\mathbb{P}^3$. Let Y be the algebraic set in $\mathbb{P}^3$ defined by the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. Let P be the point $(x, y, z, w) = (0, 0, 0, 1)$, and let φ denote the projection from P to the plane $w = 0$. Show that φ induces an isomorphism of $Y - P$ with the plane cubic curve $y^2z - x^3 + xz^2 = 0$ minus the point $(1, 0, -1)$. Then show that Y is an irreducible nonsingular curve. It is called the *elliptic quartic curve* in $\mathbb{P}^3$. Since it is defined by two equations it is another example of a complete intersection (Ex. 2.17).

PROOF. [Incomplete] □

Hartshorne I.5.12 *Quadric Hypersurfaces.* Assume char $k \neq 2$, and let f be a homogeneous polynomial of degree 2 in $x_0, \ldots, x_n$.
(a) Show that after a suitable linear change of variables, f can be brought into the form $f = x_0^2 + \ldots + x_r^2$ for some $0 \leqslant r \leqslant n$.
(b) Show that f is irreducible if and only if $r \geqslant 2$.
(c) Assume $r \geqslant 2$, and let Q be the quadric hypersurface in $\mathbb{P}^n$ defined by f. Show that the singular locus $Z = \operatorname{Sing} Q$ of Q is a *linear* variety (Ex. 2.11) of dimension $n - r - 1$. In particular, Q is nonsingular if and only if $r = n$.
(d) In case $r < n$, show that Q is a cone with axis Z over a nonsingular quadric hypersurface $Q' \subseteq \mathbb{P}^r$. (This notion of cone generalizes the one defined in (Ex. 2.10). If Y is a closed subset of $\mathbb{P}^r$, and if Z is a linear subspace of dimension $n-r-1$ in $\mathbb{P}^n$, we embed $\mathbb{P}^r$ in $\mathbb{P}^n$ so that $\mathbb{P}^r \cap Z = \varnothing$, and define the *cone over* Y *with axis* Z to be the union of all lines joining a point of Y to a point of Z.)

PROOF. (a) Follows from the method of completing the square.
(b) Clearly, it is reducible if $r = 0, 1$. For irreducibility in $r \geq 2$, it is reducible iff f factors into a product of two linear polynomials. Then the problem reduces down to the determination of a system of equations. But one can check that this system of equations cannot be solved.
(c) Do a change of variables as in (a). The singular locus is going to be the solution set of equations

$$2x_0 = 0, 2x_1 = 0, \ldots, 2x_r = 0, f(x) = 0.$$

So the singular locus is $V(x_0, \ldots, x_r)$ i.e. it is of dimension $n - r - 1$. This is clearly a linear variety.

Now Q is nonsingular iff $r = n$ because this ensures that the singular locus is empty e.g. it is of dimension -1.
(d) [Incomplete] □

Hartshorne I.5.13 It is a fact that any regular local ring is an integrally closed domain (Matsumura [2, Th. 36, p. 121]). Thus we see from (5.3) that any variety has a nonempty open subset of normal points (Ex. 3.17). In this exercise, show directly (without using (5.3))

that the set of nonnormal points of a variety is a proper closed subset (you will need the finiteness of integral closure: see (3.9A)).

PROOF. The indirect proof is straightforward from the problem statement so we do not bother with the direct proof. □

Hartshorne I.5.14 *Analytically Isomorphic Singularities.*
(a) If $P \in Y$ and $Q \in Z$ are analytically isomorphic plane curve singularities, show that the multiplicities $\mu_P(Y)$ and $\mu_Q(Z)$ are the same (Ex. 5.3).
(b) Generalize the example in the text (5.6.3) to show that if $f = f_r + f_{r+1} + \ldots \in k[[x, y]]$, and if the leading form f_r of f factors as $f_r = g_s h_t$, where g_s, h_t are homogeneous of degrees s and t respectively, and have no common linear factor, then there are formal power series

$$g = g_s + g_{s+1} + \ldots$$
$$h = h_t + h_{t+1} + \ldots$$

in $k[[x, y]]$ such that $f = gh$.
(c) Let Y be defined by the equation $f(x, y) = 0$ in $\mathbb{A}^2$, and let $P = (0, 0)$ be a point of multiplicity r on Y, so that when f is expanded as a polynomial in x and y, we have $f = f_r +$ higher terms. We say that P is an ordinary r-fold point if f_r is a product of r *distinct* linear factors. Show that any two ordinary double points are analytically isomorphic. Ditto for ordinary triple points. But show that there is a one-parameter family of mutually nonisomorphic *ordinary 4-fold points*.
*(d) Assume char $k \neq 2$. Show that any double point of a plane curve is analytically isomorphic to the singularity at $(0, 0)$ of the curve $y^2 = x^r$, for a uniquely determined $r \geqslant 2$. If $r = 2$ it is a node (Ex. 5.6). If $r = 3$ we call it a *cusp*; if $r = 4$ a *tacnode*. See (V, 3.9.5) for further discussion.

PROOF. [Incomplete] □

Hartshorne I.5.15 *Families of Plane Curves.* A homogeneous polynomial f of degree d in three variables x, y, z has $\binom{d+2}{2}$ coefficients. Let these coefficients represent a point in $\mathbb{P}^N$, where $N = \binom{d+2}{2} - 1 = \frac{1}{2}d(d + 3)$.
(a) Show that this gives a correspondence between points of $\mathbb{P}^N$ and algebraic sets in $\mathbb{P}^2$ which can be defined by an equation of degree d. The correspondence is $1-1$ except in some cases where f has a multiple factor.
(b) Show under this correspondence that the (irreducible) nonsingular curves of degree d correspond 1-1 to the points of a nonempty Zariski-open subset of $\mathbb{P}^N$.
[Hints: (1) Use elimination theory (5.7A) applied to the homogeneous polynomials $\partial f/\partial x_0, \ldots, \partial f/\partial x_n$; (2) use the previous (Ex. 5.5, 5.8, 5.9) above.]

PROOF. [Incomplete] □

6. Nonsingular Curves

Hartshorne I.6.1 Recall that a curve is rational if it is birationally equivalent to $\mathbb{P}^1$ (Ex. 4.4). Let Y be a nonsingular rational curve which is not isomorphic to $\mathbb{P}^1$.

(a) Show that Y is isomorphic to an open subset of $\mathbb{A}^1$.

(b) Show that Y is affine.

(c) Show that $A(Y)$ is a unique factorization domain.

PROOF. **(a)** From Hartshorne Proposition I.6.7, we know Y is isomorphic to an abstract nonsingular curve. That is, $Y \subseteq C_K$ where $K := K(Y)$ is the function field of Y. As $K(Y)$ is birational to $\mathbb{P}^1$, $K = K(\mathbb{P}^1)$. By Hartshorne Theorem I.6.9, we know that $C_K \cong \mathbb{P}^1$ which means that $Y \subsetneq \mathbb{P}^1$ is an open subset. Since Y is not isomorphic to $\mathbb{P}^1$, it is a proper subset. Therefore, $Y \subseteq \mathbb{P}^1 - \{p\} \cong \mathbb{A}^1$ is an open subset.

(b) Since $Y \cong U \subseteq \mathbb{A}^1$, and we can write $U = \bigcup_{f_i \in k[x], i=1,\dots,r} D(f_i) = D(f_1 f_2 \dots f_r)$. Therefore, $U = \operatorname{Spec} k[x]_{f_1 \dots f_r}$ which is certainly affine (recall Hartshorne says "affine" to mean isomorphic to some affine variety).

(c) From (a) and (b), we know Y is an affine open subset of $\mathbb{A}^1$ and that its coordinate ring is $k[x]_{f_1 \dots f_r}$. The localization of a UFD is always a UFD so we are done. □

Hartshorne I.6.2 *An Elliptic Curve.* Let Y be the curve $y^2 = x^3 - x$ in $\mathbb{A}^2$, and assume that the characteristic of the base field k is $\neq 2$. In this exercise we will show that Y is not a rational curve, and hence $K(Y)$ is not a pure transcendental extension of k.

(a) Show that Y is nonsingular, and deduce that $A = A(Y) \simeq k[x, y]/(y^2 - x^3 + x)$ is an integrally closed domain.

(b) Let $k[x]$ be the subring of $K = K(Y)$ generated by the image of x in A. Show that $k[x]$ is a polynomial ring, and that A is the integral closure of $k[x]$ in K.

(c) Show that there is an automorphism $\sigma : A \to A$ which sends y to $-y$ and leaves x fixed. For any $a \in A$, define the norm of a to be $N(a) = a \cdot \sigma(a)$. Show that $N(a) \in k[x], N(1) = 1$, and $N(ab) = N(a) \cdot N(b)$ for any $a, b \in A$.

(d) Using the norm, show that the units in A are precisely the nonzero elements of k. Show that x and y are irreducible elements of A. Show that A is not a unique factorization domain.

(e) Prove that Y is not a rational curve (Ex. 6.1). See (II, 8.20.3) and (III, Ex. 5.3) for other proofs of this important result.

PROOF. **(a)** To see that Y is nonsingular, we take $f = y^2 - x^3 + x$ and consider the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} -3x^2 + 1 & 2y. \end{pmatrix}$$

and this not of rank 1 iff $-3x^2 + 1 = 2y = 0$ which is impossible unless $\operatorname{char} k = 2$.

To see that A is an integrally closed domain, we apply Atiyah-Macdonald Proposition 5.12. In this case $A_{\mathfrak{p}}$ is a regular local ring so it is a UFD which means it is integrally closed domain.

(b) It is easy to deduce that $K \cong \operatorname{Frac}(k[x, y]/(y^2 - x^3 + x)) \cong k(x^2, x^3, y)/(y^2 - x^3 + x)$. Given x, we look at the subring it generates in A. Since there is no relation given by a polynomial purely in x, we deduce that the subring is isomorphic to the polynomial ring $k[x]$ which justifies the notation.

The integral closure of $k[x]$ in K consists of of all integral elements. It suffices to show that x and y are integral over $k[x]$ since $\frac{1}{x}$ and $\frac{1}{y}$ are clearly not. Clearly x is integral over $k[x]$ while for y, it is the solution of the equation

$$f(t) = t^2 - x^3 + x \in k[x][t].$$

(c) Certainly, $N(1) = 1 \cdot \sigma(1) = 1 \cdot 1 = 1$. Also

$$N(f(x,y)) = f(x,-y)$$

which means $f(x,y)$ is a polynomial purely in x because we eliminate all even powers of y via $y^2 = x^3 - x$. So $N(a) \in k[x]$. Further,

$$N(f(x,y)g(x,y)) = f(x,y)g(x,y)f(x,-y)g(x,-y) = N(f(x,y))N(g(x,y)).$$

(d) First, we show $A^\times \cong k^\times$. Indeed, $f \in A^\times$ iff there is an inverse f^{-1} iff $1 = N(ff^{-1})$ iff $N(f)$ is invertible iff $N(f) \in (k[x])^\times$ iff $N(f) \in k^\times$. This occurs iff $f \in k^\times$.

We show x is irreducible in A. Let $x = ab$. Then $N(x) = N(a)N(b)$ which means $x^2 = N(a)N(b)$. So WLOG either $N(a) = x$ or $N(a) = x^2$. We cannot get $N(a) = x$ by definition of N so that means $N(a) = x^2$. That means $N(b) = 1$. That means $b \in A^\times$.

For irreducibility of y, we get $N(y) = -y^2 = x - x^3 = x(1+x)(1-x)$. So if $y = ab$, then WLOG $N(a)$ is one of $1, x, 1+x, 1-x$ (there are two factors $N(a)$ and $N(b)$ so at least one of them must be of degree ≤ 1). For degree reasons and definition of N, only $N(a) = 1$ is possible. Contradiction.

To see A is not a UFD, observe that

$$y^2 = y \cdot y = x(1-x)(1+x)$$

is a factorization of y^2 into a product of irreducibles in two different ways.

(e) From (a) and (d), we know that $A(Y)$ is not a UFD. But Hartshorne I.6.1(c) asserts that any nonsingular rational curve will have $A(Y)$ a UFD. □

Hartshorne I.6.3 Show by example that the result of (6.8) is false if either (a) $\dim X \geqslant 2$, or (b) Y is not projective.

Proof. **(a)** Take the quotient map $\mathbb{A}^2 - \{(0,0)\} \to \mathbb{P}^1$ defined by $(x,y) \mapsto [x:y] \in \mathbb{P}^1$. This has no extensions because if it extended to $(0,0)$, the quotient map would say every point is identified to $(0,0)$ which is not possible.
(b) Take the map $\varphi : \mathbb{P}^1 - ([1:0)] = \mathbb{A}^1 \to \mathbb{A}^1$ defined by the quotient $[x:y] \to \frac{x}{y}$. There is no meaningful extension because if one examines the regular function $f : \mathbb{A}^1 \to k$ defined by $f(x) = x$, then we get the composite

$$f \circ \varphi : [1:w] \mapsto \frac{1}{w} \xrightarrow{f} \frac{1}{w}$$

which shows that there is no way to extend φ to $[1:0]$ which would preserve the regular function f. □

Hartshorne I.6.4 Show by example that the result of (6.8) is false if either (a) $\dim X \geqslant 2$, or (b) Y is not projective.

Proof. [Incomplete] □

Hartshorne I.6.5 Let X be a nonsingular projective curve. Suppose that X is a (locally closed) subvariety of a variety Y (Ex. 3.10). Show that X is in fact a closed subset of Y. See (II, Ex. 4.4) for generalization.

PROOF. [Incomplete] □

Hartshorne I.6.6 *Automorphisms of* $\mathbf{P}^1$. Think of $\mathbf{P}^1$ as $\mathbf{A}^1 \cup \{\infty\}$. Then we define a *fractional linear transformation* of $\mathbf{P}^1$ by sending $x \mapsto (ax+b)/(cx+d)$, for $a, b, c, d \in k$, $ad - bc \neq 0$.
(a) Show that a fractional linear transformation induces an *automorphism* of $\mathbf{P}^1$ (i.e., an isomorphism of $\mathbf{P}^1$ with itself). We denote the group of all these fractional linear transformations by PGL(1).
(b) Let Aut $\mathbf{P}^1$ denote the group of all automorphisms of $\mathbf{P}^1$. Show that Aut $\mathbf{P}^1 \simeq$ Aut $k(x)$, the group of k-automorphisms of the field $k(x)$.
(c) Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that PGL(1) $\to$ Aut $\mathbf{P}^1$ is an isomorphism.

PROOF. [Incomplete] □

Hartshorne I.6.7 Let $P_1, \ldots, P_r, Q_1, \ldots, Q_s$ be distinct points of $\mathbb{A}^1$. If $\mathbb{A}^1 - \{P_1, \ldots, P_r\}$ is isomorphic to $\mathbb{A}^1 - \{Q_1, \ldots, Q_s\}$, show that $r = s$. Is the converse true? Cf. (Ex. 3.1).

PROOF. Suppose $f : \mathbb{A}^1 - \{P_1, \ldots, P_r\} \to \mathbb{A}^1 - \{Q_1, \ldots, Q_s\}$ is an isomorphism with inverse g. This gives a birational map of $\mathbb{A}^1$ to itself. We can embed $\mathbb{A}^1$ into $\mathbb{P}^1_k$ and view it as a projective variety. In this case, we get a birational map of nonsingular projective curves which is always an isomorphism. The only way this can occur is if $\{P_1, \ldots, P_s, \infty\}$ gets mapped bijectively to $\{Q_1, \ldots, Q_r, \infty\}$ and so $r = s$.

The converse is false. It is know that $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}(1)$. If we take $r = s = 4$, then an isomorphism would have to map $\{P_1, \ldots, P_4\}$ bijectively to $\{Q_1, \ldots, Q_4\}$. Now the automorphism group of $\mathbb{P}^1$ is determined by the linear fractional transformations. This is the quotient of $\mathrm{GL}(2, k)$ by k^* and it is known that fractional linear transformations are determined by where it maps any three points. So by choosing the fourth point Q_4 accordingly (and using the fact that $\mathbb{A}^1_k$ is infinite), there cannot be an isomorphism in general. □

7. Intersections in Projective Space

Hartshorne I.7.1 (a) Find the degree of the d-uple embedding of $\mathbf{P}^n$ in $\mathbf{P}^N$ (Ex. 2.12). [Answer: d^n]
(b) Find the degree of the Segre embedding of $\mathbf{P}^r \times \mathbf{P}^s$ in $\mathbf{P}^N$ (Ex. 2.14). [Answer: $\binom{r+s}{r}$].

PROOF. (a) This can be checked directly, but let me use the language of sheaves. The Hilbert polynomial of the d-uple embedding is the Hilbert polynomial of the sheaf $\mathcal{O}(d)$ and so

$$H(m) = \dim_k H^0(\mathbb{P}^n, \mathcal{O}(md)) = \binom{md+n}{n}.$$

So,

$$H(m) = \frac{1}{n!}(md+n)(md+n-1)\ldots(md+1)$$

So the degree is $n!$ times the leading coefficient and the leading coefficient is $\frac{1}{n!}d^n$, I get degree d^n.

(b) The Segre embedding corresponds to the line bundle $p_1^*(\mathcal{O}_{\mathbb{P}^r}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^s}(1))$.

□

Hartshorne I.7.2 Let Y be a variety of dimension r in $\mathbf{P}^n$, with Hilbert polynomial P_Y. We define the *arithmetic genus* of Y to be $p_a(Y) = (-1)^r (P_Y(0) - 1)$. This is an important invariant which (as we will see later in (III, Ex. 5.3)) is independent of the projective embedding of Y.
(a) Show that $p_a(\mathbf{P}^n) = 0$.
(b) If Y is a plane curve of degree d, show that $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.
(c) More generally, if H is a hypersurface of degree d in $\mathbf{P}^n$, then $p_a(H) = \begin{pmatrix} d-1 \\ n \end{pmatrix}$.
(d) If Y is a complete intersection (Ex. 2.17) of surfaces of degrees a, b in $\mathbf{P}^3$, then $p_a(Y) = \frac{1}{2}ab(a+b-4)+1$.
(e) Let $Y^r \subseteq \mathbf{P}^n, Z^s \subseteq \mathbf{P}^m$ be projective varieties, and embed $Y \times Z \subseteq \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^N$ by the Segre embedding. Show that

$$p_a(Y \times Z) = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z).$$

Proof. (a) We know $P_{\mathbb{P}^n}(m) = \binom{m+n}{n}$. So $P_{\mathbb{P}^n}(0) = 1$ and so $p_a(\mathbb{P}^n) = 0$.
(b) From the proof of Hartshorne Proposition I.7.6, we have the SES

$$0 \to S(-d) \xrightarrow{f} S \to S/(f) \to 0$$

which gives

$$H_Y(m) = \binom{m+2}{2} - \binom{m-d+2}{2}.$$

Setting $m = 0$, we get $1 - \binom{2-d}{2}$ and so

$$H_Y(0) = 1 - \frac{(2-d)(2-d-1)}{2} = 1 - \frac{(2-d)(1-d)}{2} = \frac{2-(2-3d+d^2)}{2} = \frac{-d^2+3d}{2} = \frac{-d(d-3)}{2}.$$

Then,

$$p_a(Y) = -1\frac{-d(d-3)-2}{2} = \frac{d(d-3)+2}{2} = \frac{d^2-3d+2}{2} = \frac{(d-1)(d-2)}{2}.$$

(c) This is clear from the previous work because

$$p_a(H) = (-1)^{n-1}\left(\binom{n}{n} - \binom{-d+n}{n} - 1\right) = (-1)^n \binom{-d+n}{n} = (-1)^{n-1}\frac{(-d+n)(-d+n-1)+\ldots(-d+}{n!}$$
$$= \frac{(d-1)(d-2)(d-3)\ldots(d-n)}{n!} = \binom{d-1}{n}.$$

(d) Let Y be the complete intersection of hypersuraces of degree a and b in $\mathbb{P}^3$ cut out by equations f, g respectively. Then,

$$0 \to (S/(f))(-b) \xrightarrow{g} S/(f) \to S/(f+g) \to 0.$$

[Incomplete]
(e) **[Incomplete]**

□

Hartshorne I.7.3 *The Dual Curve.* Let $Y \subseteq \mathbf{P}^2$ be a curve. We regard the set of lines in $\mathbf{P}^2$ as another projective space, $(\mathbf{P}^2)^*$, by taking (a_0, a_1, a_2) as homogeneous coordinates of the line $L : a_0x_0 + a_1x_1 + a_2x_2 = 0$. For each nonsingular point $P \in Y$, show that there is a unique line $T_P(Y)$ whose intersection multiplicity with Y at P is > 1. This is the *tangent line* to Y at P. Show that the mapping $P \mapsto T_P(Y)$ defines a *morphism* of Reg Y (the set of nonsingular points of Y) into $(\mathbf{P}^2)^*$. The closure of the image of this morphism is called the dual curve $Y^* \subseteq (\mathbf{P}^2)^*$ of Y.

Proof. [Incomplete] □

Hartshorne I.7.4 Given a curve Y of degree d in $\mathbf{P}^2$, show that there is a nonempty open subset U of $(\mathbf{P}^2)^*$ in its Zariski topology such that for each $L \in U, L$ meets Y in exactly d points.
[Hint: Show that the set of lines in $(\mathbf{P}^2)^*$ which are either tangent to Y or pass through a singular point of Y is contained in a proper closed subset.] This result shows that we could have defined the degree of Y to be the number d such that almost all lines in $\mathbf{P}^2$ meet Y in d points, where "almost all" refers to a nonempty open set of the set of lines, when this set is identified with the dual projective space $(\mathbf{P}^2)^*$.

Proof. [Incomplete] □

Hartshorne I.7.5 (a) Show that an irreducible curve Y of degree $d > 1$ in $\mathbf{P}^2$ cannot have a point of multiplicity $\geqslant d$ (Ex. 5.3).
(b) If Y is an irreducible curve of degree $d > 1$ having a point of multiplicity $d - 1$, then Y is a rational curve (Ex. 6.1).

Proof. (a) Suppose not and let P be the point of multiplicity $m \geq d$. Then take a line L that crosses through the point of multiplicity $\geq d$. By Bezout's Theorem, we should get intersections totaling to multiplicity d. But intersecting a line through a point of multiplicity $m \geq d$ gives m intersections. So we get a contradiction if $m > d$. Now if $m = d$, then the line L intersects only at this point.

Let $m = d$. Now take a conic C that crosses through this point. By Bezout's theorem there should be $2d$ intersections and the intersection with this point accounts for d intersects. So the conic must intersect the curve Y at a total of d other points (with multiplicity). Of course, these d other points Q_i cannot possibly be points of degree > 1 on Y, since we could then take a line through P and Q to get a contradiction. So the conic must intersection Y at d other points.

[Incomplete]
(b) There is likely more than one way to do this (I did one proof in a homework set for Math 763 in Algebraic Geometry I at UW-Madison). Here's another way. Consider the projection away from this point of multiplicity $d - 1$ (this needs $d > 1$ clearly). The lines L through this point intersect at most 1 other point of Y. The collection of such lines is parameterized by $\mathbb{P}^1$ so there is a map $Y - P \to \mathbb{P}^1$ defined by sending $Q \in Y - P$ to the line $L = PQ$. This is clearly a birational map. So Y is a rational curve. □

Hartshorne I.7.6 *Linear Varieties.* Show that an algebraic set Y of pure dimension r (i.e., every irreducible component of Y has dimension r) has degree 1 if and only if Y is a linear variety (Ex. 2.11).
[Hint: First, use (7.7) and treat the case $\dim Y = 1$. Then do the general case by cutting with a hyperplane and using induction.]

PROOF. If Y is a linear variety, then Y is clearly of degree 1 and pure dimension r (cf. Hartshorne I.2.11) via an inductive argument.

For the other direction, we use the hint. Assume $\deg Y = 1$. Taking irreducible components, we may assume Y is irreducible.

For $\dim Y = 1$, we know that Y is a curve. We shall show that there is a hyperplane that contains Y. For any hypersurface H not containing Y, we know that there is $\deg H$ many intersection with Y. So for various hyperplanes not containing Y, there is a single intersection point with Y.

Suppose not and all hyperplanes intersecting with Y only ever intersect Y at a single point. Now we consider the space of all hyperplanes intersecting Y. They are determined as follows: pick a point $P \in Y$, then pick n linearly independent vectors that will span the hyperplane but none of the n linear independent vectors cannot sit on the line nor the subspace spanned by the previous vectors. In total, there is an n-dimensional family of hyperplanes intersecting Y.

[Incomplete]

□

Hartshorne I.7.7 Let Y be a variety of dimension r and degree $d > 1$ in $\mathbf{P}^n$. Let $P \in Y$ be a nonsingular point. Define X to be the closure of the union of all lines PQ, where $Q \in Y, Q \neq P$.
(a) Show that X is a variety of dimension $r + 1$.
(b) Show that $\deg X < d$. [Hint: Use induction on $\dim Y$].

PROOF. (a) One can check that X is birational to the cone over Y and so $\dim X = \dim Cone(Y) = \dim Y + 1 = r + 1$.
(b) **[Incomplete]** □

Hartshorne I.7.8 Let $Y^r \subseteq \mathbf{P}^n$ be a variety of degree 2 . Show that Y is contained in a linear subspace L of dimension $r+1$ in $\mathbf{P}^n$. Thus Y is isomorphic to a quadric hypersurface in $\mathbf{P}^{r+1}$ (Ex. 5.12).

PROOF. **Claim 1.** From the previous exercise, Y is contained in the closure of the union of all lines $Q \in Y$ and $Q \neq P$ which we call X. We also know that X it then a variety of dimension $r + 1$ with degree $\deg X = 1$. Now from Hartshorne I.7.6, X itself must be a linear variety.

Claim 2. We want to show Y is a quadric hypersurface in $\mathbb{P}^{r+1}$. First off, Y is clearly a hypersurface, so we check $\deg(Y) = 2$ inside $\mathbb{P}^{r+1}$. This is clear from the hypotheses as well. □

8. What is Algebraic Geometry

One day, I hope to write an answer to this question that can mirror a fraction of Hartshorne's answer's excellency.

CHAPTER 2

Chapter II Exercises

1. Sheaves

Hartshorne II.1.1. Let A be an abelian group, and define the constant presheaf associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.

Proof. It is more efficient to verify that $\mathscr{A}$ satisfies the condition in Hartshorne Proposition-Definition II.1.2. Let $\mathscr{F}$ be the constant presheaf associated to A. Then by definition,

$$\mathscr{F}^+(U) := \left\{ f : U \to \bigcup_{P \in U} \mathscr{F} = \bigcup_{P \in U} A = A : \forall P \in U, \begin{array}{c} f(P) \in \mathscr{F}_P = A, \\ \exists U \supseteq V \ni p, \exists t \in \mathscr{F}(U) \text{ s.t.} \\ \forall Q \in V, t(Q) = f(Q). \end{array} \right\}.$$

We shall show that $\mathscr{A}$ has precisely the stated properties on the RHS.

Each $s \in \mathscr{A}(U)$ is a continuous function $s : U \to A$ with A given the discrete topology. The second condition in $\mathscr{F}^+(U)$ ensures that the functions $f : U \to A$ are continuous. Next, given a $P \in U$, let $f(P) = R$ and $V := f^{-1}(R) \cap U$. Clearly, V is open and choose $t := R \in \mathscr{F}(U) = A$. For all $Q \in V$,

$$R(Q) = R_Q = R = f(Q).$$

This essentially shows that $\mathscr{A}(U) \subseteq \mathscr{F}^+(U)$. To show the other inclusion, we note that any element of $\mathscr{F}^+(U)$ is going to be precisely a continuous function $f : U \to A$ for A with the discrete topology given the assumptions.

□

Remark 8. This approach is easier than showing that $\mathscr{A}$ has the universal property. Hartshorne essentially did all of the hard work when it comes to checking the universal property (or at least, he gave that hard work to the reader).

Proof. Let $\mathscr{F}$ be the constant presheaf associated to A i.e. the presheaf associated to $U \mapsto \mathscr{F}(U) = A$. We claim that $\mathscr{F}^+$ is the same as the constant sheaf defined in the chapter. It suffices to prove that $\mathscr{F}^+$ is isomorphic to $\mathscr{A}$. So, we show that $\mathscr{A}$ satisfies the universal property of $\mathscr{F}^+$ and by uniqueness, we are done.

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\varphi} & \mathscr{G} \\ {\scriptstyle\theta}\downarrow & \nearrow_{\exists! \psi} & \\ \mathscr{A} & & \end{array} \qquad \begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ {\scriptstyle\theta(U)}\downarrow & \nearrow_{\psi(U)} & \\ \mathscr{A}(U) & & \end{array}$$

We wish to define $\theta : \mathscr{F} \to \mathscr{A}$ and show that for any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism $\psi : \mathscr{A} \to \mathscr{G}$ such that $\varphi = \psi \circ \theta$.

Define $\theta : \mathscr{F} \to \mathscr{A}$ in the obvious way. Suppose $f \in \mathscr{A}(U)$. So, $f = \theta(U)(a)$ for some $a \in A$ and $f : U \to A$ is a continuous map. Consider the collection $\{f^{-1}(a)\}_{a \in A}$ which forms an open cover of U. We know it forms an open cover since $f : U \to A$ is continuous and A was given the discrete topology.

We wish to have

$$\varphi(U)(a) = \psi(U)(\theta(U)(a)) \text{ for } a \in \mathscr{F}(U) = A.$$

So, we define $\psi(U) : \mathscr{A}(U) \to \mathscr{G}(U)$ by sending a to the element obtained by gluing the sections $\varphi(U)(a) = \psi(U)(f)|_{f^{-1}(a)}$ to a section t. By the sheaf structure on $\mathscr{G}$, this t is unique and does not depend on the choice of open cover of U. Then,

$$\psi(U)(\theta(U)(a)) = \psi(U)(f) = t$$

and $t|_{f^{-1}(a)} = \varphi(U)(a)$. This defines a morphism $\psi : \mathscr{A} \to \mathscr{G}$. To see that $\mathscr{A}$ is the unique sheaf that satisfies the universal property, we have $\psi : \mathscr{A} \to \mathscr{F}^+$ by taking $\mathscr{G} = \mathscr{F}^+$ and by the universal property of $\mathscr{F}^+$, we have a morphism $\delta\mathscr{F}^+ \to \mathscr{A}$. Since they ψ and δ must inverse morphisms, $\mathscr{F}^+ \cong \mathscr{A}$ as desired. □

Hartshorne II.1.2.

(a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, show that for each point $P, (\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$

(b) Show that φ is injective (respectively, surjective) if and only if the induced map on the stalks φ_P is injective (respectively, surjective) for all P.

(c) Show that a sequence ... $\mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \dots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof. Fix $\varphi : \mathscr{F} \to \mathscr{G}$ a morphism of sheaves. Throughout, we will assume $P \in U$ where U is an open subset of the underlying topological space.

(a) Suppose $\overline{s} \in (\ker \varphi)_P$. Then, it is represented by a pair $\langle s, U\rangle$ for some open U and $s \in \ker \varphi(U)$. In this case, $\varphi(s) = 0 \in \mathscr{G}(U)$. So, φ_P sends $\overline{s}$ to $\overline{0}$ in $\mathscr{G}_P$. Hence, $\overline{s} \in \ker(\varphi_P)$.

Conversely, let $\overline{s} \in \ker(\varphi_p)$ and $s \in \mathscr{F}(U)$ be its pullback. Let $t = \varphi(U)(s)$ and since $\overline{s} \in \ker(\varphi_P)$, $\overline{t} = 0$ in $\mathscr{G}_P$. So, for a neighborhood of p, $V \subseteq U$, we have $t|_V = 0$. So, $\varphi(V)(s) = 0$. So, $\overline{s} \in \ker(\varphi)_P$ by restriction to a smaller set. So, $\ker \varphi_P \subseteq (\ker \varphi)_P$. So, $\ker \varphi_P = (\ker \varphi)_P$.

For the second part, $\operatorname{im}(\varphi)_P \subseteq \operatorname{im}(\varphi_P)$ is obvious. Indeed, if $\overline{s} \in \operatorname{im}(\varphi)_P$, then $\overline{s}$ is represented by a pair $\langle s, U\rangle$ where $s \in \operatorname{im} \varphi(U)$. Choose a $t \in \mathscr{F}(U)$ s.t. $\varphi(U)$ maps t to s. Then, $\overline{t} \in \mathscr{F}_p$ gets mapped to $\overline{s} \in \operatorname{im}(\varphi_P)$. In which case, $\overline{s} \in \operatorname{im}(\varphi_P)$.

Let $\overline{t} \in \operatorname{im}(\varphi_P)$. Let $\overline{s} \in \mathscr{F}_P$ such that $\varphi_P(\overline{s}) = \overline{t}$. So, $\varphi(U)(s) = t$ and t is such that $t \mapsto \overline{t} \in \operatorname{im}(\varphi_P) \subseteq \mathscr{G}_P$. So, $\overline{t} \in (\operatorname{im} \varphi)_P$.

(b) We have

$$\varphi \text{ is injective} \iff \ker \varphi = 0 \iff (\ker \varphi)_P = 0 \iff (\ker \varphi_P) = 0 \iff \varphi_P \text{ is injective,}$$

for all $P \in X$. Note the second iff follows from Hartshorne Proposition II.1.1. Also,

$$\varphi \text{ is surjective} \iff \operatorname{im} \varphi = \mathscr{G} \iff (\operatorname{im} \varphi)_P = \mathscr{G}_P \iff (\operatorname{im} \varphi_P) = \mathscr{G}_P \iff \varphi_P \text{ is surjective,}$$

for all $P \in X$ and the same proposition.

(c) The proof that a sequence of sheaf morphisms is exact iff the induced sequence of stalks is exact follows immediately from (a) and (b).

Suppose we have a sequence

$$\ldots \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \to \ldots$$

of sheaves. It is exact iff each short sequence of sheaves

$$0 \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \to 0$$

is exact. We get a short exact sequence iff φ^{i-1} is injective and φ^i is surjective and $\operatorname{im}\varphi^{i-1} = \ker\varphi^i$. By (b), the first two conditions occur iff φ_P^{i-1} is injective and φ_P^i is surjective. On the other hand, we know that

$$\operatorname{im}\varphi^{i-1} = \ker\varphi^i \iff (\operatorname{im}\varphi^{i-1})_P = (\ker\varphi)_P \iff \operatorname{im}(\varphi_P^{i-1}) = \ker\varphi_P^i.$$

But this means for all P, the sequence

$$0 \to \mathscr{F}_P^{i-1} \xrightarrow{\varphi_P^{i-1}} \mathscr{F}_P^{i} \xrightarrow{\varphi_P^{i}} \mathscr{F}_P^{i+1} \to 0$$

is exact. Each such short sequence is exact iff

$$\ldots \to \mathscr{F}_P^{i-1} \xrightarrow{\varphi_P^{i-1}} \mathscr{F}_P^{i} \xrightarrow{\varphi_P^{i}} \mathscr{F}_P^{i+1} \to \ldots$$

is an exact sequence.

□

Hartshorne II.1.3

(a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{v_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, and an open set U such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

Proof. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.

(a) Assume φ is a surjective morphism of sheaves. So, $\operatorname{im}\varphi = \mathscr{G}$. By the preceding exercise, we know $\operatorname{im}\varphi_P = \mathscr{G}_P$. Let $U \subseteq X$ be an open subset, and let $s \in \mathscr{G}(U)$. For $P \in U$, consider the image of s in the stalk $s_P \in \mathscr{G}_P$ Since $\operatorname{im}\varphi_P = \mathscr{G}_P$, choose a $t_P \in \mathscr{F}_P$ s.t. $\varphi_P(t_P) = s_P \in \mathscr{G}_P$. Then, t_P is represented by the pair $\langle t(P), U_P, \rangle$ where $t(P) \in \mathscr{G}(U_P)$ and s_P is represented by $\langle s(P), U_P\rangle$. with $\varphi(t(P)) = s(P)$. The collection of all such U_P covers U. By construction, $\varphi(t(P)) = s|_{U_P}$.

Now, we prove the converse. We wish to show that $\operatorname{im}\varphi = \mathscr{G}$ since it will imply that the stalks are equal. It suffices to show that $\mathscr{G}(U) \subseteq \operatorname{im}\varphi(U)$ for every open set U. Let $s \in \mathscr{G}(U)$ be a section. By the hypothesis, let $\{U_i\}$ be an open covering of U and choose sections $t_i \in \mathscr{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$. Then, $s|_{U_i} \in \mathscr{G}(U_i)$ and in particular, $s|_{U_i} \in \operatorname{im}\varphi(U_i)$. We wish to glue to a section $t \in \mathscr{F}(U)$ s.t. $\varphi(U)(t) = s$ which would show that $s \in \operatorname{im}\varphi(U)$. To simplify our notation, let $s_i = s|_{U_i}$. Certainly, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all i, j. So, we have $t_i|_{U_i\cap U_j} = t_j|_{U_i\cap U_j}$ since we need $\varphi(t_i|_{U_i\cap U_j}) = \varphi(t_j|_{U_i\cap U_j})$. Thus, we can glue to a section $t \in \mathscr{F}(U)$ and $t|_{U_i} = t_i$ for all i. Hence, $\varphi(t) = s$ because $\varphi(t)|_{U_i} = \varphi(t_i) = s_i$ for all i and we can use the other sheaf axiom to conclude that $\varphi(t) = s$.

(b) [1] Consider $\mathbb{C} \setminus \{0\}$ with the standard topology. Let $\mathcal{O}$ denote the sheaf of holomorphic functions i.e. it is the sheaf associated to the presheaf $U \mapsto \mathcal{O}(U) := \{f : U \to \mathbb{C} : f \text{ holomorphic}\}$. Let $\mathcal{O}^*$ denote the sheaf of multiplicative holomorphic functions i.e. the sheaf associated to the presheaf $U \mapsto \mathcal{O}^*(U) := \{f : U \to \mathbb{C} : f \text{ holomorphic and } f(x) \neq 0 \ \forall x \in U\}$. In fact, the presheaves here are actually sheaves. Define a map $\mathcal{O} \to \mathcal{O}^*$ which sends f to $e^{2\pi i f}$. This is a morphism of sheaves.

However, $z \in \mathcal{O}^*(\mathbb{C} \setminus \{0\})$ is not the image of any $\mathcal{O}(\mathbb{C} \setminus \{0\})$ since the natural log is not holomorphic. However, $\mathcal{O}_z \to \mathcal{O}_z^*$ is a surjection since locally, a branch of logarithms always exist. So, there is a surjection on the stalks. By the previous exercise, we have a surjective morphism of sheaves.

□

Remark 9. In part (b), we have a crucial condition. A surjective morphism of sheaves $\mathscr{F} \to \mathscr{G}$ does not guarantee that $\mathscr{F}(U) \to \mathscr{G}(U)$ is surjective. For instance, $\mathcal{O}(\mathbb{C} \setminus \{0\}) \to \mathcal{O}^*(\mathbb{C} \setminus \{0\})$ is not surjective, but all of its stalks are. Even worse, there is no guarantee that $\Gamma(\mathscr{F}, X) \to \Gamma(\mathscr{G}, X)$ is surjective when $\mathscr{F} \to \mathscr{G}$ (take the restriction of $\mathcal{O}$ an $\mathcal{O}^*$ to $\mathbb{C} \setminus \{0\}$).

Hartshorne II.1.4

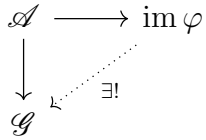
(a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Show that the induced map $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$of associated sheaves is injective.

(b) Use part (a) to show that if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then im φ can be naturally identified with a subsheaf of $\mathscr{G}$, as mentioned in the text.

Proof. As a lemma, we prove $\mathscr{F}_P^+ = \mathscr{F}_P$ for any presheaf $\mathscr{F}$ and point P (stated in the proof of Hartshorne Proposition-Definition II.1.2). It is obvious $\mathscr{F}_P^+ \cong \varinjlim_{U \supseteq P} \mathscr{F}^+(U)$ satisfies the universal property for the direct limit in the directed diagram given for $\mathscr{F}_P$.

(a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves s.t. $\varphi(U)$ is injective for each open set U. Since $\mathscr{F}_P = \mathscr{F}_P^+$ and $\mathscr{G}_P^+ = \mathscr{G}_P^+$ for every P, we have $\varphi_P^+ = \varphi_P$. Since the map on the stalks are injective (because each $\varphi(U)$ is injective), we conclude that the induced map $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ is injective.

(b) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. By the universal property[2], we have a natural map $\operatorname{im}\varphi \to \mathscr{G}$. Indeed, if $\mathscr{A}$ is the presheaf $U \mapsto \operatorname{im}(\varphi(U))$, then the maps $\mathscr{A}(U) \to \mathscr{G}(U)$ give a morphism of presheaves which induces the desired map



For any open set U, we have $\operatorname{im}(\varphi(U)) \subseteq \mathscr{G}(U)$ i.e. $\operatorname{im}(\varphi(U)) \hookrightarrow \mathscr{G}(U)$ is an injective map. Since we have an injective map $\operatorname{im}(\varphi(U)) \hookrightarrow \mathscr{G}(U)$ for all U, the induced map of sheaves $\operatorname{im}\varphi \hookrightarrow \mathscr{G}^+ = \mathscr{G}$ is injective. Hence, $\operatorname{im}\varphi$ can be identified with a subsheaf of $\mathscr{G}$.

[1] I believe that this is the standard example and that it is not very easy to find an example not using complex analysis.

[2] We refer to p. 64 of Hartshorne and the universal property of the sheaf associated to the presheaf image of φ.

□

Hartshorne II.1.5 Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

PROOF. By Hartshorne Proposition II.1.1, a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is an isomorphism iff the induced map on the stalks $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is an isomorphism. The map $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is an isomorphism iff it is both injective and surjective. This means that $\mathrm{im}(\varphi_P) = \mathscr{G}_P$ and $\ker(\varphi_P) = 0$. By Hartshorne II.1.2b, these are equivalent to saying that φ is injective and surjective respectively. So we are done; a morphism of sheaves is an isomorphism iff it is both injective and surjective. □

Hartshorne II.1.6

(a) Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$$

(b) Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and that $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.

PROOF. **(a)** Let $\phi : \mathscr{F}' \to \mathscr{F}$ be the inclusion map i.e. $\mathscr{F}'(U) \hookrightarrow \mathscr{F}(U)$ for all U. Define $\psi : \mathscr{F} \to \mathscr{F}/\mathscr{F}'$ by $\mathscr{F}(U) \to \mathscr{F}(U)/\mathscr{F}'(U)$ for every open set U. Then the induced map on stalks $\psi_P : \mathscr{F}_P \to (\mathscr{F}/\mathscr{F}')_P$ is surjective because $(\mathscr{F}/\mathscr{F}')_P \cong \mathscr{F}_P/\mathscr{F}'_P$. Furthermore, $\mathrm{im}\,\varphi_P = \ker\psi_P$. Since we have exact sequences

$$0 \to \mathscr{F}'_P \overset{\phi_P}{\to} \mathscr{F}_P \overset{\psi_P}{\to} \mathscr{F}_p/\mathscr{F}'_P \to 0$$

for all P, Hartshorne II.1.2 tells us the given sequence is exact.
(b) Suppose $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is exact. Then the induced sequence of stalks is exact. Because that implies $\mathscr{F}'_P \to \mathscr{F}_P$ is injective, we have $\mathscr{F}' \to \mathscr{F}$ injective. Then the fact that $\mathscr{F}' \to \mathscr{F}$ is injective iff $\mathscr{F}'(U) \to \mathscr{F}(U)$ is injective for all open U implies $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$. Part (a) gives us a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}/\mathscr{F}' & \longrightarrow 0 \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}'' & \longrightarrow 0 \longrightarrow & 0
\end{array}$$

where the map $\mathscr{F}/\mathscr{F}' \to \mathscr{F}''$ is obtained from the given SES. Applying the 5-lemma, the map $\mathscr{F}/\mathscr{F}' \to \mathscr{F}''$ is actually an isomorphism (assuming the independent fact that the category of sheaves of abelian groups is an abelian category).

If we did not want to use the 5-lemma here, we note that the natural map $\mathscr{F}/\mathscr{F}' \to \mathscr{F}''$ has an isomorphism on stalks at every point. □

Hartshorne II.1.7 Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.

(a) Show that $\mathrm{im}\,\varphi \cong \mathscr{F}/\ker\varphi$.
(b) Show that $\mathrm{coker}\,\varphi \cong \mathscr{G}/\,\mathrm{im}\,\varphi$.

PROOF. **(a)** The isomorphism $\operatorname{im}\varphi \cong \mathscr{F}/\ker\varphi$ follows from how

$$0 \to \ker\varphi \to \mathscr{F} \to \operatorname{im}\varphi \to 0$$

is an exact sequence since the induced sequence on stalks is exact. Indeed, we have

$$0 \to (\ker\varphi)_P \to \mathscr{F}_P \to (\operatorname{im}\varphi)_P \to 0$$

which is equivalent, by Hartshorne II.1.2, to (an obviously exact sequence)

$$0 \to \ker(\varphi_P) \to \mathscr{F}_P \to \operatorname{im}(\varphi_P) \to 0.$$

The existence of the maps $\ker\varphi \to \mathscr{F}$ and $\mathscr{F} \to \operatorname{im}\varphi$ follow by sheafifying the corresponding presheaves and using the fact that the sheafification functor is exact.
(b) The isomorphism $\operatorname{coker}\varphi \cong \mathscr{G}/\operatorname{im}\varphi$ follows by considering $0 \to \operatorname{im}\varphi \to \mathscr{G} \to \operatorname{coker}\varphi \to 0$ and an argument similar to the above. First, the induced morphism of stalks is

$$0 \to (\operatorname{im}\varphi)_P \to \mathscr{G}_P \to (\operatorname{coker}\varphi)_P \to 0.$$

Then by Hartshorne II.1.2, the definition of $\operatorname{coker}\varphi$, and how $(\mathscr{F}/\mathscr{F}')_P = \mathscr{F}_P/\mathscr{F}'_P$ for any sheaves $\mathscr{F}, \mathscr{F}'$, shows that the following sequence is exact

$$0 \to \operatorname{im}(\varphi_P) \to \mathscr{G}_P \to \mathscr{G}_P/\operatorname{im}(\varphi_P) \to 0.$$

The existence of $\operatorname{im}\varphi \to \mathscr{G}$ follows by Hartshorne II.1.4(b) while the map $\mathscr{G} \to \operatorname{coker}\varphi$ follows by considering the corresponding presheaf. Considering corresponding sequence of presheaves and then sheafifying completes the construction. □

Hartshorne II.1.8 For any open subset $U \subseteq X$, show that the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e., if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U, \mathscr{F}') \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$ is an exact sequence of groups. The functor $\Gamma(U, \cdot)$ need not be exact; see (Ex. 1.21) below.

PROOF. Let $0 \to \mathscr{F}' \to^{\varphi} \mathscr{F} \to^{\psi} \mathscr{F}''$ be exact. So, φ is an injection. By the remarks in the text, $\varphi(U)$ is an injection.

Let $\operatorname{im}\varphi(U) \subseteq \ker\psi(U)$. Given $s \in \mathscr{F}'(U)$. We see that

$$((\psi(U) \circ \varphi(U))(s))_P = (\psi_P \circ \varphi_P)(s_P) = 0.$$

So, since we have a sheaf , $\psi_U \circ \varphi_U = 0$.

To show the reverse inclusion $t \in \ker\psi(U)$. So, $\psi(U)(t) = 0$. Given $t \in \ker\psi(U)$. Hence $\psi(U)(t) = 0$. By exactness, let $t_p \in \ker\psi_P$ and choose $s_P \in \mathscr{F}'_P$ such that $\varphi_P(s_P) = t_P$. Since $s_P \in \mathscr{F}'_P$, let $s_P = (V_P, s(P))$ where $s(P) \in \mathscr{F}(V_P)$. So, $\varphi(s(P)) = t|_{V_P}$. Take $\{V_p\}$ to cover U and $s(P)$ associate to each V_P with intersection being sent to t. By the sheaf condition, there is an $s \in \mathscr{F}'(U)$ such that $s|_{V_P} = s(P)$ because $\varphi(s_P|_{V_P \cap V_Q}) = \varphi(s_Q|_{V_P \cap V_Q}) = t|_{V_P \cap V_Q}$ implies $s_P|_{V_P \cap V_Q} = s_Q|_{V_P \cap V_Q}$. So, $\varphi(s) = t$. Hence, $t \in \operatorname{im}\varphi(U)$.

Finally, the failure of exactness is a consequence of how $\Gamma(U, \cdot)$ is not a right exact functor. An example can be found by using the example in the solution to Hartshorne II.1.3b[3]. □

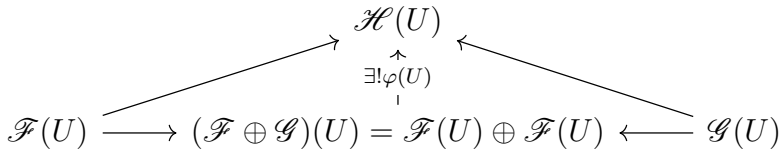
Hartshorne II.1.9 *Direct Sum.* Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf. It is called the *direct sum* of $\mathscr{F}$ and $\mathscr{G}$, and is denoted by $\mathscr{F} \oplus \mathscr{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X.

[3]Or one could use Nathan's monstrous example, or Hartshorne's exercise.

PROOF. Suppose $s \oplus t|_{V_i} = 0$ on each V_i of a covering $\{V_i\}$. Since $s \oplus t|_{V_i} = s|_{V_i} \oplus t|_{V_i}$ and $\mathscr{F}$ and $\mathscr{G}$ are sheaves, $s = 0$ and $t = 0$. So, $s \oplus t = 0$.

If $s_i \oplus t_i \in \mathscr{F}(V_i) \oplus \mathscr{G}(V_i)$ agrees with $s_j \oplus t_j \in \mathscr{F}(V_j) \oplus \mathscr{G}(V_j)$ on intersections, there is a unique $s \oplus t$ that agrees with $s_i \oplus t_i$ on V_i by making an appropriate choice of s and t by the sheaf axioms on $\mathscr{F}$ and $\mathscr{G}$.

The fact that $\mathscr{F} \oplus \mathscr{G}$ plays the role of direct sum in the category of sheaves of abelian groups on X is immediate from the universal properties. Indeed, suppose we have a sheaf $\mathscr{H}$ such that there are maps from $\mathscr{F}$ and $\mathscr{G}$. Then, use the universal property of direct sums in the category of abelian groups to get a unique morphism $\varphi(U) : \mathscr{F} \oplus \mathscr{G}(U) \to \mathscr{H}(U)$ s.t. the following diagram commutes



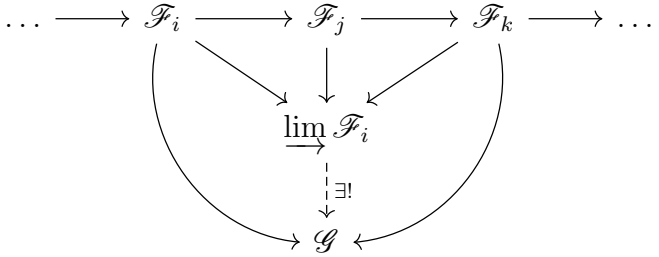
The maps $\varphi(U)$ then give a morphism $\varphi : \mathscr{H} \to \mathscr{F} \oplus \mathscr{G}$ which satisfies the desired universal property. This proof extends naturally to the direct sum of an arbitrary collection $\{\mathscr{F}_k\}$ of sheaves. For the direct product, the proof in the case of finitely many $\mathscr{F}_k$ is the same.

Note that for the case where there are arbitrarily many elements, we must sheafify. That is, $\bigoplus_{k \in K} \mathscr{F}_k$ is the sheafification of the presheaf $U \mapsto \bigoplus_{k \in K} \mathscr{F}_k(U)$. As Hartshorne is only asking for the case of finitely many sheaves, we will write no more[4].

□

Hartshorne II.1.10 *Direct Limit.* Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the direct limit of the system $\{\mathscr{F}_i\}$, denoted $\varinjlim \mathscr{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathscr{G}$, and a collection of morphisms $\mathscr{F}_i \to \mathscr{G}$, compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathscr{F}_i \to \mathscr{G}$ such that for each i, the original map $\mathscr{F}_i \to \mathscr{G}$ is obtained by composing the maps $\mathscr{F}_i \to \varinjlim \mathscr{F}_i \to \mathscr{G}$.

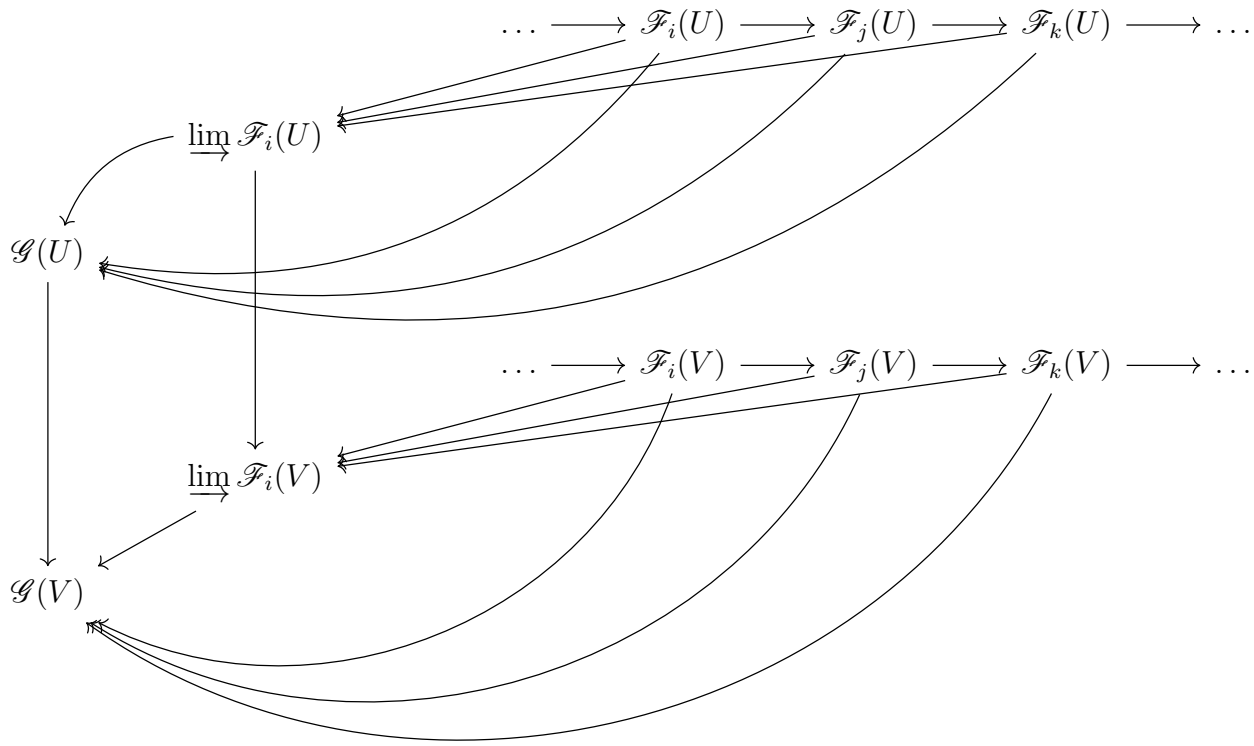
PROOF. The universal property we want to show is that following diagram commutes



We know that for any direct system $\{\mathscr{F}_i(U)\}$ of sections of sheaves, $\varinjlim \mathscr{F}_i(U)$ is the direct limit of that system. Additionally, the restriction maps of each $\mathscr{F}_i$ gives us restriction maps $\mathscr{F}_i(U) \to \mathscr{F}_i(V)$ whenever $V \subseteq U$. So, we have a morphism of direct systems that is induced by restriction maps $\{\mathscr{F}_i(U)\} \to \{\mathscr{F}_i(V)\}$. Each of the direct system has a direct limit and so, we have a morphism $\varinjlim \mathscr{F}_i(U) \to \varinjlim \mathscr{F}_i(V)$ of direct limits.

[4]In fact, I expect that the two do not coincide for arbitrary collections of sheaves.

Now, since each of the maps commute and since a given sheaf $\mathscr{G}$ would have maps $\mathscr{F}_i(U) \to \mathscr{G}(U)$ commuting with the restriction maps for all i, we get maps $\varinjlim \mathscr{F}_i(U) \to \mathscr{G}(U)$ by the universal property and which commute with the restriction maps. So, we have a morphism $\varinjlim \mathscr{F} \to \mathscr{G}$ of sheaves which is unique by the uniqueness of each $\varinjlim \mathscr{F}_i(U) \to \mathscr{G}(U)$ and s.t. $\mathscr{F}_i \to \varinjlim \mathscr{F}_i \to \mathscr{G}$.



This diagram may or may not be helpful to see how Hartshorne II.1.10. works out. □

Remark 10. Hint if stuck: use the universal properties of direct limits and sheafification.

Hartshorne II.1.11 Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In $\Gamma\left(X, \varinjlim \mathscr{F}_i\right) = \varinjlim \Gamma\left(X, \mathscr{F}_i\right)$.

Proof. Omitted.

For the first part, $U \mapsto \mathscr{F}_i(U)$ can be seen to be a sheaf by verifying the sheaf axioms. This will require use of X being a noetherian topological space and the definition of a direct limit.

Let U be an open set and $\{V_i\}$ be a covering of U. Let $s \in \varinjlim \mathscr{F}_i(U)$ be an element s.t. $s|_{V_i} = 0$ for all i. We will show $s = 0$. An element of

$$\varinjlim \mathscr{F}_i(U)$$

is represented by a pair (s, j) where $s \in \mathscr{F}_j(U)$. Because the topological space we are working with is quasi-compact, we may choose a finite subcover $\{V_i\}$. Since $s|_{V_i} = 0$ for all i, we know that $(s, j)|_{V_i} = 0$ for some j depending on i. Choose j sufficiently large so that $(s, j) = 0$ on each V_i. But this implies that $(s, j) = 0$ on U. Thus, $s = 0$.

Let U be an open set and $\{V_i\}$ be a covering of U. Let $s_i \in \varinjlim \mathscr{F}_i(V_i)$ for each i and suppose $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. We show that there is an element $s \in \varinjlim \mathscr{F}_i(U)$ s.t. $s|_{V_i} = s_i$. By quasi-compactness, we may assume there are only finitely many open sets $\{V_i\}$. Next, each s_i is represented by (s_i, k_i) where $s_i \in \mathscr{F}_{k_i}(V_i)$. By the sheaf axioms on each $\mathscr{F}_{k_i}$, we may glue to an $s_{k_i} \in \mathscr{F}_{k_i}(U)$ s.t. $s_{k_i}|V_i = s_i$. Then, take $s \in \varinjlim \mathscr{F}_i(U)$ to be the element represented by (s_{k_i}, k_i) where $s_{k_i} \in \mathscr{F}_{k_i}(U)$. By construction, we have $s_{k_i}|V_i = s_i$. **[Incomplete]**[5]

□

Hartshorne II.1.12 *Inverse Limit.* Let $\{\mathscr{F}_i\}$ be an inverse system of sheaves on X. Show that the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is a sheaf. It is called the inverse limit of the system $\{\mathscr{F}_i\}$, and is denoted by $\varinjlim \mathscr{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

Proof. Given

$$s = (s_i)_{i \in I} \in \varprojlim \mathscr{F}_i(U) = \left(\prod_{i \in I} \mathscr{F}_i(U) \right) / \{\varphi_{ij}(s_j) = (s_i) : \forall i \leq j\}.$$

Suppose $\{V_k\}$ is an open covering of U and that $s|_{V_k} = 0$ for all k. Then that means $s_i|_{V_k} = 0$ for all $i \in I$. By the sheaf axioms, $s_i = 0$ for all i and therefore $s = 0$.

Now suppose we had coherent sequences $(s_i)_k \in \varprojlim \mathscr{F}_i(V_k)$ in which $(s_i)_k|_{V_k \cap V_l} = (s_i)_l|_{V_k \cap V_l}$. But this implies that each coordinate of the sequence agree on the intersection so we may glue to a new section $S_i \in \mathscr{F}_i(U)$ s.t. $S_i|_{V_k} = s_i|_k$. Furthermore, the sequence $(S_i)_{i \in I}$ is coherent since $\varphi_{ij}(S_i) = S_j$ holds when restricted to each V_k.

To show that $\varinjlim \mathscr{F}_i$ is an inverse limit in the category of sheaves, we show it satisfies the universal property. Name $\varphi_i : \varinjlim \mathscr{F}_i \to \mathscr{F}_i$ so that $\varphi_{ij} \circ \varphi_i = \varphi_j$. Suppose $\theta_i : \mathscr{H} \to \mathscr{F}_i$ and $\varphi_{ij} \circ \theta_i = \theta_j$ for all i, j pairs. From universal property of inverse limits for abelian groups, we obtain maps $\Phi(U) : \mathscr{H}(U) \to \varinjlim \mathscr{F}_i(U)$ s.t. $\varphi_{ij}(U) \circ \Phi(U) = \theta_j(U)$. These map, being natural, thereby yields a sheaf morphism $\Phi \to \varinjlim \mathscr{F}_i$ satisfying $\varphi_i \circ \Phi = \theta_i$. □

Remark 11. It might be interesting to see why Hartshorne II.1.12 does not require X to be noetherian. For that, see [11].

Remark 12. Remember RAPL! Right Adjoints Preserve Limits. This is very useful for remembering that inverse limits of sheaves don't require a noetherian topological space. In particular, the sheafification functor is a left adjoint and therefore preserves colimits (and it is left adjoint to the forgetful functor). As the forgetful functor is a right adjoint, it preserves limits. This is why we do not need to sheafify.

Remark 13. Note that the sheafification functor is right adjoint and so, it will preserve limits (which are inverse limits). It will not preserve colimits (which are direct limits).

Hartshorne II.1.13 *Espace Étale of a Presheaf.* (This exercise is included only to establish the connection between our definition of a sheaf and another definition often found in the literature. See for example Godement [1, Ch. II, §1.2].) Given a presheaf $\mathscr{F}$ on X, we define a topological space Spé($\mathscr{F}$), called the espace étalé of $\mathscr{F}$, as follows. As a set, Spé($\mathscr{F}$) $= \bigcup_{P \in X} \mathscr{F}_P$. We define a projection map $\pi :$ Spé($\mathscr{F}$) $\to X$ by sending $s \in \mathscr{F}_P$ to P. For each open set $U \subseteq X$ and each section $s \in \mathscr{F}(U)$, we obtain a map $\bar{s} : U \to$ Spé($\mathscr{F}$) by sending $P \mapsto s_P$, its germ at P. This map has the property that $\pi \circ \bar{s} = \mathrm{id}_U$, in

[5] I need to clean up this verification carefully.

other words, it is a "section" of π over U. We now make Spé($\mathscr{F}$) into a topological space by giving it the strongest topology such that all the maps $\bar{s} : U \to$Spé($\mathscr{F}$) for all U, and all $s \in \mathscr{F}(U)$, are continuous. Now show that the sheaf $\mathscr{F}^+$associated to $\mathscr{F}$ can be described as follows: for any open set $U \subseteq X, \mathscr{F}^+(U)$ is the set of continuous sections of Spé($\mathscr{F}$) over U. In particular, the original presheaf $\mathscr{F}$ was a sheaf if and only if for each $U, \mathscr{F}(U)$ is equal to the set of all continuous sections of Spé($\mathscr{F}$) over U.

Proof. Let $s \in \mathscr{F}^+(U)$. Then s is a function $U \to \bigcup_{P \in X} \mathscr{F}_P$ given by $P \mapsto s(P) \in \mathscr{F}_P$. In particular, there exists a section $t \in \mathscr{F}(V)$ for $U \supseteq V \ni P$ s.t. $s(P) = t_P$. Then, it is clear that $\pi \circ s(P) = \pi \circ (t_P) = P$ for all $P \in U$ which means $\pi \circ s = \mathrm{id}_U$. By definition of the topology on the espace étale, s is a continuous section of Spé($\mathscr{F}$) over U. On the other hand, if we have a continuous section $s \in \mathscr{F}(U)$ of Spé($\mathscr{F}$) over U, we naturally get $s \in \mathscr{F}^+(U)$. Indeed, $s(P) = s_P \in \mathscr{F}_P$ and we may choose s for our choice of section t for any $U \supseteq V \ni P$.

The final fact follows from the above. The original presheaf was a sheaf iff $\mathscr{F}^+(U) = \mathscr{F}(U)$ iff $\mathscr{F}(U)$ equals the set of all continuous sections of Spé($\mathscr{F}$) over U. □

Hartshorne II.1.14 *Support.* Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The support of s, denoted Supp s, is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that Supp s is a closed subset of U. We define the support of $\mathscr{F}$, Supp $\mathscr{F}$, to be $\{P \in X \mid \mathscr{F}_P \neq 0\}$. It need not be a closed subset.

Proof. It suffices to show that $U \setminus \mathrm{Supp}\, s$ is open. Suppose $q \in U \setminus \mathrm{Supp}\, s$. Then $s_q = 0$ and so, by definition, there is a neighborhood $V \subseteq U$ s.t. $s_r = 0$ for all $r \in V$. So, $U \setminus \mathrm{Supp}\, s$ is open as desired.

Now we show Supp $\mathscr{F}$ need not be a closed set. Let $U \subseteq X$ be an open subset of a topological space and define a sheaf on U as follows. Assign to each open $V \subseteq U$, $\mathscr{F}(V)$ if $V \subset U$ and 0 if $V \not\subset U$.

Now, this sheaf has the property that $\mathscr{F}_x = 0$ if $x \notin U$ and $\mathscr{F}_x \neq 0$ if $x \in U$. So, Supp $\mathscr{F} = U$ as desired. □

Hartshorne II.1.15 *Sheaf $\mathscr{H}om$.* Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$ show that the set Hom $(\mathscr{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf $U \mapsto$ Hom $(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf. It is called the sheaf of local morphisms of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and is denoted $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

Proof. The set $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ has a natural structure of an abelian group if we define

$$(f+g)(V) = f(V) + g(V) : \mathscr{F}|_U(V) \to \mathscr{G}|_U(V)$$

for any pair of sheaf morphisms $f, g : \mathscr{F}|_U \to \mathscr{G}|_U$. It is clear that $f + g$ is a morphism of sheaves.

$$\begin{array}{ccc} \mathscr{F}|_U(V) & \xrightarrow{(f+g)(V)} & \mathscr{G}|_U(V) \\ \downarrow{\scriptstyle \rho_{VW}} & & \downarrow{\scriptstyle \rho_{VW}} \\ \mathscr{F}|_U(W) & \xrightarrow{(f+g)(W)} & \mathscr{F}|_U(W) \end{array}$$

The computation follows via

$$\begin{aligned}((f+g)(W))(\rho_{VW}) &= f(W)\rho_{VW} + g(W)\rho_{VW} = \rho_{VW}f(V) + \rho_{VW}g(V)\\ &= \rho_{VW}(f(V)+g(V)) = \rho_{VW}((f+g)(V)).\end{aligned}$$

To show that $U \mapsto \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf, we verify the identity and gluing axioms because it is clearly a presheaf.

Suppose U can be covered by $\{V_i\}$ and there is an $\phi \in \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ s.t. $\phi|_{V_i} = 0$ for all V_i. So, $\phi_p : \mathscr{F}_p \to \mathscr{G}_p$ for all $p \in U$ and therefore $\phi = 0$ by properties of stalks.

Assume U is covered by $\{V_i\}$. Suppose there are $\phi_i \in \mathrm{Hom}(\mathscr{F}|_{V_i}, \mathscr{G}|_{V_i})$'s which agree on overlaps i.e. $\phi_i|_{V_i\cap V_j} = \phi_j|_{V_i\cap V_j}$ as maps from $\mathscr{F}(V_i \cap V_j) \to \mathscr{G}(V_i \cap V_j)$ (we shall omit U for ease). Let $s \in \mathscr{F}(U)$ and so $\phi_i(s)|_{V_i\cap V_j} = \phi_j(s)|_{V_i\cap V_j}$. Then the sections $\phi_i(s) \in \mathscr{G}(V_i)$ glue to a section which we call $t_s \in \mathscr{G}(U)$. Then, we define $\phi \in \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ by $\phi(s) = t_s$. This gives a well-defined map because of the sheaf axioms. Furthermore, $\phi|_{V_k} = \phi_k$ for all k because

$$\phi|_{V_k}(s) = (\phi_{V_k}(s))|_{V_k} = (t_s)|_{V_k} = \phi_k(s).$$

Since $\phi_i \in \mathrm{Hom}(\mathscr{F}_{V_i}, \mathscr{G}_{V_i})$, and by the the sheaf axioms, s_i is well-defined. Indeed, since we can glue morphisms of abelian groups, I can find such a desired s. □

Hartshorne II.1.16 *Flasque Sheaves.* A sheaf $\mathscr{F}$ on a topological space X is flasque if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

(a) Show that a constant sheaf on an irreducible topological space is flasque. See (I, §1) for irreducible topological spaces.

(b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ is flasque, then for any open set U, the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ of abelian groups is also exact.

(c) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flasque, then $\mathscr{F}''$ is flasque.

(d) If $f : X \to Y$ is a continuous map, and if $\mathscr{F}$ is a flasque sheaf on X, then $f_*\mathscr{F}$ is a flasque sheaf on Y

(e) Let $\mathscr{F}$ be any sheaf on X. We define a new sheaf $\mathscr{G}$, called the sheaf of discontinuous sections of $\mathscr{F}$ as follows. For each open set $U \subseteq X, \mathscr{G}(U)$ is the set of maps $s : U \to \bigcup_{P\in U} \mathscr{F}_P$ such that for each $P \in U, s(P) \in \mathscr{F}_p$. Show that $\mathscr{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathscr{F}$ to $\mathscr{G}$.

Proof. (a) From Hartshorne Example 1.0.3, we know how the restriction maps of a constant sheaf $\mathscr{A}$ are defined. Because an irreducible topological space is necessarily connected, we know $\mathscr{A}(U) \cong A$ for every nonempty open $U \subseteq X$. Then, $\rho_{UV} : \mathscr{A}(U) \to \mathscr{A}(V)$ has to be surjective since they are isomorphisms of groups.

(b) [6] The global sections functor is already left exact by Hartshorne II.1.8. So, we want to show

$$\mathscr{F}'(U) \overset{\phi(U)}{\to} \mathscr{F}(U) \overset{\psi(U)}{\to} \mathscr{F}''(U) \to 0 \text{ is exact.}$$

It is clear that $\operatorname{im}\phi(U) = \ker\psi(U)$ so we need to show that $\psi(U)$ is surjective.

Let $s \in \mathscr{F}''(U)$ be a section. By Hartshorne II.1.3a, there exists a covering $\{U_i\}_{i\in I}$ (we omit the indexing set from here on) and sections $t_i \in \mathscr{F}(U_i)$ s.t. $\psi(t_i) = s|_{U_i}$. We may assume WLOG that $U_i \subseteq U$ for all i. Given this covering, we consider the set

$$\mathfrak{S} := \{(U_i, t_i) : U_i \in \{U_i\}_{i\in I}, t_i \in \mathscr{F}(U_i), \psi(t_i) = s|_{U_i}\}$$

(set of all possible coverings obtained from Hartshorne II.3.1a) and give it a partial ordering defined by $(U_i, t_i) \leq (U_j, t_j)$ if $t_j|_{U_i} = t_i$ and $U_i \subseteq U_j$. The set $\mathfrak{S}$ is clearly nonempty because we may assume U is nonempty. Given a chain $\{(U_{i_\alpha}, t_{i_\alpha})\}_{\alpha\in A}$. Because $t_{i_\alpha}|_{U_{i_\beta}} = t_{i_\beta}$ whenever $\beta \leq \alpha$, we know the sections are compatible so we may glue the t_{i_α} to a section t and consider the pair $\left(\bigcup_{\alpha\in A} U_{i_\alpha}, t\right)$ which is in the chain because $U_{i_\beta} \subseteq U_{i_\alpha}$ whenever $\beta \leq \alpha$ and how we constructed the section t.

Let (V, r) denote the maximal element of $\mathfrak{S}$.

We claim $V = U$. Suppose not and $V \subsetneq U$. Then choose an open set $U_j \in \{U_i\}$ s.t. $V \subsetneq V \cup U_j \subseteq U$. Then, $r|_{U_j} = t_j$ which implies $r|_{U_j\cap V} = t_j|_{U_j\cap V}$ and therefore,

$$r|_{U_j\cap V} - t_j|_{U_j\cap V} = 0.$$

This means $r|_{U_j\cap V} - t_j|_{U_j\cap V}$ in the kernel of $\psi(U_k \cap V)$. From exactness, $r|_{U_j\cap V} - t_j|_{U_j\cap V}$ in the image of $\phi(U_k \cap V)$. So, choose a $q \in \mathscr{F}'(U_k \cap V)$ s.t. $\phi(U_k \cap V)(q) = r|_{U_j\cap V} - t_j|_{U_j\cap V}$. By flasqueness, choose a $p \in \mathscr{F}'(V)$ s.t. p restricts to q. Set $z := \phi(V)(p) \in \mathscr{F}(V)$ which means $\psi(V)(z) = 0$. By construction,

$$z|_{V\cap U_j} = r|_{V\cap U_j} - t_j|_{U_k\cap V} \qquad \Longrightarrow \qquad (r-z)|_{V\cap U_j} = t_j|_{V\cap U_j}.$$

By the sheaf axioms, glue to a section $u \in \mathscr{F}(V \cup U_k)$ s.t. $u|_V = (r - z)$ and $u|_{U_j} = t_j$. Furthermore,

$$\psi(V \cup U_j)(u) = s|_{V\cup U_j}$$

because on U_j, we get t_j which equals $s|_{U_j}$ while on V, we get $(r-z)$ which restricts to $(r-z)|_V$ and recalling that $z \in \ker\psi(V)$, this means we $\psi(V)((r-z)) = r|_V = s|_V$. This computation proves the above equality. This means $(V, r) \leq (V \cup U_j, u) \in \mathfrak{S}$ which contradicts the maximality of (V, r).

At last, we know $V = U$. But by contradiction, $\psi(r) = s|_U = s$ which implies $\psi(U)(r) = s$. So, we have shown that $r \in \operatorname{im}(\psi(U))$.

(c) Let $V \subseteq U$. From the hypothesies, we have the following diagram

[6] A reference can also be found in the Stacks Project Lemma 09SY.
There is also a post here. However, the proof there doesn't seem to be entirely correct. The definition of $\mathfrak{S}$ shouldn't based on a single open covering obtained from Hartshorne II.1.8. It should consider all possible coverings, and maybe like what the Stacks project seems to do, consider open sets and section pairs only. Then, just use Hartshorne II.1.8 to show that the set is nonempty.
Another way to do this exercise is to show that $\mathscr{F}/\mathscr{F}'$ is flasque – see this post.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}'(U) & \longrightarrow & \mathscr{F}(U) & \longrightarrow & \mathscr{F}''(U) & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \vdots & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'(V) & \longrightarrow & \mathscr{F}'(V) & \longrightarrow & \mathscr{F}''(V) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

where the vertical maps are the restriction maps and maps with the trivial group are the obvious ones. By application of the Five Lemma[7], the dotted map, which is the restriction map $\rho_{UV} : \mathscr{F}''(U) \to \mathscr{F}''(V)$, is also surjective.

(d) Let $V' \subseteq U'$ be an inclusion of open subsets of Y. Then, $f^{-1}(V') \subseteq f^{-1}(U')$ is an inclusion in X. Then, the restriction map is

$$f_*\mathscr{F}(U') = \mathscr{F}(f^{-1}(U')) \to \mathscr{F}(f^{-1}(V')) = f_*\mathscr{F}(V')$$

which is surjective.

(e) By definition of $\mathscr{G}(U) = \{s : U \to \bigcup_{P\in U} \mathscr{F}_P | s(P) \in \mathscr{F}_P,\ \forall P \in U\}$, $\mathscr{G}(U) \to \mathscr{G}(V)$ is a surjection. Indeed, if $s \in \mathscr{G}(V)$, then $s(Q) \in \mathscr{F}_Q$ for all $Q \in V$ and in particular, $s \in \mathscr{G}(U)$ by taking $s(P) = 0$ for all $P \in U \setminus V$.

For $\mathscr{F} \hookrightarrow \mathscr{G}$, we need to define $\mathscr{F}(U) \hookrightarrow \mathscr{G}(U)$ for all U. For each $s \in \mathscr{F}(U)$, send s to $s : U \to \bigcup_{P\in U} \mathscr{F}_P$ which is the function sending P to $s(P)$ for $P \in U$. This is injective since if $s = t$ in $\mathscr{G}(U)$, then they define the same $s(P) = t(P)$ for all P. By the identity axiom of sheaves, $s = t$.

□

Hartshorne II.1.17 *Skyscraper Sheaves.* Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U, 0$ otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \{P\}^-$, and 0 elsewhere, where $\{P\}^-$denotes the closure of the set consisting of the point P Hence the name "skyscraper sheaf." Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\{P\}^-$, and $i : \{P\}^- \to X$ is the inclusion.

Proof. We claim $i_P(Q) = A$ for all $Q \in \{P\}^-$ and 0 elsewhere. Indeed, such a Q has a neighborhood containing P so that $i_P(A)(U) = A$. Otherwise, it is zero since $i_P(A)(V) = 0$ for all sufficiently small neighborhoods V of Q where $Q \notin \{P\}^-$.

The characterization $i_*(A)$ where $i : \{P\}^- \to X$ where A is the constant sheaf on $\{P\}^-$ is clear. Indeed, $i_*(A)(U) = A$ if $P \in U$ since $i^{-1}(U)$ would contain P. Otherwise, $i_*(A)(U) = A(i^{-1}(U))$ where $P \notin i^{-1}(U)$ since $P \notin U$ and hence, is zero. □

Hartshorne II.1.18 *Adjoint Property of* f^{-1}. Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$, and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y

$$\mathrm{Hom}_X\left(f^{-1}\mathscr{G}, \mathscr{F}\right) = \mathrm{Hom}_Y\left(\mathscr{G}, f_*\mathscr{F}\right)$$

Hence we say that f^{-1} is a left adjoint of f_*, and that f_* is a right adjoint of f^{-1}.

Proof. It is helpful to recall that the inverse image sheaf $f^{-1}\mathscr{G}$ is defined by sheafifying the presheaf $U \mapsto \lim_{V\supseteq f(U)} \mathscr{G}(V)$ and that the direct image sheaf is defined by $V \mapsto \mathscr{F}(f^{-1}(V))$.

[7]Specifically, one uses the special case of the four lemma which is the first four lemma on the following page.

Putting this together,

$$f^{-1}f_*\mathscr{F}(U) = \lim_{V \supseteq f(U)} f_*\mathscr{F}(V) = \lim_{V \supseteq f(U)} \mathscr{F}(f^{-1}(V)).$$

So, we utilize sheafification and the universal property to define a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$.

First, we prove the existence of a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$. Define map of presheaves on X,

$$\phi(U) : \left[U \mapsto \lim_{V \supseteq f(U)} \mathscr{F}(f^{-1}(V))\right] \to \mathscr{F}(U)$$

which exists because there is a always a map $\mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$. Notably, $f^{-1}(V) \supseteq f^{-1}f(U) \supseteq U$ so the map is just the restriction map.

By sheafification (Hartshorne Proposition-Definition II.1.2), we obtain morphism $\phi^+ : f^{-1}f_*\mathscr{F} \to \mathscr{F}$ making the following diagrams commute:

$$\begin{array}{ccc} \left[U \mapsto \lim_{V \supseteq f(U)} \mathscr{F}(f^{-1}(V))\right] & \longrightarrow & f^{-1}f_*\mathscr{F} \\ & \searrow^{\phi} & \downarrow{\scriptstyle \phi^+} \\ & & \mathscr{F} \end{array}$$

$$\begin{array}{ccc} U \mapsto \lim_{V \supseteq f(U)} \mathscr{F}(f^{-1}(V)) & \longrightarrow & f^{-1}f_*\mathscr{F}(U) \\ & \searrow^{\phi(U)} & \downarrow{\scriptstyle \phi^+(U)} \\ & & \mathscr{F}(U) \end{array}$$

Next, given $g \in \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$, define $g \mapsto \phi^+ \circ (f^{-1}g)$ which sends

$$f^{-1}g : f^{-1}\mathscr{G} \to f^{-1}f_*\mathscr{F} \qquad \to \qquad \phi^+ \circ (f^{-1}g) : f^{-1}\mathscr{G} \to \mathscr{F}$$

using the natural map. So, we get an induced morphism $\phi^* : \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) \to \mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$.

Second, we show the existence of a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Note that for any open $U \subseteq Y$,

$$f_*f^{-1}\mathscr{G}(U) = f^{-1}\mathscr{G}(f^{-1}(U)) = \lim_{V \supseteq f(f^{-1}(U))} \mathscr{G}(V).$$

Define a map of presheaves (which exists since $f(f^{-1}(U)) \supseteq U$)

$$\psi : \mathscr{G}(U) \to \left[V \mapsto \lim_{V \supseteq f(f^{-1}(U))} \mathscr{G}(V)\right]$$

and composing with the sheafification map $\lim_{V \supseteq f(f^{-1}(V))} \mathscr{G}(V) \to f_*f^{-1}\mathscr{G}$, we have our desired natural map $\psi^+ : \mathscr{G} \to f_*f^{-1}\mathscr{G}$. Now, given $h \in \mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$, define $h \mapsto (f_*h) \circ \psi^+$ which sends

$$f^{-1}\mathscr{G} \to \mathscr{F} \qquad \to \qquad f_*f^{-1}\mathscr{G} \xrightarrow{h} f_*\mathscr{F}$$

and then precomposes with $\mathscr{G} \to f_*f^{-1}\mathscr{G}$ to get

$$f^{-1}\mathscr{G} \to \mathscr{F} \qquad \to \qquad \mathscr{G} \to f_*\mathscr{F}.$$

This induces a map $\psi^* : \mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$.

To show bijectivity, we check that these constructions are inverse to each other i.e.

$$\phi^* \circ \psi^* = \mathrm{id}_{\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})} \qquad \& \qquad \psi^* \circ \phi^* = \mathrm{id}_{\mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})}.$$

Suppose $g \in \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$. Applying ϕ_* and ψ^*, we have a commutative diagram for any $V \subseteq Y$ open,

$$\begin{array}{ccc} \Gamma(V,\mathscr{G}) & \xrightarrow{g(V)} & \Gamma(V, f_*\mathscr{F}) \\ \downarrow & & \downarrow \wr \\ \Gamma(f^{-1}(V), f^{-1}\mathscr{G}) & \xrightarrow{\phi^*(g)} & \Gamma(f^{-1}(V), \mathscr{F}) \\ \downarrow & & \downarrow \wr \\ \Gamma(V,\mathscr{G}) & \xrightarrow{\psi^*\phi^*(g)} & \Gamma(V, f_*\mathscr{F}) \end{array}$$

which shows that the morphisms on stalks agree. A similar computation can be done for $\phi^* \circ \psi^*$ and this completes the proof, omitting a check for functorality since we only needed to show bijectivity.

Of course, if one desires, functoriality follows from functoriality at each of our steps. The functors f^{-1}, f_* and our maps $f^{-1}f_*\mathscr{F} \to \mathscr{F}$ and $\mathscr{G} \to f_*f^{-1}\mathscr{G}$ are all functorial which implies the identifications were already functorial.

Remark. See also pp. 15-16 of Bredon's *Sheaf Theory 2nd Edition* for more details.

□

Hartshorne II.1.19 *Extending a Sheaf by Zero.* Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.

(a) Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z, 0$ if $P \notin Z$. Hence we call $i_*\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathscr{F}$ instead of $i_*\mathscr{F}$, and say "consider $\mathscr{F}$ as a sheaf on X," when we mean "consider $i_*\mathscr{F}$."

(b) Now let $\mathscr{F}$ be a sheaf on U. Let $j_!(\mathscr{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U, V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathscr{F}))_P$ is equal to $\mathscr{F}_P$ if $P \in U, 0$ if $P \notin U$, and show that $j_!\mathscr{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$. We call $j_!\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside U.

(c) Now let $\mathscr{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X

$$0 \to j_!\left(\mathscr{F}|_U\right) \to \mathscr{F} \to i_*\left(\mathscr{F}|_Z\right) \to 0.$$

Proof. (a) Suppose $P \in Z$. Then, $i_*(\mathscr{F})_P$ equals $\mathscr{F}(U)$ where U is an open set containing P. So, $(i_*\mathscr{F})_P = \mathscr{F}_P$. If $P \notin Z$, then $\mathscr{F}(\emptyset)$ and hence, $(i_*\mathscr{F})_P = 0$.

If U is an open subset containing P, then $(i_*\mathscr{F})(U) = \mathscr{F}(Z \cap U)$. If $P \notin Z$, then $(i_*\mathscr{F})(U) = 0$ which means $(i_*\mathscr{F})_P = 0$. If $P \in Z$, we get $Z \cap U \neq \emptyset$ and the stalk is $\mathscr{F}_P$ since we take a direct limit.

(b) Suppose $P \in U$. Then to compute $(j_!(\mathscr{F}))_P$, note that the direct limit is taken to include all open subsets V s.t. $P \subseteq V \subseteq U$ and so, $(j_!(\mathscr{F}))_P = \mathscr{F}_P$. If $P \notin U$, then $(j_!(\mathscr{F}))_P = 0$ since the only open subset of X between P and U is $\emptyset$ (this is simply by empty set shenanigans).

(c) First, we prove that the maps actually exist. Let $V \subseteq X$ be an open subset. Then the maps $j_!(\mathscr{F}|_U) \to \mathscr{F}$ and $\mathscr{F} \to i_*(\mathscr{F}|_Z)$ are specified by

$$j_!(\mathscr{F}|_U)(V) = \begin{cases} \mathscr{F}|_U(V) = \mathscr{F}(U \cap V) = \mathscr{F}(V) \to \mathscr{F}(V) & \text{if } V \subseteq U \\ 0 \to \mathscr{F}(V) & \text{if } V \not\subseteq U \end{cases}$$

$$\text{and} \qquad \mathscr{F}(V) \to i_*(\mathscr{F}|_Z)(V) = \mathscr{F}|_Z(i^{-1}(V)) = \mathscr{F}(Z \cap V).$$

To show that the sequence is exact, it suffices to check exactness on stalks. This is immediate from (a) and (b). If $P \in U$, then $P \notin Z$ and we have a sequence

$$0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0 \to 0$$

which is exact. If $P \in Z$, then $P \notin U$ and we have

$$0 \to 0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0.$$

□

Hartshorne II.1.20 *Subsheaf with Supports.* Let Z be a closed subset of X, and let $\mathscr{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathscr{F})$ to be the subgroup of $\Gamma(X, \mathscr{F})$ consisting of all sections whose support (Ex, 1.14) is contained in Z.

(a) Show that the presheaf $V \mapsto \Gamma_{Z \cap V}(V, \mathscr{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathscr{F}$ with supports in Z, and is denoted by $\mathscr{H}^0_Z(\mathscr{F})$.

(b) Let $U = X - Z$, and let $j : U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$$

Furthermore, if $\mathscr{F}$ is flasque, the map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is surjective.

Proof. (a) Given $U \subseteq X$ with $\{V_i\}$ an open cover of U. Suppose $s \in \mathscr{G}(U)$ and $s|_{V_i} = 0$ for each V_i. Since $s \in \mathscr{G}(U)$, $\operatorname{Supp} s \subseteq Z \cap U$. Since $s|_{V_i} = 0$, by $\mathscr{F}$ sheaf on U, this implies $s = 0$.

Suppose $s_i \in \mathscr{G}(V_i)$ s.t. s_i agrees on intersections. So, by $\mathscr{F}$ sheaf, I can glue to an $s \in \mathscr{F}(U)$ and by $s_i \in \mathscr{G}(V_i)$, $\operatorname{Supp} s \subseteq Z \cap U$.

(b) Fix $U := X - Z$, $j : U \hookrightarrow X$, and consider the short exact sequence,

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} j_*(\mathscr{F}|_U).$$

First, ϕ is defined by the obvious inclusion, and ψ is defined by mapping $\mathscr{F}(V) \to \mathscr{F}(j^{-1}(V)) = \mathscr{F}(V \cap U)$ by the restriction map.

This is an exact sequence since we shall show it is exact on stalks. Suppose $P \in U$. We have $0 \to 0 \to \mathscr{F}_P \to \mathscr{F}_P$. If $P \in Z$, then $0 \to 0 \to \mathscr{F}_P \to 0$ if $\operatorname{Supp} P \not\subseteq Z$ but $0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0$ if $\operatorname{Supp} P \subseteq Z$.

Now, suppose $\mathscr{F}$ is flasque. Then $\mathscr{F}(U) \to \mathscr{F}(V)$ is a surjection. But $\mathscr{F} \to^{\psi} j_*(\mathscr{F}|_U)$ gives $\mathscr{F}_P \to \mathscr{F}_P$ surjection and $\mathscr{F}_P \to 0$ surjection for $P \in U$ and $P \notin U$ respectively. □

Hartshorne II.1.21 *Some Examples of Sheaves on Varieties.* Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathcal{O}_X$ be the sheaf of regular functions on X(1.0.1). (a) Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathscr{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show

that the presheaf $U \mapsto \mathscr{I}_Y(U)$ is a sheaf. It is called the sheaf of ideals $\mathscr{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.
(b) If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathscr{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.
(c) Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Then there is an exact sequence of sheaves on X, where $\mathscr{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$,

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathscr{F} \to 0.$$

Show however that the induced map on global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathscr{F})$ is not surjective. This shows that the global section functor $\Gamma(X, \cdot)$ is not exact (cf. (Ex. 1.8) which shows that it is left exact).
(d) Again let $X = \mathbb{P}^1$, and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathscr{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathscr{K}$. Show that the quotient sheaf $\mathscr{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P \in X} i_P(I_P)$, where I_P is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf (Ex. 1.17) given by I_P at the point P.
(e) Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}) \to 0$$

is exact. (This is an analogue of what is called the "first Cousin problem" in several complex variables. See Gunning and Rossi [1, p. 248].)

Proof. (a) Let $U \subseteq X$ be open and $\{V_i\}$ be an open covering of U. Let $s \in \mathscr{I}_Y(U)$ be s..t $s|_{V_i} = 0$ fo rall i. I will show that $s - 9$.

Since s is regular on U, $s = \frac{f}{g}$ where $f = 0$ on $V_i \cap Y$ and $g \neq 0$ on U. Now, f vanishes on each $V_i \cap Y$ and so, $f = 0$ on all of $Y \cap U$. So, $s = 0$.

Similarly, locally regular and vanishing on each $Y \cap V_i$ implies that we can take f to be the product of monomials in the obvious way. So $\mathscr{I}_Y$ is a sheaf. By definition, it is a subsheaf.

(b) Let Y be a subvariety. I claim that $\mathcal{O}_X/\mathscr{I}_Y = i_*(\mathcal{O}_Y)$.

Define a morphism $\varphi : \mathcal{O}_X/\mathscr{I}_Y \to i_*(\mathcal{O}_Y)$ as follows. Given $s \in \mathcal{O}_X/\mathscr{I}_Y(U)$. It is a regular function not vanishing on Y.

[Incomplete]

(c) **[Incomplete]**
(d) **[Incomplete]**
(e) **[Incomplete]**

□

Remark 14. This exercise is known as the "first Cousin problem" in complex analysis. It asks that, on a complex manifolds M, if we are given holomorphic functions $f_i - f_j$ for each i, j, what conditions might be needed to ensure that there exists a meromorphic function f s.t. $f - f_i$ is holomorphic on U_i.

Hartshorne II.1.22 *Glueing Sheaves.* Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j\Big|_{U_i \cap U_j}$ such that (1) for each $i, \varphi_{ii} = \mathrm{id}$, and (2) for each $i, j, k, \varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathscr{F}$ on X, together

with isomorphisms $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$ such that for each $i, j, \psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathscr{F}$ is obtained by glueing the sheaves $\mathscr{F}_i$ via the isomorphisms φ_{ij}

PROOF. Let X be a topological space, $\{U_i\}$ be an open cover of X, and suppose for each i we are given a sheaf $\mathscr{F}_i$ on U_i and that for each i, j there is an isomorphism $\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$ subjected to the conditions that $\varphi_{ii} = \mathrm{id}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

We now construct a unique sheaf $\mathscr{F}$ on X with isomorphisms $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$ s.t. $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$.

We define our sheaf $\mathscr{F}$. Let I be an indexing set consisting of the i's. Let $U \subseteq X$ be an open set. Then, we take

$$\mathscr{F}(U) = \left\{(s_i)_{i \in I} : s_i \in \mathscr{F}_i(U \cap U_i), \varphi_{ij}(s_i|_{U_i \cap U_j}) = s_j|_{U_i \cap U_j}\right\} =: \left(\prod_{i \in I} \mathscr{F}(U \cap U_i)\right) / \sim$$

The idea is that we want compatible sequences for our sections. The conditions imposed on $\mathscr{F}(U)$ ensures that we have our desired compatibility relations. Let us verify that this is indeed a sheaf.

Let U be an open set. Let $\{V_j\}$ be an open cover for U. Let $(s_i)_{i \in I} \in \mathscr{F}(U)$. Suppose $(s_i)|_{V_j} = 0$ for all V_j. Then, $s_i|_{V_j} = 0$ for each V_j. By the sheaf axioms on $\mathscr{F}_i$, one has $s_i = 0$ in $\mathscr{F}_i(U)$. Thus, $s_i = 0$ for all i. Hence, $(s_i) = 0$.

Let U and $\{V_j\}$ be as before. Suppose one has sections $(s_i)_j \in \mathscr{F}(V_j)$ s.t. $(s_i)_j|_{V_j \cap V_k} = (s_i)_k|_{V_j \cap V_k}$. By the sheaf axioms on $\mathscr{F}_i$, there exists a section t_i for each $\mathscr{F}_i$. These t_i's yield a section $(t_i) \in \mathscr{F}(U)$ satisfying the property that $(t_i)|_{V_j} = (s_i)_j$.

We now want to define our isomorphisms $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$. There is an obvious map $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$ defined by sending the section $(s_i)_{i \in I}$ to s_i. It suffices to defined an inverse map. Let $s_i \in \mathscr{F}_i(U)$ and we need to define $\psi_i^{-1}(s_i) = s \in \mathscr{F}(U)$. We know $\varphi_{ii}(s_i) = s_i$. We want $s = (s_i)_{i \in I}$ to be a compatible sequence i.e. satisfying the conditions for sections of $\mathscr{F}(U)$. So, we define $s_j = \varphi_{ji}^{-1}(s_i)$. We need $\varphi_{ij}(s_i|_{U_i \cap U_j}) = s_j|_{U_i \cap U_j}$. For the sequence to be compatible, $\varphi_{jk}(s_j) = s_k$ for each k. Thus, we need use the cocycle condition to get $\varphi_{ik}(s_i) = \varphi_{ij} \circ \varphi_{jk}(s_i)$. By our construction, we now have an inverse map since $\psi_i : \mathscr{F}_i \to \mathscr{F}_i$ maps $(s_i)_{i \in I}$ to s_i and the other maps s_i to $(s_i)_{i \in I}$ in way mapping $(s_i)_{i \in I}$ a compatible sequence.

Remark. The cocycle condition is needed for the statement regarding ψ_i being an isomorphism. On the other hand, we can construct such a sheaf $\mathscr{F}$ without the isomorphisms ψ_i by not requiring the cocycle condition. □

2. Schemes

Hartshorne II.2.1 Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $\left(D(f), \mathcal{O}_{X|_{D(f)}}\right)$ is isomorphic to $\operatorname{Spec} A_f$.

PROOF. We define a morphism of locally ringed spaces $\varphi : \operatorname{Spec} A_f \to (D(f), \mathcal{O}_{X|_{D(f)}})$ and show φ is an isomorphism. First, $A \to A_f$ is a ring homomorphism inducing $\operatorname{Spec} A_f \to \operatorname{Spec} A$ a scheme morphism. A prime ideal $\mathfrak{p}$ in A_f pulls back[8] to a prime $\mathfrak{q}$ not containing f in A. So, we get a scheme morphism $\varphi : \operatorname{Spec} A_f \to (D(f), \mathcal{O}_{X|_{D(f)}})$ sending $\mathfrak{p} \mapsto \mathfrak{q}$.

[8]See Eisenbud's Chapter 2.

We claim that φ is a homeomorphism. It is continuous: if $D(fg) \subseteq D(f)$ is a distinguished open affine subset, then $\varphi^{-1}(D(fg)) = \operatorname{Spec} A_{fg}$ which is open and by the arbitrary choice of g, φ is continuous.

For the inverse, define $\varphi^{-1} : (D(f), \mathcal{O}_{X|_{D(f)}}) \to \operatorname{Spec} A_f$ on the level of sets by $\mathfrak{q} \mapsto \mathfrak{q}A_f$. This is a continuous morphism: it suffices to check on the basis of the topology and if $D(fg) \subseteq D(f)$, then $\varphi^{-1}(D(fg)) = D(fg) \cap D(f)$ which is an open subset of $(D(f), \mathcal{O}_X|_{D(f)}$. Furthermore, φ and φ^{-1} are inverse to each other.

Consider $\varphi^{\#} : \mathcal{O}_{X|_{D(f)}} \to \varphi_* \mathcal{O}_{\operatorname{Spec} A_f}$. To see that $\varphi^{\#}$ is an isomorphism, it suffices to check that the induced morphism on stalks is an isomorphism. Indeed, $\varphi^{\#}_P : \mathcal{O}_{X|_{D(f)},\varphi(P)} \to (A_f)_P$ since $(A_f)_P \cong A_P$ and $\mathcal{O}_{X|_{D(f)},\varphi(P)} \cong A_P$ where the latter is immediate by the definition of $\mathcal{O}_{X|_{D(f)}}$ and both require $f \notin P$. □

Hartshorne II.2.2 Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. We call this the *induced scheme structure* on the open set U, and we refer to $(U, \mathcal{O}_X|_U)$ as an *open subscheme* of X.

Proof. We claim $(U, \mathcal{O}_{X|U})$ is a scheme where $U \subseteq X$ is open. Clearly, $\mathcal{O}_X|_U$ is a sheaf of rings because $\Gamma(V, \mathcal{O}_X|_U) = \Gamma(V, \mathcal{O}_X)$ for any $V \subseteq U$.

Let $x \in U$. Since $x \in U \subseteq X$, there is an open affine $(V, \mathcal{O}_X|_V) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ subscheme of X. Now, $W := (V \cap U, (\mathcal{O}_X|_U)|_V)$ is an open subscheme of $(V, \mathcal{O}_X|_V)$. As $(V, \mathcal{O}_X|_V)$ has a base of topology given by distinguished open subsets $D(f) \subseteq \operatorname{Spec} A$, we may find a $D(f)$ s.t. $x \in D(f) \subseteq W$. Indeed, a base a of a topology contains a neighborhood base at every point. □

Hartshorne II.2.3 *Reduced Schemes.* A scheme $(X, \mathcal{O}_x)$ is reduced if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

(a) Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.

(b) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{\text{red}}$, where for any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme. We call it the reduced scheme associated to X, and denote it by X_{red}. Show that there is a morphism of schemes $X_{\text{red}} \to X$, which is a homeomorphism on the underlying topological spaces.

(c) Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{\text{red}}$ such that f is obtained by composing g with the natural map $Y_{\text{red}} \to Y$.

Proof. **(a)** Assume $\mathcal{O}_{X,P}$ is reduced for each P. Let $s \in \mathcal{O}(U)$ be a nilpotent element. That is, $s^n = 0$. Then, $s^n_P = 0$ in every stalk. So, $s_P = 0$ for all $P \in U$ and $s = 0$ by the sheaf axioms[9].

Conversely, suppose $\mathcal{O}_X(U)$ is reduced for all U. If $s_P \in \mathcal{O}_{X,P}$, then s_P may be represented by some s' and U. Assume $s_P \in \mathcal{O}_{X,P}$ is nilpotent which means $s^n_P = 0$. Then, $(s')^n = 0$ in $\mathcal{O}_X(U)$ for some U sufficiently small. Therefore, $s' = 0$ in $\mathcal{O}_X(U)$ with $U \ni P$. By definition of a germ, the germ s' represents is zero. That is, $s_P = 0$.

[9]If $s_P = 0$, we know that s_P is represented by $\langle S, t_S \rangle$ where $t_S = 0$. Since this holds for all P, we can cover U by open sets S. In this case, $s|_S = 0$ for each open set S. Then, use the axioms of sheaves to conclude that $s = 0$.

(b) For ease, we shall write $\mathcal{O}_{X_{red}}$ instead of $(\mathcal{O}_X)_{red}$. We begin by constructing an isomorphism $(\operatorname{Spec} A_{red}, \mathcal{O}_{\operatorname{Spec} A_{red}}) \to (\operatorname{Spec} A_{red}, \mathcal{O}_{(\operatorname{Spec} A)_{red}})$. Let the underlying map on topological spaces be the homeomorphism $\mathrm{id} : \operatorname{Spec} A_{red} \to \operatorname{Spec} A$ which is obtained from the ring homomorphism $A \to A_{red}$.

For obtaining a map of sheaves $\mathcal{O}_{\operatorname{Spec} A_{red}} \to i_*\mathcal{O}_{(\operatorname{Spec} A)_{red}} = \mathcal{O}_{(\operatorname{Spec} A)_{red}}$, we shall glue morphisms of sheaves together. We may define it on the base $D(f)$ and here, we know that

$$\mathcal{O}_{\operatorname{Spec} A_{red}}(D(f)) \cong (A_{red})_f \cong (A/\operatorname{nilrad}(A))_f \cong A_f/\operatorname{nilrad}(A)_f$$
$$\mathcal{O}_{(\operatorname{Spec} A)_{red}}(D(f)) \cong \mathcal{O}_{(\operatorname{Spec} A)}(D(f))/\operatorname{nilrad}(\mathcal{O}_{(\operatorname{Spec} A)}(D(f))) \cong A_f/\operatorname{nilrad}(A_f).$$

As $\operatorname{nilrad}(A)_f = \operatorname{nilrad}(A_f)$ and as the RHS rings are the same, we obtain a map $\mathcal{O}_{\operatorname{Spec} A_{red}}(D(f)) \to \mathcal{O}_{(\operatorname{Spec} A)_{red}}(D(f))$. These maps are preserved by restrictions (i.e. they agree on intersections) so we may glue to a morphism $\mathcal{O}_{\operatorname{Spec} A_{red}} \to \mathcal{O}_{(\operatorname{Spec} A)_{red}}$. The fact this is an isomorphism follows by checking the stalks.

Now returning to the general case, as $(X, \mathcal{O}_X)$ is a scheme, it has a cover by open affine subsets. Let $(\varphi, \varphi^\#) : (U, \mathcal{O}_{X|U}) \xrightarrow{\cong} (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ be an open affine cover. We shall show that $(U, \mathcal{O}_{X_{red}|U}) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A_{red}})$ and then this gives an affine cover of $(X, (\mathcal{O}_X)_{red})$. On the level of topological spaces, we already use the homeomorphism from before. For sheaves,

$$\begin{aligned}
\mathcal{O}_{X_{red}|U}(V) &\cong \mathcal{O}_{X|U}(V)/\operatorname{nilrad}(\mathcal{O}_{X|U}(V)) \cong \mathcal{O}_X(U \cap V)/\operatorname{nilrad}(\mathcal{O}_X(U \cap V)) \\
&\cong \mathcal{O}_{\operatorname{Spec} A}(\varphi^{-1}(U \cap V))/\operatorname{nilrad}(\mathcal{O}_{\operatorname{Spec} A}(\varphi^{-1}(U \cap V))) \\
&\cong (\mathcal{O}_{\operatorname{Spec} A})_{red}(\varphi^{-1}(U \cap V)) \\
&\cong (\mathcal{O}_{\operatorname{Spec} A_{red}})(\varphi^{-1}(U \cap V)).
\end{aligned}$$

and the last isomorphism follows by our previous work. As each step was an isomorphism, we have an isomorphism of sheaves and $(U, \mathcal{O}_{X_{red}|U}) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A_{red}})$.

Now we construct the desired morphism $X_{red} \to X$. As the underlying topological spaces are the the same, let $\mathrm{id} : X_{red} \to X$ be the identity map on the underlying topological space. For the morphism of sheaves, we define $\mathcal{O}_X \to \mathrm{id}_*((\mathcal{O}_X)_{red})$ by the quotient $\mathcal{O}_X(U) \to \mathcal{O}_X(U)/\operatorname{nilrad}(\mathcal{O}_X(U))$ on each open set U. This gives the desired morphism of schemes.

(c) Let $f : X \to Y$ be a morphism of schemes, X a reduced scheme, and we will show there is a unique morphism $g : X \to Y_{red}$ s.t. f is a composition of g with $Y_{red} \to Y$. First, the map $Y_{red} \to Y$ arises naturally from (b). To determine g, we first define $g : X \to Y$ on the underlying topology space to be the same as that of f's. At the level of sheaves, we have a factorization of ring homorphisms (i.e. the existence of the second map)

$$\mathcal{O}_Y(U) \to \mathcal{O}_Y(U)/(\operatorname{nilrad} \mathcal{O}_Y(U)) \to \mathcal{O}_X(f^{-1}(U))$$

by the the following result from ring theory: if A is a ring and it maps to the reduced ring B, then the map must factor uniquely through A_{red}. It is also clear that these maps induce the desired morphisms of sheaves (as we can then sheafify). □

Hartshorne II.2.4 Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f : X \to \operatorname{Spec} A$, we have an associated map on sheaves $f^\# : \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map

$$\alpha : \operatorname{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec} A) \to \operatorname{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, (_X)).$$

Show that α is bijective (cf. (I, 3.5) for an analogous statement about varieties).

My Preferred Proof. [**Incomplete**][10] □

Vakil's Proof. The result is true for the affine case. Cover the scheme and use the affine case to get morphisms. These agree on intersections. Glue to a morphism for the general case. □

Sketchy Proof. Hartshorne essentially gives us one direction. All that is left is to do the other.

Suppose we have a morphism $A \to \Gamma(X, \mathcal{O}_X)$ of rings.

We first show that case that X is affine. If X is affine and equal to $\operatorname{Spec} B$, then the morphism is really just $A \to B$ which induces a map $\operatorname{Spec} B \to \operatorname{Spec} A$ as desired.

Now we do the general case. Suppose X is a scheme. Then we get morphisms $A \to \Gamma(U, \mathcal{O}_{X|U})$ for U open affine. This induces morphisms $U \to \operatorname{Spec} A$. Since we can cover X by affine U_i's, it suffices to consider the morphisms $U_i \to \operatorname{Spec} A$. Since morphisms of sheaves can glue, and it is a trivial thing to verify that the morphisms agree on overlaps, we get a morphisms $X \to \operatorname{Spec} A$ as desired.

Alternatively, we could use the contravariant functor from scheme to ring and look at exact sequences. □

More Precise Proof. We first note that the result holds in the affine case. Indeed, to give a morphism $X \to \operatorname{Spec} A$ is give a ring homomorphism $A \to \Gamma(X, \mathcal{O}_X)$ and if $X = \operatorname{Spec} B$, we know that $\Gamma(X, \mathcal{O}_X) = B$.

Cover X by open affines U_i and cover the overlaps $U_i \cap U_j$ with $U_{ij,k}$. By the assumptions, we get an exact sequence of rings

$$0 \to \Gamma(X, \mathcal{O}_X) \to^g \prod_i \Gamma(U_i, \mathcal{O}_{U_i}) \to^f \prod_{i,j,k} \Gamma(U_{ij,k}\mathcal{O}_{U_{ij,k}})$$

where the morphism g is given by $s \mapsto \prod_i s|_{U_i}$ and f is given by $\coprod_i s_i \mapsto \prod_{i,j,k}(s_i - s_j)|_{U_{ij,k}}$. Now, we want to consider the $\operatorname{Hom}(A, \cdot)$. This is because an element of $\operatorname{Hom}(A, \Gamma(X, \mathcal{O}_X)$ would correspond to an element of $\operatorname{Hom}(X, \operatorname{Spec} A)$. Indeed, if we take $\operatorname{Hom}(A, \cdot)$, we get

$$0 \to \operatorname{Hom}(A, \Gamma(X, \mathcal{O}_X)) \to \prod_i \operatorname{Hom}(A, \Gamma(U_i, \mathcal{O}_{U_i})) \to \prod_{i,j,k} \operatorname{Hom}(A, \Gamma(U_{ij,k}, \mathcal{O}_{U_{ij,k}}))$$

an exact sequence due to Hom being left exact. This corresponds to an exact sequence

$$0 \to \operatorname{Hom}(X, \operatorname{Spec} A) \to \prod_i \operatorname{Hom}(U_i \operatorname{Spec} A) \to \prod_{i,j,k} \operatorname{Hom}(U_{ij,k}, \operatorname{Spec} A).$$

Now, we get bijections $\prod_{i,j,k} \operatorname{Hom}(U_{ij,k}, \operatorname{Spec} A) \to \prod_{i,j,k} \operatorname{Hom}(A, \Gamma(U_{ij,k}, \mathcal{O}_{U_{ij,k}}))$ and $\prod_i \operatorname{Hom}(U_i \operatorname{Spec} A) \to \prod_i \operatorname{Hom}(A, \Gamma(U_i, \mathcal{O}_{U_i}))$ which, by exactness of sequences, induces a bijection,

$$\operatorname{Hom}(X, \operatorname{Spec} A) \to \operatorname{Hom}(A, \Gamma(X, \mathcal{O}_X))$$

as desired. □

Proof [4]. A more general result ends up holding with X a locally ringed space. See Proposition 3.4 in [4].

We prove the case where X is a scheme. The result has been shown for the affine case in the text. So, consider an affine covering $\{U_i\}$ of X and we have

$$\operatorname{Hom}(U_i, Y) \to \operatorname{Hom}(A, \Gamma(U_i, \mathcal{O}_X))$$

[10] See your notes and EGA.

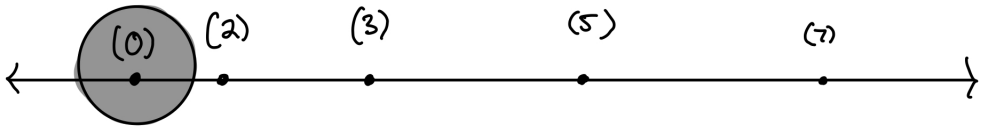


FIGURE 1. A pictorial representation of Spec($\mathbb{Z}$). The shaded area around (0) indicates that we have a nonclosed point.

is a bijection. For an affine open set $V \subseteq U_i \cap U_j$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(U_i, Y) & \longrightarrow & \mathrm{Hom}(A, \Gamma(U_i, \mathcal{O}_X)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(V, Y) & \longrightarrow & \mathrm{Hom}(A, \Gamma(V, \mathcal{O}_X)) \end{array}$$

which is commutative because $\Gamma(-, \mathcal{O}_X)$ is a natural transformation. Then, use the following result

" If X, Y are LRS, $U \subseteq X$ is open, $\mathrm{Hom}(U, Y)$ set of morphisms of LRS, then $U \mapsto \mathrm{Hom}(U, Y)$ is a sheaf of sets. "

which shows that we must have a unique morphism $X \to \operatorname{Spec} A$. □

Hartshorne II.2.5 Describe $\operatorname{Spec} \mathbb{Z}$, and show that it is a final object for the category of schemes, i.e., each scheme X admits a unique morphism to $\operatorname{Spec} \mathbb{Z}$.

PROOF. The closed points of $\operatorname{Spec} \mathbb{Z}$ are the prime ideals $p\mathbb{Z}$ for p prime and nonzero. The reason is that $p\mathbb{Z}$ is maximal if p is prime and these are the only maximal ideals are possible. There is a generic point (0) since (0) is contained in every ideal $n\mathbb{Z}$ of $\mathbb{Z}$. The basic open sets $D(n)$ consists of prime ideals $p\mathbb{Z}$ such that p does not divide n. The local ring $\mathbb{Z}_{(p)}$ is the stalk of $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}$ at $p\mathbb{Z}$ while $\mathbb{Z}_{(0)} = \mathbb{Q}$ is the stalk of $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}$ at (0). Furthermore[11], the residue field $p\mathbb{Z}$ on $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}$ is $\mathbb{Z}_{(p)}/(p)\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z}$ i.e. the finite field with p-elements.

There is a unique homomorphism from any scheme X to $\operatorname{Spec} \mathbb{Z}$. This is because there is a ring homomorphism from $\mathbb{Z}$ to any ring A defined by sending $1 \in \mathbb{Z}$ to $1 \in A$. In particular, there is a ring homomorphism from A to $\Gamma(\mathcal{O}_X, X)$. Then, Hartshorne II.2.4 shows that there is a morphism of schemes $X \to \operatorname{Spec} \mathbb{Z}$ determined by the ring homomorphism. In particular,

$$\#\operatorname{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec} \mathbb{Z})\} = \#\operatorname{Hom}_{\mathfrak{Rings}}(A, \Gamma, \mathcal{O}_X))\} = 1$$

which implies that map $X \to \operatorname{Spec} \mathbb{Z}$ is unique.

□

Hartshorne II.2.6 Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1. Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R.)

[11] See Hartshorne II.2.7.

PROOF. Recall that an initial object I in a category C is an element such that for all $X \in C$, there exists a morphism $\phi : I \to X$ that is unique.

Consider Spec(0) where 0 denotes the zero ring. The scheme is just the empty scheme since there are no prime ideals in the ring 0 (recall that a prime ideal must be a proper subset of a ring). That is, $\text{Spec}(0) = \emptyset$ and it has a sheaf of rings $\mathcal{O}_{\text{Spec}(0)}(\emptyset) = 0$.

Incidentally, there is a unique map $f : \text{Spec}(0) \to X$ of topological spaces for any scheme X. To get a morphism of schemes, we need to define a map $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_{\text{Spec}(0)}$. As $f^-(U) = \emptyset$ for all $U \subseteq X$ open, we need to only specify a map $f^\#(U) : \mathcal{O}_X(U) \to \mathcal{O}_{\text{Spec}(0)}(\emptyset) = 0$ for every U. Of course, that forces $f^\#(U)$ to always be the zero map. It is then clear that this is the only map $f : \text{Spec}(0) \to X$ and every scheme X has this map. So, Spec(0) is an initial object for $\mathfrak{Sch}$. □

Hartshorne II.2.7 Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the residue field of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of Spec K to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$

PROOF. ($\implies$) : Suppose $f : \text{Spec}(K) \to X$ is a given morphism of schemes. Let $x \in X$. Since X is a scheme, WLOG, assume $x \in U$ such that $U \cong \text{Spec}\, A$ is an affine open subset for some ring A. By restricting to U, obtain an induced morphism $f^\# : \mathcal{O}_{\text{Spec}\, A} \to f_*\mathcal{O}_{\text{Spec}\, K}$. Taking stalks, obtain an induced morphism $f^\#_x : \mathcal{O}_{\text{Spec}\, A,x} \to (f_*\mathcal{O}_{\text{Spec}\, K})_{f^{-1}(x)}$. Since $\mathcal{O}_{\text{Spec}\, A,x} \cong A_x$ and $(f_*\mathcal{O}_{\text{Spec}\, K})_{f^{-1}(x)} = \varinjlim_{f^{-1}(x) \subseteq f^{-1}(V)} \mathcal{O}_{\text{Spec}\, K}(f^{-1}(V)) = K$ due to $\emptyset$ and K being the only open subsets of Spec K, one gets a morphism $f^\#_x : A_x \to K$. As K is a field, we get an inclusion $A_x/\mathfrak{m}_x = k(x) \hookrightarrow K$ (a homorphism of fields is always injective if it is not zero).

To finish the proof, we need to check that $A/\mathfrak{m}_x \to K$ is not trivial[12]. Suppose not and so $\phi := f^\#_x : A_x \to K$ is trivial. That means $\phi^{-1}((0)) = A_x$ which is not a prime ideal. This is absurd.

($\impliedby$) : Assume $k(x) \hookrightarrow K$ is given for some $x \in X$. This gives a ring morphism $A_x \to k(x) \to K$. So,

$$(g, g^\#) : (\text{Spec}\, K, \mathcal{O}_{\text{Spec}\, K}) \to (\text{Spec}\, A_x, \mathcal{O}_{\text{Spec}\, A_x})$$

exists. By composing with the morphism $\text{Spec}(A_x) \to X$, we get a morphism $f : \text{Spec}\, K \to X$ as desired. The existence of $\text{Spec}(A_x) \to X$ is shown in the following lemma.

Lemma. Given a point $x \in X$ and a open affine subset Spec $A \subseteq X$, there is a morphism Spec $A_x \to X$.

PROOF. As Spec $A \subseteq X$, there is an inclusion homomorphism $i_A : \text{Spec}\, A \hookrightarrow X$. On the other hand, there is a ring map $A \to A_x$ which induces a morphism $f : \text{Spec}\, A_x \to \text{Spec}\, A$. So, $i_A \circ f : \text{Spec}\, A_x \to X$ is the desired map. □

□

Hartshorne II.2.8 Let X be a scheme. For any point $x \in X$, we define the Zariski tangent space T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the ring of dual numbers over k. Show that to give a

[12] A reference for this fact is [10, Section 07BH] but we write out a proof.

k-morphism of $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, rational over k (i.e., such that $k(x) = k$), and an element of T_x.

PROOF. Suppose we have a k-morphism of schemes $h : \operatorname{Spec} k[\epsilon]/(\epsilon^2) \to X$. Now, $k[\epsilon]/(\epsilon^2)$ is a local ring with one prime ideal (ϵ). There is a morphism of sheaves $\phi : \mathcal{O}_X \to h_*\mathcal{O}_{\operatorname{Spec} k[\epsilon]/(\epsilon^2)} = \mathcal{O}_{\operatorname{Spec} k[\epsilon]/(\epsilon^2)}$. Let $x \in X$ s.t. $h((\epsilon)) = x$. Taking stalks, $\phi_x : \mathcal{O}_{X,x} \to (k[\epsilon]/(\epsilon^2))_{(\epsilon)}$ and

$$(k[\epsilon]/(\epsilon^2))_{(\epsilon)} \cong k[\epsilon]/(\epsilon^2)$$

since we localized at a maximal ideal. In particular, we inverted elements outside of the maximal ideal of a local ring and those elements are always units. So, we have a map $\phi_x : \mathcal{O}_{X,x} \to k[\epsilon]/(\epsilon^2)$. Since ϕ_x is a local ring homomorphism, $\phi_x(\mathfrak{m}_x) = (\epsilon)$ and then we have a map of quotients $k(x) \to (k[\epsilon]/(\epsilon^2))/(\epsilon) \cong k$ which is an injective map of fields.

By Hartshorne II.2.7, we have equivalently given a map $\operatorname{Spec} k \to X$ which sends the unique point of k to the point x. So, we have a map of sheaves $\mathcal{O}_X \to \mathcal{O}_{\operatorname{Spec} k}$. Taking the stalk, we have a map $\mathcal{O}_{X,x} \to k$ and hence, a map $k(x) \to k$ which is nonzero. As we have a injective maps of fields $k(x) \to k$ and $k \to k(x)$, we have an isomorphism $k \cong k(x)$. Therefore, x is a rational point.

Now, we want to give a morphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$. Let's define $\mathfrak{m}_x \to k$ first and then pass to quotients. Send $z \in \mathfrak{m}_x$ to $\phi_x(z)/(\epsilon)$ (since ϕ_x maps to $k[\epsilon]/(\epsilon^2)$ and so, $\phi_x(z)/(\epsilon) \in k$). In this case, if $z \in \mathfrak{m}_x^2$, we get $\phi_x(z)/(\epsilon) = 0$ since $\phi_x(z) \subseteq (\epsilon)^2$ which is modded out in $(k[\epsilon]/(\epsilon^2))/(\epsilon)$.

Suppose we are given a rational point x and an $\ell : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$ in T_x. We want to define $\operatorname{Spec} k[\epsilon]/(\epsilon^2) \to X$. This comes down to defining a morphism[13] $f_x : \mathcal{O}_{X,x} \to k[\epsilon]/(\epsilon^2)$. Define f_x by $a + z \mapsto a + \ell(z)$ for $a \in k$ and $z \in \mathfrak{m}_x$[14]. Now, $f : \operatorname{Spec} k[\epsilon]/(\epsilon^2) \to X$ should send (ϵ) to x.

In more details, we get $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_{\operatorname{Spec} k[\epsilon]/(\epsilon^2)}$ defined by $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \xrightarrow{f_x} k[\epsilon]/(\epsilon^2)$ where the first morphism is given by mapping into direct system of $\mathcal{O}_{X,x}$. Altogether, we have a morphism of schemes

$$(f, f^\#) : (\operatorname{Spec} k[\epsilon]/(\epsilon^2), \mathcal{O}_{\operatorname{Spec} k[\epsilon]/(\epsilon^2)}) \to (X, \mathcal{O}_X).$$

Recall that X is a scheme over k and $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$ is naturally a scheme over k. So, we check that the diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec} k[\epsilon]/(\epsilon^2) & \xrightarrow{f} & X \\ & \searrow & \downarrow \\ & & \operatorname{Spec}(k) \end{array}$$

[13]This is because $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$ is a one-point scheme and so the morphism is determined by where (ϵ) gets sent. Since we want a morphism of schemes, we want to determine what the induced morphism on stalks is and what the maximal ideal is.

[14]This decomposition needs to be justified. You need to use the fact that we have an exact sequence

$$0 \to \mathfrak{m}_x \to \mathcal{O}_{X,x} \to k \to 0$$

and since we have k-linear morphisms, the sequence splits i.e. $\mathcal{O}_{X,x} \cong \mathfrak{m}_x \oplus k$. However, with this, we can actually write down the morphism.

Certainly, the diagram commutes because we have k-homomorphisms (of rings) throughout by construction.

□

Hartshorne II.2.9 If X is a topological space, and Z an irreducible closed subset of X, a generic point for Z is a point ζ such that $Z = \{\zeta\}^-$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

PROOF. [15] First off, we prove the case for when $X = \operatorname{Spec}(A)$ is an affine scheme. The closed subsets $V(\mathfrak{p})$ are isomorphic to affine schemes $\operatorname{Spec}(A/\mathfrak{p})$ which are irreducible iff $\mathfrak{p}$ was a minimal prime ideal. In this case, $\operatorname{Spec}(A/\mathfrak{p})$ has generic point corresponding to the prime $\overline{\mathfrak{p}} = (\overline{0})$ and it is unique.

Now, for the case of an arbitrary scheme X. We define a map between a point x its closure $\overline{\{x\}}$. We show this is a bijection between the collection of irreducible closed subsets and the collection of generic points of irreducible closed subsets.

First, the map is well-defined. For if x is a point, then the closure $\overline{\{x\}}$ is irreducible since any open subset $U \subseteq \overline{\{x\}}$ must contain x. Indeed, if $\overline{\{x\}} = Z_1 \cup Z_2$, then WLOG $x \in Z_1$ and so $Z_1 = \overline{\{x\}}$.

Next, this map is surjective. Let $Z \subseteq X$ be irreducible and closed. Take an affine open U and assume $U \cap Z$ is nonempty (as X is a scheme). Then $U \cap Z$ is a closed subset of U and it is irreducible as Z is. Therefore, it has a generic point ζ as a closed subset of U. Then, ζ will also serve as the generic point of Z since $U \cap Z$ is dense in Z due to Z irreducible (Hartshorne I.1.7).

Finally, this map is injective. Suppose $\overline{\{x\}} = \overline{\{y\}}$. Then, we can take an affine open neighborhood U of x and consider the intersections $U \cap \overline{\{x\}} = U \cap \overline{\{y\}}$. Since $x \in \overline{\{y\}}$, we have U is an open subset of $\overline{\{y\}}$ and so, $U \cap y \neq \emptyset$. Hence, $y \in U$. Since affine schemes have a unique generic point, we conclude that $x = y$. □

Hartshorne II.2.10 Describe $\operatorname{Spec} \mathbb{R}[x]$. How does its topological space compare to the set $\mathbb{R}$? To $\mathbb{C}$?

PROOF. In $\operatorname{Spec} \mathbb{R}[x]$, we have the maximal ideals $(x - a)$ for $a \in \mathbb{R}$ as well as those of the form $(x^2 + ax + b)$ where $x^2 + ax + c$ is irreducible. We also have the prime ideal (0). Compared to $\mathbb{R}$, $\operatorname{Spec} \mathbb{R}[x]$ has more "points" than $\mathbb{R}$. The spectrum of $\mathbb{R}[x]$ has a bijection with $\mathbb{C}$. In particular, we can let $(x-a)$ correspond to elements in $\mathbb{R}$ and primes (x^2+ax+b) correspond to elements in $\mathbb{C} - \mathbb{R}$. □

Hartshorne II.2.11 Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe $\operatorname{Spec} k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

PROOF. In this case, there are two kinds of elements of $\operatorname{Spec} k[x]$. The zero ideal (0) and the ideal (f) for f irreducible polynomial. Since $\mathbb{F}_p[x]$ is a PID, we only need one such f irreducible.

The residue field of (0) is $\mathbb{F}_p(x)$ since $\mathbb{F}_p[x]_{(0)}/(0)\mathbb{F}_p[x]$ is just going to invert x.

The residue field for (f) is trickier. Indeed, it is going to be $\mathbb{F}_{p^n}$ where n is the degree of f. □

[15] We take inspiration from Chapter 4-5 of [9]. In fact, Exercise 5.1.B in [9] is more general than the statement here.

Hartshorne II.2.12 *Glueing Lemma.* Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possible infinite). For each $i \neq j$ suppose given an open subset $U_{ij} \subseteq X_i$, and let it have the induced scheme structure (Hartshorne II.2.2). Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ such that (1) for each $i, j, \varphi_{ji} = \varphi_{ij}^{-1}$, and (2) for each i, j, k $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$. Then show that there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that
(1) ψ_i is an isomorphism of X_i onto an open subscheme of X,
(2) the $\psi_i(X_i)$ cover X
(3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and
(4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

We say that X is obtained by glueing the schemes X_i along the isomorphisms φ_{ij}. An interesting special case is when the family X_i is arbitrary, but the U_{ij} and φ_{ij} are all empty. Then the scheme X is called the disjoint union of the X_i, and is denoted $\coprod X_i$.

Proof. Define at the level of sets $X := \left(\coprod_{i\in I} X_i\right) / \sim$ where $x_{ij} \sim \varphi_{ij}(x_{ij})$ for each $x_{ij} \in U_{ij}$. For any $V \subseteq X$, we say that V is open iff $\psi_i^{-1}(V)$ is open for each i. This is precisely the quotient topology:

(a) The sets $\emptyset$ and X are open since $\psi_i^{-1}(\emptyset) = \emptyset$ and $\psi_i^{-1}(X) = X_i$ for all i.
(b) If $\{U_\alpha\}_{\alpha\in A}$ is a collection of open sets, then $\psi_i^{-1}\left(\bigcup_{\alpha\in A} U_\alpha\right) = \bigcup_{\alpha\in A} \psi_i^{-1}(U_\alpha)$ which is a union of open sets in each i.
(c) If $\{U_i\}_{i=1}^n$ is a finite collection of open sets, then $\psi_i^{-1}\left(\bigcap_{i=1}^n U_i\right) = \bigcap_{i=1}^n \psi_i^{-1}(U_i)$.

Define $\psi_i : X_i \to X$ by inclusion into the disjoint union. Next, we define the sheaf of rings on X. For any open $V \subseteq X$, we define

$$\mathcal{O}_X(V) := \left\{(s_i)_{i\in I} : s_i \in \mathcal{O}_{X_i}(\psi_i^{-1}(X)) \text{ and } \varphi_{ij}(s_i|_{\psi_i^{-1}(V)\cap U_{ij}}) = s_j|_{\psi_j^{-1}(V)\cap U_{ji}}\right\}$$

A priori this defines a presheaf. To verify that this defines a sheaf, one would use the sheaf axioms on each X_i. The verification is immediate [16] given the hypotheses. Indeed, the assumed conditions (1) and (2) imply that we may glue sheaves by Hartshorne II.1.22. The sheaf we constructed is precisely the one built from the presheaf structure defined above.

Now we check that $\psi_i : X_i \to X$ is an isomorphism onto an open subscheme of X. Indeed, $\psi_i(X_i)$ is an open subset of X which we given the induced open subscheme structure. Since $\psi_i(U)$ is open for every i and $\psi_i^{-1}(V)$ is open whenever i is open, we deduce that ψ_i is bicontinuous. Furthermore, ψ_i is a bijection of X_i onto its image so we have a homeomorphism. On the other hand, we have an isomorphism of sheaves which is due to our construction (or by Hartshorne II.1.22).

By construction, each of the $\psi_i(X_i)$ are open and they cover X. Also, we have

$$\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j) = \psi_j(X_j) \cap \psi_i(X_i) = \psi_j(U_{ji})$$

because the subsets $U_{ij} \subseteq X_i$ and $U_{ji} \subseteq X_j$ are identified via the isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$. Consequently,

$$\psi_i(U_{ij}) = \psi_j(U_{ji}) = \psi_j(\varphi_{ij}(U_{ij})) = \psi_j \circ \varphi_{ij}(U_{ji}).$$

□

[16] Alternatively, we use the method of gluing sheaves in Hartshorne II.1.22 and recognizing that the hypotheses for gluing sheaves are already fulfilled. Indeed, the conditions on the U_{ij} ensure this. We also get sheaf isomorphisms $\psi_i^\# : \mathcal{O}_X|_{X_i} \to \mathcal{O}_{X_i}$ from the ψ_i.

Hartshorne II.2.13 A topological space is *quasi-compact* if every open cover has a finite subcover.
(a) Show that a topological space is noetherian (I, §1) if and only if every open subset is quasi-compact.
(b) If X is an affine scheme, show that $\mathrm{sp}(X)$ is quasi-compact, but not in general noetherian. We say a scheme X is quasi-compact if $\mathrm{sp}(X)$ is.
(c) If A is a noetherian ring, show that $\mathrm{sp}(\operatorname{Spec} A)$ is a noetherian topological space.
(d) Give an example to show that $\mathrm{sp}(\operatorname{Spec} A)$ can be noetherian even when A is not.

PROOF. (a) Assume a topological space is noetherian. Let U be an open subset. We claim that U is quasi-compact. Let $\{V_i\}$ be a covering. Then, we have a chain

$$V_1 \subset V_1 \cup V_2 \subset V_1 \cup V_2 \cup V_3 \subset \dots$$

which terminates. Hence, U has a finite cover.

Assume every open subset is quasi-compact. Let

$$U_1 \subset U_2 \subset U_3 \subset \dots$$

be an ascending chain of open subsets. This chain must terminate for it not, then we get an open set $\cup U_i$ which does not have a finite subcover from $\{U_i\}$.

(b) Assume X is an affine scheme. We show that $\mathrm{sp}(X)$ is quasi-compact.

It actually suffices to show that X has a finite cover by affine open subschemes due to Exercise 5.1.D of [9].

We take a covering by distinguished open sets $\cup D(f_i)$. As it turns out, one only needs a finite union since $\sum a_i f_i = 1$ for finitely many nonzero $a_i \in A$.

In general, $\operatorname{Spec} A$ does not need to be noetherian. Take for example A the ring $\mathbb{R}[x_0, x_1, x_2, \dots]$ of polynomials in infinitely may variables. Then, we can get a chain of prime ideals $(x_0) \subseteq (x_0, x_1) \subseteq (x_0, x_1, x_2) \dots$ that never terminates.

(c) Assume A is noetherian. Let

$$V(\mathfrak{b}_1) \supset V(\mathfrak{b}_2) \supset \dots$$

be a descending chain of closed subsets of $\mathrm{sp}\operatorname{Spec} X$. This gives us an ascending chain of ideals

$$\mathfrak{b}_1 \subset \mathfrak{b}_2 \subset \mathfrak{b}_3 \subset \dots$$

which must eventually terminate. Hence, the descending chain of closed subsets of $\mathrm{sp}\operatorname{Spec} X$ must also terminate.

(d) We will make use of Hartshorne II.2.2 in constructing our example.

Take $A = k[x_1, x_2, \dots]/(x_1^2, x_2^2, \dots)$. Notice that A is homeomorphic to $A/(x_1, x_2, x_3, \dots)$ by Hartshorne II.2.2. Then, $A/(x_1, x_2, x_3, \dots) \cong k$ and so, $\operatorname{Spec} A$ has just one point. So, $\operatorname{Spec} A$ is noetherian. However, A is clearly not noetherian.

□

Hartshorne II.2.14
(a) Let S be a graded ring. Show that $\operatorname{Proj} S = \emptyset$ if and only if every element of S_+ is nilpotent.

(b) Let $\varphi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of $\operatorname{Proj} T$ and show that φ determines a natural morphism $f : U \to \operatorname{Proj} S$.

(c) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

(d) Let V be a projective variety with homogeneous coordinate ring S (I,§2). Show that $t(V) \cong \operatorname{Proj} S$

PROOF. **(a)** We know $\operatorname{Proj} S = \emptyset$ iff every homogeneous prime ideal of S contains S_+ iff $S_+ \subseteq \bigcap_{\mathfrak{p} \text{ homogeneous}} \mathfrak{p}$. Completing the proof entails showing that the RHS is contained in the nilradical. Let s be in the RHS but it is not nilpotent. Let Σ be the set of homogeneous ideals $\mathfrak{a}$ with property $n > 0 \Rightarrow s^n \notin \mathfrak{a}$. We know Σ is nonempty since $0 \in \Sigma$ by hypothesis. Order Σ by inclusion and the arbitrary union of homogeneous ideals is also a homogeneous ideal. So there is a maximal element $\mathfrak{p} \in \Sigma$.

We can show $\mathfrak{p}$ is a homogeneous prime ideal. Let $x, y \notin \mathfrak{p}$. Then $\mathfrak{p} \subsetneq (x) + \mathfrak{p}, (y) + \mathfrak{p}$ meaning the $(x) + \mathfrak{p}, (y) + \mathfrak{p} \notin \Sigma$. Therefore,

$$f^m \in (x) + \mathfrak{p} \qquad \& \qquad f^l \in (y) + \mathfrak{p}$$

for some $n, m \in \mathbb{N}$. It follows

$$f^{m+n} \in ((x) + \mathfrak{p})((y) + \mathfrak{p})) = (xy) + (x)\mathfrak{p} + (y)\mathfrak{p} + \mathfrak{p}^2 \subseteq (xy) + \mathfrak{p}.$$

That means $xy \notin \mathfrak{p}$. It follows that s not nilpotent implies $s \notin \mathfrak{p}$ for some homogeneous prime ideal. The cotrapositive is if $s \in \bigcap_{\mathfrak{p} \text{ homogeneous}} \mathfrak{p}$, then s is nilpotent.

(b) We show $\operatorname{Proj} T \setminus U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \supseteq \varphi(S_+)\}$ is closed. Indeed, $\operatorname{Proj} T \setminus U = V((\varphi(S_+)))$ where $(\varphi(S_+))$ is the homogeneous ideal generated by $\varphi(S_+)$.

To see that φ determines a natural morphism $f : U \to \operatorname{Proj} S$, define f by sending $\mathfrak{p} \in U$ to $\varphi^{-1}(\mathfrak{p})$. By definition of U, $\varphi^{-1}(\mathfrak{p}) \not\supseteq S_+$ and so, $\varphi^{-1}(\mathfrak{p}) \in \operatorname{Proj} S$. This is continuous since $f^{-1}(D_+(g)) = D_+(\varphi(g)) = U \cap D_+(\varphi(g))$ (see the lemma below).

The the map of sheaves follows by restricting the morphism of sheaves induced by $\operatorname{Proj} T \to \operatorname{Proj} S$. See the second lemma below.

We have the following useful lemma:

Lemma 2.1. Under $f : U \to \operatorname{Proj} S$, we have $f^{-1}(D_+(g)) = U \cap D_+(\varphi(g))$ for all $g \in S_+$. This implies f is continuous.

PROOF. Let $\mathfrak{p} \in U$. We have

$$f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}) \in D_+(g) \qquad \iff \qquad g \notin f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}) \qquad \iff \qquad \varphi(g) \notin \mathfrak{p}.$$

This last statement means $\mathfrak{p} \in D_+(\varphi(g))$. □

We have the following lemma completes the proof of this part.

Lemma 2.2. First off, the sets $D_+(\varphi(s))$ for $s \in S_+$ homogeneous covers U.

Secondly, for each s, there is a morphism of affine schemes

$$D_+(\varphi(s)) = \operatorname{Spec}(S_{(\varphi(s))}) \to \operatorname{Spec}(S_{(s)}) = D_+(s)$$

which agree on overlaps. These then glue to a morphism $f : U \to \operatorname{Proj} S$.

PROOF. For the first part, the $D_+(\varphi(s))$ cover U iff every $\mathfrak{p} \in U$ has $\mathfrak{p} \in D_+(\varphi(s))$ for some s homogeneous.

Let $\mathfrak{p} \in U$. Assume $\varphi(s) \notin \mathfrak{p}$ for $s \in S_+$. Let $s = \sum_{d>0} s_d$ be its decomposition into homogeneous parts. Then $\sum_{d>0} \varphi(s_d) \notin \mathfrak{p}$ and by contradiction, there is some e s.t. $\varphi(s_e) \notin \mathfrak{p}$.

We now check the second part. The morphism is induces from the ring map

$$S_{(s)} \to T_{(\varphi(s))}$$

induced from the degree 0 part of the $\mathbb{Z}$-graded map $\varphi : S_s \to T_{\varphi(s)}$. The morphisms agree on overlaps

$$D_+(\varphi(s)) \cap D_+(\varphi(s')) = D_+(\varphi(s)\varphi(s')) = D_+(\varphi(ss'))$$

because the ring maps $S_s \to T_{\varphi(s)}$ and $S_{s'} \to T_{\varphi(s')}$ (obtained from $\varphi : S \to T$) induce the same map $S_{ss'} \to T_{\varphi(ss')}$. Taking degree 0 parts, we see that the morphisms induced in degree 0 agree on overlaps. This means we can glue to the morphism $f : U \to \operatorname{Proj} S$ taken by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. □

(c) We have $\varphi(S_+) \subseteq T_+$ since φ is a graded homomorphism. Now, $U \subseteq \operatorname{Proj} T$ is obvious. So we prove $\operatorname{Proj} T \subseteq U$. Given $\mathfrak{p} \in \operatorname{Proj} T$. Since $T_+ \not\subseteq \mathfrak{p}$, and $\varphi_d : S_d \to T_d$ are isomorphisms for $d \geq d_0$, we get $\varphi(S_+) \not\subseteq \mathfrak{p}$. Indeed, suppose $\mathfrak{p} \not\supseteq T_+$, but $\mathfrak{p} \supseteq \varphi(S_+)$. Then there is some $t_r \notin \mathfrak{p}$ where $t_r \in T_r$. For sufficiently large s, we get $t_r^s \in \varphi(S_{rs}) = T_{rs}$ which implies $t_s^r \notin \mathfrak{p}$. So, $\mathfrak{p} \not\supseteq \varphi(S_+)$.

We begin by showing $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a homeomorphism. Let $t \in T_+$ be a homogeneous element of degree r with r large. Then $t^d \in T_{rd} = S_{rd}$ for large d. The associated affine open sets $\operatorname{Spec}(S_{(b^d)})$ cover $\operatorname{Proj}(S)$. There are then subrings

$$S_{(t^d)} \subseteq T_{t^d} = T_t \qquad \& \qquad T_{(t)} \subseteq T_t$$

which coincide as subrings. Indeed, $S_{(t^d)} = T_{(t^d)} = T_{(t)}$. It follows that $D_+(t) \subseteq \operatorname{Proj}(T)$ is mapped to $D_+(\varphi^{-1}(t)^d) \subseteq \operatorname{Proj} S$ homeomorphically. Since the $D_+(t)$ cover $\operatorname{Proj} T$ and $D_+(\varphi^{-1}(t)^d)$ cover $\operatorname{Proj} T$, map $f : \operatorname{Proj} T \to \operatorname{Proj} S$ itself was a homeomorphism.

But the identifications above in conjunction with the second lemma show that we have an isomorphism on the structure sheaves. Now just take $d = 1$ for our result. What we've actually shown is much stronger:

Lemma 2.3. If there exists an integer $d \geq 1$ s.t. $\varphi_r : S_r \to T_r$ is bijective for all large $r \in d \cdot \mathbb{N}$, then $U = \operatorname{Proj} T$ and $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

(d) Omitted. This just says that the functor t applies to projective varieties and that the Proj construction agrees with how we understand projective varieties. One can cover projective varieties by the affine covers U_i and then check the isomorphisms via t to $D_+(s_i)$.

Indeed, $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is covered by the open affine subsets $(D_+(x_i), \mathcal{O}|_{D_+(x_i)}) \cong \operatorname{Spec} S_{(x_i)}$ for $x_i \in k[x_0, \ldots, x_n]/I(V)$. The functor t identifies $(D_+(x_i), \mathcal{O}|_{D_+(x_i)})$ with the affine variety $U_i \cap V$ where U_i is the subset of $\mathbb{P}^n$ where x_i does not vanish. That is, $t(V \cap U_i) \cong D_+(x_i)$ via the isomorphism β in Hartshorne Proposition II.2.6. This gives morphisms $\beta_i : t(V \cap U_i) \to \operatorname{Proj} S$ and gluing morphisms gives the isomorphism $t(V) \cong \operatorname{Proj} S$. □

Hartshorne II.2.15 (a) Let V be a variety over the algebraically closed field k. Show that a point $P \in t(V)$ is a closed point if and only if its residue field is k.
(b) If $f : X \to Y$ is a morphism of schemes over k, and if $P \in X$ is a point with residue field k, then $f(P) \in Y$ also has residue field k.
(c) Now show that if V, W are any two varieties over k, then the natural map

$$\mathrm{Hom}_{\mathfrak{R}_{\mathfrak{Var}}}(V, W) \to \mathrm{Hom}_{\varepsilon_{cb/k}}(t(V), t(W))$$

is bijective. (Injectivity is easy. The hard part is to show it is surjective.)

Remark 15. This is a problem that requires use of Hartshorne Proposition II.2.6. It stumped me when first reading it. Additionally, we will require use of Zariski's Lemma. Refer to [9] for more on Zariski's Lemma.

Proof. (a) WLOG, let us assume V is an affine variety. Indeed, Hartshorne Proposition 4.3 allows us to cover any variety with open affine subsets. This allows us to assume $t(V) = \operatorname{Spec} A$ for $A = k[x_1, \ldots, x_n]/I$.

With this reduction, let us assume that the residue field of P is already k. Since P is a point of $t(V)$, let us write $P = \mathfrak{p}$. Now, we get morphisms $k \to A/\mathfrak{p} \to (A/\mathfrak{p})_{\mathfrak{p}} \cong k$. So, $k = A/\mathfrak{p}$. This implies that $\mathfrak{p}$ is maximal and hence, $\mathfrak{p}$ is a closed point.

For the converse, let $\mathfrak{p}$ be a closed point. Zariski's Lemma says that $k(\mathfrak{p}) = A/\mathfrak{p}$ is finite. Hence, $k(\mathfrak{p})$ is a finite field extension of k. So, $k = k$ since k is algebraically closed.

(b) We get an induced morphism of local rings $\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$.

Note that if $B \to A$ is a morphism of local k-algebras and $A/\mathfrak{m} \cong k$, then $B/\mathfrak{n} \cong k$. So, in this case, if P has residue field k, then $\mathcal{O}_{Y,f(P)}/\mathfrak{m}_{f(P)} \cong k$ and we conclude that $f(P)$ has the same residue field.

(c) **[Incomplete]**

□

Hartshorne II.2.16 Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$

(a) If $U = \operatorname{Spec} B$ is an open affine subscheme of X, and if $\bar{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

(b) Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$ [Hint: Use an open affine cover of X.]

(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. (This hypothesis is satisfied, for example, if $\mathrm{sp}(X)$ is noetherian.) Let $b \in \Gamma\left(X_f, \mathcal{O}_{X_f}\right)$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

(d) With the hypothesis of (c), conclude that $\Gamma\left(X_f, \mathcal{O}_{X_f}\right) \cong A_f$.

Proof. **(a)** Let $x \in U \cap X_f = \{x \in \operatorname{Spec} B | f_x \notin \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}\}$. For ease, write $\mathfrak{p}$ for x when considering it as a prime ideal of B. In this case, $U \cap X_f = \{\mathfrak{p} \in \operatorname{Spec} B | f_{\mathfrak{p}} \notin \mathfrak{p}B_{\mathfrak{p}}\}$. The condition $f_{\mathfrak{p}} \notin \mathfrak{p}B_{\mathfrak{p}}$ implies $\overline{f} \notin \mathfrak{p}$. So, $x \in D(\overline{f})$ showing $U \cap X_f \subseteq D(\overline{f})$.

Now if $x \in D(\overline{f})$, that means $\overline{f} \notin \mathfrak{p}$. But this means $f_{\mathfrak{p}} \notin \mathfrak{p}B_{\mathfrak{p}}$ and $x \in U \cap X_f$. Hence, $U \cap X_f = D(\overline{f})$.

To show X_f is open, take an open affine cover $\{\operatorname{Spec} A_i\}$ of X. The intersections $X_f \cap \operatorname{Spec} A_i$ are open subsets of $\operatorname{Spec} A_i$. Then X_f is a union of these open sets and must be open as well.

(b) Cover X by finitely many affines $U_i = \operatorname{Spec} A_i$. By part (a), $U_i \cap X_f = D(f_i)$ where we write f_i to mean the restriction of f to U_i. The restrictions of a to each X_f is zero imply a_i is zero on $D(f_i)$. So there is an $r_i > 0$ so that $f^{r_i} a = 0$ in A_i. Since we have finitely many affines, we take r to be the maximum. Hence, $af^r = 0$ on each U_i. By the sheaf axioms, $af^r = 0$.

(c) If $b \in \Gamma(X_f, \mathcal{O}_{X_f})$, then we can consider the restrictions $b|_{X_f \cap U_i}$ as X_f is open. Clearly,

$$b|_{X_f \cap U_i \cap U_j} - b|_{X_f \cap U_j - U_i} = 0$$

and so (b) says there is an $n > 0$ s.t.

$$f^n (b|_{U_i \cap U_j} - b|_{U_j \cap U_i}) = 0.$$

Since the n depends on i, j and there are only finitely many open affines, we may take n to be the maximum over all choices of n. Thus, $f^n b|_{U_i}$ are sections whose restrictions to the $U_i \cap U_j$ agree. We can then glue the $f^n b|_{U_i}$ to a section $a \in \Gamma(X, \mathcal{O}_X)$. That is, $a|_{U_i} = f^n b|_{U_i}$. But then that means $a|_{X_f} = f^n b$.

(d) Define the map $A_f \mapsto \Gamma(X_f, \mathcal{O}_X)$ by $\frac{a}{f^n} \mapsto \frac{a|_{X_f}}{f^n|_{X_f}}$ because $A = \Gamma(X, \mathcal{O}_X)$. This is clearly a ring homomorphism.

Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Then (c) says $f^n b = a|_{X_f}$ for some $n > 0$ and $a \in A$. Now, $\frac{a}{f^n} \mapsto \frac{a|_{X_f}}{f^n|_{X_f}} = b$ establishing surjectivity.

Suppose $\frac{a|_{X_f}}{f^n|_{X_f}} = 0 \in \Gamma(X_f, \mathcal{O}_{X_f})$ and we show $\frac{a}{f^n} = 0$. As $f^n|_{X_f}$ is a unit in $\Gamma(X_f, \mathcal{O}_{X_f})$, $\frac{a|_{X_f}}{f^n|_{X_f}} = 0 \Rightarrow a|_{X_f} = 0$. As X is compact (the finite union of compact sets is compact), we deduce that there is an k s.t. $f^k a = 0$. Therefore, $\frac{a}{f^n} = 0$.

□

Hartshorne II.2.17 *A Criterion for Affineness.*

(a) Let $f : X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

(b) A scheme X is affine if and only if there is a finite set of elements $f_1, \ldots, f_r \in A = \Gamma(X, \mathcal{O}_X)$, such that the open subsets X_{f_i} are affine, and $f_1, \ldots, f_r$ generate the unit ideal in A. [Hint: Use (Ex. 2.4) and (Ex. 2.16d) above.]

Proof. **(a)** The sets $f^{-1}(U_i)$ and U_i are given the induced open subscheme structure. If the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism, we know there is an inverse $g_i : U_i \to f^{-1}(U_i)$ morphism of schemes. Define the inverse morphism $g : Y \to X$ as follows. At the level of topological spaces, let $g : Y \to X$ be the homeomorphism obtained by gluing local inverses (which agree on intersections and $f^{-1}(U_i) \setminus f^{-1}(U_j)$ is disjoint from $f^{-1}(U_j) \setminus f^{-1}(U_i)$ for

all i, j pairs). In addition, g is the continuous inverse of f. For the morphism of sheaves $g^\# : \mathcal{O}_X \to g_*\mathcal{O}_Y$, start by defining

$$g^\#(U_i) := (f^\#)^{-1}(U_i) : \mathcal{O}_X(U_i) \to \mathcal{O}_Y(g^{-1}(U_i)).$$

As

$$g_i^\#(U_i \cap U_j), g_j^\#(U_i \cap U_j) : \mathcal{O}_Y(U_i \cap U_j) \to \mathcal{O}_X(f^{-1}(U_i \cap U_j))$$

are both inverses of $f^\#(f^{-1}(U_i \cap U_j)) : \mathcal{O}_X(f^{-1}(U_i \cap U_j)) \to \mathcal{O}_Y(U_i \cap U_j)$, they are the same maps due to uniqueness of inverses. Therefore, defining $g^\#(U_i)$ determines the map of sheaves $g^\# : \mathcal{O}_X \to g_*\mathcal{O}_Y$. Additionally,

$$f^\# \circ g^\#(U_i) : \mathcal{O}_X(U_i) \to \mathcal{O}_Y(g^{-1}(U_i)) \to \mathcal{O}_X(f^{-1}g^{-1}(U_i)) = \mathcal{O}_X(U_i)$$

is the identity morphism for all i. Taking stalks at $P \in U_i$, we deduce

$$(f^\# \circ g^\#)_P : \mathcal{O}_{X,P} \to \mathcal{O}_{X,P}$$

is an isomorphism. Therefore, $f^\#$ is inverse to $g^\#$ as a morphism of sheaves. As both $f_P^\#$ and $g_P^\#$ are local homomorphisms, the composition is as well. This proves that $g \circ f : X \to X$ is the identity. The proof that $f \circ g = \mathrm{id}_Y$ follows similarly. Everything we did above applies equally well for showing $g^\# \circ f^\#$ is the identity. In conclusion, $(g, g^\#)$ is a two-sided inverse to $(f, f^\#)$ in the category of schemes.

(b) ($\implies$): If $X = \operatorname{Spec} A$, then $A = \Gamma(X, \mathcal{O}_X)$ with $X_{f_i} \cong D(f_i)$ affine by Hartshorne II.2.16(a). Additionally, by X quasicompact, we may choose finitely many $f_i \in A$ so that $(f_1, \ldots, f_m) = 1$ and $X_{f_i} = D(f_i)$ covers X for $i = 1, \ldots, m$.

($\impliedby$): Set $A := \Gamma(X, \mathcal{O}_X)$. By Hartshorne II.2.4, we get a morphism $g : X \to \operatorname{Spec} A$. Cover $\operatorname{Spec} A$ by the finitely many affine open sets $\operatorname{Spec} A_{f_i}$. Now we examine the induced maps $f^{-1}(\operatorname{Spec} A_{f_i}) \to \operatorname{Spec} A_{f_i}$. We claim that $f^{-1}(\operatorname{Spec} A_{f_i}) \cong X_{f_i}$ in a natural way. Indeed, $f_i \in \Gamma(X, \mathcal{O}_X)$ and the primes of $f^{-1}(\operatorname{Spec} A_{f_i})$ are exactly those mapping to primes x of $\operatorname{Spec} A_{f_i}$ under f. But those primes are exactly those that do not contain f_i and hence, are exactly those $x = \mathfrak{p}$ in which $f_x \notin \mathfrak{m}_x = \mathfrak{p}A_\mathfrak{p}$. Therefore, the induced maps $f^{-1}(\operatorname{Spec} A_{f_i}) \to \operatorname{Spec} A_{f_i}$ are isomorphisms and (a) shows that $f : X \to \operatorname{Spec} A$ is an isomorphism. Therefore, X is affine.

By Hartshorne II.2.16, $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$. By quasi-compactness of $\operatorname{Spec}(A)$, we may choose finitely many f_i generating the unit ideal. Hence, the hypotheses of (a) are satisfied and f is an isomorphism. So, $X = \operatorname{Spec} A$ is an affine scheme.

□

Hartshorne II.2.18 In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.

(a) Let A be a ring, $X = \operatorname{Spec} A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

(b) Let $\varphi : A \to B$ be a homomorphism of rings, and let $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} \dot{A}$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is dominant, i.e., $f(Y)$ is dense in X.

(c) With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

(d) Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then φ is surjective. [Hint: Consider $X' = \operatorname{Spec}(A/\ker\varphi)$ and use (b) and (c).]

PROOF. **(a)** f is nilpotent iff f is contained in every prime ideal iff $D(f) = \emptyset$.

(b) The map $\varphi : A \to B$ is injective iff $\varphi_\mathfrak{p} : A_\mathfrak{p} \to B_\mathfrak{p} =: B \otimes A_\mathfrak{p}$ is injective for all $\mathfrak{p} \in \operatorname{Spec}(A)$ by Proposition 3.9 in [2] (where we view B as an A-module). But $f^\#_\mathfrak{p} : \mathcal{O}_{X,\mathfrak{p}} \to (f_*\mathcal{O}_Y)_\mathfrak{p}$ is precisely the map $\varphi_\mathfrak{p}$ since $(f_*\mathcal{O}_Y)_\mathfrak{p} = \mathcal{O}_{Y,\mathfrak{p}} = B \otimes A_\mathfrak{p} = B_\mathfrak{p}$ (here, we used closedness of $Y \subseteq X$).

For the second statement[17], set $f(Y) := f(V(\operatorname{nilrad} B))$. Then the closure is[18],

$$\overline{f(Y)} = \overline{f(V(0))} = V(\varphi^{-1}(0)) = V(0) = \operatorname{Spec} A = X.$$

where the third equality used injectivity of φ.

(c) If φ surjective, then $B \cong A/\ker\varphi$. This shows that f is a homeomorphism of Y onto a closed subset of X, in particular, f induces a homeomorphism between Y and $V(A/\ker\varphi)$.

Next, $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is a surjection. Taking stalks, we have

$$f^\#_\mathfrak{p} : \mathcal{O}_{X,\mathfrak{p}} \to (f_*\mathcal{O}_Y)_\mathfrak{p}$$

and since we can identity Y with a closed subset of X, $(f_*\mathcal{O}_Y)_\mathfrak{p} = \mathcal{O}_{Y,\mathfrak{p}}$. So, the map above is the surjective map $f^\#_\mathfrak{p} : A_\mathfrak{p} \to B_\mathfrak{p}$ since localization is exact.

(d) The solution we present proves a little more than is asked of us. Indeed, it shows that $\varphi : A \to B$ is a surjection and that $B \cong A/\ker\varphi$. First,

$$A \overset{\psi}{\twoheadrightarrow} A/\ker\phi \overset{\phi}{\hookrightarrow} B \qquad \Longrightarrow \qquad Y \overset{h}{\to} X' \overset{g}{\to} X$$

where $\phi \circ \psi = \varphi$ and $g \circ h = f$.

From (c), $g : X' \to X$ is a homeomorphism of X' onto a closed subset of X with $g^\# : \mathcal{O}_X \to g_*\mathcal{O}_{X'}$ is surjective. From (b), $h : Y \to X'$ is a dominant map with $h^\# : \mathcal{O}_{X'} \to h_*\mathcal{O}_Y$ injective. Since the composition $f = g \circ h$ is a homeomorphism onto a closed subset,

$$g \circ h(Y) = \overline{g \circ h(Y)} = g \circ (\overline{h(Y)}) = g(X').$$

The middle equality is clear by g a homeomorphism onto a closed subset (if $f : X \to Y$ is a homeomorphism, then it is always true $f(\overline{U}) = \overline{f(U)}$). This implies $h(Y) = X'$ is a homeomorphism onto X'. Indeed, $f = g \circ h$ implies $g^{-1} \circ f = h$. OTOH, the composition is surjective

$$\mathcal{O}_X \longrightarrow g_*\mathcal{O}_{X'} = \mathcal{O}_{X'} \longrightarrow f_*\mathcal{O}_Y (g \circ h)_*\mathcal{O}_Y = h_*g_*\mathcal{O}_Y = h_*\mathcal{O}_Y$$

From (b), $\mathcal{O}_{X'} \to h_*\mathcal{O}_Y$ is injective and from (c), the map $\mathcal{O}_X \to \mathcal{O}_Y$ is surjective. But the entire composition is surjective which means $\mathcal{O}_{X'} \to h_*\mathcal{O}_Y$ is also surjective which implies $\mathcal{O}_{X'} \to h_*\mathcal{O}_Y$ is surjective as well. It follows $h^\# : \mathcal{O}_{X'} \to h_*\mathcal{O}_Y$ is an isomorphism (note that we must translate everything to the level of stalks). Taking global sections,

$$h^\#(X') := A/\ker\varphi \to B$$

[17] A stronger statement can be found in Exercise 1.21 of [2]. A full proof can be found here.
[18] See Exercise 1.21 (iii) of [2]

is an isomorphism. Since the map $\mathcal{O}_X \to \mathcal{O}_Y$ is surjective, taking global sections gives a surjective map

$$\varphi : A \twoheadrightarrow B \cong A/\ker\varphi.$$

Since $f : Y \to X$ is a homeomorphism of Y onto a closed subset, $h(Y) \subseteq X'$ is a homeomorphism onto a closed subset of X'. From (b), $h(Y)$ is dense in X' and $h^\# : \mathcal{O}_{X'} \to h_*\mathcal{O}_Y$ is injective. So, $h(Y) = X'$. From (c), g is a homeomorphism of X' onto a closed subset of X and $g^\# : \mathcal{O}_X \to g_*\mathcal{O}_{X'}$ is surjective. This gives

$$\mathcal{O}_X \to g_*\mathcal{O}_{X'} \to g_*h_*\mathcal{O}_Y$$

in which $\mathcal{O}_X \to g_*h_*\mathcal{O}_Y$ is surjective. Since f is a homeomorphism onto a closed subset, and g is a homeomorphism of X' onto a closed subset, h is a homeomorphism. The map $h^\# : X' \to h_*\mathcal{O}_Y$ is an isomorphism since (b) shows it is injective, and the above factorization shows it is surjective[19]. Taking global sections, $\phi : A/\ker\phi \to B$ is surjective. That means, $A/\ker\phi = \varphi(A) = B$ and φ is surjective. □

Hartshorne II.2.19 Let A be a ring. Show that the following conditions are equivalent:

(i) Spec A is disconnected;

(ii) there exist nonzero elements $e_1, e_2 \in A$ such that $e_1e_2 = 0, e_1^2 = e_1, e_2^2 = e_2, e_1+e_2 = 1$ (these elements are called orthogonal idempotents);

(iii) A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.

Proof. (i)⇒(iii): Let $U_1 \cup U_2 = \operatorname{Spec} A$ disconnect $\operatorname{Spec} A$. Then $\mathcal{O}(\operatorname{Spec} A) = \mathcal{O}(U_1) \oplus \mathcal{O}(U_2)$ which means $A \cong \mathcal{O}(U_1) \times \mathcal{O}(U_2)$.
(iii)⇒(ii): Set $e_1 := (1, 0)$ and $e_2 = (0, 1)$.
(ii)⇒(i): Let $U_1 := D(e_1)$ and $U_2 := D(e_2)$. Then $U_1 \cap U_2 = D(e_1e_2) = D(0) = \emptyset$ and $U_1 \cup U_2 = D(e_1 + e_2) = D(1) = \operatorname{Spec} A$. □

3. First Properties of Schemes

Hartshorne II.3.1 Show that a morphism $f : X \to Y$ is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

Proof. We will make use of a few results not covered Hartshorne[20].

($\Longrightarrow$): Assume $f : X \to Y$ is of locally finite type. Let $V = \operatorname{Spec} B \subseteq Y$ be an open affine. We claim that V can be covered by $\operatorname{Spec} B_i$ open affines of Y such that $f^{-1}(\operatorname{Spec} B_i)$ can be covered by $\operatorname{Spec} A_{ij}$ where A_{ij} is a finitely generated B_i-algebra.

[19]The composition of maps is surjective, so the precomposed map is surjective i.e. the last map.

[20]We will make use of Proposition 5.3.1 and Proposition 5.3.3 of [9]. We will just call the former the **Affine Trick** or **Nike's Trick**. See also [13].
Proposition 5.3.1 states: "Suppose $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are affine open subschemes of a scheme X. Then $\operatorname{Spec} A \cap \operatorname{Spec} B$ is the union of open sets that are simultaneously distinguished open subschemes of $\operatorname{Spec} A$ and $\operatorname{Spec} B$.
Proposition 5.3.3 states: "Suppose A is a ring, and $(f_1, \ldots, f_n) = A$.
(a) If A is a Noetherian ring, then so is A_{f_i}. If each A_{f_i} is Noetherian, then so is A.
(b) Suppose B is a ring, and A is a B-algebra. (Hence A_g is a B-algebra for all $g \in A$.) If A is a finitely generated B-algebra, then so is A_{f_i}. If each A_{f_i} is a finitely generated B-algebra, then so is A."

By Nike's Trick, the intersection $\operatorname{Spec} B \cap \operatorname{Spec} B_i$ can be covered by open sets which are distinguished open sets in both $\operatorname{Spec} B$ and $\operatorname{Spec} B_i$. So, $\operatorname{Spec} B$ can be covered by $\operatorname{Spec}(B_i)_{b_i}$ localized at some $b_i \in B_i$. By Proposition 5.3.3, $f^{-1}(\operatorname{Spec}(B_i)_{b_i})$ can be covered by $\operatorname{Spec} A_{ij}$ which are finitely generated $(B_i)_{b_i}$-algebras and hence, finitely generated B_i-algebras.

We are now able to reduce to the case where $Y = \operatorname{Spec} B$ (since if we prove this case, then the preceding paragraph gives us the result for any scheme Y). By locally finite type, we can cover Y with $\operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i)$ can be covered by $\operatorname{Spec} A_{ij}$ which are B_i-algebras. By Nike's Trick, take intersections $\operatorname{Spec} B \cap \operatorname{Spec} B_i$ and consider a covering of Y by $\operatorname{Spec} B_{b_i}$. The $f^{-1}(\operatorname{Spec} B_{b_i})$ can be covered by $\operatorname{Spec} A_{ij}$ which are finitely-generated B-algebras by the preceding paragraph.

We are not done! Recall that we had to show that every open affine subset of Y had the property. In this case, every open affine subset U of Y corresponds to an affine open subset $f^{-1}(U) \subseteq X$.

Let $\operatorname{Spec} A \subseteq X$ be arbitrary. By Nike's Trick, take $\operatorname{Spec} A_{a_k}$ which are distinguished open in both $\operatorname{Spec} A$ and $\operatorname{Spec} A_{ij}$. We can cover $\operatorname{Spec} A$ by the $\operatorname{Spec} A_{a_k}$'s. Now, we wish to show that $\operatorname{Spec} A$ is a finitely generated B-algebra. This comes easily since A_{a_k} is a localization of A_{ij} and so, A_{a_k} is a finitely-generated B-algebra. Since the ideal generated by the a_k's must be A, Proposition 5.3.3 let's us conclude that A is a finitely generated B-algebra.

(Since each $\operatorname{Spec} A$ is a finitely generated B-algebra, we can conclude that the every affine open subset U of Y can be covered by spectras that have a preimage in X that can be covered by finitely generated algebras.)

($\Longleftarrow$): In some sense, this direction is obvious. We have a scheme Y so it is locally an affine scheme. Let $\{V_i\}$ be affine open subsets with $V_i = \operatorname{Spec} A_i$ covering Y. Now, consider $f^{-1}(V_i)$ that cover X. By the assumptions, $f^{-1}(V_i)$ can be covered by affine open subsets $U_{ij} = \operatorname{Spec} A_{ij}$ such that A_{ij} are finitely generated B_i-algebras. This is precisely the definition of of locally of finite type and so, we are done. □

Hartshorne II.3.2 A morphism $f : X \to Y$ of schemes is quasi-compact if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasicompact if and only if for every open affine subset $V \subseteq Y, f^{-1}(V)$ is quasi-compact.

Proof. ($\Longrightarrow$): Let $V = \operatorname{Spec} B \subseteq Y$ be an affine open subset. We will show that $f^{-1}(V)$ is quasi-compact. By assumption, we have that $Y = \bigcup_i \operatorname{Spec} A_i$ such that $f^{-1}(\operatorname{Spec} A_i)$ is quasi-compact for each i. Let $f^{-1}(\operatorname{Spec} A_i) = \bigcup_{i,j} \operatorname{Spec} A_{i,j}$ where the index in j is finite. We can cover V by finitely many $\operatorname{Spec} B_{b_i}$ where $\operatorname{Spec} B_{b_i}$ is distinguished open in $\operatorname{Spec} A_i$ and $\operatorname{Spec} B$ by Nike's Trick [21]. Now, $f^{-1}(\operatorname{Spec} B_{b_i}) = \bigcup_{i,j} f^{-1} \operatorname{Spec}(A_{i,j})_{b_i}$ which is a finite union. So, $f^{-1}(\operatorname{Spec} B_{b_i})$ is quasi-compact. Recall that $\operatorname{Spec} B = \cup_i \operatorname{Spec} B_{b_i}$ is a finite union, and so, $f^{-1}(\operatorname{Spec} B) = \bigcup_{i,j} (\operatorname{Spec} A_{i,j})_{b_i}$ is a finite union of affine open subsets.

($\Longleftarrow$): This direction is obvious. Take an affine open cover $\{V_i\}$ of Y. By hypothesis, $f^{-1}(V_i)$ is quasi-compact for each i. But then that is the desired cover by open affines needed for quasi-compactness of f. □

Hartshorne II.3.3

(a) Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

[21] b_i is not indexed by i, this is just poor notation choice. I probably should've written b_k.

(b) Conclude from this that f is of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B -algebra.

(c) Show also if f is of finite type, then for every open affine subset $V = \operatorname{Spec} B \subseteq Y$, and for every open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V)$, A is a finitely generated B -algebra.

PROOF. (a) ($\Longrightarrow$): Assume $f : X \to Y$ is of finite type. By definition, it is of locally finite type. The only nontrivial part is that $f : X \to Y$ is quasi-compact. By $f : X \to Y$ of finite type, cover Y with open affines $\operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i)$ is covered by finitely many $\operatorname{Spec} A_{ij}$ which are finitely generated B_i-algebras. So, $f : X \to Y$ is quasicompact since the $f^{-1}(\operatorname{Spec} B_i)$ are quasi-compact (since we covered it by finitely many $\operatorname{Spec} A_{ij}$).

($\Longleftarrow$): Assume that $f : X \to Y$ is locally of finite type and is quasi-compact. Then, $f : X \to Y$ can be covered by $\operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i)$ is covered by $\operatorname{Spec} A_{ij}$ which are finitely generated B_i-algebras. By quasi-compact, we can take a finite subcover by $\operatorname{Spec} A_{ij}$ to be finite.

(b) We now know that f is of finite type iff it is locally of finite type and quasi-compact iff for every open subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact and we can cover $f^{-1}(V)$ where $V = \operatorname{Spec} B$ by $\operatorname{Spec} A_j$ which are finitely generated B-algebras (by Hartshorne II.3.1 and Hartshorne II.3.2).

(c) Let $f : X \to Y$ be of finite type and $\operatorname{Spec} B \subseteq Y$ and $\operatorname{Spec} A \subseteq f^{-1}(V)$ be arbitrary. We will show that A is a finitely generated B-algebra.

We can cover $f^{-1}(V)$ by $\operatorname{Spec} A_i$ which are all finitely-generated B-algebras. By Nike's Trick, cover $\operatorname{Spec} A$ by open affine sets $\operatorname{Spec} A_{a_k}$ which are distinguished in both $\operatorname{Spec} A$ and some $\operatorname{Spec} A_i$. Since f is finite type, f is quasi-compact and we can take $\operatorname{Spec} A = \bigcup_k \operatorname{Spec} A_{a_k}$ which is a finite union. So, the a_k's generate A. Also, each of that A_{a_k}'s are f.g. B-algebras since they are localizations of a f.g. B-algebra. By Proposition 5.3.3 of [9], we conclude that A is a finitely generated B-algebra. □

Hartshorne II.3.4 Show that a morphism $f : X \to Y$ is finite if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine, equal to $\operatorname{Spec} A$, where A is a finite B-module.

PROOF. ($\Longleftarrow$): This direction is easy. Because Y is a scheme, cover Y by open affines $\{\operatorname{Spec} B_i\}$. By hypothesis, $f^{-1}(\operatorname{Spec} B_i)$ is affine and each is the spectra of a finite B_i-module. But this is precisely the definition of being a finite morphism.

($\Longrightarrow$): Suppose $f : X \to Y$ is a finite morphism. cover Y by V_i where $V_i = \operatorname{Spec} B_i$ and $f^{-1}(V_i) = \operatorname{Spec} A_i$ where A_i is a finite B_i-algebra.

Let $V \subseteq Y$ be an open affine $V = \operatorname{Spec} B$. we claim that $f^{-1}(V)$ is affine. By Nike's Trick, we can cover $V \cap V_i$ by $\operatorname{Spec} B_{b_{ij}}$ which is distinguished in both V and V_i. By quasicompactness, we only need finitely many j's. Thus, $f^{-1}(\operatorname{Spec} B_{b_{ij}}) = \operatorname{Spec} A_{i_{b_{ij}}}$. Hence, $\bigcup_j f^{-1}(\operatorname{Spec} B_{b_{ij}}) = \operatorname{Spec} A_i$. Since we can take finitely many A_i's, we have a finite union $f^{-1}(V) = \bigcup \operatorname{Spec} A_{i_{b_{ij}}}$ which shows that $f^{-1}(V)$ is affine.

Now, we claim that $f^{-1}(V) = \operatorname{Spec} A$ is s.t. A is a finitely generated B-module. Indeed, each $A_{i_{b_{ij}}} = A_{b_{ij}}$ and since $A_{b_{ij}}$ is a finitely generated B-module and the b_{ij}'s generate the unit ideal, A is a finitely generated B-module for each b_{ij}. □

Hartshorne II.3.5 A morphism $f : X \to Y$ is quasi-finite if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.

(a) Show that a finite morphism is quasi-finite.

(b) Show that a finite morphism is closed, i.e., the image of any closed subset is closed.

(c) Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

PROOF. (a) Let $y \in Y$ be a point and we show that $f^{-1}(y)$ is a finite set. By Hartshorne II.3.10 X_y is homemorphic to $f^{-1}(y)$. We have the following commutative diagram

$$\begin{array}{ccc} X \times_Y \operatorname{Spec} k(y) \cong X_y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k(y) & \hookrightarrow & Y \end{array}$$

By possibly taking an affine open cover, we reduce to the case where $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ (using the preceding exercise).

Now, $f^{-1}(y) = X_y$ and $X_y = \operatorname{Spec} B \times_Y \operatorname{Spec} k(y)$. By f finite morphism, $f^{-1}(Y) = X = \operatorname{Spec} B$ and B is a finite A-module. Now, the underlying ring of X_y is $B \otimes_A k(y)$. This is a vector space over $k(y)$ and since B is a finite generated A-module, $B \otimes_A k(y)$ is finite dimensional. By Proposition 6.10 of [2][22], $\operatorname{Spec}(B \otimes_A k(y))$ is finite and so $f^{-1}(y)$ is finite.[23]

(b) Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ with $f : X \to Y$. We have an induced ring homomorphism $\varphi : A \to B$. Let $V(S)$ be a closed subset of X. We claim that $f(V(S))$ is closed in B. By f finite morphism, B is a finite A-module. We will reduce to the case where $S = 0$ and φ is injective. By possibly modding out by an ideal, we can reduce to $S = 0$. By possibly modding out by the kernel, we reduce to the case where φ is injective. Now, notice that φ is a finite ring homomorphism and hence, φ is an integral ring homomorphism. Hence, $A \subseteq B$ and B is integral is over A.

By Theorem 5.10 in [2], if $\mathfrak{q}$ is a prime ideal of B, then $f(\mathfrak{q})$ gives us a prime ideal of A. So, closed sets get mapped to closed sets.

We now have the result for when X, Y are affine schemes. By Hartshorne II.3.4, we know that finite morphisms are affine morphisms i.e. the preimage of affines are affine. Let $Z \subseteq X$ be closed. Take an affine open V of Y so that $f(Z) \subseteq V$. So, $Z \subseteq f^{-1}(V)$. Thus $f(Z)$ is closed in V and thus, closed in Y.

(c) Let $k[x] \to k[x, y]/(xy - 1) \times k$ be the morphism defined by $p(x) \mapsto (p(x), p(0))$. Thus, $f : X = \operatorname{Spec}(k[x, y]/(xy - 1) \times k) \to Y = \operatorname{Spec} k[x]$. It is surjective, of finite type, and quasifinite.

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[22]Proposition 6.10 states: "For k-vector spaces V the following conditions are equivalent:
i) finite dimension;
ii) finite length;
iii) a.c.c.;
iv) d.c.c.
Moreover, if these conditions are satisfied, length = dimension."
In particular, we use the fact that $\operatorname{Spec}(V)$ is a finite set iff V is a finite dimensional k-vector space.
See this post for a proof of this result explicitly.

[23]Actually, the reference I gave can be better. See Corollary 8.2 of [2].

We know f is surjective because it is certainly a surjective map of topological spaces. The underling morphism of sheaves

$$f^{\#} : \mathcal{O}_{k[x]} \to f_* \mathcal{O}_{k[x,y]/(xy-1)\times k}$$

is surjective by Hartshorne II.2.18c.

Next, f is of finite type because $f^{-1}(\operatorname{Spec}(k[x,y]/(xy-1)\times k))$ is covered by $\operatorname{Spec} k[x]$ which is certainly a f.g. (also because $k[x,y]/(xy-1)\times k \cong k\left[x, \frac{1}{x}\right] \times k$ is a finitely generated $k[x]$-algebra.).

Furthermore, f is quasi-finite because

$$\begin{aligned} f^{-1}((x-a)) &\cong X_y \\ &\cong X \times_Y \operatorname{Spec} k(y) \\ &\cong \operatorname{Spec}((k[x,y]/(xy-1)\times k) \otimes_{k[x]} k[x]_{(x-a)}) \\ &\cong \operatorname{Spec}\left(\left(k\left[x, \frac{1}{x}\right] \times k\right) \otimes_{k[x]} k[x]_{(x-a)}\right) \\ &\cong \operatorname{Spec}((k(x)\times k) \otimes_{k[x]} k[x]_{(x-a)}) \\ &\cong \operatorname{Spec}((k(x)\times k)_{(x-a)}) \\ &\cong \operatorname{Spec}(k \times k) \end{aligned}$$

and $\operatorname{Spec}(k\times k)$ is finite because $k\times k$ is a finite dimensional k-vector space. However, $k[x,y]/(xy-1)\times k$ is not a finite $k[x]$-module (the submodule

$$\left(\frac{1}{x} \times k, \frac{1}{x^2} \times k, \frac{1}{x^3} \times k, \dots,\right)$$

is not f.g.).

Note. Other examples for (c) exist. For example, the map $X \to \mathbb{A}^1$ from the affine line. □

Hartshorne II.3.6 Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. It is called the function field of X, and is denoted by $K(X)$. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.

Proof. Let X be an integral scheme. Let ξ be the generic point of X. In the case when $X = \operatorname{Spec} A$, ξ is the nilradical. As X is integral, ξ is the zero ideal. So, $\mathcal{O}_\xi$ is localization at the zero ideal and so $\mathcal{O}_\xi$ is the field of fractions of A.

In the general case, note that taking stalks is local is $\mathcal{O}_\xi = A_\xi$ where $\operatorname{Spec} A$ is an affine open subscheme containing the generic point ξ. But that means $\mathcal{O}_\xi$ is a field since $\Gamma(\operatorname{Spec} A, \mathcal{O}_X) \cong \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \cong A$ is an integral domain.

Now, for $U = \operatorname{Spec} A$ open affine subset of X, we claim that $K(X)$ is isomorphic to the quotient field of A. The quotient field of A is just $A_{(0)}$ localization at (0). Then, $K(X)$ is isomorphic to $A_{(0)}$ since by X integral scheme, the generic point of $\operatorname{Spec} A$ is the same as the generic point of X.

Taking the stalks of the restriction homomorphism $\rho_{XU} : \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ at ξ gives

$$\Gamma(X, \mathcal{O}_X)_{(0)} \hookrightarrow \Gamma(U, \mathcal{O}_X)_{(0)} \cong A_{(0)}.$$

Now for integral schemes, we know that the restriction maps are injective and so, the map above is injective. On the other hand,

$$\Gamma(U, \mathcal{O}_X) \hookrightarrow \mathcal{O}_{X,\xi}$$

as well and the localizing at ξ means $A_{(0)} \hookrightarrow \mathcal{O}_{X,\xi}$. An injective map both ways means we have an isomorphism of rings. So every quotient field $A_{(0)}$ of A is isomorphic to $\Gamma(X, \mathcal{O}_X)_{(0)}) \cong \mathcal{O}_{X,\xi} = K(X)$.

Lemma. The restriction maps $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are injective for X an integral scheme.

Proof. Suppose $\rho_{UV}(s) = 0$. The $s|_V = 0$. We know that U is irreducible by irreducibility of X. By Hartshorne I.1.6, we know that V is dense in U. So $s|_U = 0$ as well. That is, $s_P \in \mathcal{O}_{X|U,P}$ is zero for all $P \in U$. Reducedness then implies $s = 0$. □

Example. In the preceding proof, one may wonder why $s|_U = 0$ did not immediately imply that $s = 0$. Here is why. Consider $\operatorname{Spec}(k[x]/(x^2)) = \{(\overline{x})\}$. A global section is an element of $k[x]/(x^2)$. Take the section x and it is then zero on $(\overline{x})$ (because on $\overline{x}$, we get $x = 0$). However, the section x itself is not zero. □

Second proof. One can observe that X being integral means it has a unique generic point and any affine opens subset will contain the generic point. Moreover X being integral means that $\operatorname{Spec} A$, an affine open subset, is integral with generic point $(0) = \xi$ which is that of X. So $\mathcal{O}_\xi \cong A_{(0)}$. □

Hartshorne II.3.7 A morphism $f : X \to Y$, with Y irreducible, is **generically finite** if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y. A morphism $f : X \to Y$ is **dominant** if $f(X)$ is dense in Y. Now let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite. [Hint: First show that the function field of X is a finite field extension of the function field of Y.]

Proof. We shall split the problem into a number of parts. First, let us make a reduction to the case where Y is affine. Suppose we knew the result for when Y is an open affine subset. Now consider the general case where $f : X \to Y$ are just schemes under the problem's hypotheses. Any affine open subset $\operatorname{Spec} B$ of Y is going to be integral, dense, and contains the generic point of Y because of Hartshorne I.1.6, Hartshorne II.2.3, and Hartshorne II.3.6. If we can find a dense open subset $U \subseteq \operatorname{Spec} B$ s.t. $f^{-1}(U) \to U$ is affine, then we will have found a dense open subset U of Y.

Lemma 1. The function field $K(X)$ is a finite field extension of the function field $K(Y)$ of Y.

Proof. In this situation, $f : f^{-1}(\operatorname{Spec} B) \to \operatorname{Spec} B$. By being of finite type, assume $f^{-1}(Y) = \bigcup_{i=1}^n \operatorname{Spec} A_i$ for A_i f.g. A-algebras. As $\eta \in \operatorname{Spec} B$ is the generic point, integrality of Y ensures that η corresponds to the zero idea. For each $f_i : \operatorname{Spec} A_i \to \operatorname{Spec} B$, we know that the fibre $f^{-1}(\eta) = \operatorname{Spec} A_i \otimes_B k(\eta) = A_i \otimes_B B_{(0)} = A_i \otimes_B K(Y)$ is a finite dimensional vector space. By the integrality assumption on X, $A_i \otimes_B K(Y)$ is an integral domain and hence, it is a field with field of fractions $K(\operatorname{Spec} A_i) \cong A_i \otimes_A B_{(0)}$. Being a finite dimensional vector space implies that $K(\operatorname{Spec} A_i)$ is a finite field extension of $K(Y)$ (this

part of the problem can be immensely quickened with applying Noether Normalization or Zariski's Lemma). □

Lemma 2. Knowing that there is a finite field extension $K(X)/K(Y)$, for every $\operatorname{Spec} B \subseteq Y$ and $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$, there exists a $g \in B$ s.t. $B_g \to A_g$ is a finite morphism.

Proof. There is a set-up

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes_B K(Y) & \longrightarrow & K(X) \\ \uparrow & & \uparrow & & \\ B & \longrightarrow & K(Y) & & \end{array}$$

with $K(X)/K(Y)$ finite. Set $K(X) = K(Y)(g_1, \ldots, g_m)$ and since $A \hookrightarrow K(X)$, the of finite type hypothesis tells us that $A = B[x_1, \ldots, x_n]/I$. Let $F_1, \ldots, F_n$ be the monic polynomials in coefficients $K(Y)$ that are satisfies by x_i. Scale each term so that we get polynomials $F_1', \ldots, F_n'$ all of the same denominator and $F_i'(x_i) = 0$. Let $g \in B$ be this common denominator. It is then clear that $B_g \to A_g$ because $A_g \cong B_g[x_1, \ldots, x_n]/I$ and each of the x_i is integral over B_g (apply Atiyah-Macdonald Corollary 5.2). □

It is not sufficient to take A_g and B_g because we do not know if $f^{-1}(\operatorname{Spec} B_g) = A_g$. This paragraph deals with this issue. First off, we know that $f^{-1}(\operatorname{Spec} B) = \bigcup_{i=1}^n \operatorname{Spec} A_i$ for A_i f.g. B-algebras. Now we set $U_i := \operatorname{Spec} A_i$ and observe that $(X \setminus U_1) \cap U_i$ is closed in U_i for all i. Also, we may choose g_i s.t. $B_{g_i} \to (A_i)_{g_i}$ is a finite morphism (meaning $(A_i)_{g_i}$ is a f.g. B_{g_i}-algebra that is also a finite B_{g_i}-module). Because the map $B_{g_i} \to (A_i)_{g_i}$ is finite, it is an integral homomorphism of rings. This means $\operatorname{Spec}(B_{g_i}) \to \operatorname{Spec}((A_i)_{g_i})$ is a closed mapping. Now, $f(X \setminus U_1) = \bigcup_{i \neq 1,1} f((X \setminus U_1) \cap U_i)$ is closed in Y. Now choose g s.t. $\operatorname{Spec} B_g \subseteq Y \setminus f(X \setminus U_1)$ and we have a map $B_g \to (A_1)_g$. Localize further to get a finite morphism $B_{g,h} \to (A_1)_{g,h}$.

Lemma 3. One has $\operatorname{Spec}((A_1)_{g,h}) = f^{-1}(\operatorname{Spec} B_{g,h})$.

Proof. Clearly, $\operatorname{Spec}(A_1)_{g,h} \supseteq f^{-1}(\operatorname{Spec} B_{g,h})$ because of our choice of $\operatorname{Spec} B_{g,h} \subseteq Y \setminus f(X \setminus U)$. Meanwhile, $\operatorname{Spec}(A_1)_{g,h} \subseteq f^{-1}(\operatorname{Spec} B_{g,h})$ by the following argument.

Let $\mathfrak{p} \in \operatorname{Spec}(A_1)_{g,h}$. Now, $(A_1)_{g,h}$ is a finitely generated $B_{g,h}$-algebra and is therefore integral over $B_{g,h}$. By the Lying Over theorem, there is a prime ideal $\mathfrak{q}$ of $\operatorname{Spec} B_{g,h}$ s.t. $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ (since the corresponding ring map $B_{g,h} \to (A_1)_g$ is integral). □

It is clear what we need to do then. Our desired open set is going to be $\operatorname{Spec} B_{g,h} \subseteq \operatorname{Spec} B = Y$ and $f^{-1}(\operatorname{Spec} B_{g,h}) \to \operatorname{Spec} B_{g,h}$ is a finite morphism due to $f^{-1}(\operatorname{Spec} B_{g,h})$ being affine with ring $(A_1)_{g,h}$ that is a finite $B_{g,h}$-module. □

Hartshorne II.3.8 *Normalization.* A scheme is normal if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\widetilde{A}$ be the integral closure of A in its quotient field, and let $\widetilde{U} = \operatorname{Spec} \widetilde{A}$. Show that one can glue the schemes $\widetilde{U}$ to obtain a normal integral scheme $\widetilde{X}$, called the normalization of X. Show also that there is a morphism $\widetilde{X} \to X$, having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\widetilde{X}$. If X is of finite type over a field k, then the morphism $\widetilde{X} \to X$ is a finite morphism. This generalizes (I, Ex. 3.17).

PROOF. Fix X an integral scheme. We cover X by open affines and normalize each open affine U to get $\widetilde{U} = \operatorname{Spec} \widetilde{A}$ where $\widetilde{A}$ is the integral closure of A in its field of fractions. Note that we know A is an integral domain since X is integral.

We claim gluing is possible. Suppose we covered X by open affines $\{U_i\}$ where $U_i = \operatorname{Spec} A_i$. Normalize each piece i.e. take $\widetilde{U_i}$. Now we check overlaps i.e. $\widetilde{U_i \cap U_j} = \widetilde{U_i} \cap \widetilde{U_j}$. Using Nike's trick, we could take affine sets $D(f)$ which are distinguished open in both U_i and U_j. In this case, WLOG, assume $U_i = D(f_i)$ and $U_j = D(f_j)$ are distinguished open affine sets. Then, $D(f_i) \cap D(f_j) = D(f_i f_j)$. But normalization commutes with localization by Corollary 4.13 of [5]. Apply Hartshorne II.2.12 to glue these schemes to $\widetilde{X}$.

We claim that $\widetilde{X}$ has the desired universal property. By possibly replacing X by an open affine, we will assume $X = \operatorname{Spec} A$ is affine where A is an integral domain. Let $\widetilde{A}$ be the integral closure. We have an injective ring homomorphism $A \to \widetilde{A}$. Since $f : Z \to X$ is dominant, the ring homomorphism $A \hookrightarrow \Gamma(Z, \mathcal{O}_Z)$ is an injection[24]. Since $\Gamma(Z, \mathcal{O}_Z)$ is normal, $A \hookrightarrow \Gamma(Z, \mathcal{O}_Z)$ factors through $\widetilde{A}$ uniquely (by an application of Hartshorne II.2.4). This gives us the morphism $Z \to \widetilde{X}$ we desire.

Assume that X is of finite type over a field k. Let $\phi : \widetilde{X} \to X$. Given $U = \operatorname{Spec} A \subseteq X$. Then $\phi^{-1}(U) = \operatorname{Spec} \widetilde{A}$. Since the integral closure $\widetilde{A}$ is a finite A-module by Hartshorne Theorem 1.3.9A, an application of Hartshorne II.3.4 (which says being finite is an affine local condition) completes the proof. □

Hartshorne II.3.9

The Topological Space of a Product. Recall that in the category of varieties, the Zariski topology on the product of two varieties is not equal to the product topology (I, Ex. 1.4). Now we see that in the category of schemes, the underlying point set of a product of schemes is not even the product set.

(a) Let k be a field, and let $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ be the affine line over k. Show that $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k \cong \mathbb{A}^2_k$, and show that the underlying point set of the product is not the product of the underlying point sets of the factors (even if k is algebraically closed).

(b) Let k be a field, let s and t be indeterminates over k. Then $\operatorname{Spec} k(s), \operatorname{Spec} k(t)$ and $\operatorname{Spec} k$ are all one-point spaces. Describe the product scheme $\operatorname{Spec} k(s) \times_{\operatorname{Spec} k} \operatorname{Spec} k(t)$.

PROOF. (a): First, $k[x] \otimes_k k[y] \cong k[x, y]$ which gives us the isomorphism. The product of the underlying point sets consists of $((x-a),(0))$, $((x-a),(x-b))$, $((0),(x-b))$, and $((0),(0))$ where (0) is the generic point of $\operatorname{Spec} k[x]$ and $(x-a)$ is a maximal ideal (since k is algebraically closed). The closures of these points are $(V((x-a), \operatorname{Spec} k[x])$, $(\operatorname{Spec} k[x], V((x-b))$, $(V((x-a)), V((x-b))$, and $(\operatorname{Spec} k[x], \operatorname{Spec} k[x])$.

Now consider the ideal $(xy-1) \subseteq k[x, y]$. The quotient $k[x, y]/(xy-1) \cong k\left[x, \frac{1}{x}\right]$ is an integral domain so $(xy-1)$. However, $(xy-1)$ is not equal to any of the prime ideals $(0), (x-a, y-b), (x-a, 0), (0, y-b)$ which are those corresponding to the prime ideals of the product set. So, $\operatorname{Spec} k[x, y] \neq \operatorname{Spec} k[x] \times \operatorname{Spec} k[y]$ as sets.

(b) We claim $\operatorname{Spec}(k(s) \otimes_k k(t))$ has infinitely many points. Indeed, $k(s)$ is the localization of $k[s]$ at (0). Let us write $U^{-1}k[s]$ where U is the set of nonzero elements of $k[s]$ and similarly for $V^{-1}k[t]$. We have that $k(s) \otimes k(t) = (UV)^{-1}k[s, t]$ where UV is the set of

[24]Since $A \hookleftarrow \Gamma(Z, \mathcal{O}_Z)$ is injective, the kernel is trivial. So, the pullback is dense is the spectrum of A.

polynomials entirely in terms of s or in terms of t. Of course, there are infinitely many irreducible polynomials which cannot be written as a product of $f(s) \in k[s]$ and $g(t) \in k[t]$. So, $\mathrm{Spec}((k(s) \otimes_k k(t))$ has infinitely many points. □

Hartshorne II.3.10 *Fibres of a Morphism.*

(a) If $f : X \to Y$ is a morphism, and $y \in Y$ a point, show that $\mathrm{sp}\,(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

(b) Let $X = \mathrm{Spec}\, k[s,t]/\,(s - t^2)$, let $Y = \mathrm{Spec}\, k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \to s$. If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fibre X_y consists of two points, with residue field k. If $y \in Y$ corresponds to $0 \in k$, show that the fibre X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η. (Assume k algebraically closed.)

Proof. (a). Reduce to the case $f : f^{-1}(\mathrm{Spec}\, A) \to \mathrm{Spec}\, A$ where $y \in \mathrm{Spec}\, A$ by replacing Y with $\mathrm{Spec}\, A$ and X with $f^{-1}(\mathrm{Spec}\, A)$. Now $X = \bigcup_{i \in I} U_i$ where $U_i = \mathrm{Spec}\, B_i$ is an affine covering. Then $X_y \cap U_i = \mathrm{Spec}\, B_i \otimes_A k(y)$ which is clear from the fibre product diagram. We write $\mathfrak{q}$ for the prime ideal y for ease. Therefore, $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y = A_\mathfrak{q}/\mathfrak{q}A$. Therefore, $X_y \cap U_i = \mathrm{Spec}\, B_i \otimes_A A_\mathfrak{q}/\mathfrak{q}A_\mathfrak{q}$. It suffices to replace $f : f^{-1}(\mathrm{Spec}\, A) \to \mathrm{Spec}(A)$ with $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$.

Suppose we had reduced to the affine case. Our goal is just to show that $sp(X_y)$ is homeomorphic to $f^{-1}(y)$ in the case where $X = \mathrm{Spec}\, B, Y = \mathrm{Spec}\, A$ are affine (so that $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$). Let us identify $\mathfrak{p}$ with $y \in Y$. Let $f^\# : A \to B$ be the corresponding ring homomorphism. We claim

$$f^{-1}(\mathfrak{p}) \cong \mathrm{Spec}(B \otimes_A k(\mathfrak{p})). \tag{2}$$

Notice that $f^{-1}(\mathfrak{p})$ just denotes the set

$$\begin{aligned}\{\mathfrak{q} \in X = \mathrm{Spec}\, B \mid f(\mathfrak{q}) = \mathfrak{p}\} &= \{\mathfrak{q} \in \mathrm{Spec}\, B \mid (f^\#)^{-1}(\mathfrak{q}) = \mathfrak{p}\} \\ &= \{\mathfrak{q} \in \mathrm{Spec}(B) \mid \mathfrak{q} \cap A = \mathfrak{p}\}\end{aligned}$$

Let $C := B \otimes_A k(\mathfrak{p})$. Now we need to define $g : \mathrm{Spec}(C) \to \mathrm{Spec}(B)$ in a way s.t. g is a homeomorphism onto the set described on the RHS of (2). We do this by defining $g^\# : B \to C$ in the correct way. As I will briefly note here, it turns out you just define this by $b \mapsto b \otimes 1$ (to get the correct isomorphism onto the correct image).

Notice that we can write the following equalities[25] and because the steps from the fourth equality and onwards are all due to isomorphisms of rings, we get our desired homeomorphism:

$$\begin{aligned}
\{\mathfrak{q} \in \operatorname{Spec}(B) \mid \mathfrak{q} \cap A = \mathfrak{p}\} &= \{\mathfrak{q} \in \operatorname{Spec}(B) \mid \mathfrak{q} \supset \mathfrak{p}B, f^{\#}(A-\mathfrak{p}) \cap \mathfrak{q} = \emptyset\} \\
&= \operatorname{Spec}((B/\mathfrak{p}B)_{f^{\#}(A-\mathfrak{p}))}) \\
&= \operatorname{Spec}((B/\mathfrak{p}B)_{A-\mathfrak{p}}) \\
&= \operatorname{Spec}((B/\mathfrak{p}B) \otimes_A A_{A-\mathfrak{p}}) && \text{(AM Prop 3.5)} \\
&= \operatorname{Spec}(B \otimes_A A/\mathfrak{p}A \otimes_A A_{A-\mathfrak{p}}) && \text{(AM Prop 2.14)} \\
&= \operatorname{Spec}(B \otimes_A (A/\mathfrak{p}A)_{A-\mathfrak{p}}) && \text{(AM Prop 3.5)} \\
&= \operatorname{Spec}(B \otimes_A (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})) && \text{(AM Cor. 3.4)} \\
&= \operatorname{Spec}(B \otimes_A k(\mathfrak{p})) && \text{(Def)}
\end{aligned}$$

(b) Let k be algebraically closed. The fibre over $y = (s-a) \in \operatorname{Spec} k[s]$ is going to consist of the primes $x_1 = (s-a, t-\sqrt{a})$ and $x_2 = (s-a, t+\sqrt{a})$ in X. The residue fields of x_1 is

$$k(x_1) = k[s,t]/(s-t^2)_{x_1}/\mathfrak{m}_{x_1} = k[a,\sqrt{a}]/(s-t^2)_{x_1} = k$$

and the same reasoning implies $k(x_2) = k$.

If $a = 0$, this implies that $y = (s)$. The fibre looks like

$$X \times_Y \operatorname{Spec} k(y) = \operatorname{Spec} k[s,t]/(s-t^2) \otimes_{k[s]} k[s]_{(s)}/(s) = k[s,t]/(s-t^2) \otimes_{k[s]} k = \operatorname{Spec} k[t]/(t^2)$$

since $k(y) = k(s)$. The latter is a nonreduced scheme as desired. For the case of the generic point η of $\operatorname{Spec} k[s]$. The prime ideals of $k[s,t]/(s-t^2)$ that contract to the zero ideal $(0) \subseteq k[t]$ are the prime ideals containing $(s-t^2)$ and no polynomials entirely in s (because of how we defined f). The only such ideal is $(s-t^2)$. So, fibre is

$$k[s,t]/(s-t^2) \otimes_{k[s]} k(s) \cong k(s)[t]/(s-t^2)$$

This is a field. Thus, X_η is a one-point scheme. The residue field is degree two since the residue field is $k(s)[t]/(s-t^2)$ which is a degree two extension of the residue field $k(s)$ of η. To see that this is of degree two, notice that in $k(s)[t]/(s-t^2)$, the inverse of t is just $\frac{t}{s}$ and the basis of this as a $k(s)$-vector space is $\{1, t\}$. □

Hartshorne II.3.11 *Closed Subschemes.*
(a) Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.
(b) If Y is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$. [Hints: First show that Y can be covered by a finite number of open affine subsets of the form $D(f_i) \cap Y$, with $f_i \in A$. By adding some more f_i with $D(f_i) \cap Y = \varnothing$, if necessary, show that we may assume that the $D(f_i)$ cover X. Next show that $f_1, \ldots, f_r$ generate the unit ideal of A. Then use (Ex. 2.17b) to show that Y is affine, and (Ex. 2.18d) to show that Y comes from an ideal $\mathfrak{a} \subseteq A$.] Note: We will give another proof of this result using sheaves of ideals later (5.10).
(c) Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological

[25] Read it backwards if it helps.

space, show that the closed immersion $Y \to X$ factors through Y'. We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.
(d) Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. We call Y the scheme-theoretic image of f. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

PROOF. (a) We show $f' : Y \times_X X' \to X'$ is a closed immersion.

By possibly taking a cover by open affines and noting being a closed immersion is local on target, let $X' = \operatorname{Spec} A'$ and $X = \operatorname{Spec} A$. Then it follows that $Y = \operatorname{Spec} A/\mathfrak{a}$ (using the next part of this problem and note that the proof is independent). Now, $f' : \operatorname{Spec}(A/\mathfrak{a} \otimes_A A') \to \operatorname{Spec}(A')$ is the map $\operatorname{Spec}(A'/\mathfrak{a}) \to \operatorname{Spec}(A')$ induced by $A'/\mathfrak{a}A' \to A'$. This is a closed immersion.

Lemma. Let $g : Y \to X$ be a morphism of schemes. Suppose there is a cover of X by open sets U_i s.t. $g : g^{-1}(U_i) \to U_i$ are all closed immersions. Then $g : Y \to X$ is a closed immersion.

PROOF. It is relatively clear that these local homeomorphisms imply that we have a global homeomorphism of Y onto a closed subscheme of X. Additionally, the associated map of sheaves $g_p^{\#} : \mathcal{O}_{X,p} \to (g_*\mathcal{O}_Y)_p$ equals induced map on stalks $\mathcal{O}_{X|U_i,p} \to (g_*\mathcal{O}_{Y|g^{-1}(U_i)})_p$ for $p \in U_i$. These are are all surjective maps and hence, $g^{\#}$ is surjective. □

(b) Let Y be a closed subscheme of $X = \operatorname{Spec} A$.

First, by quasi-compactness of X, take a finite covering of X by distinguished open sets $D(f_i)$. Since Y is a closed subscheme, its underlying set is a closed subset of X and hence, Y is quasicompact. So, finitely many $D(f_i) \cap Y$ cover Y. Because these $D(f_i)$ cover X, the $f_1, \ldots, f_r$ generate the unit ideal of A.

Now, we show $D(f_i) \cap Y$ can be taken to be affine open subsets. Indeed, Y is a closed subscheme and so, has an affine covering by V_j. Then Nike's Trick implies $V_j \cap D(f_i)$ can be covered by distinguished open subsets in both V_j and $D(f_i)$. So cover V_j by $\operatorname{Spec} A_{g_k}$ and the index in $1 \leq k \leq s$ is finite by quasi-compactness of Y.

By Hartshorne II.2.17b, the scheme Y is affine since the $g_1, \ldots, g_s \in A$ give rise to affine open subsets Y_{g_k} and they generate the unit ideal in A and thus, in $\Gamma(Y, \mathcal{O}_X)$.

By Hartshorne II.2.17d, there is a surjective ring homomorphism $\varphi : A \to \Gamma(Y, \mathcal{O}_X)$. This implies that $\Gamma(Y, \mathcal{O}_X) \cong A/\ker\varphi$ i.e. $\Gamma(Y, \mathcal{O}_X) \cong A/\mathfrak{a}$ for some ideal $\mathfrak{a}$.
(c) Since being a closed immersion is a local property, we reduce to the affine case.

Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/\mathfrak{a}$ be a close subscheme that is reduced with the induced scheme structure. This means that $\mathfrak{a}$ is a prime ideal. Now, suppose Y' has the same topological space as $\operatorname{Spec} A/\mathfrak{a}$. Then, $Y' = \operatorname{Spec} A/\mathfrak{b}$ where $\mathfrak{b} \subseteq \mathfrak{a}$. So, we get ring homomorphisms $A \to A/\mathfrak{b} \to A/\mathfrak{a}$ which induces the morphisms $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec}(A/\mathfrak{b}) \to \operatorname{Spec}(A)$.

Let $h : Y \to X$ be the closed immersion obtained by giving Y the reduced induced subscheme structure (the construction is Example II.3.2.6 in Hartshorne). Let Y' be a closed subscheme with same underlying topological space as Y. To prove that $Y \xrightarrow{g} Y' \xrightarrow{f} X$ is a factorization, we simply check that the maps on sheaves factor i.e. there is a $\mathcal{O}_{Y'} \to g_*\mathcal{O}_Y$

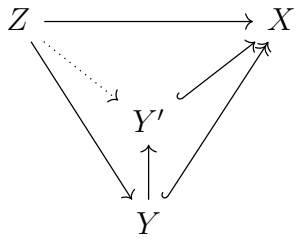
s.t.

$$\mathcal{O}_X \to f_*\mathcal{O}_{Y'} \to f_*g_*\mathcal{O}_Y$$

is the same as $\mathcal{O}_X \to h_*\mathcal{O}_Y$. We check that the factorization exists affine locally and note that the maps may be glued together. That is, given a closed immersion $\operatorname{Spec} A/\mathfrak{b} \to \operatorname{Spec} A$ and the reduced induced scheme structure $\operatorname{Spec} A/\mathfrak{a}$, there is a factorization

$$A \to A/\mathfrak{b} \to A/\mathfrak{a}$$

which follows by $\mathfrak{a} \supseteq \mathfrak{b}$.
(d) In the case Z is reduced, it is clear from the previous part and Hartshorne II.2.3(c) that Y is $\overline{f(Z)}$ with the reduced induced scheme structure. One only needs to look at the following diagram.



For the general case, give $f(Z)$ the reduced induced scheme structure and then take the closure of the topological space. This is the desired choice of Y. □

Remark 16. Statement (b) is also proven in Hartshorne Corollary II.5.10. An extremely general construction can be found in [10, Section 01R5].

Hartshorne II.3.12 *Closed Subschemes of Proj S.*
(a) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Ex. 2.14) is equal to Proj T, and the morphism $f :$ Proj $T \to$ Proj S is a closed immersion.
(b) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X =$ Proj S defined as image of the closed immersion Proj $S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geqslant d_0} I_d$. Show that I and I' determine the same closed subscheme.

We will see later (5.16) that every closed subscheme of X comes from a homogeneous ideal I of S (at least in the case where S is a polynomial ring over S_0).

Proof. (1) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings preserving degrees.

First, I claim that $U = \operatorname{Proj} T$. Hartshorne II.2.14 already showed that $U \subseteq \operatorname{Proj} T$. So, we prove the other inclusion. Suppose $\mathfrak{p} \in \operatorname{Proj} T$. Because φ is a surjective morphism preserving degrees, $\varphi^{-1}(\mathfrak{p})$ does not contain S_+. Hence, $\mathfrak{p}$ does not contain $\varphi(S_+)$. So, $\mathfrak{p} \in U$.

Now, I claim that $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion. Because φ is a surjective homomorphism, I know that $T \cong S/I$ for some ideal I. Thus, $f : \operatorname{Proj} S/I \to \operatorname{Proj} S$ and I claim that this is a closed immersion. I check that the induced morphism on sheaves $f^\#$ is a surjection. On the stalks, $f^\#_P : (S/I)_{(P)} \to S_{(P)}$ and this is a surjection. Thus, $f^\#$ is a surjection.

(2) Set $T = S/I$ and Y be closed subscheme that is the image of $\operatorname{Proj} S/I \to \operatorname{Proj} S$. Set $X = \operatorname{Proj} S$. I claim that $I' = \bigoplus_{d \geq d_0} I_d$ will determine the same closed subscheme as I.

Notice that I have ring morphisms $S \to S/I'$, $S \to S/I$ and $S/I' \to S/I$. These have corresponding morphisms of schemes. By Hartshorne II.2.14c, it was seen that $S/I' \to S/I$ being an isomorphism for degree $d \geq d_0$ implies $\operatorname{Proj} S/I' \to \operatorname{Proj} S/I$ is an isomorphism. Thus, two different ideals determine the same closed subscheme. □

Hartshorne II.3.13 *Properties of Morphisms of Finite Type.*

(a) A closed immersion is a morphism of finite type.

(b) A quasi-compact open immersion (Ex. 3.2) is of finite type.

(c) A composition of two morphisms of finite type is of finite type.

(d) Morphisms of finite type are stable under base extension.

(e) If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.

(f) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and if f is quasi-compact, and $g \circ f$ is of finite type, then f is of finite type.

(g) If $f : X \to Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.

Proof. (a) Fix a closed immersion $f : X \to Y$. That is, $sp(X)$ is homeomorphic to a closed subset of $sp(Y)$. Now, if $U \subseteq Y$ then the map $f^{-1}(U) \to U$ is a closed immersion since $f^{-1}(U)$ is homeomorphic to $U \cap f(X)$.

So, it suffices to prove the result for the case of $Y = \operatorname{Spec} A$ since we can consider an affine cover of Y. This forces $X \cong \operatorname{Spec}(A/I)$ for an ideal I since we know what closed subsets of affine schemes look like. But A/I is a finitely generated A-algebra and so by Hartshorne II.3.1 and Hartshorne II.3.3(a), f is of finite type.

(b) Since $f : X \to Y$ is an open immersion, assume $f(X) \cong U$ for an open subscheme $U \subseteq Y$. By Hartshorne II.3.3(a), it suffices to just prove f is of locally finite type. Let $\operatorname{Spec} A \subseteq Y$ be an open affine. So, $f^{-1}(\operatorname{Spec} A) = f^{-1}(U \cap \operatorname{Spec} A)$. By quasicompactness and Hartshorne II.3.2, we may cover $U \cap \operatorname{Spec} A$ with finitely many $\operatorname{Spec} A_{f_i}$. Hence, we can cover $f^{-1}(\operatorname{Spec} A)$ with the $\operatorname{Spec} A_{f_i}$ since $f(X) \cong U$. Each A_{f_i} is a finitely generated A-algebra and we are done.

(c) If we have $h = g \circ f$, let $f : X \to Y$ and $g : Y \to Z$. Consider an open affine $V \subseteq Z$. Then $g^{-1}(Z)$ is a finite union of affine open subsets of Y where each $f^{-1}(U_i)$ is a finite union of affine open subsets W_{ij} of X. Now, we have a sequence $W_{ij} \to U_i \to V$ which yields morphisms of rings with arrows going the opposite direction. This shows that W_{ij} is a finitely generated A-algebra where $\operatorname{Spec} A = V$. Since $h^{-1} = \bigcup_{i,j} W_{i,j}$, which is a finite union due to the f, g being of finite type, we are done.

(d) To show that morphisms of finite type are stable under base extension, we first note that being of finite type is a locally affine property by Hartshorne II.3.3(c).

First, assume the schemes are affine. Take $\operatorname{Spec} A' \to \operatorname{Spec} A$ as the base extension and $X = \operatorname{Spec} B$. We get a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} B \otimes_A A' & \xrightarrow{f'} & \operatorname{Spec} A' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \varphi^\#} \\ \operatorname{Spec} B & \xrightarrow[f]{} & \operatorname{Spec} A \end{array}$$

Without loss of generality, let us take $Z = \operatorname{Spec} A'$. We look at $f'^{-1}(Z)$.

First, we push down $\varphi^\#(Z) = \{\varphi^{-1}(\mathfrak{q}) | \mathfrak{q} \subseteq A'\}$ where $\varphi : A \to A'$ is the corresponding ring morphism.

We may assume $\varphi^\#(Z) = \operatorname{Spec} A$ by possibly restricting to a smaller ring. Now, $f^{-1}(\varphi^\#(Z)) = f^{-1}(\operatorname{Spec} A) = \cup_i^n \operatorname{Spec}(B_i)$ where B_i are finitely generated A-algebras for each i.

Now, $p_1^{-1}(f^{-1}(\operatorname{Spec} A)) = \cup_i^n p_1^{-1}(\operatorname{Spec} B_i) = \cup_i^n \operatorname{Spec}(B_i \otimes_A A')$ where $B_i \otimes_A A'$ is a finitely generated A'-algebra. We are done since $p_1^{-1}(f^{-1}(\operatorname{Spec} A)) = f'^{-1}(Z)$

(e) Morphisms of finite type are stable under base extension so $X \times_S Y \to Y$ is of finite type. Since $Y \to S$ is of finite type and composition of morphisms of finite type is of finite type, $X \times_S Y \to S$ is of finite type.

(f) By Hartshorne II.3.3(c), we reduce to the affine case. Let V be an affine open subset of Y. By hypothesis, $f^{-1}(V)$ is a finite union of open subsets U_i. Each U_i has sections $\mathcal{O}_X(U_i)$ that are finitely generated over A for $Z = \operatorname{Spec} A$. But $\mathcal{O}_X(U_i)$ is then finitely generated over $\mathcal{O}_Y(V)$ which implies that f is of finite type.

(g) Let $f : X \to Y$ be of finite type and Y a noetherian scheme. Recall that X is noetherian iff it is locally noetherian and quasicompact.

First, show X is quasicompact. Take a finite cover by affine schemes $\bigcup_i^n \operatorname{Spec} A_i$ for Y. Each $f^{-1}(\operatorname{Spec} A_i)$ is finitely covered by by $\operatorname{Spec}(B_{ij})$ where each B_{ij} is a finitely generated A_i-algebra. So, we can cover X with $\bigcup_{i,j} \operatorname{Spec}(B_{ij})$ which is a finite cover. If a space is covered by finitely many open quasi-compact sets, then it is quasi-compact and this implies X is quasi-compact.

Also, the B_{ij} are each noetherian by the following result of commutative algebra: If B is a finitely generated A-algebra and A is noetherian, then B is noetherian (Corollary 7.6 of [2]).

□

Hartshorne II.3.14 If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.

Proof. To prove density, we show each open subset of X has a closed point. Choose a cover of X by affine opens $U_i = \operatorname{Spec} A_i$. Each A_i is a finitely generated k-algebra and there are only finitely many U_i's. Since each A_i is a finitely generated k-algebra, A_i is a Jacobson ring. That is, every prime ideal is the intersection of maximal ideals [2].

Let U be an arbitrary open subset of X. Then, $U \cap U_i \neq \emptyset$ for some i. By the usual basis of topology, there exists a nonempty $D(f) \subseteq U \cap U_i$ distinguished open subset where $D(f) = (\operatorname{Spec} A_i) \setminus V(f)$ for some $f \in A_i$.

Thus, it suffices to show that every distinguished open subset contains a closed point. Indeed, if $D(f)$ did not contain a closed point, we can conclude that every maximal ideal of A_i contains f. So, the intersection of every maximal ideal would contain f. Indeed, this means every prime ideal contains f. Thus, f is nilpotent. However, this implies $D(f) = \emptyset$. Contradiction.

Counterexample: Consider $\mathbb{Z}_{(2)}$ the integers localized at the prime ideal (2). Now, the maximal ideal is the ideal generated by (2). The closed point of $\operatorname{Spec}\mathbb{Z}_{(2)}$ corresponds to the maximal ideal generated by (2). Now, take $(\operatorname{Spec}\mathbb{Z}_{(2)}/V((2))$ i.e. $\operatorname{Spec}\mathbb{Z}_{(2)}$ removing the closed point. This is an open subset of $\operatorname{Spec}\mathbb{Z}_{(2)}$, but it does not contain a closed point.

Take the ring $\mathbb{Z}[x]$. We understand the prime ideals and maximal ideals of $\mathbb{Z}[x]$ well. Localize at the maximal ideal (p, x) and let $X = \operatorname{Spec}\mathbb{Z}[x]_{(p,x)}$. Then the underlying set consists of those prime ideals of $\mathbb{Z}[x]$ contained inside (p, x) and the only[26] closed point of X is (p, x). However, $X \setminus V(p, x))$ is an open subset of X which contains no closed points. □

Hartshorne II.3.15 Let X be a scheme of finite type over a field k (not necessarily algebraically closed).

(a) Show that the following three conditions are equivalent (in which case we say that X is geometrically irreducible).
  (i) $X \times_k \bar{k}$ is irreducible, where $\bar{k}$ denotes the algebraic closure of k. (By abuse of notation, we write $X \times_k \bar{k}$ to denote $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$.)
  (ii) $X \times_k k_s$ is irreducible, where k_s denotes the separable closure of k.
  (iii) $X \times_k K$ is irreducible for every extension field K of k.

(b) Show that the following three conditions are equivalent (in which case we say X is geometrically reduced).
  (i) $X \times_k \bar{k}$ is reduced.
  (ii) $X \times_k k_p$ is reduced, where k_p denotes the perfect closure of k.
  (iii) $X \times_k K$ is reduced for all extension fields K of k.

(c) We say that X is geometrically integral if $X \times_k \bar{k}$ is integral. Give examples of integral schemes which are neither geometrically irreducible nor geometrically reduced.

Proof. For part (a), we shall make use of the following lemmas.

Lemma 1. (see [10, Lemma 01OM]) Let X be a scheme. TFAE:

(1) X is irreducible,
(2) X has a cover by open affine sets $\{U_i\}_{i\in I}$ with U_i irreducible, $I \neq \emptyset$, and $U_i \cap U_j \neq \emptyset$ for all i, j,
(3) $X \neq \emptyset$ and every nonempty open affine set is irreducible.

Lemma 2. (Exercise 1.19 of [2]) Let A be a ring. Then $\operatorname{Spec}(A)$ is irreducible iff the nilradical is a prime ideal.

Lemma 3. (p. 569 [5]) Let R, S be integral domains containing a field k. If $K(R)$ is separable over k and k is algebraically closed in $K(R)$, then $R \otimes_k S$ is a domain.

Lemma 4. (see [10, Lemma 037K]) Let k be a field and R a k-algebra. TFAE:

(1) for every field extension $k \subseteq K$, $\operatorname{Spec}(R \otimes_k K)$ is irreducible,
(2) for every finite separable extension $k \subseteq K$, $\operatorname{Spec}(R \otimes_k K)$ is irreducible,

[26] There are some subtleties here that one should be careful with.

(3) $\operatorname{Spec}(R \otimes_k k_s)$ is irreducible for k_s the separable algebraic closure,
(4) $\operatorname{Spec}(R \otimes_k \overline{k})$ is irreducible for $\overline{k}$ the algebraic closure.

(a) This result follows from combining Lemmas 1,2,3 and 4 together.

(b) Being reduced is local on stalks. The implications (iii) implies (ii) and (i) is clear. The other directions follow by the results in [10, Section 05DS] and [10, Lemma 030V].
(c) This example is taken out of [3] on pg. 90-91. We consider $X := \operatorname{Proj}(k[u,v,w]/(u^2 - av^2))$ with a having square root in the field k and $\operatorname{char} k \neq 2$. This is an integral scheme but over the algebraic closure $\overline{k}$, $X \times_k \overline{k} = \operatorname{Proj}(k[u,w,v]/(u - \sqrt{a}v)(u + \sqrt{a}v))$.

An idea on how to get a full example is quotient by a polynomial that has a repeated root and more than one root.

□

Remark 17. A few sources I referred to when doing this exercise are 9.5 of [9], 3.2.2 of [3], and 5.1.3 of [4]. The proof presented here follows the approach in [3].[27]

Hartshorne II.3.16 *Noetherian Induction.* Let X be a noetherian topological space, and let $\mathscr{P}$ be a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y. (In particular, $\mathscr{P}$ must hold for the empty set.) Then $\mathscr{P}$ holds for X.

The exercise provides a useful to prove results for Noetherian topological spaces. In particular, it gives a way to show that a property holds for a Noetherian topological space X by showing first that it's true on its closed subsets. We call this **Noetherian Induction**.

Proof. We claim that property $\mathscr{P}$ holds for the wholes space X. By Zorn's Lemma, it suffices to show that there is an upper bound to every chain of closed subsets having the property $\mathscr{P}$.

Indeed, we know that for any closed subset Y of X, if $\mathscr{P}$ holds for each proper closed subset of Y, then $\mathscr{P}$ holds for Y.

Let us consider a chain $\emptyset \subseteq Z_1 \subseteq Z_2 \subseteq \dots$ of closed sets that have the property. By X noetherian, this chain terminates. So, the finite union $\cup Z_i$ must cover X and it must also have this property. Hence, $\mathscr{P}$ holds for X. □

Hartshorne II.3.17 *Zariski Spaces.* A topological space X is a Zariski space if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point (Ex. 2.9). For example, let R be a discrete valuation ring, and let $T = \operatorname{sp}(\operatorname{Spec} R)$. Then T consists of two points $t_0 =$ the maximal ideal, $t_1 =$ the zero ideal. The open subsets are $\varnothing, \{t_1\}$, and T. This is an irreducible Zariski space with generic point t_1.
(a) Show that if X is a noetherian scheme, then $\operatorname{sp}(X)$ is a Zariski space.
(b) Show that any minimal nonempty closed subset of a Zariski space consists of one point. We call these *closed points*.
(c) Show that a Zariski space X satisfies the axiom T_0 :given any two distinct points of X, there is an open set containing one but not the other.
(d) If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X.
(e) If x_0, x_1 are points of a topological space X, and if $x_0 \in \{x_1\}^-$, then we say that x_1 *specializes* to x_0, written $x_1 \wedge \to x_0$. We also say x_0 is a *specialization* of x_1, or that x_1 is

[27] Which I have not yet typed up.

a *generization* of x_0. Now let X be a Zariski space. Show that the minimal points, for the partial ordering determined by $x_1 > x_0$ if $x_1 \leftrightarrow x_0$, are the closed points, and the maximal points are the generic points of the irreducible components of X. Show also that a closed subset contains every specialization of any of its points. (We say closed subsets are *stable under specialization*.) Similarly, open subsets are *stable under generization*.
(f) Let t be the functor on topological spaces introduced in the proof of (2.6). If X is a noetherian topological space, show that $t(X)$ is a Zariski space. Furthermore X itself is a Zariski space if and only if the map $\alpha : X \to t(X)$ is a homeomorphism.

PROOF. (a) If X is a noetherian scheme, then ${}^{(}X)$ is clearly noetherian by the definition.
It suffices to show every irreducible closed subset has a unique generic point. Indeed, this follows from Hartshorne II.2.9.

(b) Let Z be a minimal nonempty closed subset of a Zariski space. We claim that Z consists of one point. We call such points **closed points**.
If there is more than one point, say a and b. Without loss of generality, let us assume a is the generic point since a minimal nonempty closed subset is necessarily irreducible. Now, $\{a\}^- = Z$. By this implies that $Z = \{a\}^- \cup \{a\}$ and so, $\{a\}^-$ has to be contained in Z. This contradicts minimality.

(c) We claim that Zariski spaces are sober i.e. they are T_0 topological spaces. Let $x, y \in X$. Then since they are distinct, $\{x\}^- \neq \{y\}^-$. By the fact that X is noetherian, we may take $\cup_z \{z\}^- = X$ for elements $z \in X$ and this can be taken as a finite union (and we requite that the finite union at least use x and y). So, $X \setminus \{x\}^-$ is an open set containing y but not x.

(d) Assume X is an irreducible Zariski space. Let $\{\eta\}^- = X$ be its generic point. Let $U \subseteq X$ be nonempty. If $\eta \notin U$, we get $U \cap X = U \cap \{\eta\}^- \subseteq X$. However, X is irreducible so we must have $U \cap \{\eta\} = X$. This implies $U = X$ and so, $\eta \in U$. Contradiction.

(e) If is clear that minimal points are closed since if x_0 minimal, then we would have $x_0 = \{x_0\}^-$. If not, we would immediately find a contradiction.
Next, maximal points are generic points of irreducible components for if x_1 maximal, then $\{x_1\}^-$ is an irreducible component. If V closed and $x \in V$, then $\{x\}^- \subseteq V$. If U open, then if $y \in U$ and $y \in \{y'\}^-$, we must have $y' \in U$ by a topological argument.

(f) **[Incomplete]**

□

Remark 18. I want to remark that this exercise is also covered in [9] on page 122 and 153. Exercises 5.1.B, 5.1.C, 5.1.D, and 5.1.E in [9] were helpful since they are "baby" steps in understanding generic points.

Hartshorne II.3.18 *Constructible Sets.* Let X be a Zariski topological space. A constructible subset of X is a subset which belongs to the smallest family $\mathscr{J}$ of subsets such that (1) every open subset is in $\mathfrak{F}$, (2) a finite intersection of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the complement of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.

(a) A subset of X is locally closed if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible if and only if it can be written as a finite disjoint union of locally closed subsets.

(b) Show that a constructible subset of an irreducible Zariski space X is dense if and only if it contains the generic point. Furthermore, in that case it contains a nonempty open subset.

(c) A subset S of X is closed if and only if it is constructible and stable under specialization. Similarly, a subset T of X is open if and only if it is constructible and stable under generization.

(d) If $f : X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.

PROOF. (a) ($\implies$): Suppose U is constructible. Then, $U \in \mathfrak{F}$. Consider every open subset of U and consider the closure U^-. Let Z_i be the intersection of an open subset of U and U^-. We get a chain of closed subsets Z_i. By possibly removing those Z_i's that are not irreducible, we have a chain of irreducible closed subsets that must terminate and the union of these irreducible closed subsets is a finite union of locally closed subsets which equals U.

($\impliedby$): Suppose $U \subseteq X$ is a finite union $\cup_i^n V_i \cap Z_i$ for V_i open and Z_i closed. Then, V_i and Z_i are both in $\mathfrak{F}$. Since $\mathfrak{F}$ is closed under finite intersections and complements, this implies $U \in \mathfrak{F}$ and hence, U is constructible.

(b) ($\implies$): Let U be a dense constructible subset of X. Note that X is an irreducible Zariski space. We claim that U must contain the generic point of X.

By (a), let us write $U = \cup_1^n (V_i \cap Z_i)$ for $V_i, Z_i \in \mathfrak{F}$ where V_i open and Z_i closed. Since U dense, $U^- = X$. So, $\cup_1^n (V_i \cap Z_i)^- = X$. by X irreducible, this implies $(V_i \cap Z_i)^- = X$ for some i. This implies V_i is nonempty. So, the generic point ξ is in V_i. All that is left is to show that ξ is also in Z_i.

Notice that since $(V_i \cap Z_i)^- = X$, and by properties of closure $(V_i \cap Z_i)^- \subseteq V_i^- \cap Z_i^- = V_i^- \cap Z_i$, we can conclude that Z_i contains X. In particular, $\xi \in Z_i$.

($\impliedby$): This direction is obvious.

(c) Suppose S is closed. Then clearly, S is constructible since $X \setminus S$ is open. Also, suppose $x_0 \in \overline{\{x_1\}}$ for some $x_1 \in S$. Then x_0 lies in every closed subset about x_1. Taking the subset to be sufficiently small, S closed implies that $x_0 \in S$.

For the converse, let $x_1 \in S$. Let $S = \cup^n (U_i \cap Z_i)$ where U_i open and Z_i closed by (a). It will suffice to show that $\overline{S} = S$. To do this, suppose x_1 were a boundary point of S. Then, it is the specialization of some point $\{x_0\}$ in S. Since $x_0 \in S$, it is contained in $\cup U_i$ and $\cup Z_i$. So, $x_1 \in \cup U_i$ and $x_1 \in \cup Z_i$. So, $x_0 \in S$.

The corresponding statement for T open follows similarly.

(d) Suppose $U = \bigcup_{i=1}^n (V_i \cap Z_i)$ due to U constructible and (a). Then, $f^{-1}(U) = \bigcup_{i=1}^n (f^{-1}(V_i) \cap f^{-1}(Z_i))$ by properties of preimage and note that $f^{-1}(V_i)$ is open and $f^{-1}(Z_i)$ is closed by f continuous. By (a), $f^{-1}(U)$ is constructible.

□

Remark 19. For (b), the constructible condition is necessary.

Consider the set of all closed points of $\mathbb{A}^1_{\mathbb{C}}$. This set is stable under specialization. However, it is not closed. This is because $\mathbb{A}^1_{\mathbb{C}}$ removing the generic point is not constructible.

Similarly, the subset $\mathbb{A}^1_C$ of closed points removing those points whose corresponding point in $\mathbb{C}$ habe magnitude ≥ 1 is stable under generization. However, it is not open.

Hartshorne II.3.19 The real importance of the notion of constructible subsets derives from the following theorem of Chevalley-see Cartan and Chevalley [1, exposé 7] and see

also Matsumura [2, Ch. 2, §6] : let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then the image of any constructible subset of X is a constructible subset of Y. In particular, $f(X)$, which need not be either open or closed, is a constructible subset of Y. Prove this theorem in the following steps.

(a) Reduce to showing that $f(X)$ itself is constructible, in the case where X and Y are affine, integral noetherian schemes, and f is a dominant morphism.

*(b) In that case, show that $f(X)$ contains a nonempty open subset of Y by using the following result from commutative algebra: let $A \subseteq B$ be an inclusion of noetherian integral domains, such that B is a finitely generated A-algebra. Then given a nonzero element $b \in B$, there is a nonzero element $a \in A$ with the following property: if $\varphi : A \to K$ is any homomorphism of A to an algebraically closed field K, such that $\varphi(a) \neq 0$, then φ extends to a homomorphism φ' of B into K, such that $\varphi'(b) \neq 0$. [Hint: Prove this algebraic result by induction on the number of generators of B over A. For the case of one generator, prove the result directly. In the application, take $b = 1$.]

(c) Now use noetherian induction on Y to complete the proof.

(d) Give some examples of morphisms $f : X \to Y$ of varieties over an algebraically closed field k, to show that $f(X)$ need not be either open or closed.

PROOF. (a) First, we reduce to showing $f(X)$ is constructible. Since we want to show that any constructible U has a constructible image, we may replace $f : X \to Y$ by $f : U \to Y$ and show that $f(U)$ constructible. So, we may very have shown $f(X)$ constructible.

Next, we reduce to when X, Y are affine, integral noetherian schemes. Indeed, if X and Y are arbitrary schemes, then being constructible is a local property and so we can take an affine open subset and prove the result for that case. Also, X and Y are automatically noetherian by assumption. The hard part is seeing that X and Y are integral. Because $f : X \to Y$ is a morphism of finite type, X and Y are noetherian, we see that B is a finitely generated A-algebra for $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. By possibly modding out by an ideal we assume A and B are integral domains. Finally, to see that $f : X \to Y$ is dominant, it suffices to show that $f(X)$ contains the generic point of Y. Indeed, $f : X \to Y$ is given by a ring homomorphism $A \to B$. Since this ring homomorphism is a homomorphism of integral domains, the zero ideal of B is mapped to by the zero ideal of A. Hence, the generic point of X is mapped to the generic point of Y. That is, $f(X)$ contains the generic point of Y.

(b) **[Incomplete]**

(c) **[Incomplete]**

(d) First, we show that $f(X)$ need not be closed. Indeed, consider $\operatorname{Spec} k[x, y]/(xy - 1) \to \operatorname{Spec} k[x]$ given by $k[x] \to k[x, y]/(xy - 1)$. Then, $f(X)$ is not closed.

Next, we show that $f(X)$ need not be open. Consider the morphism $\operatorname{Spec} k[x]/(x^2) \to \operatorname{Spec} k[x]$ given by $k[x] \to k[x]/(x^2)$. In this case, $f(X)$ is not open.

Finally, we show that $f(X)$ can be neither open nor closed.

[Incomplete]

□

Remark 20. Chevally's Theorem tells us something important about morphisms of varieties. The image of a variety is not necessarily a variety. Just consider the morphism from $\mathbb{A}^2 \to \mathbb{A}^2$ which sends (x, y) to (x, xy). The image here is certainly not a variety, but it is definitely constructible.

Note that all varieties are constructible.

Hartshorne II.3.20 *Dimension.* Let X be an integral scheme of finite type over a field k (not necessarily algebraically closed). Use appropriate results from (I, §1) to prove the following.

(a) For any closed point $P \in X, \dim X = \dim \mathcal{O}_P$, where for rings, we always mean the Krull dimension.

(b) Let $K(X)$ be the function field of X (Ex. 3.6). Then $\dim X = \text{tr.d. } K(X)/k$.

(c) If Y is a closed subset of X, then $\operatorname{codim}(Y, X) = \inf\{\dim \mathcal{O}_{P,X} \mid \mathrm{P} \in Y\}$.

(d) If Y is a closed subset of X, then $\dim Y + \operatorname{codim}(Y, X) = \dim X$.

(e) If U is a nonempty open subset of X, then $\dim U = \dim X$.

(f) If $k \subseteq k'$ is a field extension, then every irreducible component of $X' = X \times_k k'$ has dimension $= \dim X$.

Proof. Fix X an integral scheme of finite type over a field k.

(a) Let us assume $X = \operatorname{Spec} B$. We are proving the result for the affine case. The justification for this is that the dimension of $\mathcal{O}_P$ is a local question.

In this case, $\dim X = \dim B =$ height $\mathfrak{p} + \dim B/\mathfrak{p}$ and since $\mathfrak{p}$ is a closed point, $\dim B/\mathfrak{p} = 0$ since $B/\mathfrak{p}$ is a field. So, $\dim X$ is equal to the height of $\mathfrak{p}$ which is just the dimension of $B_{\mathfrak{p}}$ which is equal to $\dim \mathcal{O}_{X,\mathfrak{p}}$.

(b) This result follows immediately from Hartshorne I.1.8Aa.

(c) Let $Y \subseteq X$ closed. We have

$$\operatorname{codim}(Y, X) = \inf_{Z \subseteq Y} \operatorname{codim}(Z, Y) = \inf\{\mathcal{O}_{X,p} | p \in Y\}$$

since irreducible closed subsets Z are in a one-to-one correspondence with closed points.

(d) Let $Y \subseteq X$. If Y were reducible, then Hartshorne Theorem I.1.8Ab finishes the proof.

If not, let Z be an irreducible subset of Y of largest dimension. Then,

$$\dim Y + \operatorname{codim}(Y, X) = \dim Z + \operatorname{codim}(Z, X) = \dim X$$

where the last equality follows from the irreducible case.

(e) This result follows from Hartshorne I.1.10.

(f) Let $k \subseteq k'$ be a field extension. Let $X' = X \times_k k'$. Now,

$$\dim X' = \dim(X \times_k k') = \dim X + \dim k = \dim X.$$

Note that the second to last equality follows from Hartshorne I.3.15.

□

Remark 21. Notice that this exercise shows that the usual notion of dimensions apply to schemes that are not necessarily varieties. These notions coincide with how we define dimensions for varieties. In Hartshorne Proposition II.4.10, we see that varieties are just integral schemes of finite type that are separated. So, the just reminds us of how dimension works with varieties.

Hartshorne II.3.21 Let R be a discrete valuation ring containing its residue field k. Let $X = \operatorname{Spec} R[t]$ be the affine line over $\operatorname{Spec} R$. Show that statements (a), (d), (e) of (Ex. 3.20) are false for X.

Proof. Let $R = k[[x]]$ be the formal power series ring. Any DVR containing its residue field is isomorphic to a ring of this form because one can define

$$\phi : k[[x]] \to R$$

by sending x to a choice of uniformizer and k to itself which is clearly a ring isomorphism. Let $X = \operatorname{Spec} R[t]$.

For (a), note that $\dim X = 2$ because $\dim R = 1$ and dimensions for power series rings and polynomial rings over a noetherian coefficient ring are the dimension of the coefficient ring plus one. Let P be the point defined by $(x-t)$. Then, $\mathcal{O}_P \cong k[[x]][t]_{(x-t)}$ has dimension equal to 1. Indeed, $x - t$ is neither a unit nor a zero-divisor in the ring so Krull's Hauptidealsatz tells us $\operatorname{height}_{R[t]}((x-t)) = 1$. Hence, $\dim \mathcal{O}_P = \operatorname{height}_{R[t]}((x-t)) = 1 \neq 2 = \dim X$ showing that (a) is false.

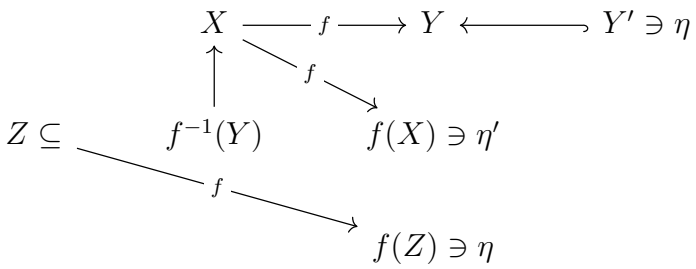
For (d), let $Y = \operatorname{Spec} R[t]/(xt-1)$ i.e. we invert t. So, $Y = \operatorname{Spec} k((t))$. Thus, $\dim Y = 0$. Also, $\operatorname{codim}(Y, X) = 1$ since the irreducible subset defined by t contains Y. However, $\operatorname{codim}(Y, X) + \dim Y = 1$ and $\dim X = 2$.

For (e), take $U = D(t) = \operatorname{Spec} R[t]_t$. Then, $\dim U = \dim R = 1$. However, $\dim X = 2$. □

Hartshorne II.3.22 *Dimension of the Fibres of a Morphism.* Let $f : X \to Y$ be a dominant morphism of integral schemes of finite type over a field k.
(a) Let Y' be a closed irreducible subset of Y, whose generic point η' is contained in $f(X)$. Let Z be any irreducible component of $f^{-1}(Y')$, such that $\eta' \in f(Z)$, and show that $\operatorname{codim}(Z, X) \leqslant \operatorname{codim}(Y', Y)$.
(b) Let $e = \dim X - \dim Y$ be the relative dimension of X over Y. For any point $y \in f(X)$, show that every irreducible component of the fibre X_y has dimension $\geqslant e$. [Hint: Let $Y' = \{y\}^-$, and use (a) and (Ex. 3.20b).]
(c) Show that there is a dense open subset $U \subseteq X$, such that for any $y \in f(U)$, $\dim U_y = e$. [Hint: First reduce to the case where X and Y are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then A is a finitely generated B-algebra. Take $t_1, \ldots, t_e \in A$ which form a transcendence base of $K(X)$ over $K(Y)$, and let $X_1 = \operatorname{Spec} B[t_1, \ldots, t_e]$. Then X_1 is isomorphic to affine e-space over Y, and the morphism $X \to X_1$ is generically finite. Now use (Ex. 3.7) above.]
(d) Going back to our original morphism $f : X \to Y$, for any integer h, let E_h be the set of points $x \in X$ such that, letting $y = f(x)$, there is an irreducible component Z of the fibre X_y, containing x, and having $\dim Z \geqslant h$. Show that (1) $E_e = X$ (use (b) above); (2) if $h > e$, then E_h is not dense in X (use (c) above); and (3) E_h is closed, for all h (use induction on $\dim X$).

(e) Prove the following theorem of Chevalley – see Cartan and Chevalley [1, exposé 8]. For each integer h, let C_h be the set of points $y \in Y$ such that $\dim X_y = h$. Then the subsets C_h are constructible, and C_e contains an open dense subset of Y.

PROOF. (a) We have the setup



See this question here.
(b) See this question here.
(c)
(d)
(e) □

Hartshorne II.3.23 If V, W are two varieties over an algebraically closed field k, and if $V \times W$ is their product, as defined in (I, Ex. 3.15, 3.16), and if t is the functor of (2.6), then $t(V \times W) = t(V) \times_{\operatorname{Spec} k} t(W)$.

PROOF. Let V and W be two varieties over k. Let $V \times W$ be their product.

Consider $t(V) \times_{\operatorname{Spec} k} t(W)$. This is separated by Hartshorne Corollary II.4.6d. IT is also integral by Hartshorne Proposition II.4.10. Lastly, it is of finite type. Thus, it is an abstract variety as defined in Hartshorne II.4. Thus, $t(V) \times_{\operatorname{Spec} k} t(W) = t(Y)$ for some variety Y by Hartshorne Proposition II.4.10. Finally, Y satisfies the universal property of Hartshorne II.3.15 and II.3.16 and hence, $Y = V \times W$. □

Remark 22. This result generalizes Hartshorne I.3.15 and I.3.16. It also tells us that our fibred product works with varieties and this is a suitable generalization.

4. Separated and Proper Morphisms

Hartshorne II.4.1 Show that a finite morphism is proper.

PROOF. By Hartshorne Corollary II.4.8(f), reduce to proving properness on a base. In particular, to show that $f : X \to Y$ is proper morphism when f is finite, let us take an affine open cover $\{V_i\}$ of Y. It will suffice to show that $f^{-1}(V_i) \to V_i$ is proper.

Let us assume $V_i = \operatorname{Spec} B_i$. By Hartshorne II.3.4, we let $f^{-1}(V_i) = \operatorname{Spec} A_i$ where A_i is a finite B_i-module. We will omit the index for ease. We show $\operatorname{Spec} A \to \operatorname{Spec} B$ is a proper morphism when A is a finite B-module. There is a finite ring homomorphism $B \to A$ induced from $\operatorname{Spec} A \to \operatorname{Spec} B$. Since a finite ring homomorphism is the same as being an integral and finite type ring homomorphism, may use Theorem 5.10[28] in [2]. So $\operatorname{Spec} A \to \operatorname{Spec} B$ is a

[28]The result state that if $A \subseteq B$ is an integral ring extension, every $\mathfrak{p} \in \operatorname{Spec} A$ has a $\mathfrak{q} \in \operatorname{Spec} B$ laying over it i.e. $\mathfrak{q} \cap A = \mathfrak{p}$.

closed immersion. By Hartshorne II.3.11, closed immersions are stable under base extension and thus, $\operatorname{Spec} A \to \operatorname{Spec} B$ is universally closed[29].

Since $\operatorname{Spec} A \to \operatorname{Spec} B$ is separated, of finite type, and universally closed, $\operatorname{Spec} A \to \operatorname{Spec} B$ is proper by definition. □

Second proof along same lines. This is Vakil's proof in 10.3.3 of [9]. If $f : X \to Y$ is finite, then Hartshorne II.3.4 says that we can reduce to the affine case and since morphism of affine schemes are always separated, $f : X \to Y$ is separated. To show that $f : X \to Y$ is of finite type, use Hartshorne II.3.4 once again. To see that $f : X \to Y$ is universally closed, $f : X \to Y$ is closed by Hartshorne II.3.5.b. So, we need to check that after a base extension, the morphism is still closed. Indeed, finite morphisms are stable under base change by 9.4.B of [9]. Thus, $f : X \to Y$ is universally closed. □

Hartshorne II.4.2 Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Show that $f = g$. Give examples to show that this result fails if either (a) X is nonreduced, or (b) Y is nonseparated. [Hint: Consider the map $h : X \to Y \times_S Y$ obtained from f and g.]

Proof. Let $h = (f, g)$ so that $h : X \to Y \times_S Y$. Notice that h agrees with (f, f) on some dense open subset D of X. Let $\Delta : Y \to Y \times_S Y$ be the diagonal morphism. Since Y is separated, Δ is a closed immersion. Now, $h(D) \subseteq Y \times_S Y$ and $h(D) \subseteq \Delta(D)$. So, $D \subseteq h^{-1}(\Delta(D))$. Since $\Delta(D)$ is closed by the fact that Δ is a closed immersion, $X \subseteq h^{-1}(\Delta(D))$ and so we have $h(X) = \Delta(D)$. Thus, $f = g$ on X.

Now, we check the morphism on the structure sheaf are equal. Since we can check the result on stalks, we may assume X and Y are affine schemes with induced ring homomorphisms $f^\#, g^\# : A \to B$ associated to f and g. If $a \in A$, let $f^\#(a) - g^\#(a) = b$. Our goal is to show that $b = 0$. If $a \in D$, then $b = 0$ by assumption. Thus, $D \subseteq V(b)$ since b vanishes on D. Since $V(b)$ is closed, $V(b) = X$. So, b must be a nilpotent element in B. But X is reduced so all nilpotents are zero. So, $b = 0$ for all a. So, the morphism on sheaves agree.

Example 1. Reducedness is necessary. Indeed, consider $\mathbb{Z}[x]/(x^2, xp)$ for $p \in \mathbb{Z}$ prime.

Let f be the morphism given by $x \mapsto x$ and g be the morphism defined by $x \mapsto 0$ (i.e. g is the evaluation map).

That is, we let $f^\#, g^\# : \mathbb{Z}[x] \to \mathbb{Z}[x]/(x^2, xp)$ be given by $f^\#(x) = x$ and $g^\#(x) = 0$ respectively. Now take the induced maps $f : \operatorname{Spec} \mathbb{Z}[x]/(x^2, xp) \to \operatorname{Spec} \mathbb{Z}[x]$ and $g : \operatorname{Spec}[x]/(x^2, xp) \to \operatorname{Spec} \mathbb{Z}[x]$. Explicitly,

$$\operatorname{Spec}(\mathbb{Z}[x]/(x^2, xp)) = \{(\overline{x}), (\overline{x}, \overline{p})\}$$

Now we compute the maps on the level of sets

$$f((\overline{x}, \overline{p})) = (f^\#)^{-1}((\overline{x}, \overline{p})) = (x, p) = (g^\#)^{-1}((\overline{x}, \overline{p})) = g((\overline{x}, \overline{p}))$$
$$f((\overline{x})) = (f^\#)^{-1}((\overline{x})) = (x) = (g^\#)^{-1}(\overline{x}) = g((\overline{x})).$$

So the two maps certainly agree on a dense open subset (in particular the whole space). However, the maps of sheaves are not the same so $f \neq g$ as maps of locally ringed spaces.

[29]Closed immersions are clearly closed map because they are a homeomorphism onto a closed subset. In particular, recall Hartshorne II.3.5b. Also see 9.4.B of [9].

Example 2. The separated condition is necessary. Let Y be the affine line with doubled origin and X the usual affine line. Let f, g be the maps of $X \to Y$ in which the origin of X is sent to different origins of Y. now $f = g$ on the dense open subset $\operatorname{Spec} k[x] \setminus \{(x)\}$. However, the two maps are clearly not equal. □

Hartshorne II.4.3 Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. Then $U \cap V$ is also affine. Give an example to show that this fails if X is not separated.

Proof. Because X is separated over an affine scheme S, $\Delta : X \to X \times_S X$ is a closed immersion and so, $\Delta(X)$ is a closed subscheme of $X \times_S X$. We look at $U \times_S V = \operatorname{Spec}(A \otimes_C B)$. So, $(U \times_S V) \cap \Delta(X) = U \cap V$ by definition of $\Delta(X)$.

Since $\Delta(X)$ is a closed subscheme, $U \times_S V$ is affine, and the intersection of an affine scheme with a closed subscheme is affine (this follows from Hartshorne II.3.11(b)), we conclude that $U \cap V$ is affine.

Example. For a counterexample, consider the affine plane with a point doubled. Two affine open subsets are the affine plane except we choose a different point for each plane from our doubled point. The intersection of two such affine planes is the affine plane with a point removed. This is clearly not affine[30]. I will just sketch why the affine plane with a point removed is not affine.

Fix U as the affine plane with origin removed. Suppose it were affine and $U = \operatorname{Spec} A$ for some A. Then, we should have $A = \Gamma(U, \mathcal{O}_{\mathbb{A}^2_k|_U}) = k[x, y]$. The latter equality follows from the fact that $k[x, y]_{(x)} \cap k[x, y]_{(y)} = k[x, y]$ and $U = D(x) \cap D(y)$. The way to understand this latter inequality is to say that functions not vanishing on the x-axis and the y-axis are precisely functions in $k[x, y]$ that have only powers of x in the denominator and only powers of y in the denominator (which forces the denominator to be 1). However, notice that there should be a bijection between closed points of $k[x, y]$ and U which does not occur. Alternatively, since $A = k[x, y]$, U must be $\mathbb{A}^2_k$ which is not possible. □

Hartshorne II.4.4 Let $f : X \to Y$ be a morphism of separated schemes of finite type over a Noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure (Ex. 3.11 d) is proper over S. We refer to this result by saying that "the image of a proper scheme is proper." [Hint: Factor f into the graph morphism $\Gamma_f : X \to X \times_S Y$ followed by the second projection p_2, and show that Γ_f is a closed immersion.]

Proof. [31] One can consider the diagram:

$$\begin{array}{ccc} f(Z) & \longrightarrow & f(Z) \times_S f(Z) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta} & Y \times_S Y \end{array}$$

Now for $f(Z) \to S$ being of finite type, we first show that $Z \to f(Z)$ is a proper morphism. By hypothesis, $Z \to S$ is a proper morphism. We have the diagram:

[30] For more details, see [9] pp. 282-283.

[31] As far as I am aware, this seems to be the only proof approach that is publicly posted online. I do not think anyone has posted a proof online that uses the hint Hartshorne has given.

$$\begin{array}{ccc} Z & \longrightarrow & f(Z) \\ & \searrow & \downarrow \\ & & S \end{array}$$

and since $f(Z) \to S$ is separated, we use conclude that $Z \to f(Z)$ is proper using Hartshorne Corollary II.4.8e. Since we know $Z \to f(Z)$ is proper, we know it is a closed immersion. Then because Z is a closed subscheme of X, $f(Z)$ is a closed subscheme of Y. So, $f(Z) \to Y$ is closed immersion which means it is of finite type (by Hartshorne II.3.13). Then $f(Z) \to Y \to S$ is a factorization. Since $Y \to S$ is a morphism of finite type, Hartshorne II.3.13c says $f(Z) \to S$ is of finite type.

Now we deal with universally closed.[32] First, $f(Z) \to S$ is closed. Indeed, $Z \to f(Z)$ and $Z \to S$ are proper maps so they are closed (the composite of closed maps is closed).

Suppose $T \to S$ is a base change. Then, we have the diagram:

$$\begin{array}{ccc} Z \times_S T & \longrightarrow & Z \\ \downarrow & & \downarrow \\ W \subseteq f(Z) \times_S T & \longrightarrow & f(Z) \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

where W is a closed subset of $f(Z) \times_S T$. The image of W in T is closed by pulling back to $Z \times_S T$ and then pushing forward using the second projection $Z \times_S T \to T$. Now, $Z \times_S T \to f(Z) \times_S T$ is surjective because surjectivity is preserved by base extension. Using the fact that $Z \to S$ is universally closed, $Z \times_S T \to T$ is also closed. □

An incomplete and likely incorrect proof. We make use of the hint. Since, X and Y are schemes of finite type over S and S is Noetherian, Hartshorne II.3.13g tells us that X and Y are Noetherian.

We show $\Gamma_f : X \to X \times_S Y$ is a closed immersion. We have the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_S Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \xrightarrow[\Delta]{} & Y \times_S Y \end{array}$$

which is obtained by taking $\Delta : Y \to Y \times_S Y$ and then taking a base extension using $X \to Y$. In other words, we get $\Gamma_f : X \times_Y Y \to X \times_Y Y \times_S Y$ which is the same as $\Gamma_f : X \to X \times_S Y$. Since Δ is a closed immersion by hypothesis (Y is separated) and closed immersions are stable under base change, Γ_f is also a closed immersion. We can factor f

$$Z \to Z \times_S f(Z) \to f(Z).$$

We know that Γ_f is a closed immersion and p_2 is a closed immersion (since the underlying morphism of rings is a surjective). Since $f : X \to Y$ is a closed map (as it is a closed immersion), $f(Z)$ is closed.

[32]This is the approach given by Professor Izadi in her email for the Fall 2020 quarter.

To show that $f(Z)$ is proper over S, notice that separatedness follows by $\Gamma_f|_Z$ being a closed immersion. In particular, we have the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\Gamma_f|_Z} & Z \times_S f(Z) \\ \downarrow & & \downarrow \\ f(Z) & \longrightarrow & f(Z) \times_S f(Z) \end{array}$$

and because $\Gamma_f|_Z$ is a closed immersion, the bottom row is also a closed immersion. But that bottom row being a closed immersion implies $f(Z)$ is separated over S.

The fact that $f(Z)$ is of finite type over S follows form $f(Z) \subseteq Y$ being a closed subscheme of a finite type scheme over S.

The last step is to prove that $f(Z)$ is universally closed. Let $W \to S$ be a morphism and we want to show $f(Z) \times_S W \to W$ is closed. Let $T \subseteq f(Z) \times_S W$ be a closed subset.

THIS LAST PART IS INCOMPLETE. □

Hartshorne II.4.5 Let X be an integral scheme of finite type over a field k, having function field K. We say that a valuation of K/k (see I, §6) has center x on X if its valuation ring R dominates the local ring $\mathcal{O}_{x,X}$.
(a) If X is separated over k, then the center of any valuation of K/k on X (if it exists) is unique.
(b) If X is proper over k, then every valuation of K/k has a unique center on X.
*(c) Prove the converses of (a) and (b). [Hint: While parts (a) and (b) follow quite easily from (4.3) and (4.7), their converses will require some comparison of valuations in different fields.]
(d) If X is proper over k, and if k is algebraically closed, show that $\Gamma(X, \mathcal{O}_X) = k$. This result generalizes (I, 3.4a). [Hint: Let $a \in \Gamma(X, \mathcal{O}_X)$, with $a \notin k$. Show that there is a valuation ring R of K/k with $a^{-1} \in \mathfrak{m}_R$. Then use (b) to get a contradiction.]
Note. If X is a variety over k, the criterion of (b) is sometimes taken as the definition of a complete variety.

Proof. *(a)* If X is separated over k, then the dotted map on the diagonal is unique if it exists

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec}(k) \end{array} .$$

Suppose there were a pair $x, y \in X$ that were centers. If $\xi \in X$ is the generic point, then $x, y \in \overline{\{\xi\}}$. Then Harshorne Lemma II.4.4 with $x_0 = x, y$ and $x_1 = \xi$ (observing that R dominates $\mathcal{O}_{x,\overline{\{\xi\}}}$ and $\mathcal{O}_{y,\overline{\{\xi\}}}$ gives two possible map on the diagonal. But $X \to \operatorname{Spec}(k)$ separated means these maps are equal. So our points x, y were equal to begin with.

(b) If $X \to \operatorname{Spec}(k)$, then the map on the diagonal always exists. But that means the ring R always dominates some $\mathcal{O}_{x,X}$ for some $x \in X$ by the valuative criterion for properness.

(c) **[Incomplete]**

(d) Let $a \notin k$ and $a \in \Gamma(X, \mathcal{O}_X)$. **[Incomplete]** □

Hartshorne II.4.6 Let $f : X \to Y$ be a proper morphism of affine varieties over k. Then f is a finite morphism. [Hint: Use (4.11A).]

PROOF 1. Observe that X, Y being affine varieties means that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ which are integral, of finite type, and separated (all affine schemes are separated) as schemes. A morphism of affine varieties is necessarily an affine morphism (see Lurie's explanation here). To help exposit Lurie's proof, one refers to the diagram

$$\begin{array}{ccccccc} & & & \overset{f}{\frown} & & & \\ X & \xrightarrow{\exists! f'} & X \times_{\mathbb{Z}} Y & \xrightarrow{\mathrm{id}_{X\times_{\mathbb{Z}}Y}} & X \times_{\mathbb{Z}} Y & \xrightarrow{f''} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \exists!} & & {\scriptstyle p_1}\downarrow & & \downarrow \\ Y & \xrightarrow{\exists! \Delta_Y} & Y \times_{\mathbb{Z}} Y & & X & \longrightarrow & \mathbb{Z} \end{array}$$

and shows that the maps where $\exists!$ is written exist and are unique via the universal property of the fibre product. This ensures that the LHS square commutes and X is actually the indicated pull-back.

Hartshorne II.3.4 reduces the problem to show that if $V = \operatorname{Spec} C \subseteq Y$, then $f^{-1}(V) = \operatorname{Spec} D$ is affine with D a finite C-module. Since properness is local, $f : \operatorname{Spec} D \to \operatorname{Spec} C$ is proper and a morphism of affine varieties over k.

Now we consider the diagram in the valuative criterion with K and R specified later

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} D \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} C \end{array}$$

Here, we want $\operatorname{Spec} K \to \operatorname{Spec} D$ which is induced from the $D \hookrightarrow K$ for K the function field of D. The ring R is just any valuation ring while $\widetilde{C}$ is the integral closure of C inside K (which we need to justify is actually possible).

For now, set $K := K(\operatorname{Spec} D)$ and $F := K(\operatorname{Spec} C)$. The commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ K(\operatorname{Spec} C) & \longrightarrow & K(\operatorname{Spec} D) \end{array}$$

tells us that $C \to K(\operatorname{Spec} C) \to K(\operatorname{Spec} D)$ can be considered with $K(\operatorname{Spec} D)/K(\operatorname{Spec} C)$ a field extension. So we can consider C as a subring of K and hence, the integral closure by use of Hartshorne Theorem II.4.11A. The theorem suggests we take $\widetilde{C} \to R$ as the inclusion map. So, $C \to R$.

By the valuative criterion, we get the diagonal part of the diagram $\operatorname{Spec} R \to \operatorname{Spec} D$. Examining the upper part of the diagram,

$$\begin{array}{ccc} K & \hookleftarrow & D \\ \uparrow & \swarrow & \\ R & & \end{array}$$

we know that $D \to R \hookrightarrow K$ is an injective map of rings and therefore, $D \to R$ must be injective. Since R was an arbitrary valuation ring, $D \subseteq \bigcap_{C \subseteq R \subseteq K \text{and } R \text{ val. ring}} R = \widetilde{C}$. So, D is integral over C. With D a finitely generated C-algebra that is also integral over C means that D is a finite C-module (see p. 60 of Atiyah-Macdonald). □

PROOF 2. We know $f : X \to Y$ is affine and since it is proper, it is universally closed. We show this implies integral and since it is already of finite type, we get finite.

Let $A \to B$ be a ring homomorphism, and consider $b \in B$. We want sto show b is integral over A. To see this, we look at the base change diagram

$$\begin{array}{ccc} \operatorname{Spec} B[x] & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ \operatorname{Spec} A[x] & \longrightarrow & \operatorname{Spec} A \end{array}$$

By universal closedness, $\operatorname{Spec} B[x] \to \operatorname{Spec} A[x]$ is a closed map. Now consider $\operatorname{Spec} B[x]/(bx-1)$ which is closed. Then its image $\operatorname{Spec} B[x]/(bx-1) \to \operatorname{Spec} A[x]$ is closed. Now we factor the ring as

$$A[x] \to A[x]/J \hookrightarrow B[x]/(bx-1)$$

where J is the ideal consisting of polynomials $a_0+a_1x+\cdots+a_dx^d$ s.t. $b^N(a_0b^d+a_1b^{d-1}+\cdots+a_d) = 0$ in B for N sufficiently large. Since the ring map $A[x]/J \hookrightarrow B[x]/(bx-1)$ is injective, the map on spectrum has dense image. So closedness implies that $\operatorname{Spec} B[x]/(bx-1) \to \operatorname{Spec} A[x]/J$ is surjective.

Now surjective maps are preserved under base change, and then

$$\begin{array}{ccc} \operatorname{Spec} B[x]/(bx-1,x) = \emptyset & \longrightarrow & \operatorname{Spec} B[x]/(bx-1) \\ \downarrow & & \downarrow \\ \operatorname{Spec} A[x]/(J,x) & \longrightarrow & \operatorname{Spec} A[x]/J \end{array}$$

implies $A[x]/(J,x) = 0$. So, J contains a polynomial with constant coefficient 1. Hence, by definition of J,

$$b^N(b^d + a_1b^{d-1} + \cdots + a_d) = b^{N+d} + a_1b^{N+d-1} + \cdots + a_db^N = 0$$

gives an integral relation for b. □

Hartshorne II.4.7 Schemes Over $\mathbf{R}$. For any scheme X_0 over $\mathbf{R}$, let $X = X_0 \times_{\mathbf{R}} \mathbf{C}$. Let $\alpha : \mathbf{C} \to \mathbf{C}$ be complex conjugation, and let $\sigma : X \to X$ be the automorphism obtained by keeping X_0 fixed and applying α to $\mathbf{C}$. Then X is a scheme over $\mathbf{C}$, and σ is a semi-linear automorphism, in the sense that we have a commutative diagram

[Incomplete]

Since $\sigma^2 = \mathrm{id}$, we call σ an involution.

(a) Now let X be a separated scheme of finite type over $\mathbf{C}$, let σ be a semilinear involution on X, and assume that for any two points $x_1, x_2 \in X$, there is an open affine subset containing both of them. (This last condition is satisfied for example if X is quasi-projective.) Show that there is a unique separated scheme X_0 of finite type over $\mathbf{R}$, such that $X_0 \times_{\mathbf{R}} \mathbf{C} \cong X$, and such that this isomorphism identifies the given involution of X with the one on $X_0 \times_{\mathbf{R}} \mathbf{C}$ described above.

For the following statements, X_0 will denote a separated scheme of finite type over $\mathbf{R}$, and X, σ will denote the corresponding scheme with involution over $\mathbf{C}$.

(b) Show that X_0 is affine if and only if X is.

(c) If X_0, Y_0 are two such schemes over $\mathbf{R}$, then to give a morphism $f_0 : X_0 \to Y_0$ is equivalent to giving a morphism $f : X \to Y$ which commutes with the involutions, i.e., $f \circ \sigma_X = \sigma_Y \circ f$.

(d) If $X \cong \mathbf{A}^1_{\mathbf{C}}$, then $X_0 \cong \mathbf{A}^1_{\mathbf{R}}$.

(e) If $X \cong \mathbf{P}^1_{\mathbf{C}}$, then either $X_0 \cong \mathbf{P}^1_{\mathbf{R}}$, or X_0 is isomorphic to the conic in $\mathbf{P}^2_{\mathbf{R}}$ given by the homogeneous equation $x_0^2 + x_1^2 + x_2^2 = 0$.

Proof. (a)

(b) If X is affine, X_0 is clearly affine. So assume X_0 is affine. [**Incomplete**]

(c) [**Incomplete**]

(d) Assume $X \cong \mathbb{A}^1_{\mathbb{C}}$. Then clearly $X \cong \mathbb{A}^1_{\mathbb{R}} \times_k \mathbb{C}$ and $\mathbb{A}^1_{\mathbb{R}}$ is an separate scheme of finite type over $\mathbb{R}$. The obvious involution preserves $\mathbb{A}^1_{\mathbb{R}}$ and uniqueness from (a) implies $X_0 \cong \mathbb{A}^1_{\mathbb{R}}$.

(e)[**Incomplete**] □

Remark 23. The hypothesis that any two points in X share a common open affine subset is necessary. Consider the situation of $X = \operatorname{Spec} \mathbb{C} \times \mathbb{C}$. Then X could be realized from any of $\mathbb{C} \times \mathbb{R}$, $\mathbb{R} \times \mathbb{C}$, or $\mathbb{R} \times \mathbb{R}$ with appropriate $\mathbb{R}$-module structures.

Hartshorne II.4.8 Let $\mathscr{P}$ be a property of morphisms of schemes such that:
(a) a closed immersion has $\mathscr{P}$;
(b) a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$;
(c) $\mathscr{P}$ is stable under base extension.

Then show that:
(d) a product of morphisms having $\mathscr{P}$ has $\mathscr{P}$;
(e) if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$;
(f) If $f : X \to Y$ has $\mathscr{P}$, then $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ has $\mathscr{P}$.
[Hint: For (e), consider the graph morphism $\Gamma_f : X \to X \times_Z Y$ and note that it is obtained by base extension from the diagonal morphism $\Delta : Y \to Y \times_Z Y$.]

Proof. (1) Suppose $f : X \to Y$ and $f' : X' \to Y'$ have $\mathscr{P}$. I want to show that $f \times f' : X \times_S X' \to Y \times_S Y'$ as $\mathscr{P}$ as well. Notice that we can write $f \times f'$ as

$$X \times_S X' \to X \times_S Y' \to Y \times_S Y'$$

and so, it suffices to show that $X \times_S X' \to X \times_S Y'$ has $\mathscr{P}$ by applying a dual argument to $X \times_S Y' \to Y \times_S Y'$ and using the fact that $\mathscr{P}$ is stable under composition.

We wish to show $X \times_S X' \to X \times_S Y'$ has $\mathscr{P}$. The morphism is precisely,

$$X \times_X (X \times_S Y') \to X \times_S Y'$$

defined by projection on the second component. In this situation, the morphism is the base extension of $X \to X$ by $X \times_S Y' \to X$. Since $\mathscr{P}$ is stable under base extension, the morphism has $\mathscr{P}$ as desired.

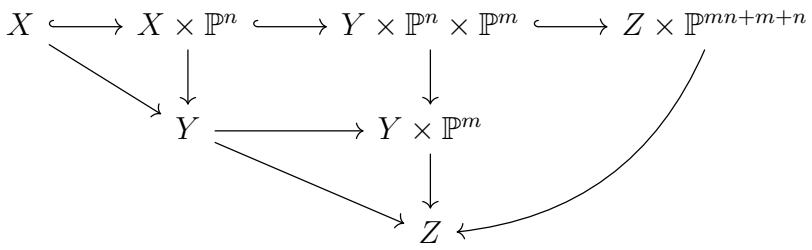
(2) Note that $\Gamma_f : X \to X \times_Z Y$ is obtained as a base extension from $\Delta : Y \to Y \times_Z Y$. Since g is separated, Δ is a closed immersion and so, Δ has $\mathscr{P}$. By stability under base extension, Γ_f has $\mathscr{P}$. Let $p_2 : X \times_Z Y \to Y$ be projection. Then, $f = p_2 \circ \Gamma_f$ and since $\mathscr{P}$ is stable under composition, f has $\mathscr{P}$ as desired.

(3) Let $i : X_{red} \to X$ and $j : Y_{red} \to Y$ be the canonical morphisms given in Hartshorne II.2.3. These morphisms are closed immersions and thus have $\mathscr{P}$. By Hartshorne II.2.3 once again, we have $f \circ i = j \circ f_{red}$ where $f_{red} : X_{red} \to Y_{red}$. Since $f \circ i$ has $\mathscr{P}$, $j \circ f_{red}$ has $\mathscr{P}$. By (2), f_{red} has $\mathscr{P}$.

□

Hartshorne II.4.9 Show that a composition of projective morphisms is projective.
[Hint: Use the Segre embedding defined in (I, Ex. 2.14) and show that it gives a closed immersion $\mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{rs+r+s}$.]
Conclude that projective morphisms have properties (a) - (f) of (Ex. 4.8) above.

Proof. We consider the commutative diagram



and since the composition of closed immersions gives a closed immersion we see that $X \hookrightarrow Z \times \mathbb{P}^{mn+m+n}$ is a closed immersion if we show that $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ is a closed immersion. This follows from [10, Tag 01WD].

Let us prove that we do have a closed immersion here so that this is complete. We clearly have a homeomorphism onto a closed subset of $\mathbb{P}^{mn+m+n}$. All that is left is check that the induced morphism of sheaves $\mathcal{O}_{\mathbb{P}^{mn+m+n}} \to \phi_* \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}$ is a surjection. This can be checked on the stalks. Alternatively, we can check that we have surjections on the ring homomorphisms of the affine covers. □

Hartshorne II.4.10 *Chow's Lemma.* This result says that proper morphisms are fairly close to projective morphisms. Let X be proper over a noetherian scheme S. Then there is a scheme X' and a morphism $g : X' \to X$ such that X' is projective over S, and there is an open dense subset $U \subseteq X$ such that g induces an isomorphism of $g^{-1}(U)$ to U. Prove this result in the following steps.
(a) Reduce to the case X irreducible.
(b) Show that X can be covered by a finite number of open subsets $U_i, i = 1, \ldots, n$, each of which is quasi-projective over S. Let $U_i \to P_i$ be an open immersion of U_i into a scheme P_i which is projective over S.
(c) Let $U = \bigcap U_i$, and consider the map

$$f : U \to X \times_S P_1 \times_S \cdots \times_S P_n$$

deduced from the given maps $U \to X$ and $U \to P_i$. Let X' be the closed image subscheme structure (Ex. 3.11d) $f(U)^-$. Let $g : X' \to X$ be the projection onto the first factor, and let $h : X' \to P = P_1 \times_S \ldots \times_S P_n$ be the projection onto the product of the remaining factors. Show that h is a closed immersion, hence X' is projective over S.
(d) Show that $g^{-1}(U) \to U$ is an isomorphism, thus completing the proof.

Proof. **[Incomplete]** □

Remark 24. For more on Chow's Lemma, see 18.9.2 of [9].

Hartshorne II.4.11 If you are willing to do some harder commutative algebra, and stick to Noetherian schemes, then we can express the valuative criteria of separatedness and properness using only discrete valuation rings.

(a) If $\mathcal{O}, \mathfrak{m}$ is a Noetherian local domain with quotient field K, and if L is a finitely generated field extension of K, then there exists a discrete valuation ring R of L dominating $\mathcal{O}$.

Prove this in the following steps. By taking a polynomial ring over $\mathcal{O}$ reduce to the case where L is a finite extension field of K.

Then show that for a suitable choice of generators $x_1, \ldots, x_n$ of $\mathfrak{m}$, the ideal $\mathfrak{a} = (x_1)$ in $\mathcal{O}' = \mathcal{O}[x_2/x_1, \ldots, x_n/x_1]$ is not equal to the unit ideal.

Then let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{a}$ and let $\mathcal{O}'_{\mathfrak{p}}$ be the localization of $\mathcal{O}'$ at $\mathfrak{p}$. This is a Noetherian local domain of dimension 1 dominating $\mathcal{O}$.

Let $\widetilde{\mathcal{O}}'_{\mathfrak{p}}$ be the integral closure of $\mathcal{O}'_{\mathfrak{p}}$ in L. Use the theorem of Krull-Akizuki (see Nagata [7, p.115]) to show that $\widetilde{\mathcal{O}}'_{\mathfrak{p}}$ is Noetherian of dimension 1.

Finally, take R to be a localization of $\widetilde{\mathcal{O}}'_{\mathfrak{p}}$ at one of its maximal ideals.

(b) Let $f : X \to Y$ be a morphism of finite type of Noetherian schemes. Show that f is separated (respectively, proper) if and only if the criterion of (4.3) (respectively, (4.7)) holds for all discrete valuation rings.

PROOF. **(a)** So that the proof is easier to read, we break down everything into parts labeled with roman numerals. For ease, we quote the theorem of Krull and Akizuki (translated from Nagata's text into modern terminology). Note that we are interested in the special case.

Theorem. Let R be a Noetherian ring with $K = \mathrm{Frac}(R)$. Let L/K be a finite algebraic extension of K. Let R' be a ring s.t. $R \subseteq R' \subseteq L$. If $\dim R = 1$ and $\mathfrak{a} \subseteq R'$ is a nonzero ideal, then $R'/\mathfrak{a}'$ is a module of finite length. In particular, R' is a Noetherian ring of dimension 1.

(i) Suppose we knew the result for the case where $L := K(\alpha_1, \ldots, \alpha_k)/K$ is of finite degree. Now suppose L were not of finite degree. Then one of α_i is does not yield a finite degree extension $K(\alpha_i)/K$. However, we may replace our ring $\mathcal{O}$ with the ring $\mathcal{O}[\alpha_i]_{\mathfrak{m}}$ which has quotient field $K(\alpha_i)$. Now if $L/K(\alpha_i)$ is finite, then there exists a DVR R of L that dominates $\mathcal{O}[\alpha_i]_{\mathfrak{m}}$. But $\mathfrak{m}\mathcal{O}[\alpha_i]$ lays over the maximal ideal of $\mathcal{O}$. So R is a DVR of L dominating $\mathcal{O}$.

From here on, we assume L/K is a finite extension.

(ii) Suppose not. Assume that $\mathfrak{m} = (x_1, \ldots, x_n)$ with n chosen minimally. Now $\mathfrak{a} = \mathcal{O}'$ means that $x_1^{-1} \in \mathcal{O}'$ exists. Write $\frac{1}{x_1} = \sum_{I \text{ multi-index}} a_I \frac{x^I}{x_1^{|I|}} =: f(X)$ with $a_I \in \mathcal{O}$. Of course, we assume each a_I is not divisible by any of the $x_2, \ldots, x_n$. Now divide by $x_1^{\deg(f(X))}$ which means for we can write

$$x_1^{\deg(f)-1} = \sum_{I \text{ multi-index}} a_I x^I x_1^{\deg f - |I|}.$$

Now divide by a sufficiently large (but minimal) power of x_1 so that we get

$$x_1^m = \left(\sum_{J \text{ multi-index}} a_I x^J \right) + dg(Y)$$

where $g(Y)$ is a polynomial whose coefficients are not divisible by x_1 and its monomials are not either (while the x^J monomials are divisible by x_1). It then follows that x_1 divides $g(Y)$. But that means x_1 was generated by the $x_2, \ldots, x_n$. This contradicts minimality of n.

(iii) Let $\mathfrak{p}$ be a minimal prime ideal over $\mathfrak{a}$. We check that $\mathcal{O}'_{\mathfrak{p}}$ is a local domain of dimension 1 dominating $\mathcal{O}$. It is Noetherian because it is the localization of a Noetherian ring (use Hilbert's Basis Theorem to see $\mathcal{O}'$ is Noetherian). Since we localized at a prime ideal, it is certainly local. The localization of a domain is also a domain. OTOH, by Krull's Hauptidealsatz, the $\mathfrak{p}$ had height 1 inside $\mathcal{O}'$ which implies that $\dim \mathcal{O}'_{\mathfrak{p}} = \text{height}\, \mathfrak{p} = 1$.

Now we check $\mathcal{O}'_{\mathfrak{p}}$ dominates $\mathcal{O}$. That is, we want to show $\mathfrak{p}\mathcal{O}'_{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{m}$. It suffices to show that $x_i \in \mathfrak{p}\mathcal{O}_{\mathfrak{p}}$ for all $2 \leq i \leq n$. Fix an i. Multiplying by x_1 implies $x_i \in \mathfrak{p}$.

(iv) Let $\widetilde{\mathcal{O}'_{\mathfrak{p}}}$ be the integral closure of $\mathcal{O}'_{\mathfrak{p}}$ in L. By the special case of the theorem of Krull and Akizuki ($\widetilde{\mathcal{O}'_{\mathfrak{p}}}$ is the ring R' as part iii showed that it dominates $\mathcal{O}$), we know $\widetilde{\mathcal{O}'_{\mathfrak{p}}}$ is Noetherian.

(v) Set $R := (\widetilde{\mathcal{O}'_{\mathfrak{p}}})_{\mathfrak{n}}$ (i.e. it is the localization) where $\mathfrak{n} \subseteq \widetilde{\mathcal{O}'_{\mathfrak{p}}}$ is a maximal ideal. We check that R is a DVR of L dominating $\mathcal{O}$.

Observe that $\widetilde{\mathcal{O}'_{\mathfrak{p}}}$ is an integrally closed domain which is noetherian and of dimension 1. Theorem 9.3[33] of [2] tells us that the localization at a prime ideal is a DVR.

We check that $\mathfrak{n}(\widetilde{\mathcal{O}'_{\mathfrak{p}}}) \cap \mathcal{O} = \mathfrak{m}$. Since $\mathfrak{n}$ is the maximal ideal of $\widetilde{\mathcal{O}'_{\mathfrak{p}}}$, it lays over the maximal ideal $\mathfrak{p}\mathcal{O}'_{\mathfrak{p}}$ of $\mathcal{O}'_{\mathfrak{p}}$ by the Going-Down Theorem (see Theorem 5.16 of [2]). So, $\mathfrak{n}(\widetilde{\mathcal{O}'_{\mathfrak{p}}})_{\mathfrak{n}} \cap \mathcal{O}'_{\mathfrak{p}} = \mathfrak{p}\mathcal{O}'_{\mathfrak{p}}$. But we know that $\mathfrak{p}\mathcal{O}'_{\mathfrak{p}} \cap \mathcal{O}' = \mathfrak{p}$ because $\mathfrak{p}\mathcal{O}'_{\mathfrak{p}}$ sits over the prime ideal $\mathfrak{p}$ of $\mathcal{O}'$. However, part iii showed that $\mathfrak{p}$ of $\mathcal{O}'$ satisfied the equation $\mathfrak{p}\mathcal{O}' \cap \mathcal{O} = \mathfrak{m}$. Putting this together, we essentially are saying

$$\mathfrak{n}(\widetilde{\mathcal{O}'}_{\mathfrak{p}})_{\mathfrak{n}} \cap \mathcal{O} = [\mathfrak{n}(\widetilde{\mathcal{O}'}_{\mathfrak{p}})_{\mathfrak{n}} \cap \mathcal{O}'_{\mathfrak{p}}] \cap \mathcal{O} = \mathfrak{p}\mathcal{O}'_{\mathfrak{p}} \cap \mathcal{O} = [\mathfrak{p}\mathcal{O}'_{\mathfrak{p}} \cap \mathcal{O}'] \cap \mathcal{O} = \mathfrak{p}\mathcal{O}' \cap \mathcal{O} = \mathfrak{m}.$$

We conclude that R is the discrete valuation ring we wanted.

(b) Let $f : X \to Y$ be a morphism of finite type and X, Y noetherian schemes.

Discrete Valuative Criterion for Separatedness. We begin with proving the statement for separated morphisms. The ($\Longrightarrow$) direction is clear. Assume the criterion in (4.3) holds with our modification. From part (a), we may consider the following diagram

$$\begin{array}{ccccc}
\operatorname{Spec} K & \longrightarrow & \operatorname{Spec} K = U & \xrightarrow{g} & X \\
\downarrow & & \downarrow{\scriptstyle i} & \nearrow{\scriptstyle \phi} & \downarrow{\scriptstyle f} \\
\operatorname{Spec} \mathcal{O} & \longrightarrow & \operatorname{Spec} R = T & \xrightarrow{h} & Y
\end{array}$$

which exists for the following reasons. First, we assume that the RHS square exists and is commutative because we wish to use Hartshorne Theorem II.4.3. The existence of the LHS square follows by part (a) with $L = K$ and $\mathcal{O}$ in our diagram being the DVR that we can find. Applying part (a) is justified by the fact that R is a valuation ring (which are always noetherian local domains). This LHS square is clearly commutative.

[33] We use the direction that the localization of a Dedekind domain at a prime ideal is a DVR.

Now suppose $\phi, \phi' : \operatorname{Spec} R \to X$ were two maps making the RHS square commute. Then these induce maps $\operatorname{Spec} \mathcal{O} \to \operatorname{Spec} R \overset{\phi,\phi'}{\to} X$ which make the square

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow\!\nearrow & \downarrow \\ \operatorname{Spec} \mathcal{O} & \longrightarrow & Y \end{array}$$

commute. By assumption, the composites are equal. So we reduce to showing that if $\operatorname{Spec} \mathcal{O} \overset{\psi}{\to} \operatorname{Spec} R \overset{\phi,\phi'}{\to} X$ are equal composites, then $\phi = \phi'$ as maps. It is a simple task of showing that $\psi : \operatorname{Spec} \mathcal{O} \to \operatorname{Spec} R$ is an epimorphism in the category of schemes. Indeed, a criterion can be found here: a morphism $f : X \to Y$ is an epimorphism if the underlying set map is surjective and $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective. Indeed, $\mathcal{O}$ dominates R, and so our map $\operatorname{Spec} \mathcal{O} \to \operatorname{Spec} R$ is surjective. This map was induced by the inclusion $R \hookrightarrow \mathcal{O}$.

It may be helpful to note that we only needed the hypothesis that X was Noetherian here. For the next part, $f : X \to Y$ being of finite type and X Noetherian are required by Hartshorne Theorem II.4.7, but the Noetherian hypothesis on Y is needed to construct an actual morphism.

Discrete Valuative Criterion for Properness. The ($\implies$) direction is clear by the usual valuative criterion. Consider the other direction. From part (a), we know that if the modified criterion holds, then f is separated. Since the map was assumed to be of finite type, we need only show it is universally closed.

The proof actually follows exactly as in Hartshorne Theorem II.4.7. The Noetherian hypothesis on Y ensures that $k(z_1)/k(y_1)$ is a finitely generated field extension (use the of finite type hypothesis). Apply part (a) in place of Hartshorne I.6.1A to get the desired DVR R of K and use our assumption. The rest of the proof follows as is. □

Remark 25. Comments from Nathan Wenger.

okay, just looked over this solution. (i) looks good. I haven't checked (ii) carefully, but I can point out something that is probably slicker. $\mathcal{O}$ is a subring of the field K, so it is dominated by a valuation ring with fraction field K (Stacks project Lemma 10.50.2). Since you only care about domination, you can just assume that $\mathcal{O}$ is a valuation ring. Then just pick x_1 to be minimal under the valuation. Then the fractions will all have nonnegative valuations, so they all lie in $\mathcal{O}$, so of course x_1 doesn't become a unit. The rest looks to be approximately what I did, but I can discuss more if that's the part you want to talk about. For (b), I'm not sure why you can't just replace the conclusion of I, 6.1A by "R is dominated by a valuation ring" and then write out the rest of the proof as is, but I haven't looked over it carefully. I should note that the "dominated by a valuation ring" thing isn't my own trick (I saw it somewhere that I can't find now, maybe stacks project) but it is very handy

Hartshorne II.4.12 *Examples of Valuation Rings.* Let k be an algebraically closed field.

(a) If K is a function field of dimension 1 over k(I, §6), then every valuation ring of K/k (except for K itself) is discrete. Thus the set of all of them is just the abstract nonsingular curve C_K of (I, §6).

(b) If K/k is a function field of dimension two, there are several different kinds of valuations. Suppose that X is a complete nonsingular surface with function field K.

(1) If Y is an irreducible curve on X, with generic point x_1, then the local ring $R = \mathcal{O}_{x_1,X}$ is a discrete valuation ring of K/k with center at the (nonclosed) point x_1 on X.
(2) If $f : X' \to X$ is a birational morphism, and if Y' is an irreducible curve in X' whose image in X is a single closed point x_0, then the local ring R of the generic point of Y' on X' is a discrete valuation ring of K/k with center at the closed point x_0 on X.
(3) Let $x_0 \in X$ be a closed point. Let $f : X_1 \to X$ be the blowing-up of x_0 (I, §4) and let $E_1 = f^{-1}(x_0)$ be the exceptional curve. Choose a closed point $x_1 \in E_1$, let $f_2 : X_2 \to X_1$ be the blowing-up of x_1, and let $E_2 = f_2^{-1}(x_1)$ be the exceptional curve. Repeat. In this manner we obtain a sequence of varieties X_i with closed points x_i chosen on them, and for each i, the local ring $\mathcal{U}_{x_{i+1},X_{i+1}}$ dominates $\mathcal{O}_{x_i,X_i}$. Let $R_0 = \bigcup_{i=0}^{\infty} \mathcal{O}_{x_i,X_i}$. Then R_0 is a local ring, so it is dominated by some valuation ring R of K/k by (I, 6.1A). Show that R is a valuation ring of K/k, and that it has center x_0 on X. When is R a discrete valuation ring?

Note. We will see later (V, Ex. 5.6) that in fact the R_0 of (3) is already a valuation ring itself, so $R_0 = R$. Furthermore, every valuation ring of K/k (except for K itself) is one of the three kinds just described.

Proof. [Incomplete] □

5. Sheaves of Modules

Hartshorne II.5.1 Let $(X, \mathcal{O}_x)$ be a ringed space, and let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the dual of $\check{\mathscr{E}}$, denoted $\mathscr{E}$, to be the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)$.

(a) Show that $(\check{\mathscr{E}}) \cong \mathscr{E}$.
(b) For any $\mathcal{O}_X$-module $\mathscr{F}$, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \cong \check{\mathscr{E}} \otimes_{\mathcal{O}_X} \mathscr{F}$.
(c) For any $\mathcal{O}_X$-modules $\mathscr{F}, \mathscr{G}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))$.
(d) (Projection Formula). If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathscr{F}$ is an $\mathcal{O}_X$-module, and if $\mathscr{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{E}) \cong f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E}$.

Proof. We begin by proving two lemmas which we shall use repeatedly. The second lemma is a generalization of the first.

Lemma 5.1. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. The map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathscr{F}) \to \Gamma(X, \mathscr{F})$$

defined by mapping $f : \mathcal{O}_X \to \mathscr{F}$ to $f(X)(1) \in \mathscr{F}(X)$ is an isomorphism of $\Gamma(X, \mathcal{O}_X)$-modules. It is also functorial in $\mathscr{F}$.

Proof. Note that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathscr{F})$ is a $\Gamma(X, \mathcal{O}_X)$-module because if $u \in \Gamma(X, \mathcal{O}_X)$ and $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathscr{F})$, we define $u \cdot \phi(V) := u|_V \cdot \phi(V) : \mathcal{O}_X(V) \to \mathscr{F}(V)$.

The map defined is clearly $\Gamma(X, \mathcal{O}_X)$-linear and functorial in $\mathscr{F}$.

If $s \in \Gamma(X, \mathscr{F})$, there is a unique morphism $w : \mathcal{O}_X \to \mathscr{F}$ s.t. $w_U(1) = s|_U \in \Gamma(U, \mathscr{F})$ for all open sets $U \subseteq X$. This is because $1|_U$ generates all of $\mathcal{O}_X(U)$. □

Lemma 5.2. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. Then the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathscr{F}) \to \Gamma(X, \mathscr{F})^n$$

defined by mapping $f : \mathcal{O}_X^n, \mathscr{F}$ to $f(X)(1) \in \mathscr{F}(X)$ is an isomorphism.

PROOF. We omit the checks from the previous proof. Let $(s_i)_{i=1}^n \in \Gamma(X, \mathscr{F})^n$. For each s_i, there is a unique morphism $w_i : \mathcal{O}_X \to \mathscr{F}$ s.t. $w_U(1) = s_i|_U$ for all open sets $U \subseteq X$. These then defines a unique morphism $\mathcal{O}_X^n \to \mathscr{F}$ which sends $(1_i)_U \to s_1|_U + s_2|_U + \cdots + s_n|_U$ for all $U \subseteq X$ open. □

(a) Let U be an open set in which $\mathscr{E}|_U \cong \mathcal{O}_X|_U^n$. We will define an isomorphism of sheaves

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X), \mathcal{O}_X)|_U \xrightarrow{\simeq} \mathcal{O}_X^n|_U.$$

It suffices to construct an isomorphism

$$\mathscr{H}om_{\mathcal{O}_X|_V}(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)|_V, \mathcal{O}_X|_V) \xrightarrow{\simeq} \mathcal{O}_X(V)^n$$

for every open subset $V \subseteq U$. Suppose we are given a $\phi : \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)|_V \to \mathcal{O}_X|_V$ which is a morphism of $\mathcal{O}_{X|V}$-modules. Let $W \subseteq V$ be an open set. Then $\phi(W) : \mathrm{Hom}_{\mathcal{O}_X|_W}(\mathcal{O}_X^n|_W, \mathcal{O}_X|_W) \to \mathcal{O}_X(W)$ is a morphism of $\mathcal{O}_X(W)$-modules. Using the second lemma, we may identify $\phi(W)$ as a morphism $\phi(W) : \mathcal{O}_X(W)^n \to \mathcal{O}_X(W)$ and such a morphism is determined entirely by where it maps the set $(1_W)_i$. It then follows that ϕ as a morphism is determined uniquely by a choice of elements $(1_V)_i \in \mathcal{O}_X(V)^n$. So we get the displayed isomorphism above.

With the isomorphism defined over each open set U and the fact that the isomorphisms are canonical, obtain an isomorphism of sheaves globally. Therefore, $(\check{\check{\mathscr{E}}}) \cong \mathscr{E}$.

(b) The idea of proof is essentially the same as that in (a). We define maps

$$\mathrm{Hom}_{\mathcal{O}_X|_V}(\mathcal{O}_X^n|_V, \mathscr{F}|_V) \to \mathrm{Hom}_{\mathcal{O}_X|_V}(\mathcal{O}_X^n|_V, \mathcal{O}_X|_V) \otimes_{\mathcal{O}_X(V)} \mathscr{F}(V) \tag{3}$$

for each $V \subseteq U$ open. From the second lemma, the RHS is isomorphic to

$$\Gamma(V, \mathcal{O}_X|_V)^n \otimes_{\mathcal{O}_X(V)} \mathscr{F}(V) \cong \mathcal{O}_X(V)^n \otimes_{\mathcal{O}_X(V)} \mathscr{F}(V) \cong \mathscr{F}(V)^n.$$

But the LHS is also isomorphic to $\mathscr{F}(V)^n$ by the second lemma. So we have an obvious canonical choice for the maps in (3). This gives the isomorphism of sheaves restricted to U and so we obtain an isomorphism of sheaves globally. That is, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \cong \check{\mathscr{E}} \otimes_{\mathcal{O}_X} \mathscr{F}$.

(c) We construct the isomorphism of $\Gamma(X, \mathcal{O}_X)$-modules. For a morphism $\phi : \mathscr{E} \otimes \mathscr{F} \to \mathscr{G}$, we have morphisms of $\mathcal{O}_X(V)$-modules given by $\phi(V) : \mathscr{E}(V) \otimes_{\mathcal{O}_X(V)} \mathscr{F}(V) \to \mathscr{G}(V)$. By Hom-tensor adjunction of modules, there is a unique morphism $\psi(V) : \mathscr{F}(V) \to \mathrm{Hom}_{\mathcal{O}_X(V)}(\mathscr{E}(V), \mathscr{G}(V))$.

We claim that

$$\mathrm{Hom}_{\mathcal{O}_X(V)}(\mathscr{E}(V), \mathscr{G}(V)) \cong \mathscr{H}om_{\mathcal{O}_X|_V}(\mathscr{E}|_V, \mathscr{G}|_V)$$

in this case. Indeed, any morphism on the RHS determines an element on the LHS by taking global sections. OTOH, let $w : \mathscr{E}(V) = \mathcal{O}_X^n(V) \to \mathscr{G}(V)$ be an element on the LHS. The morphism is determined exactly by where it maps the elements $(1_V)_{i=1}^n$. By the same argument as in Lemma 2, this means w uniquely determines a morphism $\mathcal{O}_X^n|_V \to \mathscr{G}|_V$. It is clear this process gives a $\mathcal{O}_X(V)$-module isomorphism. So $\psi(V)$ can be identified with a map $\mathscr{F}(V) \to \mathscr{H}om_{\mathcal{O}_X|_V}(\mathscr{E}|_V, \mathscr{G}|_V)$.

Observe that the above process preserves the $\Gamma(X, \mathcal{O}_X)$-module structure of $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G})$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G})$. So we have the desired isomorphism.

(d) WLOG as f_*, f^* and tensor products commute with finite direct sums, we assume WLOG that $\mathscr{E} \cong \mathcal{O}_Y$ locally (as the question itself is a local one with checks on intersections of the trivializations). It then follows that there are isomorphisms

$$f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E} \cong f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \cong f_*(\mathscr{F}) \cong f_*(\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X) \cong f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y) \cong f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{E}).$$

Note that we used $f^*\mathcal{O}_Y \cong f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X$. Globally, the trivialization glue together to give the indicated isomorphism (as all of our maps are canonical). □

Hartshorne II.5.2 Let R be a discrete valuation ring with quotient field K, and let $X = \operatorname{Spec} R$.

(a) To give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R K \to L$.

(b) That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.

Proof. **(a)** ($\Rightarrow$) : Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. Let $X := \{\xi, x\}$ be the two points of X where ξ is the generic point and x is the maximal ideal. Since $\mathscr{F}$ is an $\mathcal{O}_X$-module, there is a $\mathcal{O}_X(D(\xi))$-module homomorphism $\mathscr{F}(X) \to \mathscr{F}(D(\xi))$. The $\mathcal{O}_X$-module structure is determined by the following diagram

$$\begin{array}{ccc}
\mathcal{O}_X(X) \times \mathscr{F}(X) & \longrightarrow & \mathscr{F}(X) \\
\downarrow{\scriptstyle \rho_{X\xi}\times \mathrm{id}} & & \downarrow \\
\mathcal{O}_X(\xi) \times \mathscr{F}(X) & \xrightarrow{\phi} & \mathscr{F}(\xi) \\
\downarrow{\scriptstyle \rho_{X\xi}\times \mathrm{id}} & & \downarrow \\
\mathcal{O}_X(\xi) \times \mathscr{F}(\xi) & \longrightarrow & \mathscr{F}(\xi)
\end{array}$$

where $\rho_{X\xi}$ is the restriction map from X to $\{\xi\}$. The center map called ϕ is given by $\phi(a, s) = a \cdot \rho_{X\xi}(s)$ and this determines the $\mathcal{O}_X$-module structure once we know $\mathscr{F}(X)$ is an R-module.

This is equivalent to giving a homomorphism[34]

$$\mathscr{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(\xi) \to \mathscr{F}(\xi).$$

Of course, note that $\mathcal{O}_X(\xi) \cong \mathcal{O}_{X,\xi}$ and $\mathscr{F}(\xi) \cong \mathscr{F}_\xi$ because in the colimit for the stalk at ξ there is only the open set $\{\xi\}$. It follows $\mathcal{O}_X(\xi) = K$ and $\mathscr{F}(\xi)$ is a K-vector space.

We take $M := \mathscr{F}(X)$ and $\mathcal{O}_X(\xi) = K$ and $\mathscr{F}(\xi) = L$. In this case, we obtain the homomorphism $\rho : M \otimes_R K \to L$.

($\Leftarrow$): Let $\rho : M \otimes_R K \to L$ be given. Then set $R = \mathcal{O}_X(X)$, $K = \mathcal{O}_X(\xi) = \mathcal{O}_{X,\xi}$, $L = \mathscr{F}_\xi = \mathscr{F}(\xi)$, and $M = \mathscr{F}(X)$.

The map on top is specified by $M = \mathscr{F}(X)$ being an R-module. The map on the bottom is determined by L being a K-vector space. The map in the center is then determined by $\rho : M \otimes_R K \to L$ which equivalently specifies an R-bilinear map $M \times K \to L$ and hence, a map $\mathcal{O}_X(\xi) \times \mathscr{F}(X) \to \mathscr{F}(\xi)$ that is $\mathcal{O}_X(X)$-bilinear.

[34]This is equivalent is given as follows. An A-module homomorphism $\phi : M \to N$ is abelian group homomorphism satisfying a set of axioms. These axioms imply that $M \times A \to N$ given by $(m, a) \mapsto \phi(am)$ is an A-bilinear map. By the universal property of the tensor product, this is then determined by a tensor product map $M \otimes A \to N$.

(b) By Hartshorne Corollary II.5.5, $\mathscr{F}$ is quasi-coherent iff $\mathscr{F} \cong \widetilde{M}$ for an R-module M. This is iff $\mathcal{O}_X(\xi) \otimes_{\mathcal{O}_X(X)} \mathscr{F}(X) \to \mathscr{F}(\xi)$ is the map $K \otimes_R M \to M_\xi$ and this is equivalent to the map $R_\xi \otimes_R M \to M_\xi$ which is the isomorphism of Proposition 3.5 of Atiyah-Macdonald. □

Hartshorne II.5.3 Let $X = \operatorname{Spec} A$ be an affine scheme. Show that the functors $\sim$ and Γ are adjoint, in the following sense: for any A-module M, and for any sheaf of $\mathcal{O}_X$ -modules $\mathscr{F}$, there is a natural isomorphism

$$\operatorname{Hom}_A(M, \Gamma(X, \mathscr{F})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathscr{F})$$

Proof. Given $\varphi : M \to \Gamma(X, \mathscr{F})$. Define $\varphi_f : M_f \to \mathscr{F}(D(f))$ by $\frac{m}{f^n} \mapsto \frac{1}{f^n}\rho_{X,D(f)}(\varphi(m))$ where $\rho_{X,D(f)} : \mathscr{F}(X) \to \mathscr{F}(D(f))$ is the restriction map. From Hartshorne Proposition II.5.1, $\widetilde{M}(D(f)) = M_f$ so we have a map $\varphi_f : \widetilde{M}(D(f)) \to \mathscr{F}(D(f))$. The maps φ_f agree on overlaps; suppose $\frac{m}{f^n} = \frac{m'}{g^k} \in M_{fg}$,

$$\frac{m}{f^n} \overset{\varphi_f}{\to} \frac{\varphi(m)}{f^n} = \frac{\varphi\left(\frac{f^n}{g^k}m'\right)}{f^n} = \frac{\varphi(m')}{g^k} \overset{\varphi_g}{\leftarrow} \frac{m}{g^k}$$

Therefore, by Hartshorne II.1.22, we can glue to a morphism of sheaves $\widetilde{M} \to \mathscr{F}$.

Given a morphism of $\mathcal{O}_X$-modules $\theta : \widetilde{M} \to \mathscr{F}$, we obtain a morphism $\theta(X) : \widetilde{M}(X) \to \mathscr{F}(X)$ and this is just an A-module homomorphism $M \to \Gamma(X, \mathscr{F})$.

To check that we have an isomorphism, observe that the two constructions are inverse to each other; both are determined by morphisms on distinguished open sets $D(f)$. □

Hartshorne II.5.4 Show that a sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ on a scheme X is quasi-coherent if and only if every point of X has a neighborhood U, such that $\mathscr{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves on U. If X is noetherian, then $\mathscr{F}$ is coherent if and only if it is locally a cokernel of a morphism of free sheaves of finite rank. (These properties were originally the definition of quasi-coherent and coherent sheaves.)

Proof. *Part 1.* ($\Rightarrow$): Suppose $\mathscr{F}$ is quasicoherent. So, $\mathscr{F}|_U \cong \widetilde{M}$ for some A-module M where $U = \operatorname{Spec} A$. Since A modules M have a presentation $0 \to \bigoplus_{i\in I} \mathcal{O}_X(U) \to \bigoplus_{j\in J} \mathcal{O}_X(U) \to M \to 0$, and $\sim$ is an exact functor, the exact sequence

$$0 \to \bigoplus_{i\in I} \mathcal{O}_X|_U \overset{\varphi}{\to} \bigoplus_{j\in J} \mathcal{O}_X|_U \to \widetilde{M} \to 0$$

implies $\mathscr{F}|_U \cong \widetilde{M} \cong \operatorname{coker}\varphi$ under the correspondence of Hartshorne II.5.3.

($\Leftarrow$): Given $x \in X$ and $U \ni x$ s.t. $\mathscr{F}|_U \cong \operatorname{coker}\varphi$ where

$$0 \to \bigoplus_{i\in I} \mathcal{O}_X|_U \overset{\varphi}{\to} \bigoplus_{j\in J} \mathcal{O}_X|_U \to \mathscr{F}|_U \to 0.$$

By shrinking U, we may assume $U = \operatorname{Spec} A$ and that we have a SES

$$0 \to \bigoplus_{i\in I} \widetilde{A} \to \bigoplus_{j\in J} \widetilde{A} \to \mathscr{F}|_{\operatorname{Spec} A} \to 0.$$

Now apply Hartshorne Proposition II.5.6 to take gloabl sections and deduce that we have a SES

$$0 \to \bigoplus_{i\in I} A \xrightarrow{\phi} \bigoplus_{j\in J} A \to \Gamma(\operatorname{Spec} A, \mathscr{F}|_{\operatorname{Spec} A}) \to 0.$$

It follows that $\mathscr{F}|_{\operatorname{Spec} A} = \widetilde{\frac{\bigoplus_{j\in J} A}{\bigoplus_{i\in I} A}}$. As x and U were arbitrary, Hartshorne Proposition II.5.4 finishes the proof.

Part 2. Repeat the proof in Part 1 except we use the fact that a Noetherian A-module M always sits inside a SES

$$0 \to A^m \to A^n \to M \to 0$$

for $m, n \in \mathbb{N}_0$. □

Hartshorne II.5.5 Let $f : X \to Y$ be a morphism of schemes.

(a) Show by example that if $\mathscr{F}$ is coherent on X, then $f_*\mathscr{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.

(b) Show that a closed immersion is a finite morphism (§3).

(c) If f is a finite morphism of noetherian schemes, and if $\mathscr{F}$ is coherent on X, then $f_*\mathscr{F}$ is coherent on Y.

Proof. (a) Define $f : X \to Y$ where $X = \operatorname{Spec} k[x]$ and $Y = \operatorname{Spec} k$ induced from $\varphi : k \hookrightarrow k[x]$. Clearly, $\mathcal{O}_X$ is coherent on X. However, $f_*\mathcal{O}_Y$ is not coherent since

$$f_*\mathcal{O}_X(\operatorname{Spec} k[x]_{(x)}) = \mathcal{O}_X(f^{-1}(\operatorname{Spec} k[x]_{(x)}) \cong \mathcal{O}_X(X) = k$$

which is clearly not coherent since k is not finitely generated $k[x]_{(x)}$-module.

(b) Let $f : X \to Y$ be a closed immersion. Replace Y by the closed subset $f(X)$ and give it the induced reduced scheme structure. By Hartshorne II.3.4, replace Y by $V := \operatorname{Spec} B$ for some affine open subset $\operatorname{Spec} B \subseteq Y$ and then restrict $f|_{f^{-1}(V)} : f^{-1}(V) \to V$. We show $f^{-1}(V)$ is affine and equal to $\operatorname{Spec} A$ where A is a finite B-module. Since $f^{-1}(V)$ can be regarded as a closed subscheme of V, Hartshorne Corollary II.5.4 gives $f^{-1}(V) = \operatorname{Spec}(B/\mathfrak{a})$ for some ideal $\mathfrak{a}$ and $B/\mathfrak{a}$ is certainly a f.g. B-module.

(c) Since f is finite, $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ where A is a f.g. B-module for any affine open subset $\operatorname{Spec} B \subseteq Y$. Then, $f_*\mathscr{F}|_{\operatorname{Spec} B} = \mathscr{F}(f^{-1}(\operatorname{Spec} B)) = \mathscr{F}(\operatorname{Spec} A) = \widetilde{N}$ where N is an A-module. Cover $\operatorname{Spec} B$ by finitely many $\operatorname{Spec} B_{b_i}$ and the b_i's generate the unit ideal. So, $\mathscr{F}(f^{-1}(\operatorname{Spec} B_{b_i})) = \mathscr{F}(\operatorname{Spec} A_{b_i}) = \widetilde{N_{b_i}}$ where each N_{b_i} is a f.g. A_{b_i}-module. Since each N_{b_i} is a f.g. A_{b_i}-module, we deduce that N is a f.g. A-module[35]. Since A was a f.g. B-module, N is a f.g. B-module as well and $f_*\mathscr{F}$ is coherent.

□

Hartshorne II.5.6 *Support.* Recall the notions of support of a section of a sheaf, support of a sheaf, and subsheaf with supports from (Ex. 1.14) and (Ex. 1.20).

(a) Let A be a ring, let M be an A-module, let $X = \operatorname{Spec} A$, and let $\mathscr{F} = \widetilde{M}$. For any $m \in M = \Gamma(X, \mathscr{F})$, show that $\operatorname{Supp} m = V(\operatorname{Ann} m)$, where $\operatorname{Ann} m$ is the annihilator of $m = \{a \in A \mid am = 0\}$.

[35]See this Stacks Project page.

(b) Now suppose that A is noetherian, and M finitely generated. Show that $\operatorname{Supp} \mathscr{F} = V(\operatorname{Ann} M)$.

(c) The support of a coherent sheaf on a noetherian scheme is closed.

(d) For any ideal $\mathfrak{a} \subseteq A$, we define a submodule $\Gamma_{\mathfrak{a}}(M)$ of M by $\Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$. Assume that A is noetherian, and M any A-module. Show that $\Gamma_{\mathfrak{a}}(M)^{\sim} \cong \mathscr{H}^0_Z(\mathscr{F})$, where $Z = V(\mathfrak{a})$ and $\mathscr{F} = \widetilde{M}$. [Hint: Use (Ex. 1.20) and (5.8) to show a priori that $\mathscr{H}^0_Z(\mathscr{F})$ is quasi-coherent. Then show that $\Gamma_{\mathfrak{a}}(M) \cong \Gamma_Z(\mathscr{F})$.]

(e) Let X be a noetherian scheme, and let Z be a closed subset. If $\mathscr{F}$ is a quasicoherent (respectively, coherent) $\mathcal{O}_X$-module, then $\mathscr{H}^0_Z(\mathscr{F})$ is also quasicoherent (respectively, coherent).

PROOF. (a) I claim that $\operatorname{Supp} m = V(\operatorname{Ann} m)$. Indeed, if $P \in \operatorname{Supp} m$, I know that $m_P \neq 0$. This implies that $\mathscr{F}_P$ must not have inverted $\operatorname{Ann} m$. That is, $\operatorname{Ann} m \subseteq P$. For the converse, if $P \in V(\operatorname{Ann} m)$, then $m_P \neq 0$ because we do not invert the annihilator of m.

(b) Assume A is noetherian and M is finitely generated. I claim that $\operatorname{Supp} \mathscr{F} = V(\operatorname{Ann} M)$.

Suppose $P \in \operatorname{Supp} \mathscr{F}$. Then $\mathscr{F}_P \neq 0$. There is at least one $m \in M$ s.t. $m_P \neq 0$. So, $\mathscr{F}_P$ should not have $\operatorname{Ann} M$ inverted. That is, $\operatorname{Ann} M \subseteq P$.

For the converse, suppose $P \in V(\operatorname{Ann} M)$. Because A is noetherian and M is finitely generated, $\mathscr{F}$ is coherent. So, it suffices to show that $M_P \neq 0$. Since $P \in V(\operatorname{Ann} M)$, $M_P = M \otimes_A A_P$. Since we do not invert P, $M_P \neq 0$.

[Incomplete]

(c) I claim that the support of a coherent sheaf on a noetherian scheme is closed. Suppose $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module where X is a noetherian scheme. Consider $\operatorname{Supp} \mathscr{F}$. Then, it is equal to $V(\operatorname{Ann}(M))$ by (b) since we can reduce to an open affine subset $U = \operatorname{Spec} A \subseteq X$. Now, $V(\operatorname{Ann} M)$ is closed in X. So, $\operatorname{Supp} \mathscr{F}$ is closed.

(d) **[Incomplete]**

(e) **[Incomplete]**

□

Hartshorne II.5.7 Let X be a noetherian scheme, and let $\mathscr{F}$ be a coherent sheaf.

(a) If the stalk $\mathscr{F}_x$ is a free $\mathcal{O}_x$ -module for some point $x \in X$, then there is a neighborhood U of x such that $\mathscr{F}|_U$ is free.

(b) $\mathscr{F}$ is locally free if and only if its stalks $\mathscr{F}_x$ are free $\mathcal{O}_x$ -modules for all $x \in X$.[36]

(c) $\mathscr{F}$ is invertible (i.e., locally free of rank 1) if and only if there is a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \cong \mathcal{O}_X$. (This justifies the terminology invertible: it means that $\mathscr{F}$ is an invertible element of the monoid of coherent sheaves under the operation $\otimes$.)

PROOF. *(a)* We may immediately reduce to the affine case $X = \operatorname{Spec} A$ and assume $\mathscr{F} \cong \widetilde{M}$ for M a f.g. A-module $M = (x_1, x_2, \ldots, x_n)A$. Next, $\mathscr{F}_x := M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module and

[36] As per Professor Izadi's comments, there is a stronger result. A sheaf $\mathscr{F}$ is locally free of rank n iff each stalk $\mathscr{F}_x$ are free $\mathcal{O}_x$-modules of rank n as well.

choose basis $m_1, m_2, \ldots, m_\ell \in M_{\mathfrak{p}}$. We may assume WLOG that $m_1, \ldots, m_\ell \in M$ by clearing denominators. Then,

$$\frac{x_i}{1} = \sum_{j=1}^{\ell} \frac{a_{ij}}{b_{ij}} m_{ij} \text{ for all } 1 \leq i \leq n. \tag{4}$$

Set $b := \prod_{1 \leq i \leq n, 1 \leq j \leq \ell} b_{ij}$. Next, define a map

$$\bigoplus_{i=1}^{\ell} A_b \to M_b \qquad \text{by sending } (a_1, \ldots, a_\ell) \mapsto a_1 m_1 + a_2 m_2 + \cdots + a_\ell m_\ell$$

and this is a surjective map because we can always generate the x_i by (4). There is a SES

$$0 \to N \to \bigoplus_{i=1}^{\ell} A_b \to M_b \to 0 \tag{5}$$

and since N is necessarily f.g., we may choose generators generators $n_1, n_2, \ldots, n_k \in N \subseteq \bigoplus_{i=1}^{\ell} A_b$. Since they lay in the kernel of the map, there are $s_1, s_2, \ldots, s_k \in A_b$ s.t. $s_1 n_1 = s_2 n_2 = \cdots = s_k n_k = 0$. now take $c = b \times \prod_{i=1}^{k} s_i$. Localizing (5),

$$0 \to N_c = 0 \to \bigoplus_{i=1}^{\ell} A_c \to M_c \to 0.$$

So $M_c \cong \bigoplus_{i=1}^{\ell} A_c$ is a free A_c-module. It follows that we should take U to be $D(c)$.

(a) Version 2 Let us describe the method of proof: one first reduces to the affine case and then constructs a localization of the module that is free a module with ring the corresponding localization of the free.

Choose an open affine $\operatorname{Spec} A \ni x$ to reduce the problem to $\mathscr{F}_x \cong M_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec} A$ and $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$. Assume $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module.

Choose generators $v_1, \ldots, v_t$ of $M_{\mathfrak{p}}$ as a free $A_{\mathfrak{p}}$-module. Since $v_i \in M_{\mathfrak{p}}$, we write $v_i = \frac{m_i}{r_i}$ where $m_i \in M$ and $r_i \in A \setminus \mathfrak{p}$. Then elements of $M_{\mathfrak{p}}$ are finite linear combinations

$$\sum_{i=1}^{t} \frac{a_i}{s_i} v_i \qquad \text{where} \qquad \frac{a_i}{s_i} \in A_{\mathfrak{p}} \qquad \& \qquad M_{\mathfrak{p}} = \bigoplus_{i=1}^{t} A_{\mathfrak{p}} v_i = \bigoplus_{i=1}^{t} A_{\mathfrak{p}} \frac{m_i}{r_i}. \tag{6}$$

Next, we show that $v_1, \ldots, v_t$ generate $M_{r_1 r_2 \ldots r_t}$. We can clear denominators and reduce to showing that the m_i generate. Since M is a f.g. module, we can assume that the $m_1, \ldots, m_r$ generated the image of generators of M inside $M_{\mathfrak{p}}$. This establishes that $v_1, \ldots, v_t$ are generators of $M_{r_1 \ldots r_t}$.

Fix $s = r_1 \ldots r_t$. From this, there is an exact sequence of A_s-modules

$$0 \to N \hookrightarrow \bigoplus_{i=1}^{t} A v_i \twoheadrightarrow M_s \to 0 \tag{7}$$

where $\bigoplus_{i=1}^{t} A v_i \twoheadrightarrow M_s$ maps $v_i \to v_i$ and N is the kernel. Since we assumed coherent and noetherian, N is a f.g. A-module with generators say $w_1, w_2, \ldots, w_l$. Localizing at $\mathfrak{p}$ gives a

SES

$$0 \to N_{\mathfrak{p}} \hookrightarrow \bigoplus_{i=1}^{t} A_{\mathfrak{p}} v_i \twoheadrightarrow (M_{\mathfrak{p}})_s \to 0.$$

Notice from (6) that $(M_{\mathfrak{p}})_s = M_{\mathfrak{p}}$ so the map is an isomorphism. So $N_{\mathfrak{p}} = 0$ and we have $\frac{w_i}{1} = 0$ for all i. Choose $a_i \in A$ s.t. $a_i w_i = 0$. Localizing (7) at the a_i,

$$0 \to N_{a_1 \dots a_l} \to \bigoplus_{i=1}^{t} A_{a_1 \dots a_l} v_i \to M_{s a_1 \dots a_l} \to 0.$$

and clearly $N_{a_1 \dots a_l} = 0$. Hence, $\bigoplus_{i=1}^{t} A_{a_1 \dots a_l} v_i \cong M_{s a_1 \dots a_l}$ which means

$$\mathscr{F}(\operatorname{Spec} A_{s a_1 \dots a_t}) \cong M_{s a_1 \dots a_l} \cong \bigoplus_{i=1}^{t} A_{a_1 \dots a_l} v_i \cong \bigoplus_{i=1}^{t} A_{s a_1 \dots a_l} v_i.$$

is free. It follows that $\mathscr{F}|_U$ is a free $\mathcal{O}_{\operatorname{Spec} A}|_U$-module when we take $U = D(s a_1 a_2 \dots a_l)$.

(b) If all of its stalks $\mathscr{F}_x$ are free $\mathcal{O}_x$-modules, part (a) implies there is a covering of X by open sets $U_x \ni x$ s.t. $\mathscr{F}|_U$ is free. It follows $\mathscr{F}$ is locally free.

Conversely, if $\mathscr{F}$ is locally free, then there cover X by open affines $\{U_i = \operatorname{Spec} A_i\}_{i \in I}$ s.t. $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$ for M_i a free A_i-module. Taking stalks at a point $\mathfrak{p} \in U_i$, $\mathscr{F}_{\mathfrak{p}} \cong \widetilde{M_i}_{\mathfrak{p}} \cong (M_i)_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module.

(c) ($\implies$): If $\mathscr{F}$ is invertible, then the desired coherent sheaf $\mathscr{G}$ is just the dual sheaf $\mathscr{F}^{\smile}$ and the isomorphism follows from Hartshorne II.5.2.
($\impliedby$): Assume $\mathscr{G}$ coherent and $\mathscr{G}|_U \cong \widetilde{N}$ and $\mathscr{F}|_U \cong \widetilde{M}$. Because $\sim$ is well-behaved w.r.t tensor products, we know $\widetilde{M} \otimes \widetilde{N} \cong \widetilde{M \otimes N} \cong \mathcal{O}_X|_U$. Assume $U = \operatorname{Spec} A$ is an open affine and $M \otimes_A N \cong A$. Consider the localization at some $\mathfrak{p} \in \operatorname{Spec} A$ which we identify $\mathfrak{p}$ with x. Then $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ and taking the quotient by $\mathfrak{p}A_{\mathfrak{p}}$, $\frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}} \otimes \frac{N_{\mathfrak{p}}}{\mathfrak{p}N_{\mathfrak{p}}} \cong \frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}}$ is an isomorphism of vector spaces. The dimension of the tensor product of vector spaces is the product of dimensions and so, $\frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}}$ and $\frac{N_{\mathfrak{p}}}{\mathfrak{p}N_{\mathfrak{p}}}$ are a one-dimensional vector spaces. Therefore, if $\overline{m}$ is a generator of $\frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}}$ as a $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ vector space, the lift of m to $M_{\mathfrak{p}}$ generates $M_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$-module. Therefore, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module (see the lemma below). Therefore, $\mathscr{F}_x \cong \mathcal{O}_{X,x}$ and by part (b) and x being arbitrary, $\mathscr{F}$ is locally free of rank 1.

Lemma 5.3. In the context above, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module by lifting the basis element of $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ which is a 1-dimensional $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$-vector space.

Proof. Let x be the lifted generator and $\overline{x}$ be the basis element. Suppose it were not free. Then there is an $\frac{a}{s} \in A_{\mathfrak{p}}$ s.t. $\frac{a}{s}x = 0$ where $\frac{a}{s}$ is nonzero. Multiply by s to get $ax = 0$. If a is not in $\mathfrak{p}$, then a is a unit in $A_{\mathfrak{p}}$ which then means $x = \frac{1}{a}ax = 0$. This contradicts the fact that $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is 1-dimensional.

So assume $a \in \mathfrak{p}A_{\mathfrak{p}}$ is a nonzero element s.t. $ax = 0$. From $\mathscr{F} \otimes \mathscr{G} \cong \mathcal{O}_X$, we know

$$M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \cong A_{\mathfrak{p}} \implies a(xA_{\mathfrak{p}} \otimes_{A\mathfrak{p}} N_{\mathfrak{p}}) \cong aA_{\mathfrak{p}} \implies axA_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \cong aA_{\mathfrak{p}} \implies 0 \otimes N_{\mathfrak{p}} \cong aA_{\mathfrak{p}} \implies 0 \cong aA_{\mathfrak{p}}.$$

However, this means $a = 0$ leading to a contradiction. □

□

Remark 26. My solution to Hartshorne II.5.7c looks correct to me, but here is a solution that is presented in [**cais**].

The forward direction is as follows. If $\mathscr{F}$ is locally free of rank 1, then $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{F}^\smile \cong \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})$. Explicitly showing that $\mathscr{H}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F}) \cong \mathcal{O}_X$ is somewhat involved. Take a cover of X by open affines U_i. Then $\mathscr{F}|_{U_i} \cong \mathcal{O}_X$. So, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})(U_i) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X|_{U_i}, \mathcal{O}_X|_{U_i})$. Define $\varphi : \mathcal{O}_X|_{U_i} \to \mathcal{O}_X|_{U_i}$ in the obvious way. This gives an isomorphism of $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F}) \cong \mathcal{O}_X$.

For the converse, suppose such a $\mathscr{G}$ exists. By Hartshorne II.5.7b, it suffices to check the stalk. Using coherence, there is an isomorphism $M \otimes_A N \cong A$. It then suffices to show that $M \cong A$ and $N \cong A$ when A is a local ring. Indeed,

$$M/PM \otimes_k N/PM \cong (M \otimes_A N) \otimes_A k \cong k$$

where $k = A/P$ where P is the unique maximal ideal of A. So, M/PM is of rank 1. By Nakayama's Lemma, M is of rank 1. In particular, it is free of rank 1.

Hartshorne II.5.8 Again let X be a noetherian scheme, and $\mathscr{F}$ a coherent sheaf on X. We will consider the function

$$\varphi(x) = \dim_{k(x)} \mathscr{F}_x \otimes_{\mathcal{O}_x} k(x)$$

where $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following results.

(a) The function φ is upper semi-continuous, i.e., for any $n \in \mathbb{Z}$, the set $\{x \in X \mid \varphi(x) \geq n\}$ is closed.

(b) If $\mathscr{F}$ is locally free, and X is connected, then φ is a constant function.

(c) Conversely, if X is reduced, and φ is constant, then $\mathscr{F}$ is locally free.

Proof. We will make use of Nakayama's Lemma and its corollaries[37].

(a) It suffices to show $\{x \in X \mid \varphi(x) < n\}$ is open. Let $x \in \{x \in X \mid \varphi(x) < n\}$ and choose an open affine $\operatorname{Spec} A \ni x$ s.t. x corresponds to a $\mathfrak{p} \in \operatorname{Spec} A$. Since $\mathscr{F}$ is coherent, let $\mathscr{F}|_{\operatorname{Spec} A} = \widetilde{M}$ and assume M is generated by $m_1, \ldots, m_l$ as an A-module. Furthermore,

$$\varphi(x) = \dim_{k(x)} \mathscr{F}_x \otimes_{\mathcal{O}_x} k(x) = \dim_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}} M_\mathfrak{p} \otimes_{A_\mathfrak{p}} (A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) = \dim_{k(x)} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}.$$

Clearly, $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ is a vector space over $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ and since M was finitely-generated, this is a finite dimensional vector space. Choose a basis $v_1, v_2, \ldots, v_k$ and by $\varphi(x) < n$, we know $k < n$.

By Nakayama's Lemma, these v_i's lift to a generating set $w_1, \ldots, w_k$ of $M_\mathfrak{p}$. For each $m_j \in M_\mathfrak{p}$, we can write

$$m_j = \frac{m_j}{1} = \sum_{1 \leq i \leq k} \frac{a_{ij}}{f_{ij}} w_i, \qquad \text{where } \frac{a_{ij}}{f_{ij}} \in A_\mathfrak{p} \text{ for all } 1 \leq i \leq k \text{ and } 1 \leq j \leq l.$$

Set $D(f) := D\left(\prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}} f_{ij}\right)$. Clearly, $x = \mathfrak{p} \in D(f)$ since each $f_{ij} \in A \setminus \mathfrak{p}$. We show $D(f) \subseteq \{x \in X \mid \varphi(x) < n\}$ which shows each point of the RHS set is an interior point.

[37] See Chapter 2 of [2] or [5] In particular, we use the followings facts: (1) if $\{x_i\}_{1 \leq i \leq n} \in M$ are elements of a f.g. M over a local ring $(A, \mathfrak{m})$, s.t. $\overline{x_i} \in M/\mathfrak{m}M$ form a basis, then $M = (x_1, x_2, \ldots, x_n)A$ and (2) if M is a f.g. A-module, $N \subseteq M$ a submodule, $\mathfrak{a}$ is an ideal inside the Jacobson radical, then $M = \mathfrak{a}M + N \Rightarrow M = N$.

Let $\mathfrak{q} \in D(f)$. Then $M_{\mathfrak{q}}$ is generated by the elements

$$m_j = \frac{m_j}{1} = \sum_{i,j} \frac{a_{ij}}{f_{ij}} w_i$$

in $M_{\mathfrak{q}}$ since $f_{ij} \in A \setminus \mathfrak{q}$ by $\mathfrak{q} \in D(f)$. Since the w_i's generate the m_j's, the vector space $M_{\mathfrak{q}}/\mathfrak{q}M_{\mathfrak{q}}$ may be generated by the residue classes $\overline{w_1}, \overline{w_2}, \ldots, \overline{w_k}$. Therefore,

$$\varphi(y) = \dim_{k(y)} \mathscr{F}_y \otimes_{\mathcal{O}_x} k(y) = \dim_{A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}} M_{\mathfrak{q}}/\mathfrak{q}M_{\mathfrak{q}} \leq k < n.$$

Hence, $y \in \{x \in X : \varphi(x) < n\}$. By $y = \mathfrak{q} \in D(f)$ arbitrary, we get $D(f) \subseteq \{x \in X : \varphi(x) < n\}$.

(b) Suppose $\varphi(x)$ were not constant. For each value $n \in \mathbb{N}$ that φ takes, consider the nonempty sets $\varphi^{-1}(n)$. For every $x \in \varphi^{-1}(n)$, Hartshorne II.5.7b shows that $\mathscr{F}_x$ is free of rank n as an $\mathcal{O}_x$-module.

For each x, choose a neighborhood U_x s.t. $\mathscr{F}|_{U_x}$ is free of rank n. Define V_n to be the union over all such U_x's. That is, $V_n := \bigcup_{x \in \varphi^{-1}(n)} U_x$. Then, $X := \bigcup_{n \in \mathbb{N}} V_n$ and since $\varphi(x)$ is not constant, there is more than one n for which $V_n \neq \emptyset$. Furthermore, $V_n \cap V_m = \emptyset$ whenever $n \neq m$ by Hartshorne II.5.7. So we have written X as a union of disjoint open sets which contradicts X connected.

(c) Assume X is reduced and assume $\varphi(x)$ is constant. To show $\mathscr{F}$ is locally free, fix a point $x \in X$ and associate $U \ni x =: \mathfrak{p}$ an open affine neighborhood. Fix an A-module M which is generated by $m_1, \ldots, m_l$ s.t. $\mathscr{F}|_{\operatorname{Spec} A} = \widetilde{M}$. Our goal is to show that there is a cover on which $\mathscr{F}$ is free.

First, we repeat the technique of part (a). View $\frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}}$ as a vector space over $\frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}}$ and choose generators $v_1, \ldots, v_k$. Since $\varphi(x)$ is constant, this means we have fixed $\varphi(X) = k$. By Nakayama's Lemma, we lift these generators to $w_1, \ldots, w_k \in M_{\mathfrak{p}}$ which generate $M_{\mathfrak{p}}$. Then, we can solve for the relations, since the m_i's generate M,

$$m_j = \frac{m_j}{1} = \sum \frac{a_{ij}}{f_{ij}} w_i$$

and set $D(f) := D\left(\prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}} f_{ij}\right)$. Clearly, $x = \mathfrak{p} \in D(f)$.

Next, we prove some auxiliary results which we use to finish the proof.

For every $y = \mathfrak{q} \in D(f)$, we view $\frac{M_{\mathfrak{q}}}{\mathfrak{q}M_{\mathfrak{q}}}$ as a $k(y)$-vector space. Certainly[38], the elements $w_1, \ldots, w_k$ generate $\frac{M_{\mathfrak{q}}}{\mathfrak{q}M_{\mathfrak{q}}}$ as a $k(y)$-vector space and since $\varphi(X) = k$, the w_i's form a basis for the vector space.

At last, we claim that M_f is actually free over A_f i.e. $\mathscr{F}|_{D(f)}$ is a free $\mathcal{O}_X|_{D(f)}$-module. From the first step, we know that the w_i's generate M_f as an A_f-module. We claim that they actually generate it freely. Suppose

$$\sum_{i=1}^{k} \frac{b_i}{g_i} w_i = 0, \qquad \text{for } \frac{b_i}{g_i} \in A_f.$$

Since $\mathfrak{q} \in D(f)$, we can view the above equation as a relation in $\frac{M_{\mathfrak{q}}}{\mathfrak{q}M_{\mathfrak{q}}}$ as a $k(y)$-vector space. Since a basis for a vector space is linearly independent, $\frac{b_i}{g_i}$ must be zero i.e. $\frac{b_i}{g_i} \in \mathfrak{q}A_{\mathfrak{q}}$.

[38]Consider the denominators to see this.

This holds for every $\mathfrak{q} \in D(f)$ and hence, $\frac{b_i}{g_i} \in \text{Nilrad}(A_f)$. But we assumed X was reduced and so, $\text{nilrad}(A_f) = (\text{nilrad}(A))_f = 0_f = 0$. Hence, $\frac{b_i}{g_i} = 0$ as an element of A_f.

Therefore, $\mathscr{F}|_{D(f)}$ is a free $\mathcal{O}_X|_{D(f)}$-module.

Our choice of $x \in X$ was arbitrary. So, we can find $D(f_x)$ for every $x \in X$ s.t. $\mathscr{F}|_{D(f_x)}$ is a free $\mathcal{O}_X|_{D(f_x)}$-module. But then we can cover X by these $D(f_x)$ and hence, $\mathscr{F}$ is a locally free $\mathcal{O}_X$-module.

□

Hartshorne II.5.9 Let S be a graded ring, generated by S_1 as an S_0 -algebra, let M be a graded S module, and let $X = \operatorname{Proj} S$.

(a) Show that there is a natural homomorphism $\alpha : M \to \Gamma_*(\widetilde{M})$.

(b) Assume now that $S_0 = A$ is a finitely generated k -algebra for some field k, that S_1 is a finitely generated A -module, and that M is a finitely generated S -module. Show that the map α is an isomorphism in all large enough degrees, i.e., there is a $d_0 \in \mathbb{Z}$ such that for all $d \geqslant d_0, \alpha_d : M_d \to \Gamma(X, \widetilde{M}(d))$ is an isomorphism. [Hint: Use the methods of the proof of (5.19).]

(c) With the same hypotheses, we define an equivalence relation $\approx$ on graded S -modules by saying $M \approx M'$ if there is an integer d such that $M_{\geqslant d} \cong M'_{\geqslant d}$ Here $M_{\geqslant d} = \bigoplus_{n \geqslant d} M_n$. We will say that a graded S -module M is quasifinitely generated if it is equivalent to a finitely generated module. Now show that the functors $\sim$ and Γ_* induce an equivalence of categories between the category of quasi-finitely generated graded S-modules modulo the equivalence relation $\approx$, and the category of coherent $\mathcal{O}_X$-modules.

PROOF. **(a)** First off, we prove there are natural morphisms $\alpha_d : M_d \to \Gamma(X, \widetilde{M}(d))$. For any $f \in S$, there is always a map

$$M_{(f)} \otimes_{S_{(f)}} S(d)_{(f)} \to (M_d)_{(f)}$$

given by multiplication (and this needs not be an isomorphism). Applying $\sim$, we get a map

$$\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)|_{D_+(f)} = \widetilde{M_{(f)}} \otimes_{\mathcal{O}_{\operatorname{Spec} S_{(f)}}} \widetilde{S(d)_{(f)}} = [M_{(f)} \otimes_{S_{(f)}} S(d)_{(f)}]^\sim \longrightarrow \widetilde{(M_d)_{(f)}}.$$

Now if we examine the LHS carefully,

$$\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)|_{D_+(f)} = \widetilde{M}(d)|_{D_+(f)}$$

as in the definition on p. 117 of Hartshorne. Everything thus far behaves w.r.t. restriction and intersections of $D_+(f)$. Therefore, the maps

$$\widetilde{M}(d)|_{D_+(f)} \to \widetilde{(M_d)_{(f)}}$$

induce maps $\widetilde{M}(d)|_{D_+(f)} \to \widetilde{M}_d$ and then a map

$$\widetilde{M}(d) \to \widetilde{M}_d.$$

Taking global sections, we obtain a map

$$\alpha_d : M_d \to \Gamma(X, \widetilde{M}(d)).$$

Taking the direct sum over all α_d gives a map

$$\alpha : \bigoplus_{d\in\mathbb{Z}} M_d = M \to \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \widetilde{M}(d)) = \Gamma_*(\widetilde{M}).$$

(b)[39] First, fix a finite filtration $0 \subseteq M^1 \subseteq M^2 \subseteq \cdots \subseteq M^r =: M$, where $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)$ using Hartshorne Proposition I.7.4 (note from the hypotheses and Atiyah-Macdonald Proposition 10.7 that the ring S is a Noetherian graded ring).

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{i-1} & \longrightarrow & M^i & \longrightarrow & (S/\mathfrak{p}_i)(n_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(X, \widetilde{M^{i-1}}) & \longrightarrow & \Gamma(X, \widetilde{M^i}) & \longrightarrow & \Gamma(X, \widetilde{(S/\mathfrak{p}_i)(n_i)})) & \longrightarrow & 0 \end{array}$$

Assume for now that for every $(S/\mathfrak{p})(n)$, one could always find an e s.t. for all $d \geq e$, one has an isomorphism $[(S/\mathfrak{p})(n)]_d \to \Gamma(X, \widetilde{(S/\mathfrak{p}_i)}(n)(d))$.

We claim that for every $0 \leq i \leq r$, we can always find a d_i s.t. for all $d \geq d_i$, we have an isomorphism $M^i_d \to \Gamma(X, \widetilde{M^i}(d))$ and we prove this using induction. First, the base case[40] with $i = 1$ yields a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & M^1_d & \longrightarrow & M^1_d & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow\cong & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \Gamma(X, M^1(d)) & \longrightarrow & \Gamma(X, \widetilde{(S/\mathfrak{p}_1)}(n_1)(d))) & \longrightarrow & 0 \end{array}$$

In the diagram, we chose $d \geq e_1$ s.t. $[(S/\mathfrak{p}_1)(n_1)]_d \to \Gamma(X, \widetilde{S/\mathfrak{p}_1}(n_1)(d))$ is an isomorphism. By the 5-lemma, the map in the middle is an isomorphism for all $d \geq e_1$.

Now assume the result were true for all $0 \leq i \leq r-1$ (and we can continue by setting $M^n = M^r$ for $n \geq r+1$). Call this assumption (**). Look at the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & M^{r-1}_{d_{r-1}} & \longrightarrow & M^{r-1}_{d_{r-1}} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \Gamma(X, M^{r-1}(d)) & \longrightarrow & \Gamma(X, \widetilde{S/\mathfrak{p}_r}(n_r)(d)) & \longrightarrow & 0 \end{array}$$

To get the result for r, repeat what we just did above except with M^1 replaced by M^r and the left vertical map with $M^{r-1}_{d_{r-1}} \to \Gamma(X, M^{r-1}(d))$ and d_{r-1} chosen s.t. $M^{r-1}_{d_{r-1}} \to \Gamma(X, M^{r-1}(d))$ is an isomorphism when $d \geq d_{r-1}$ and e_r chosen s.t. $[(S/\mathfrak{p}_r)(n_r)]_d \to \Gamma(X, \widetilde{S/\mathfrak{p}_r}(n_r)(d))$ is an isomorphism when $d \geq e_r$.

Now choose $d_r := \max\{e_r, d_{r-1}\}$ and this makes $M^r_d \to \Gamma(X, M^r(d))$ an isomorphism whenever $d \geq d_r$ by the 5-lemma. Since $M^r = M$, we are done.

[39] Some hints for this can be found here. I, unfortunately, still do not understand the hint too well so have barely written anything here. I will need to elaborate more on the details of this proof and clarify my understanding of finite filtrations and why they even exist in this situation.

[40] We could have done $i = 0$, but that is a trivial case $M_0 = 0$ and does not provide insight.

All that remains is to show (**). Because $\widetilde{(S/\mathfrak{p})}(n)(d) = \widetilde{(S/\mathfrak{p})}(n+d)$ and $[(S/\mathfrak{p})(n)]_d = (S/\mathfrak{p})(n+d)$, assume WLOG $n = 0$. We reduce to proving

$$\text{There exists } e \in \mathbb{Z} \text{ s.t. for all } d \geq e,\ (S/\mathfrak{p})_d \cong \Gamma(X, \widetilde{(S/\mathfrak{p})}(d)).$$

We are done due to the proof of Hartshorne Theorem II.5.19. In particular, the rest of the proof starting from the second line up from the bottom of Hartshorne p. 122. The only modification we need to make is that instead of S, we have $S/\mathfrak{p}$ and instead of $\operatorname{Proj} S$, we have $\operatorname{Proj} S/\mathfrak{p}$.

(c) We show an equivalence of categories

$$\mathscr{C} = \left\{ \begin{matrix} \text{coherent} \\ \mathcal{O}_X\text{-modules} \end{matrix} \right\} \underset{\sim}{\overset{\Gamma_*}{\rightleftarrows}} \left\{ \begin{matrix} \text{quasi-finitely generated} \\ \text{graded } S\text{-modules} \\ \text{modulo } \approx \end{matrix} \right\} = \mathscr{D}$$

and there are three properties we must check. We check that Γ_* and $\sim$ are functors mapping to the appropriate categories. Then, we check that there are natural isomorphisms $\epsilon : \Gamma_*(\sim(-)) \to \mathrm{id}_{\mathscr{D}}$ and $\eta : \sim(\Gamma_*(-)) \to \mathrm{id}_{\mathscr{C}}$.

Let $M \in \mathscr{D}$. We may assume that M is a f.g. graded S-module. Then $\widetilde{M}$ is an $\mathcal{O}_X$-module. It is coherent by Hartshorne Proposition II.5.11(c).

Let $\mathscr{F} \in \mathscr{C}$ be a coherent $\mathcal{O}_X$-module. From the methods in the first paragraph of the proof of Hartshorne Theorem II.5.19 we see $\mathscr{F} \cong \widetilde{M'}$ where M' is a f.g. graded S-module. Now applying $\Gamma_*(-)$ and using part (b), the map $\alpha : M' \to \Gamma_*(\mathscr{F})$ is an isomorphism. Therefore, $\Gamma_*(\mathscr{F})$ is a quasi-finitely generated graded S-module.

We now show that $\widetilde{\Gamma_*(-)} \cong \mathrm{id}_{\mathscr{C}}$ and $\Gamma_*(\ \widetilde{(-)}\ ) \cong \mathrm{id}_{\mathscr{D}}$. The first natural isomorphism follows immediately from Hartshorne Proposition II.5.15. To prove the second, let M be a quasi-finitely generated graded S-module. From part (b), we know $\alpha : M \to \Gamma_*(\widetilde{M})$ is an isomorphism. It is easy to see that this isomorphism is natural because we modulo the equivalence $\approx$. □

Hartshorne II.5.10 Let A be a ring, let $S = A[x_0, \ldots, x_r]$ and let $X = \operatorname{Proj} S$. We have seen that a homogeneous ideal I in S defines a closed subscheme of X(Ex. 3.12), and that conversely every closed subscheme of X arises in this way (5.16) .

(a) For any homogeneous ideal $I \subseteq S$, we define the saturation $\bar{I}$ of I to be $\{s \in S \mid$ for each $i = 0, \ldots, r$, there is an n such that $x_i^n s \in I\}$. We say that I is saturated if $I = \bar{I}$. Show that $\bar{I}$ is a homogeneous ideal of S.

(b) Two homogeneous ideals I_1 and I_2 of S define the same closed subscheme of X if and only if they have the same saturation.

(c) If Y is any closed subscheme of X, then the ideal $\Gamma_*(\mathscr{I}_Y)$ is saturated. Hence it is the largest homogeneous ideal defining the subscheme Y.

(d) There is a $1-1$ correspondence between saturated ideals of S and closed subschemes of X.

Proof. (a) Suppose $s \in \overline{I}$. Then $x_i^n s \in I$ for some n. If $s = \bigoplus_{d \geq 0} s_d$, then $x_i^n s = \bigoplus_{d \geq 0} x_i^n s_d$ means $x_i^n s_d \in S_{n+d}$ for each i and so, $s_d \in \overline{I}$.

(b) ($\Rightarrow$). Suppose I_1 and I_2 define the same closed subscheme of X. Then $\operatorname{Proj}(S/I_1) = \operatorname{Proj}(S/I_2)$. Towards a contradiction, suppose $\overline{I_1} \setminus \overline{I_2} \neq \emptyset$ and choose an element

$s \in \overline{I_1} \setminus \overline{I_2}$. Choose an $\mathfrak{p} \in \operatorname{Proj}(S/I_1)$, $x_i^n s \in I_1 \subseteq \mathfrak{p}$, and assume $x_i^m s \notin I_2 \subseteq \mathfrak{p}$ for all m. Then $s, x_i^m \notin \mathfrak{p}$ for all m and $x_i^n \in \mathfrak{p}$ which means $x_i^{m-n} x_i^n \in \mathfrak{p}$. However, $x_i^{m-n} \cdot x_i^n \notin \mathfrak{p}$ choosing $m > n$. But we cannot have $x_i^m \in \mathfrak{p}$ and $x_i^m \notin \mathfrak{p}$ at the same time. Contradiction.

($\Leftarrow$). Assume $\overline{I_1} = \overline{I_2}$. It suffices to show that $\overline{I}$ and I define the same closed subscheme.

Lemma 5.4. The homogeneous ideal $\overline{I}/I$ of S/I defines a closed immersion $\operatorname{Proj} S/\overline{I} \to \operatorname{Proj} S/I$.

PROOF. This follows easily from the commutativity of the corresponding triangle diagram of commutative graded rings. □

Lemma 5.5. $f : \operatorname{Proj} S/\overline{I} \to \operatorname{Proj} S/I$ defines an isomorphism.

PROOF. It is easy to show that the underlying sets can be identified i.e. $\mathfrak{p}$ contains $\overline{I}$ iff it contains I. See also Atiyah-Macdonald's exercises on saturated ideals.

We know that there is a homeomorphism on the underlying topological spaces. We show there is an isomorphism of sheaves $\mathcal{O}_{\operatorname{Proj} S/I} \to f_* \mathcal{O}_{\operatorname{Proj} S/\overline{I}}$. Surjectivity is clear since we have a closed immersion from Hartshorne II.3.12(a). To show injectivity, we check injectivity over each $D_+(f) \subseteq \operatorname{Proj} S/I$. The map

$$(S/I)_{(f+I)} \to (S/\overline{I})_{(f+\overline{I})}$$

is clearly injective. Indeed, if $\frac{s+\overline{I}}{(f+\overline{I})^m} = 0$, then $(s+\overline{I})(f+\overline{I})^M = 0$ for some M. This says $sf^M \in \overline{I}$.

For sufficiently large n, we have $sf^M x_i^n \in I$ for each i. But for any $N \geq \frac{n(r+1)}{\deg f}$, we have $f^N \in (x_0^n, \ldots, x_r^n)$. That means $sf^{M+N} \in I$. So, $\frac{s+I}{\frac{s+I}{f+I}} = 0$ in $(S/I)_{(f+I)}$ and so the map is indeed injective. The injectivity then follows by Hartshorne II.2.18(b). □

(c) Let Y be a closed subscheme. Let $s \in \overline{\Gamma_*(\mathscr{I}_Y)}$. Suppose for all i, there is an n_i s.t. $x_i^{n_i} s \in \Gamma_*(\mathscr{I}_Y)$. So $x_i^{n_i} s_n \in \Gamma(X, \mathscr{I}_Y(n+n_i)$ where $s_n \in S_n$. But

$$\Gamma(X, \mathscr{I}(n+n_i)) = \Gamma(X, \mathscr{I}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(n+n_i)) = \Gamma(X, \mathscr{I}_Y(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n_i)) = I_Y(n) \otimes S(n_i).$$

Hence, $x_i^{n_i} \in S(n_i)$ and $s_n \in I_Y(n)$. So, $s = \bigoplus_{n \geq 0} s_n \in \Gamma_*(\mathscr{I}_Y)$.

(d) Let $\mathfrak{S}$ be the collection of saturated ideals and $\mathfrak{T}$ the collection of closed subschemes of X and define $\mathfrak{S} \to \mathfrak{T}$ using the fact that each saturated ideal is homogeneous and hence, defines a closed subscheme. From (b), we have injectivity. From (c), we have surjectivity.

□

Hartshorne II.5.11 Let S and T be two graded rings with $S_0 = T_0 = A$. We define the Cartesian product $S \times_A T$ to be the graded ring $\bigoplus_{d \geqslant 0} S_d \otimes_A T_d$. If $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} T$, show that $\operatorname{Proj}(S \times_A T) \cong X \times_A Y$, and show that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}(S \times_A T)$ is isomorphic to the sheaf $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ on $X \times Y$.

The Cartesian product of rings is related to the Segre embedding of projective spaces (I, Ex. 2.14) in the following way. If $x_0, \ldots, x_r$ is a set of generators for S_1 over A, corresponding to a projective embedding $X \subset \mathbb{P}_A^r$, and if $y_0, \ldots, y_s$ is a set of generators for T_1, corresponding to a projective embedding $Y \subset \mathbb{P}_A^s$, then $\{x_i \otimes y_j\}$ is a set of generators for $(S \times_A T)_1$, and

hence defines a projective embedding $\operatorname{Proj}(S \times_A T) \hookrightarrow \mathbb{P}^N_A$, with $N = rs + r + s$. This is just the image of $X \times Y \subseteq \mathbb{P}^r \times \mathbb{P}^s$ in its Segre embedding.

PROOF. First off, for $f \in S_+$ and $g \in T_+$ of the same degree, we know that

$$D_+(f) \times_A D_+(g) \cong \operatorname{Spec} S_{(f)} \otimes T_{(g)}.$$

OTOH, if $(f \otimes g) \in (S \times_A T)_+$, then

(8)
$$\left[\bigoplus_{d\geq 0}(S_d \otimes_A T_d)\right]_{(f\otimes g)} \cong \bigoplus_{d\geq 0}(S_d \otimes_A T_d)_{(f\otimes g)} \cong \bigoplus_{d\geq 0}(S_d)_{(f)} \otimes_A (T_d)_{(g)} \cong \bigoplus_{d\geq 0}(S_d)_{(f)} \otimes_A \bigoplus_{d\geq 0}(T_d)_{(g)}$$

which follows from the identifications

$$\frac{\sum a_d \otimes_A b_d}{(f\otimes g)^n} \to \sum \frac{a_d \otimes_A b_d}{(f\otimes g)^n} \to \sum \frac{a_d}{f^n} \otimes_A \frac{b_d}{g^n}$$

of degree 0 elements. If we vary $f \otimes g \in (S \times_A T)_+$, we have a cover of $\operatorname{Proj}(S \times_A T)$ and the identification of (8) shows $D_+(f \otimes g) \cong D_+(f) \times_A D_+(g)$. Furthermore, the isomorphisms above agree on overlaps since

$$D_+(f_1 \otimes g_1) \cap D_+(f_2 \otimes g_2) = D_+(f_1 f_2 \otimes g_1 g_2).$$

Recalling that $\{D_+(f) \times_A D_+(g)\}$ covers the fibre product due to $\{D_+(f)\}$ and $\{D_+(g)\}$ covering X and Y respectively, we conclude that we may now glue to an isomorphism $\operatorname{Proj}(S \times_A T) \to \operatorname{Proj} S \times_{\operatorname{Spec} A} \operatorname{Proj} T$.

For the twisting sheaf, let $p_1 : X \times_A Y \to X$ and $p_2 : X \times_A Y \to Y$ be the projection maps in the fibre product diagram. Computing,

$$\mathcal{O}(1) \cong \widetilde{(S \times_A T)}(1) \cong \widetilde{S_1 \otimes_A T_1} \cong \widetilde{S}_1 \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \widetilde{T}_1 \tag{9}$$

OTOH, we know

$$p_1^*(\mathcal{O}_X(1)) \cong p_1^*(\widetilde{S(1)}) \cong p_1^{-1}\widetilde{S(1)} \otimes_{p_1^{-1}\mathcal{O}_{\operatorname{Proj} S}} \mathcal{O}_{X\times_A Y}. \tag{10}$$

By definition, $p_1^{-1}(\widetilde{S(1)})(U) = \varinjlim_{V \supseteq p_1(U)} \mathcal{O}_X(1)(V)$. We know that for every $D_+(f)$, there is an isomorphism $\mathcal{O}_X(1)|_{D_+(f))} \cong \widetilde{S(1)_{(f)}}$. Because $D_+(f) \cup D_+(g) = D_+(f+g)$ and $D_+(f) \cap D_+(g) \cong D_+(fg)$, we conclude that $p_1^{-1}(\widetilde{S(1)}) \cong \mathcal{O}_S(1) \cong \widetilde{S}_1$. From (10),

$$p_1^*(\mathcal{O}_X(1)) \cong \widetilde{S}_1 \otimes_{p_1^{-1}\mathcal{O}_{\operatorname{Proj} S}} \mathcal{O}_{X\times_A Y}. \tag{11}$$

Working over the open sets $D_+(f \otimes g)$, we deduce that

$$p_1^{-1}\mathcal{O}_{\operatorname{Proj} S} \cong \mathcal{O}_{X\times_A Y}.$$

From 11, we conclude that $p_1^*(\mathcal{O}_X(1)) \cong \widetilde{S}_1$. In which case, (9) implies

$$\mathcal{O}(1) \cong p_1^{-1}\mathcal{O}_X(1) \otimes_{\mathcal{O}_{\operatorname{Spec} A}} p_2^{-1}\mathcal{O}_Y = p_1^{-1}\mathcal{O}_X(1) \otimes p_2^{-1}\mathcal{O}_Y.$$

□

Hartshorne II.5.12 (a) Let X be a scheme over a scheme Y, and let $\mathscr{L}, \mathscr{M}$ be two very ample invertible sheaves on X. Show that $\mathscr{L} \otimes \mathscr{M}$ is also very ample.
[Hint: Use a Segre embedding.]

(b) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes. Let $\mathscr{L}$ be a very ample invertible sheaf on X relative to Y, and let $\mathscr{M}$ be a very ample invertible sheaf on Y relative to Z. Show that $\mathscr{L} \otimes f^*\mathscr{M}$ is a very ample invertible sheaf on X relative to Z.

Proof. Before starting the proof, we must prove a few things: we show that the product of two immersions is necessarily an immersion and the composition of two immersions is an immersion (but see my comments in (b)). In particular, we change the Hartshorne's definition of an immersion and say $f : X \to Y$ is an immersion iff $f = j \circ i$ for j an open immersion and i a closed immersion.

Lemma 5.6. Immersions are stable under composition.

Proof. Noting our change in definition, let $f : X \to Y$ and $g : Y \to Z$ be immersions. So find a $V \subseteq Y$ and $U \subseteq Z$ open subschemes s.t. $f(X) \subseteq V$ and $g(Y) \subseteq U$ are closed subsets.

Now observe $Y \subseteq g^{-1}(U)$ so that the topology on Y is induced from that of U. So find an open subset $W \subseteq Z$ open s.t. $g^{-1}(W) = V$. Then $X \to V = g^{-1}(W) \to W$ is a composition of closed immersions and therefore is a closed immersion. □

Lemma 5.7. The diagonal morphism is an immersion $X \to X \times_S X$.

Proof. If $X \to Y \to Z$ is an immersion, then it is easy to see that $X \to Y$ is also an immersion. But $X \to X \times_S X \to X$ is the identity map which is an immersion implying $X \to X \times_S X$ is an immersion. □

Lemma 5.8. Immersions are stable under fibre products.

Proof. See [10, Tag 01JO]. □

(a) Fix $\mathscr{L} \cong i^*(\mathcal{O}(1))$ where $i : X \to \mathbb{P}^n_Y$ and $\mathscr{M} \cong j^*(\mathcal{O}(1))$ where $j : X \to \mathbb{P}^s_Y$. Note that the maps i and j are immersions. We have a diagram:

$$X \xrightarrow{i \times j} \mathbb{P}^r_Y \times \mathbb{P}^s_Y \xrightarrow{g} \mathbb{P}^{rs+r+s}_Y, \qquad X \xrightarrow{h} \mathbb{P}^{rs+r+s}_Y$$

where the second map g is the Segre embedding. The composition of an immersion with an embedding is necessarily an immersion. The product of two immersions is also an immersion[41]. So, we show $\mathscr{L} \otimes \mathscr{M} \cong h^*(\mathcal{O}(1))$ over $\mathbb{P}^{rs+r+s}_Y$.

First off, $h^* = (g \circ i \times j)^* = (i \times j)^* g^*$. It follows that

$$g^*\mathcal{O}(1) \cong p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$$

from the previous exercise. Now that f^* of a tensor product is the tensor product of the individual f^*. It follows that

$$h^*\mathcal{O}(1) \cong (i \times j)^* p_1^*\mathcal{O}(1) \otimes (i \times j)^* p_2^*\mathcal{O}(1) \cong i^*\mathcal{O}_{\mathbb{P}^r_Y}(1) \otimes j^*\mathcal{O}_{\mathbb{P}^s_Y}(1)) \cong \mathscr{L} \otimes \mathscr{M}.$$

[41] A reference can be found here.

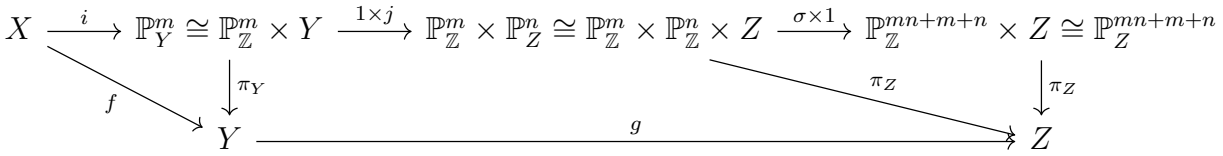
(b) *Note.* With changes to the problem made at the start, the following proof here suffices.

We provide the details of the proof ourselves. Let $i : X \hookrightarrow \mathbb{P}^m_Y$ and $j : Y \hookrightarrow \mathbb{P}^n_Z$ be the projective morphisms induced from very ampleness. The base extension of an immersion is an immersion.

It follows that

$$X \xrightarrow{i} \mathbb{P}^m_Y \cong \mathbb{P}^m_{\mathbb{Z}} \times Y \overset{1\times j}{\hookrightarrow} \mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_Z \cong \mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}} \times Z \overset{\sigma\times 1}{\to} \mathbb{P}^{mn+m+n}_{\mathbb{Z}} \times Z \cong \mathbb{P}^{mn+m+n}_Z$$

is a composition of immersions (here σ denotes the Segre embedding $\mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}} \to \mathbb{P}^{mn+m+n}_{\mathbb{Z}}$). Let h be the composite. It is clear from properties of fibre products that the following diagram commutes



Now we have a sheaf computation. Note that inverse images commute with tensor products and are contravariant. Next, let $\phi : \mathbb{P}^m_Y \to Y$ be the projection onto Y. Let $\pi_1 : \mathbb{P}^{mn+m+n}_{\mathbb{Z}} \to \mathbb{P}^m_{\mathbb{Z}}$ and $\pi_2 : \mathbb{P}^{mn+m+n}_{\mathbb{Z}} \to \mathbb{P}^n_{\mathbb{Z}}$ be the projections from the Segre embedding (i.e. it is restricted to the fibre product $\mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}}$). Indeed,

$$\begin{aligned}
h^*\mathcal{O}_{\mathbb{P}^{mn+m+n}_Z} &\cong i^*(1\times j)^*[\pi_1^*\mathcal{O}_{\mathbb{P}^{mn+m+n}_Z}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^{mn+m+n}_Z}(1)] \\
&\cong i^*(\mathcal{O}_{\mathbb{P}^m_Y}(1) \otimes \mathcal{O}_{\mathbb{P}^n_Y}(1)) \\
&\cong i^*(\mathcal{O}_{\mathbb{P}^m_Y}(1) \otimes \phi^*\mathscr{M}) \\
&\cong i^*\mathcal{O}_{\mathbb{P}^m_Y}(1) \otimes i^*\phi^*\mathscr{M} \\
&\cong i^*\mathcal{O}_{\mathbb{P}^m_Y}(1) \otimes (\phi\circ i)^*\mathscr{M} \\
&\cong \mathscr{L} \otimes f^*\mathscr{M} \\
&\cong f^*\mathscr{M} \otimes \mathscr{L}.
\end{aligned}$$

We explain the isomorphisms in more detail. The first isomorphis follows from properties of the Segre embedding. The compositions

$$(1\times j)^*\pi_1^* = (\pi_1 \circ 1 \times j)^* = \left(\mathbb{P}^m_{\mathbb{Z}} \times Y \overset{1\times j}{\to} \mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}} \times Z \overset{\pi_1}{\to} \mathbb{P}^m_{\mathbb{Z}} \times Z\right)^*$$

and

$$(1\times j)^*\pi_2^* = (\pi_2 \circ 1 \times j)^* = \left(\mathbb{P}^m_{\mathbb{Z}} \times Y \overset{1\times j}{\to} \mathbb{P}^m_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}} \times Z \overset{\pi_2}{\to} \mathbb{P}^n_{\mathbb{Z}} \times Z\right)^*$$

give the second isomorphism. The third isomorphism follows from $\mathscr{M} \cong j^*\mathcal{O}_{\mathbb{P}^n_Z}(1)$ and

$$\phi^*j^* = (j\circ\phi)^* = (\mathbb{P}^m_Y \to Y \to \mathbb{P}^n_Z)^*$$

The fourth isomorphism follows from commutativity of tensor products and inverse image functors. The fifth isomorphism is due to contravariance if $(-)^*$. The second to last line follows from our choice of i and if we can show $\phi \circ i = f$. This is easy:

there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{P}^m_Y \\ & \searrow^{f} & \downarrow{\scriptstyle \phi} \\ & & Y \end{array}$$

Clearly the final isomorphism is due to our commutative setting.

□

Remark 27. Look at p. 229 of Vakil 8.1.M for Hargun's comments. It is true that Hartsrhorne's definition implies the locally closed embedding definition of an immersion.

Hartshorne II.5.13 Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d > 0$, let $S^{(d)}$ be the graded ring $\bigoplus_{n\geqslant 0} S^{(d)}_n$ where $S^{(d)}_n = S_{nd}$. Let $X = \operatorname{Proj} S$. Show that $\operatorname{Proj} S^{(d)} \cong X$, and that the sheaf $\mathcal{O}(1)$ on Proj $S^{(d)}$ corresponds via this isomorphism to $\mathcal{O}_X(d)$

This construction is related to the d-uple embedding (I, Ex. 2.12) in the following way. If $x_0, \ldots, x_r$ is a set of generators for S_1, corresponding to an embedding $X \subset \mathbb{P}^r_A$, then the set of monomials of degree d in the x_i is a set of generators for $S^{(d)}_1 = S_d$. These define a projective embedding of Proj $S^{(d)}$ which is none other than the image of X under the d-uple embedding of $\mathbb{P}^r_A$.

Proof. We begin by constructing the morphism $\operatorname{Proj} S \to \operatorname{Proj} S^{(d)}$ locally. That is, we construct a morphism from a cover $D_+(f)$ for $f \in S_+$ homogeneous. We may require the cover to have $f \in S_d$ since S is generated by S_1 as an S_0-algebra and using the fact that $D_+(f) = D_+(f^k)$ for $k > 0$.

As an element, $f \in S_d = S^{(d)}_1$ so it makes sense to attempt to construct a map $S^{(d)}_{(f)} \to S_{(f)}$. Of course, there an obvious map: for any $\frac{a}{f^n} \in S^{(d)}_{(f)}$, send $\frac{a}{f^n}$ to itself. This is an isomorphism because every element in $S_{(f)}$ is represented by a fraction $\frac{a}{f^n}$ with $\deg(a) = n \deg(f) = nd$ meaning that $a \in S_{nd}$. This makes sense since $a \in S_{nd} = S^{(d)}_n$.

Note that the morphism we defined above is well-defined under restriction to $D_+(fg)$ for another $g \in S_d$. Of course, the morphism it glues to is not induced from a morphism of graded rings[42]. The morphism it glues to is also an isomorphism because the open sets $D_+(f)$ cover $\operatorname{Proj} S$ and $\operatorname{Proj}(S^{(d)})$.

Let $F : \operatorname{Proj} S^{(d)} \to \operatorname{Proj} S$ denote the isomorphism (changing the direction is easy from above by letting the local isomorphism go $S_{(f)} \to S^{(d)}_{(f)}$). Locally, we know

$$\mathcal{O}(1)|_{D_+(f)} = \widetilde{S^{(d)}_1}|_{D_+(f)} \cong \widetilde{(S_d)_{(f)}} = \mathcal{O}_X(d)|_{D_+(f)}$$

and as these isomorphisms are well-behaved under restriction to $D_+(fg)$, we deduce that $\mathcal{O}(1)$ on $\operatorname{Proj} S^{(d)}$ corresponds to $\mathcal{O}_X(d)$.

Remark/Warning. The grading of the rings matter. In this exercise, Hartshorne has $S^{(d)}_n = S_{nd}$ which is different from the situation where $S^{(d)}_n$ is zero if $n \not\equiv 0 \bmod d$ and S_{id} if $n = id$. More details are given below.

[42]Other examples exist of course see this.

The isomorphism $\operatorname{Proj} S^{(d)} \cong X$ with the "wrong grading" can be proven using the results at the end of this handout. The solution of Hartshorne Exercise II.2.14 contained within it the proof of Lemma 4.2 in the handout:

Lemma 5.9. If there exists an integer $d \geq 1$ s.t. $\varphi_r : S_r \to T_r$ is bijective for all large $r \in d \cdot \mathbb{N}$, then $U = \operatorname{Proj} T$ and $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism (where f is the morphism defined in Exercise II.2.14).

View $S^{(d)} := \oplus_{n \geq 0} S_{nd}$ as the graded ring whose ith-graded component is zero if i is not a multiple of d and S_{nd} if $i = nd$. Now define a map of graded rings $\varphi : S \to S^{(d)}$ by $\varphi_i = 0$ if i is not a multiple of d and $\varphi_i = \mathrm{id}_{S_{nd}}$ if $i = nd$. We know that φ_n is bijective all $n \in d\mathbb{N}$ so the lemma shows that the map $f : \operatorname{Proj} S^{(d)} \to \operatorname{Proj} S$ is an isomorphism.

In this situation, $\mathcal{O}_{\operatorname{Proj} S^{(d)}}(1) = \widetilde{S_1^{(d)}} = \widetilde{0}$ is just the constant zero sheaf. Then, $\mathcal{O}_X(1)$ corresponds to this sheaf which is *not* the sheaf $\mathcal{O}_X(d)$.

But if we have the grading that Hartshorne states, then $\mathcal{O}_{\operatorname{Proj} S^{(d)}}(1) = \widetilde{S_1^{(d)}} = \widetilde{S(d)} = \mathcal{O}_X(d)$. □

Hartshorne II.5.14 Let A be a ring, and let X be a closed subscheme of $\mathbb{P}^r_A$. We define the homogeneous coordinate ring $S(X)$ of X for the given embedding to be $A[x_0, \ldots, x_r]/I$, where I is the ideal $\Gamma_*(\mathscr{I}_X)$ constructed in the proof of (5.16) . (Of course if A is a field and X a variety, this coincides with the definition given in (I, §2)!) Recall that a scheme X is *normal* if its local rings are integrally closed domains. A closed subscheme $X \subseteq \mathbb{P}^r_A$ is *projectively normal* for the given embedding, if its homogeneous coordinate ring $S(X)$ is an integrally closed domain (cf. (I, Ex. 3.18)). Now assume that k is an algebraically closed field, and that X is a connected, normal closed subscheme of $\mathbb{P}^r_k$. Show that for some $d > 0$, the d-uple embedding of X is projectively normal, as follows.

(a) Let S be the homogeneous coordinate ring of X, and let $S' = \bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{O}_X(n))$. Show that S is a domain, and that S' is its integral closure.

[Hint: First show that X is integral. Then regard S' as the global sections of the sheaf of rings $\mathscr{S} = \bigoplus_{n \geqslant 0} \mathcal{O}_X(n)$ on X, and show that $\mathscr{S}$ is a sheaf of integrally closed domains.]

(b) Use (Ex. 5.9) to show that $S_d = S'_d$ for all sufficiently large d.

(c) Show that $S^{(d)}$ is integrally closed for sufficiently large d, and hence conclude that the d-uple embedding of X is projectively normal.

(d) As a corollary of (a), show that a closed subscheme $X \subseteq \mathbb{P}^r_A$ is projectively normal if and only if it is normal, and for every $n \geqslant 0$ the natural map $\Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ is surjective.

Proof. This problem provides an opportunity for the reader to understand the construction of $\Gamma_*(\mathscr{I}_X)$. We provide a more details for it than is given in Hartshorne Corollary II.5.16.

Lemma 5.10. $\Gamma_*(\mathscr{I}_Y) \subseteq \Gamma_*(\mathcal{O}_X) = A[x_0, \ldots, x_r]$ is a S-submodule for $X := \mathbb{P}^r_A$.

Proof. [43] Let $\mathscr{I}_Y$ be the ideal sheaf of $i : Y \subseteq \mathbb{P}^r_A$ i.e. it is the kernel sheaf of the map $i^\# : \mathcal{O}_{\mathbb{P}^r_A} \to i_* \mathcal{O}_Y$. In this way, $\mathscr{I}_Y$ is a subsheaf of $\mathcal{O}_{\mathbb{P}^r_A} := \mathcal{O}_X$. The twisting functor is exact

[43] Due to the length and nature of the problem, it may be beneficial for a first try to look at this post and this post which include a a rough sketch on some aspects of the problem.

so $\mathscr{I}_Y(m) \subseteq \mathcal{O}_X(m)$ is a subsheaf for all $m \in \mathbb{Z}$. The global section functor is left exact so $\Gamma(X, \mathscr{I}_Y(m)) \subseteq \Gamma(X, \mathcal{O}_X(m))$ is always an S-submodule. Therefore,

$$\Gamma_*(\mathscr{I}_Y) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathscr{I}_Y(n)) \subseteq \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(n)) = \Gamma_*(\mathcal{O}_X)$$

is always an S-submodule. □

(a) The idea is to follow the hint. Fix $X \subseteq \mathbb{P}^r_k$ a connected normal closed subscheme.

Lemma 5.11. X is an integral scheme and this implies $\Gamma_*(\mathscr{I}_X)$ is a prime ideal.

Proof. First off, X being an integral scheme is equivalent to being reduced and irreducible. It is reduced because $\mathcal{O}_{X,P}$ must be a reduced ring for each P by the fact that $\mathcal{O}_{X,P}$ is an integrally closed domain. OTOH, if X is reducible, there is a point x laying in two different irreducible components. It follows that $\mathcal{O}_{X,x}$ has zero divisors which contradicts normality.

By definition, $\Gamma_*(\mathscr{I}_X) = \bigoplus_{n\geq 0} \Gamma(\mathbb{P}^r_k, \mathscr{I}_X(n))$. Now assume that $a, b \in \bigoplus_{n\geq 0} \Gamma(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(n))$ are homogeneous sections s.t. $ab \in \Gamma_*(\mathscr{I}_X)$ and is of degree n. This means $i^\#$ sends ab to zero in $\Gamma(\mathbb{P}^r_k, \mathcal{O}_X(n))$. By integrality of $\mathcal{O}_X(n)$, this implies that a or b were already zero. So therefore, a or b already were in $\Gamma_*(\mathscr{I}_X)$. □

Lemma 5.12. $(X, \mathcal{O}_X) \cong (\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ for $S = k[x_0, \ldots, x_r]/\Gamma_*(\mathscr{I}_X)$.

Proof. Follows from Hartshorne Corollary II.5.16(a)'s proof. □

We may regard S' as the global sections of $\mathscr{S} = \bigoplus_{n\geq 0} \mathcal{O}_X(n)$ because the global sections functor is left exact (so it preserves direct sums).

Lemma 5.13. $\mathscr{S}$ is a sheaf of integrally closed domains.

Proof. A sheaf is a sheaf of integrally closed domains iff its stalks are integrally closed. Its stalks are

$$\mathscr{S}_{\mathfrak{p}} = \bigoplus_{n\geq 0} \mathcal{O}_X(n)_{(\mathfrak{p})} = \bigoplus_{n\geq 0} S(n)_{(\mathfrak{p})} = \left\{ \frac{f}{g} : \deg(f) \geq \deg(g), g \notin \mathfrak{p}, \frac{f}{g} \in S_{\mathfrak{p}} \right\}.$$

Any element integral over $\mathscr{S}_{\mathfrak{p}}$ is then integral over $S_{\mathfrak{p}}$ and therefore is in $S_{\mathfrak{p}}$ by normality of $\mathcal{O}_{X,\mathfrak{p}}$. Indeed, normality of $\mathcal{O}_{X,\mathfrak{p}} \cong S_{(\mathfrak{p})}$ only implies normality for the degree zero elements. So we need to obtain normality of $S_{\mathfrak{p}}$ from that of $S_{(\mathfrak{p})}$ which is the content of the next lemma.

Now that any element s integral over $\mathscr{S}_{\mathfrak{p}}$ is inside $S_{\mathfrak{p}}$, we show that $s \in \mathscr{L}_{\mathfrak{p}}$. To do this, it suffices to show that $\deg(s)$ is always nonnegative. if $\deg(s) < 0$, then there can be no integral relation from a monic polynomial. Hence, $s \in \mathscr{S}_{\mathfrak{p}}$. □

Lemma 5.14. Normality of $S_{\mathfrak{p}}$ is implied by normality of $S_{(\mathfrak{p})}$

Proof. See the proof here. □

The last step is to show that S' is the integral closure of S. Since S is a domain and S' is integrally closed, we are now required to show the integral closure $\overline{S}$ contains S'. We knew

$$\Gamma(X, \mathscr{L}) = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(n)) \cong S'.$$

which means that S' itself is an integrally closed domain (being the global sections of a sheaf of integrally closed domains).

Lemma 5.15. Let $\overline{S}$ denote the integral closure of S inside its field of fractions. Then $S' \subseteq \overline{S}$.

PROOF. From Hartshorne pp. 122-123, we know S' is integral over S. Therefore, $S' \subseteq \overline{S}$. □

(b) Under sufficiently nice conditions, we have isomorphisms

$$\alpha_d : S_d \to \Gamma(X, \widetilde{S}(d)) = \Gamma(X, \mathcal{O}_X(d))$$

and we show the equality. Observe $\widetilde{S}(d) = \mathcal{O}_X(d)$ follows if we can show $\widetilde{S} = \mathcal{O}_X$. This follows from the closed subscheme correspondence result on Proj.

Furthermore, $S = k[x_0, \ldots, x_r]/I$ is a ring with $S_0 = \overline{k}$ and S_1 finitely generated as a $\overline{k}$-module. Furthermore, $\Gamma(X, \mathcal{O}_X(d)) \cong S_d$ is a f.g. S-module.

(c) We know $S_d = S'_d$ for sufficiently large d and as S has integral closure S', we conclude $S^{(d)}$ is integrally closed if we choose d large.

Now the d-uple embedding is already known to be an embedding. Since $S^{(d)}$ is integrally closed for large d, it suffices to show that $S^{(d)}$ is the homogeneous coordinate ring of the d-uple embedding.

Lemma 5.16. $S^{(d)}$ is the homogeneous coordinate ring of the d-uple embedding of X (whose homogeneous coordinate ring is S).

PROOF. From Hartshorne II.5.13, if $X = \operatorname{Proj} S$, then $\operatorname{Proj} S^{(d)} \cong X$ and the remarks of that exercise state that $\operatorname{Proj} S^{(d)}$ is the image of X under the d-uple embedding. □

(d) ($\Rightarrow$): Assume X is projectively normal. This means $S(X)$ is an integrally closed domain and hence, $S(X) = S'$. So $S(X) = (\bigoplus_{n\geq 0} \Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)))/I = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(n))$ gives the surjectivity condition indicated. For normality, we need to show that each of the local rings is an integrally close domain.

By projectively normal, $S(X)$ is an integrally closed domain and so is $S(X)_{\mathfrak{p}}$. Note that $\widetilde{S(X)} \cong \operatorname{Proj}(X)$ this gives the desired result.

($\Leftarrow$): From $\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ surjective for $n \geq 0$, we know $A[x_0, \ldots, x_r] \to S'$ is surjective. But it then follows that $S = A[x_0, \ldots, x_r]/I \cong S'$ is an isomorphism showing that S is projectively normal. The normality condition is used to show that S' is integrally closed. □

Hartshorne II.5.15 *Extension of Coherent Sheaves.* We will prove the following theorem in several steps: Let X be a noetherian scheme, let U be an open subset, and let $\mathscr{F}$ be a coherent sheaf on U. Then there is a coherent sheaf $\mathscr{F}'$ on X such that $\mathscr{F}'|_U \cong \mathscr{F}$.

(a) On a noetherian affine scheme, every quasi-coherent sheaf is the union of its coherent subsheaves. We say a sheaf $\mathscr{F}$ is the union of its subsheaves $\mathscr{F}_\alpha$ if for every open set U, the group $\mathscr{F}(U)$ is the union of the subgroups $\mathscr{F}_\alpha(U)$.

(b) Let X be an affine noetherian scheme, U an open subset, and $\mathscr{F}$ coherent on U. Then there exists a coherent sheaf $\mathscr{F}'$ on X with $\mathscr{F}'|_U \cong \mathscr{F}$.

[Hint: Let $i : U \to X$ be the inclusion map. Show that $i_*\mathscr{F}$ is quasi-coherent, then use (a).]

(c) With $X, U, \mathscr{F}$ as in (b), suppose furthermore we are given a quasi-coherent sheaf $\mathscr{G}$ on X such that $\mathscr{F} \subseteq \mathscr{G}|_U$. Show that we can find $\mathscr{F}'$ a coherent subsheaf of $\mathscr{G}$, with $\mathscr{F}'|_U \cong \mathscr{F}$.

[Hint: Use the same method, but replace $i_*\mathscr{F}$ by $\rho^{-1}(i_*\mathscr{F})$, where ρ is the natural map $\mathscr{G} \to i_*(\mathscr{G}|_U)$.]

(d) Now let X be any noetherian scheme, U an open subset, $\mathscr{F}$ a coherent sheaf on U, and $\mathscr{G}$ a quasi-coherent sheaf on X such that $\mathscr{F} \subseteq \mathscr{G}|_U$. Show that there is a coherent subsheaf $\mathscr{F}' \subseteq \mathscr{G}$ on X with $\mathscr{F}'|_U \cong \mathscr{F}$. Taking $\mathscr{G} = i_*\mathscr{F}$ proves the result announced at the beginning.

[Hint: Cover X with open affines, and extend over one of them at a time.]

(e) As an extra corollary, show that on a noetherian scheme, any quasi-coherent sheaf $\mathscr{F}$ is the union of its coherent subsheaves. [Hint: If s is a section of $\mathscr{F}$ over an open set U, apply (d) to the subsheaf of $\mathscr{F}|_U$ generated by s.]

Proof. Note Hartshorne's use of the noetherian hypothesis. If V is an open affine subset of X, let $\mathscr{F}|_V \cong \widetilde{M_V}$ for M_V a finitely generated $\mathcal{O}_X(V)$-module.

(a) Fix $X := \operatorname{Spec} A$ a a noetherian affine scheme. Let $\mathscr{F} \cong \widetilde{M}$ be a quasi-coherent sheaf in this part. Now as a module, M is a union of its f.g. submodules N_α. Each of the N_α give rise to a subsheaf $\widetilde{N_\alpha} \subseteq \widetilde{M}$ that is coherent. These are precisely the coherent subsheaves by the correspondence given by the $\sim$ functor.

(b) The sheaf $i_*\mathscr{F}$ is a sheaf on X. For quasi-coherence, observe that

$$i_*\mathscr{F}|_{D(f)} \cong \mathscr{F}|_{D(i^{-1}(f)} \cong \widetilde{M_{i^{-1}(f)}}$$

if $i^{-1}(f)$ exists and $i_*\mathscr{F}|_{D(f)} \cong 0$ when $i^{-1}(f)$ does not exist. This proves quasi-coherence and using (a), we know $i_*\mathscr{F}$ is a union of its coherent subsheaves. Let $\mathscr{G}_\alpha$ its coherent subsheaves. We need to find a finite union $\bigcup_{i\in I} \mathscr{G}_{\alpha_i}$ whose restriction to U is isomorphic to $\mathscr{F}$. Using the fact that $\mathscr{F}$ is coherent on U and that U is quasi-compact (which follows from $\operatorname{Spec} X$ being noetherian and Hartshorne I.1.7), choosing a finite index set I is always possible.

(c) First off, we have to make sure ρ^{-1} is well-defined as this has not been covered in Hartshorne yet.

Definition 5.17. If $\rho : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves (no conditions on them), then at the level of presheaves,

$$\Gamma(U, \rho^{-1}\mathscr{F}) \cong \rho_U^{-1}(\mathscr{G}(U))$$

for $\rho_U : \mathscr{F}(U) \to \mathscr{G}(U)$ the induced map on sections. Now we show that this gives a sheaf $\rho^{-1}\mathscr{F}$.

Let s be a section over some open set and assume there is a cover that open set s.t. $s|_{U_i} = 0$. Then that means $s|_{U_i} = 0$ as an element of $\Gamma(U_i, \mathscr{F})$ for all i. By the sheaf axioms on $\mathscr{F}$, we know $s = 0$.

For the gluing axiom, suppose (s_i) were sections s.t. $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all i, j pairs. Then the gluing axiom on $\mathscr{F}$ lets us glue to an s s.t. $s|_{U_i} = s_i$. It is necessary to check that $\rho_U(s) \in \mathscr{G}(U)$. Observe that $\rho_U(s)|_{U_i} = \rho_U(s_i) \in \mathscr{G}(U_i)$ for all i. It then follows that after gluing, our element lays inside $\rho_U(\mathscr{F}(U)) \in \mathscr{G}(U)$.

Our work shows that $\rho^{-1}(i_*\mathscr{F})$ is a sheaf on X.

Lemma 5.18. $\rho^{-1}(i_*\mathscr{F})$ is a quasi-coherent sheaf on X.

PROOF. Since $\mathscr{F}$ is a coherent sheaf, it is at least quasi-coherent and so $i_*\mathscr{F}$ is quasi-coherent by Hartshorne Proposition II.5.8(c).

It suffices to show that ρ^{-1} of a quasi-coherent sheaf is also one. Note that $\rho : \mathscr{G} \to i_*(\mathscr{G}|_U)$ is a morphism of quasi-coherent sheaves. Restricting to an open affine scheme V gives $\rho_V : \mathscr{G}|_V \to \mathscr{G}|_{U\cap i^{-1}(V)}$ which by quasi-coherence is a morphism $\rho_V : \widetilde{N} \to \widetilde{N}|_{U\cap i^{-1}(V)}$. The preimage of this is then quasi-coherent because it is the tilde of the submodule $(\rho_V(V)^{-1}\Gamma(i^{-1}(V)\cap U, \widetilde{N})$. □

Now we follow our proof in (b) to write $\rho^{-1}i_*\mathscr{F}$ as a union of its coherent subsheaves. A finite union of these subsheaves will always give a subsheaf of $\mathscr{G}|_U$ at least. But since we had $i_*\mathscr{F}$ and $\mathscr{F} \subseteq \mathscr{G}|_U$, there is a finite union of these subsheaves which when restricted to U is isomorphic to $\mathscr{F}$. Of course, this finite union is coherent.

(d) Cover X with open affine schemes V_i. From quasi-compactness, there are only finitely many V_i's. Also, these open affine schemes are necessarily noetherian.

Since $\mathscr{F}$ is coherent on U, it is coherent on $V_i \cap U$. Since $\mathscr{G}$ is quasi-coherent on X, it is quasi-coherent on V_i. It is clear that $\mathscr{F}|_{U\cap V_i} \cong \mathscr{F}|_{V_i} \subseteq \mathscr{G}|_{U\cap V_i}$ and $U \cap V_i \subseteq V_i$ is an open subset. So we can find a coherent subsheaf $\mathscr{F}_i' \subseteq \mathscr{G}|_{V_i}$ s.t. $\mathscr{F}_i'|_{V_i} \cong \mathscr{F}_i'|_{V_i\cap U} \cong \mathscr{F}|_{V_i}$. We call $\mathscr{F}_i'$ the extension of $\mathscr{F}$ to V_i.

Let $\mathscr{F}_i'$'s be the extensions of $\mathscr{F}$ to V_i and assume WLOG that $i = 1, 2, \ldots, l$. It is clear that the $\mathscr{F}_i'$'s agree on overlaps $V_i \cap V_j$ because

$$\mathscr{F}_i'|_{V_i\cap V_j} \cong \mathscr{F}|_{V_i\cap V_j} \cong \mathscr{F}_j'|_{V_j\cap V_i}.$$

It is also clear that the sheaves agree on the triple intersections as indicated in Hartshorne II.1.22 so we may glue to a sheaf $\mathscr{F}'$ with restrictions $\mathscr{F}'|_{V_i} \cong \mathscr{F}_i'$. It then follows $\mathscr{F}'|_{U\cap V_i} \cong \mathscr{F}|_{V_i}$ for all i and hence, $\mathscr{F}'|_U \cong \mathscr{F}$.

Taking $\mathscr{G} = i_*\mathscr{F}$ and noting $\mathscr{G}$ is a quasi-coherent sheaf on X, we get the stated result at the beginning.

(e) If $s \in \mathscr{F}(U)$, we can look at the subsheaf of $\mathscr{F}|_U$ generated by s. From (d), we can extend s to a coherent subsheaf $\mathscr{F}_s \subseteq \mathscr{F}$ on X whose restriction to U is precisely the sheaf generated by s. Now consider the extensions $\mathscr{F}_{(s_1,\ldots,s_n)}$ of the subsheaf of $\mathscr{F}|_U$ generated by $s_1, \ldots, s_n$. As there are only finitely many open subsets U that we need to consider, it is then clear from (a) and by requiring our open set to be affine that the statement of the theorem holds. □

Hartshorne II.5.16 *Tensor Operations on Sheaves.* First we recall the definitions of various tensor operations on a module.

Let A be a ring, and let M be an A-module.

Let $T^n(M)$ be the tensor product $M \otimes \ldots \otimes M$ of M with itself n times, for $n \geqslant 1$. For $n = 0$ we put $T^0(M) = A$. Then $T(M) = \bigoplus_{n\geqslant 0} T^n(M)$ is a (noncommutative) A -algebra, which we call the **tensor algebra** of M.

We define the **symmetric algebra** $S(M) = \bigoplus_{n\geqslant 0} S^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes y - y \otimes x$, for all $x, y \in M$. Then $S(M)$ is a commutative A -algebra. Its component $S^n(M)$ in degree n is called the n**th**

symmetric product of M. We denote the image of $x \otimes y$ in $S(M)$ by xy, for any $x, y \in M$. As an example, note that if M is a free A -module of rank r, then $S(M) \cong A[x_1, \ldots, x_r]$.

We define the **exterior algebra** $\bigwedge(M) = \bigoplus_{n \geqslant 0} \bigwedge^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes x$ for $x \in M$. Note that this ideal contains all expressions of the form $x \otimes y + y \otimes x$ so that $\bigwedge(M)$ is a **skew commutative graded A -algebra**. This means that if $u \in \bigwedge^r(M)$ and $v \in \bigwedge^s(M)$, then $u \wedge v = (-1)^{rs} v \wedge u$ (here we denote by $\wedge$ the multiplication in this algebra; so the image of $x \otimes y$ in $\bigwedge^2(M)$ is denoted by $x \wedge y$). The nth component $\bigwedge^n(M)$ is called the **nth exterior power** of M

Now let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the **tensor algebra, symmetric algebra, and exterior algebra of** $\mathscr{F}$ by taking the sheaves associated to the presheaf, which to each open set U assigns the corresponding tensor operation applied to $\mathscr{F}(U)$ as an $\mathcal{O}_X(U)$ -module. The results are $\mathcal{O}_X$ -algebras, and their components in each degree are $\mathcal{O}_X$ -modules.

(a) Suppose that $\mathscr{F}$ is locally free of rank n. Then $T^r(\mathscr{F})$, $S^r(\mathscr{F})$, and $\bigwedge^r(\mathscr{F})$ are also locally free, of ranks n^r, $\binom{n+r-1}{n-1}$, and $\binom{n}{r}$ respectively.

(b) Again let $\mathscr{F}$ be locally free of rank n. Then the multiplication map $\bigwedge^r \mathscr{F} \otimes \bigwedge^{n-r} \mathscr{F} \to \bigwedge^n \mathscr{F}$ is a perfect pairing for any r, i.e., it induces an isomorphism of $\bigwedge^r \mathscr{F}$ with $\left(\bigwedge^{n-r} \mathscr{F}\right)^{\smile} \otimes \bigwedge^n \mathscr{F}$. As a special case, note if $\mathscr{F}$ has rank 2 then $\mathscr{F} \cong \mathscr{F}^{\smile} \otimes \bigwedge^2 \mathscr{F}$.

(c) Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of locally free sheaves. Then for any r there is a finite filtration of $S^r(\mathscr{F})$

$$S^r(\mathscr{F}) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong S^p(\mathscr{F}') \otimes S^{r-p}(\mathscr{F}'')$$

for each p.

(d) Same statement as (c), with exterior powers instead of symmetric powers. In particular, if $\mathscr{F}', \mathscr{F}, \mathscr{F}''$ have ranks n', n, n'' respectively, there is an isomorphism $\bigwedge^n \mathscr{F} \cong \bigwedge^{n'} \mathscr{F}' \otimes \bigwedge^{n''} \mathscr{F}''$

(e) Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be an $\mathcal{O}_Y$ -module. Then f^* commutes with all the tensor operations on $\mathscr{F}$, i.e., $f^*(S^n(\mathscr{F})) = S^n(f^*\mathscr{F})$ etc.

PROOF. **(a)** Suppose $\mathscr{F}$ is locally free of rank n. Suppose $\{U_i\}$ is a cover of X on which $\mathscr{F}|_{U_i} \cong \mathcal{O}_X(U_i)^n$.

First, $T^r(\mathscr{F})$ is locally free of rank N because over each $U \in \{U_i\}$, we have

$$T^r(\mathscr{F})|_U \cong \mathscr{F}|_U \otimes \cdots \otimes \mathscr{F}|_U$$

where there are r copies of $\mathscr{F}|_U$ and so, the RHS is a tensor product of r copies of modules of rank n. Therefore, the $\mathcal{O}_X|_U$-module on the RHS is of rank n^r which means $T^r(\mathscr{F})$ is locally free of rank n^r.

For the remaining two, for ease, we delegate the proof to repeating the same idea as above (consider $S^r(\mathscr{F})$ and $\bigwedge^r(\mathscr{F})$ over each open $U \in \{U_i\}$ and use the fact that $(-)|_U$ commutes with $\otimes$ and $\oplus$) and the using the fact that the rth symmetric power and rth exterior power of a free module has the given rank. For a reference, see Corollary A2.3 of

Eisenbud's Commutative Algebra text or Section 11.5 of Dummit and Foote (they prove the results for vector spaces, but those can generalize to free modules in general which they note as well). A helpful online reference can be found in these notes or in Keith Conrad's blurb.

Warning. Though each $T^r(\mathscr{F})$, $S^r(\mathscr{F})$, and $\bigwedge^r \mathscr{F}$ are locally free when $\mathscr{F}$ is locally free, this needs not be the case for $T(\mathscr{F}), S(\mathscr{F}), \bigwedge \mathscr{F}$. The proof itself can be found at [10, Lemma 01CL].

(b) Being pedantic, one must check that the multiplication map can be defined. Restricting to an open affine subset $U = \operatorname{Spec} A$, we may define $\left(\bigwedge^r \mathscr{F} \otimes \bigwedge^{n-r} \mathscr{F}\right)|_U \to \left(\bigwedge^n \mathscr{F}\right)|_U$ as follows. One any distinguished open subset $D(f) \subseteq U$, we define the map on sections $\bigwedge^r A_f^m \otimes_{A_f} \bigwedge^{n-r} A_f^m \to \bigwedge^n A_f^m$ by $(u_1 \wedge \cdots \wedge u_r) \otimes (v_1 \wedge \cdots \wedge v_{n-r}) \mapsto u_1 \wedge \cdots \wedge u_r \wedge v_1 \wedge \cdots \wedge v_{n-r}$ where $u_i, v_j \in A_f^m$.

We show $\bigwedge^r \mathscr{F} \otimes \bigwedge^{n-r} \mathscr{F} \to \bigwedge^n \mathscr{F}$ induces an isomorphism. This can be done locally in a canonical manner and thus, we get the isomorphism globally. Indeed, there is the following lemma.

Lemma 5.19. Let M be an A-module. Then the multiplication map

$$\bigwedge^r M \otimes_A \bigwedge^{n-r} M \to \bigwedge^n M$$

defined by mapping

$$(m_1 \wedge \cdots \wedge m_r) \otimes (m_1' \wedge \cdots \wedge m_{n-r}') \mapsto m_1 \wedge \cdots \wedge m_r \wedge m_1' \wedge \cdots \wedge m_{n-r}'$$

is a perfect pairing. That is, there is an isomorphism $\bigwedge^r M \to \left(\bigwedge^{n-r} M\right)^\smile \otimes_A \bigwedge^n M$.

Proof. This is a standard result in the literature. Combine Proposition 22.1 of this note and the proof of Lemma A.29 of this appendix note. □

In the stated special case, we take $n = 2$ and $1 =$ and so,

$$\mathscr{F}^\smile \otimes \bigwedge^2 \mathscr{F} \cong \left(\bigwedge^{2-1} \mathscr{F}\right)^\smile \otimes \bigwedge^2 \mathscr{F} \cong \bigwedge^1 \mathscr{F} \cong \mathscr{F}.$$

(c)[44] Choose an open affine set $U = \operatorname{Spec} A$, where we might shrink as needed, such that $\mathscr{F}'$, $\mathscr{F}$, and $\mathscr{F}''$ are free of rank n on U. Then, we get an exact sequence

$$0 \to \mathcal{O}_{X|U}^l \to \mathcal{O}_{X|U}^m \to \mathcal{O}_{X|U}^n \to 0 \qquad \longleftrightarrow \qquad 0 \to A^l \to A^m \to A^n \to 0$$

and this sequence splits so that we have an isomorphism $\mathcal{O}_{X|U}^m \cong \mathcal{O}_{X|U}^n \oplus \mathcal{O}_{X|U}^l \cong \mathcal{O}_{X|U}^{l+n}$ or just $A^m \cong A^{l+n}$. Choose basis $(e_1, \ldots, e_l)$ and $(f_1, \ldots, f_n)$ for A^l and A^n and identify them with the images and preimages $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ in A^m which form a basis for A^m. From this, we have[45]

$$S^r(\mathscr{F})(U) \cong S^r(A^m) \cong \bigoplus_{i=0}^r S^i(A^l) \otimes S^{r-i}(A^n) \cong \bigoplus_{i=0}^r S^i(\mathscr{F}')(U) \otimes_{\mathcal{O}_X(U)} S^{r-i}(\mathscr{F}'')(U).$$

[44] For this exercise, I referred to Vakil's FOAG p. 385 which outlined two approaches to this problem only one of which I felt comfortable tackling.

[45] For a quick reference on this isomorphism: here.

Now define $F^p(U) := \bigoplus_{i=p}^r S^i(\mathscr{F}')(U) \otimes_{\mathcal{O}_X(U)} S^{r-i}(\mathscr{F}'')(U)$ and it is clear that $F^p/F^{p+1}(U)$ has the desired isomorphism. Furthermore, it is immediate from how we define F^p that

$$S^r(\mathscr{F})(U) = F^0(U) \supseteq F^1(U) \supseteq \cdots \supseteq F^r(U) \supseteq F^{r+1}(U) = 0.$$

We must show that this construction is independent of the splitting we chose. Suppose $(e'_1, \ldots, e'_l)$ and $(f'_1, \ldots, f'_n)$ were another choice of splitting. Notice that by definition, $F^{r+1}(U) = 0$ so this is certainly independent of our splitting. Furthermore, notice that $F^r(U) \cong S^r(A^l) \otimes S^{r-r}(A^n) \cong S^r(A^l) \otimes A \cong S^r(A^l)$ which is also independent of our choice of splitting[46]. This deals with the base case (and we considered $F^1(U)$ to clarify what is going on).

Now, suppose we were able to determine the choices of $F^{k+1}(U), F^{k+2}(U), \ldots, F^r(U), F^{r+1}(U)$ and showed that these were independent of splitting. We claim that once we construct $F^k(U)$, it is independent of the choice of splitting. Since we constructed $F^k(U)$ to satisfy $F^k/F^{k+1}(U) \cong S^k(A^l) \otimes S^{r-k}(A^m)$, we know that there is a natural multiplication map

$$S^k(A^l) \otimes S^{r-k}(A^m) \to S^r(A^n)$$

defined by $(e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(k)}, f_{\sigma(k+1)} \otimes \cdots \otimes f_{\sigma(r)}) \to e_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(r)}$ (where the σ is meant to indicate that we may have permuted the indices) and then extended to $S^k(A^l) \otimes S^{r-k}(A^m)$ using the universal property of tensor products[47]. Since $F^{k+1}(U) \subseteq F^k(U) \subseteq S^r(A^n)$ are submodules, we can consider

$$F^k/F^{k+1}(U) \twoheadrightarrow S^r(A^n)/F^{k+1}(U).$$

given by the isomorphism we constructed, then composed with the multiplication map and then with the surjection onto the quotient. Then $F^k(U)$ is defined by taking the preimage of the above map.

Now, we show that the choice of splitting does not matter in doing this. Indeed, a choice of a different splitting would yield an isomorphism of $\mathscr{F}$ which leaves $S^r(\mathscr{F})$ and the F^i invariant.

(d) The statement we want to show is: if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of locally free sheaves, then for all r, there exists a finite filtration

$$\bigwedge^r(\mathscr{F}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^{r+1} = 0$$

and for all p,

$$F^p/F^{p+1} \cong \bigwedge^p(\mathscr{F}') \otimes \bigwedge^{r-p}(\mathscr{F}'').$$"

The proof is to do same thing as in part (c).

(e) First, we show that f^* commutes with the tensor algebra of $\mathscr{F}$ when given $f : X \to Y$ a morphism of ringed spaces with $\mathscr{F}$ an $\mathcal{O}_Y$-module. Because $T(\mathscr{F}) = \bigoplus_{n \geq 0} T^n(\mathscr{F})$, it suffices to show that f^* commutes with $T^n(\mathscr{F})$ for *every* n. We proceed by induction on $n \in \mathbb{N}_0$. For the base case, $T^0(\mathscr{F})$ is defined by $T^0(\mathscr{F})(U) \cong A$ over any affine $U = \operatorname{Spec} A$ on which $\mathscr{F}$ is free and so, $T^0(\mathscr{F}) \cong \mathcal{O}_Y$. In particular,

$$f^*T^0(\mathscr{F}) \cong f^{-1}T^0(\mathscr{F}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong f^{-1}\mathcal{O}_Y \otimes \mathcal{O}_X \cong \mathcal{O}_X \cong T^0(\mathcal{O}_X) \cong T^0(f^*\mathscr{F}).$$

[46] Just identify $(e_1, \ldots, e_n)$ with $(e'_1, \ldots, e'_n)$ with an isomorphism $A^l \cong A^l$.

[47] Need to check bilinearity, but it looks to be true.

Now suppose the result held true for some $n \in \mathbb{N}_0$ i.e. $f^*T^n(\mathscr{F}) \cong T^n(f^*\mathscr{F})$. Then,

$$\begin{aligned}
f^*T^{n+1}(\mathscr{F}) &\cong f^{-1}(T^{n+1}(\mathscr{F})) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong f^{-1}(\mathscr{F}^{\otimes n+1}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\
&\cong f^{-1}(\mathscr{F} \otimes \mathscr{F}^{\otimes n}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\
&\cong f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathscr{F}^{\otimes n}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X && (*) \\
&\cong f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} f^*(T^n(\mathscr{F})) \\
&\cong f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} T^n(f^*\mathscr{F}) && \text{(inductive hypothesis)} \\
&\cong f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \otimes_{\mathcal{O}_X} T^n(f^*\mathscr{F}) \\
&\cong f^*\mathscr{F} \otimes_{\mathcal{O}_X} T^n(f^*\mathscr{F}) \\
&\cong T^{n+1}(f^*\mathscr{F}).
\end{aligned}$$

The step at (*) uses the fact that f^{-1} commutes with tensor products. Now we prove the result for $S(\mathscr{F})$ and $\bigwedge \mathscr{F}$. We observe a few facts.

Lemma 5.20. Let M be an A-module and B an A-algebra. Then there is a natural isomorphism

$$\mathrm{Hom}_{A-\mathrm{algebra}}(S(M), B) \cong \mathrm{Hom}_A(M, B).$$

Lemma 5.21. Let $\varphi : A \to B$ be a ring homomorphism. Then

$$S_A(M) \otimes_A B \cong S_B(M \otimes_A B)$$

for any A-module M.

Proof. From the previous lemma, we have

$$\begin{aligned}
\mathrm{Hom}_{B-\mathrm{algebra}}(S_B(M \otimes_A B), C) &\cong \mathrm{Hom}_B(M \otimes_A B, C) \\
&\cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(B, C)) \cong \mathrm{Hom}_A(M, C) \\
&\cong \mathrm{Hom}_{A-\mathrm{algebra}}(S(M), C) \\
&\cong \mathrm{Hom}_{B-\mathrm{algebra}}(S(M) \otimes_A B, C)
\end{aligned}$$

for any B-algebra C. The last isomorphism follows from the fact C is assumed to be a B-algebra so the extension of scalars identifies the two Hom-sets. □

Lemma 5.22. Because f^{-1} is an exact functor, $f^{-1}S(\mathscr{F}) \cong S(f^{-1}\mathscr{F})$ and $f^{-1}(\bigwedge \mathscr{F}) \cong \bigwedge f^{-1}\mathscr{F}$.

For $f^*S(\mathscr{F}) \cong S(f^*\mathscr{F})$, it suffices to prove the result locally. We utilize Hartshorne Proposition II.5.2 and II.5.8(b). Choose an open affine subsets $\mathrm{Spec}\, A \subseteq X$ and $\mathrm{Spec}\, B \subseteq Y$ so that $f|_{\mathrm{Spec}\, A} : \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ and so that $\mathscr{F}|_{\mathrm{Spec}\, B} \cong \widetilde{M}$ and $f_*(\mathscr{F})|_{\mathrm{Spec}\, A} \cong \widetilde{M \otimes_B A}$. Then,

$$\begin{aligned}
S(f^*\mathscr{F})|_{\mathrm{Spec}\, A} &\cong S(f^*\mathscr{F}|_{\mathrm{Spec}\, A}) \cong S(\widetilde{M \otimes_B A}|_{\mathrm{Spec}\, A}) \cong S(\widetilde{M \otimes_B A})\ |_{\mathrm{Spec}\, A} \\
&\cong S_B(\widetilde{M}) \otimes_B A\ |_{\mathrm{Spec}\, A} \cong \widetilde{S_B(M)} \otimes_{f^{-1}\mathcal{O}_{\mathrm{Spec}\, B}} \mathcal{O}_{\mathrm{Spec}\, A}\ |_{\mathrm{Spec}\, A} \\
&\cong f^{-1}S(\widetilde{M}) \otimes_{f^{-1}\mathcal{O}_{\mathrm{Spec}\, B}} \mathcal{O}_{\mathrm{Spec}\, A}\ |_{\mathrm{Spec}\, A} \\
&\cong f^{-1}S(\mathscr{F}) \otimes_{f^{-1}\mathcal{O}_{\mathrm{Spec}\, B}} \mathcal{O}_{\mathrm{Spec}\, A}\ |_{\mathrm{Spec}\, A} \\
&\cong f^*S(\mathscr{F})\ |_{\mathrm{Spec}\, A}.
\end{aligned}$$

Note that we used Lemma 5.21 for the fifth isomorphism. All the isomorphisms above are canonical so they induce the isomorphism globally. Now the result for $\bigwedge \mathscr{F}$ follows in a similar fashion if we note the following lemma (which is an analogue of Lemma 5.21 for exterior algebras) is true.

Lemma 5.23. Let $\varphi : A \to B$ be a ring homomorphism. Let M be an A-module. Then

$$\left(\bigwedge_A^n M\right) \otimes_A B \cong \bigwedge_B^n (M \otimes_A B).$$

□

Remark 28. In case I want to check my work, I think [4] has a proof for part (b). The approach I took for (c) and (d) is from Exercise 13.5.E of [9]. The method of proof for (e) came from Görtz and Wedhorn's book "Algebraic Geometry I: Schemes with Examples and Exercises".

Hartshorne II.5.17 *Affine Morphisms.* A morphism $f : X \to Y$ of schemes is *affine* if there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is affine for each i

(a) Show that $f : X \to Y$ is an affine morphism if and only if for every open affine $V \subseteq Y$, $f^{-1}(V)$ is affine.

[Hint : Reduce to the case Y affine, and use (Ex. 2.17).]

(b) An affine morphism is quasi-compact and separated. Any finite morphism is affine.

(c) Let Y be a scheme, and let $\mathscr{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras (i.e., a sheaf of rings which is at the same time a quasi-coherent sheaf of $\mathcal{O}_Y$-modules). Show that there is a unique scheme X, and a morphism $f : X \to Y$, such that for every open affine $V \subseteq Y$, $f^{-1}(V) \cong \operatorname{Spec} \mathscr{A}(V)$, and for every inclusion $U \subset V$ of open affines of Y, the morphism $f^{-1}(U) \hookrightarrow f^{-1}(V)$ corresponds to the restriction homomorphism $\mathscr{A}(V) \to \mathscr{A}(U)$. The scheme X is called $\mathbf{Spec}\,\mathscr{A}$.

[Hint: Construct X by glueing together the schemes Spec $\mathscr{A}(V)$, for V open affine in Y.]

(d) If $\mathscr{A}$ is a quasi-coherent $\mathcal{O}_Y$-algebra, then $f : X = \mathbb{S}_!\mathscr{A} \to Y$ is an affine morphism, and $\mathscr{A} \cong f_*\mathcal{O}_X$. Conversely, if $f : X \to Y$ is an affine morphism, then $\mathscr{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, and $X \cong \mathbb{S}_!\mathscr{A}$

(e) Let $f : X \to Y$ be an affine morphism, and let $\mathscr{A} = f_*\mathcal{O}_X$. Show that f_* induces an equivalence of categories from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of quasi-coherent $\mathscr{A}$-modules (i.e., quasi-coherent $\mathcal{O}_{\mathbb{Y}}$-modules having a structure of $\mathscr{A}$-module).

[Hint: For any quasi-coherent $\mathscr{A}$-module $\mathscr{M}$, construct a quasi-coherent $\mathcal{O}_X$ -module $\widetilde{\mathscr{M}}$, and show that the functors f_* and $\tilde{}$ are inverse to each other.]

Proof. **(a)** The ($\Leftarrow$) direction is obvious so we prove the converse. Assume $f : X \to Y$ is an affine morphism. Let $V \subseteq Y$ be an arbitrary open affine and $\{V_i\}$ be the affine cover of Y s.t. $f^{-1}(V_i)$ is affine for each i. Looking at $V \cap V_i$, Nike's trick implies $V \cap V_i = \bigcup_j D(f_{ij})$ where $D(f_{ij})$ is distinguished in both V and V_i. It follows that $f^{-1}(D(f_{ij})) \subseteq f^{-1}(V_i)$ are

open affine schemes. Writing V as a union of $D(f_{ij})$, it follows that $f : f^{-1}(V) \to V$ is an affine morphism as well. So, we reduce to the case Y is affine.

Assume $f : X \to \operatorname{Spec}(B) =: Y$. It suffices to show that $f^{-1}(\operatorname{Spec}(B))$ is affine. Now use Hartshorne II.2.17(b) to show $f^{-1}(\operatorname{Spec}(B))$ is affine.

Fix $A := \Gamma(f^{-1}(Y), \mathcal{O}_X|_{f^{-1}(Y)})$. First, cover $\operatorname{Spec}(B)$ by finitely many open affine sets $\{V_i\}$ with $f^{-1}(V_i)$ affine. For each $f^{-1}(V_i) \subseteq f^{-1}(Y)$, we cover $f^{-1}(V_i)$ with finitely many distinguished open sets $D(f_{ij})$. Using Hartshorne II.2.16, $\Gamma(f^{-1}(Y)_{f_{ij}}, \mathcal{O}_{f^{-1}(Y)}|_{D(f_{ij})}) \cong A_{f_{ij}}$ and the f_{ij} generate the unit ideal in A because the $D(f_{ij})$ cover $f^{-1}(Y)$.

(b) Using (a), for any cover of Y by open affines V_i, the subset $f^{-1}(V_i)$ is affine and hence, quasicompact. For separatedness, the morphisms $f^{-1}(V_i) \to V_i$ are always a morphism of affine schemes and Hartshorne Proposition 4.1 shows this morphism is separated. By Hartshorne Corollary II.4.6(f), $f : X \to Y$ is separated.

Let $f : X \to Y$ be a finite morphism. Hartshorne II.3.4 says that for every open affine $V = \operatorname{Spec} B \subseteq Y$, we have $f^{-1}(V) = \operatorname{Spec} A$. From part (a), this means $f : X \to Y$ is affine.

(c) We follow the hint and construct X via gluing together the schemes $\operatorname{Spec} \mathscr{A}(V)$ for any open affine $V \subseteq Y$. Note that $\operatorname{Spec} \mathscr{A}(V)$ makes sense because $\mathscr{A}(V)$ is a ring. Let $\operatorname{Spec} A_i$ be an open affine cover of Y so that $\mathscr{A}(\operatorname{Spec} A_i) \cong \widetilde{M_i}$ where M_i is an A_i-algebra.

To glue schemes, let $U, V, W \subseteq Y$ be open affine schemes. Define the isomorphism $\varphi_{UV} : \operatorname{Spec} \mathscr{A}(U) \cap \operatorname{Spec} \mathscr{A}(V) \to \operatorname{Spec} \mathscr{A}(V) \cap \operatorname{Spec} \mathscr{A}(U)$ in the obvious sense. It is clear that $\varphi_{UV} = \varphi_{VU}^{-1}$. Also,

$$\begin{aligned}&\varphi_{UV}(\operatorname{Spec} \mathscr{A}(U) \cap \operatorname{Spec} \mathscr{A}(V) \cap \operatorname{Spec} \mathscr{A}(U) \cap \operatorname{Spec} \mathscr{A}(W))\\ &\quad = \operatorname{Spec} \mathscr{A}(V) \cap \operatorname{Spec} \mathscr{A}(U) \cap \operatorname{Spec} \mathscr{A}(V) \cap \operatorname{Spec} \mathscr{A}(W).\end{aligned}$$

and it is easy to see that $\varphi_{UW} = \varphi_{UV} \circ \varphi_{VW}$ on the scheme $\operatorname{Spec} \mathscr{A}(U) \cap \operatorname{Spec} \mathscr{A}(V) \cap \operatorname{Spec} \mathscr{A}(U) \cap \operatorname{Spec} \mathscr{A}(W)$. From Hartshorne II.2.12, we can glue to a scheme X with isomorphisms $\psi_V : \operatorname{Spec} \mathscr{A}(V) \to X$ satisfying some conditions. Covering X by these $\operatorname{Spec} \mathscr{A}(V)$, we always have a morphism

$$f_V : \operatorname{Spec} \mathscr{A}(V) \cong f^{-1}(V) \to V$$

as follows: $\mathscr{A}|_V \cong \widetilde{M_V}$ for M_V an A-algebra for $V = \operatorname{Spec} A$ which means there is a natural morphism $A \to M_V =: \mathscr{A}(V)$ which then induce a morphism $\operatorname{Spec} \mathscr{A}(V) \to \operatorname{Spec} A$ going the other direction.

It is clear that the morphisms f_V behave well w.r.t. the restriction maps. Indeed, let $U \subseteq V$ be open affine sets. Then there is a diagram

$$\begin{array}{ccc} \mathscr{A}(V) \cong M_V & \longleftarrow & \mathcal{O}_Y(V) \cong B \\ \downarrow & & \downarrow \\ \mathscr{A}(U) \cong M_U & \longleftarrow & \mathcal{O}_Y(U) \cong A \end{array}$$

and applying $\operatorname{Spec}(-)$, we get morphisms going the other direction. This shows that the f_V behave well w.r.t. restrictions and also that $f^{-1}(U) \hookrightarrow f^{-1}(V)$ corresponds to the restriction homomorphism $\mathscr{A}(V) \to \mathscr{A}(U)$.

Uniqueness of the scheme X follows from the condition that $f^{-1}(V) \cong \operatorname{Spec} \mathscr{A}(V)$ for all open affine $V \subseteq Y$.

(d) To see that $f : X \to Y$ is an affine morphism, we use (a). For any open affine $V \subseteq Y$, we have $f^{-1}(V) = \operatorname{Spec} \mathscr{A}(V)$ which is affine.

To show $\mathscr{A} \cong f_*\mathcal{O}_X$, observe that for any open affine $V \subseteq Y$,

$$\mathscr{A}(V) \cong \mathcal{O}_X(f^{-1}(V)) \cong \mathcal{O}_{\operatorname{Spec} \mathscr{A}(V)}(V) \cong \mathscr{A}(V).$$

It is then clear how to define the morphism and it is an easy check from (b) that these morphisms agree on restrictions. Letting V vary over an open cover of Y, we get the desired isomorphism $\mathscr{A} \cong f_*\mathcal{O}_X$.

Conversely, let $f : X \to Y$ be an affine morphism. Set $\mathscr{A} := f_*\mathcal{O}_X$ as a sheaf on Y. We know from (b) that f is quasi-compact and separated and Hartshorne Proposition II.5.8 implies[48] $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules.

Now we prove the isomorphism $X \cong \mathbf{Spec}\, \mathscr{A}$. This follows from the uniqueness of $\mathbf{Spec}\, \mathscr{A}$ if we can show X satisfies the properties in (c).

For $V \subseteq Y$ be an open affine, then $f^{-1}(V) = \operatorname{Spec} A$ is an affine scheme. Now $f^{-1}(V) \cong \operatorname{Spec} \mathscr{A}(V)$ because $\mathscr{A}(V) \cong \mathcal{O}_X(f^{-1}(V)) \cong A$. It is then clear that $f^{-1}(U) \hookrightarrow f^{-1}(V)$ corresponds to the restriction $\mathscr{A}(V) \to \mathscr{A}(U)$.

(e) Everything we did from above can be summarized in the proof of the following lemma.

Lemma 5.24. Let Y be a scheme. There is an anti-equivalence of categories

$$\left\{\text{schemes affine over } Y\right\} \leftrightarrow \left\{\begin{matrix}\text{quasi-coherent sheaves of} \\ \mathcal{O}_Y\text{-algebras}\end{matrix}\right\}$$

which associates to $f : X \to Y$ the module $f_*\mathcal{O}_Y$.

Proof. From above, $f_*\mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_Y$-algebra. The map from right to left associates to a quasi-coherent $\mathcal{O}_Y$-algebra $\mathscr{M}$ the scheme $f : \mathbf{Spec}\, \mathscr{M} \to Y$. Applying this to $f_*\mathcal{O}_Y$ recovers $\mathscr{M}$ from (d).

Going the other way, an $\mathscr{M}$ on the RHS is mapped to $g : \mathbf{Spec}\, \mathscr{M} \to Y$. Then the second part of (d) gives $g_*\mathcal{O}_Y \cong \mathscr{M}$. □

Now we prove the stated equivalence of categories:

Lemma 5.25. Fixing $f : X \to Y$ an affine morphism, $\mathscr{A} = f_*\mathcal{O}_X$, there is an equivalence of categories induced by f_*

$$\left\{\text{quasi-coherent } \mathcal{O}_X\text{-modules}\right\} \leftrightarrow \left\{\text{quasi-coherent } \mathscr{A}\text{-modules}\right\}.$$

Proof. Define $\widetilde{\mathscr{M}}$ for any $\mathscr{M}$ on the RHS as follows. A priori it is an $f_*\mathcal{O}_X$-module. For any affine $V \subseteq Y$, let $f^{-1}(V) = \operatorname{Spec} B$. Then,

$$\mathscr{M} = f_*\mathcal{O}_X|_V = \mathcal{O}_X|_{f^{-1}(V)} = \mathcal{O}_X|_{\operatorname{Spec} B} = \widetilde{M_B}$$

for some B-module M_B. Now cover Y by affine subsets V_i. These then give a cover $f^{-1}(V_i)$ of X which determine $\mathscr{M}|_{f^{-1}(V_i)}$. As the $f^{-1}(V_i)$ cover X and $\mathscr{M}|_{f^{-1}(V_i)}$ all agree on restrictions, we may glue to a sheaf $\widetilde{\mathscr{M}}$. From this construction, $\widetilde{\mathscr{M}}$ gets an $\mathcal{O}_X$-module structure.

Define $\widetilde{\mathscr{M}}$ for any $\mathscr{M}$ on the RHS as follows. Cover Y by open affines $V_i = \operatorname{Spec} A_i$ and assume $U_i = f^{-1}(V_i) = \operatorname{Spec} A_i$. There is a map $B_i \to A_i$ from f and by quasi-coherence, $\mathscr{M}|_{V_i} = \widetilde{N_i^B}$ for N_i^B a B-module. We then define $\widetilde{M}|_{f^{-1}(V_i)} = \widetilde{N_i^A}$ for N_i^A denoting the restriction of scalars of N_i^B to A_i. It is clear from the fact that $\operatorname{Spec}(A_i)_g$ corresponds to

[48] Clearly, $\mathcal{O}_X$ is always a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

$\operatorname{Spec}(B_i)_g$ that the $\widetilde{M}|_{f^{-1}(V_i)}$ will glue together to give a sheaf $\widetilde{M}$. Since there is a cover by open affines s.t. the sheaf restricted to V_i is the tilde of $\mathcal{O}_X(f^{-1}(V_i))$, we know $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module.

It follows that $f_*\widetilde{\mathscr{M}} \cong \mathscr{M}$ and $\widetilde{f_*\mathscr{L}} \cong \mathscr{L}$ for $\mathscr{L}$ in the LHS via a stalk computation and using the canonical maps. □

□

Remark 29. This exercise gives an explicit way to view X when $f : X \to Y$ is an affine morphism. See [10, Tag 01S5] for more. Corollary 1.3.2 of [**EGA**] II is also helpful. A closed immersion is an affine morphism and so, this ties in with Hartshorne Corollary II.5.10.

Hartshorne II.5.18 *Vector Bundles.* Let Y be a scheme. *A* **(geometric) vector bundle of rank** n over Y is a scheme X and a morphism $f : X \to Y$, together with additional data consisting of an open covering $\{U_i\}$ of Y, and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbb{A}^n_{U_i}$, such that for any i, j, and for any open affine subset $V = \operatorname{Spec} A \subseteq U_i \cap U_j$ the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{A}^n_V = \operatorname{Spec} A[x_1, \ldots, x_n]$ is given by a **linear automorphism** θ of $A[x_1, \ldots, x_n]$, i.e., $\theta(a) = a$ for any $a \in A$, and $\theta(x_i) = \sum a_{ij}x_j$ for suitable $a_{ij} \in A$.

An **isomorphism** $g : (X, f, \{U_i\}, \{\psi_i\}) \to (X', f', \{U_i'\}, \{\psi_i'\})$ of one vector bundle of rank n to another one is an isomorphism $g : X \to X'$ of the underlying schemes, such that $f = f' \circ g$, and such that X, f, together with the covering of Y consisting of all the U_i and U_i', and the isomorphisms ψ_i and $\psi_i' \circ g$, is also a vector bundle structure on X.

(a) Let $\mathscr{E}$ be a locally free sheaf of rank n on a scheme Y. Let $S(\mathscr{E})$ be the symmetric algebra on $\mathscr{E}$, and let $X = \mathbf{Spec} S(\mathscr{E})$, with projection morphism $f : X \to Y$. For each open affine subset $U \subseteq Y$ for which $\mathscr{E}|_U$ is free, choose a basis of $\mathscr{E}$, and let $\psi : f^{-1}(U) \to \mathbb{A}^n_U$ be the isomorphism resulting from the identification of $S(\mathscr{E}(U))$ with $\mathcal{O}(U)[x_1, \ldots, x_n]$. Then $(X, f, \{U\}, \{\psi\})$ is a vector bundle of rank n over Y, which (up to isomorphism) does not depend on the bases of $\mathscr{E}_U$ chosen. We call it the **geometric vector bundle associated to** $\mathscr{E}$, and denote it by $\mathbb{V}(\mathscr{E})$.

(b) For any morphism $f : X \to Y$, a **section of** f over an open set $U \subseteq Y$ is a morphism $s : U \to X$ such that $f \circ s = \mathrm{id}_U$. It is clear how to restrict sections to smaller open sets, or how to glue them together, so we see that the presheaf $U \mapsto \{$ set of sections of f over $U\}$ is a sheaf of sets on Y, which we denote by $\mathscr{S}(X/Y)$. Show that if $f : X \to Y$ is a vector bundle of rank n, then the sheaf of sections $\mathscr{S}(X/Y)$ has a natural structure of $\mathcal{O}_Y$ -module, which makes it a locally free $\mathcal{O}_Y$-module of rank n.

[Hint: It is enough to define the module structure locally, so we can assume $Y = \operatorname{Spec} A$ is affine, and $X = \mathbb{A}^n_Y$. Then a section $s : Y \to X$ comes from an A -algebra homomorphism $\theta : A[x_1, \ldots, x_n] \to A$, which in turn determines an ordered n-tuple $\langle \theta(x_1), \ldots, \theta(x_n) \rangle$ of elements of A. Use this correspondence between sections s and ordered n-tuples of elements of A to define the module structure.]

(c) Again let $\mathscr{E}$ be a locally free sheaf of rank n on Y, let $X = \mathbb{V}(\mathscr{E})$, and let $\mathscr{S} = \mathscr{S}(X/Y)$ be the sheaf of sections of X over Y. Show that $\mathscr{S} \cong \mathscr{E}^\smile$, as follows.

Given a section $s \in \Gamma(V, \mathscr{E}^\smile)$ over any open set V, we think of s as an element of $\operatorname{Hom}(\mathscr{E}|_V, \mathcal{O}_V)$. So s determines an $\mathcal{O}_V$-algebra homomorphism $S(\mathscr{E}|_V) \to \mathcal{O}_V$.

This determines a morphism of spectra $V = \mathbf{Spec}\mathcal{O}_V \to \mathbf{Spec}S\left(\mathscr{E}|_V\right) = f^{-1}(V)$, which is a section of X/Y.

Show that this construction gives an isomorphism of $\mathscr{E}^\smile$ to $\mathscr{S}$.

(d) Summing up, show that we have established a one-to-one correspondence between isomorphism classes of locally free sheaves of rank n on Y, and isomorphism classes of vector bundles of rank n over Y. Because of this, we sometimes use the words "locally free sheaf" and "vector bundle" interchangeably, if no confusion seems likely to result.

PROOF. **(a)** First, we check that this indeed gives a vector bundle of rank n.

Fix a set of affine open coverings $\{U\}$ on which $\mathscr{E}$ trivializes i.e. $\mathscr{E}|_U \cong \mathcal{O}_X|_U^n$. Fix a basis of elements $e_1, \ldots, e_n$ and so

$$\mathscr{E}(U) \cong \mathcal{O}_X(U)(e_1, \ldots, e_n) \cong \mathcal{O}_X(U)^n$$

is the $\mathcal{O}_X(U)$-module generated by the basis $e_1, \ldots, e_n$. The identification

$$\mathcal{O}(U)[x_1, \ldots, x_n] \to S(\mathscr{E}(U)) \qquad x_1 \mapsto e_1, x_2 \mapsto e_2, \ldots, x_n \mapsto e_n$$

follows from the a standard fact about symmetric algebras. From Hartshorne II.5.17, we know that $f^{-1}(U) \cong \operatorname{Spec} S(\mathscr{E}(U))$ so the contravariance of $\operatorname{Spec}(-)$ gives

$$\psi : f^{-1}(U) = \operatorname{Spec} S(\mathscr{E}(U)) \to \operatorname{Spec} \mathcal{O}(U)[x_1, \ldots, x_n] = \mathbb{A}^n_U.$$

Let us call this map ψ_U for precision. Our goal is to show $\psi_V \circ \psi_U^{-1}$ is a given by a linear automorphism when $W = \operatorname{Spec} A \subseteq U \cap V$.

First off, there is an identification $\psi_W : f^{-1}(W) = \operatorname{Spec} S(\mathscr{E}(W)) \to \mathbb{A}^n_W$. From the restriction maps, we have a diagram

$$\begin{array}{ccc}
\operatorname{Spec} \mathcal{O}_X(U)([x_1, \ldots, x_n]) & & \operatorname{Spec} \mathcal{O}_X(V)[x_1, \ldots, x_n] \\
\uparrow{\scriptstyle \psi_U} & & \uparrow{\scriptstyle \psi_V} \\
f^{-1}(U) \cong \operatorname{Spec} S(\mathscr{E}(U)) & & f^{-1}(V) \cong \operatorname{Spec} S(\mathscr{E}(V)) \\
& \nwarrow \quad \nearrow & \\
& \operatorname{Spec} S(\mathscr{E}(W)) &
\end{array}$$

Now if we restricted ψ_U to $\operatorname{Spec} S(\mathscr{E}(W)) = f^{-1}(W) \subseteq f^{-1}(U)$, the map ψ_U being linear means that it is linear on $f^{-1}(W)$. Similarly, ψ_V is linear on $f^{-1}(W)$ and it being an isomorphism means ψ_V^{-1} is also linear on $f^{-1}(W)$. Hence, $\psi_U \circ \psi_V^{-1}$ is linear on $f^{-1}(W) \cong \operatorname{Spec} S(\mathscr{E}(W)) \cong \operatorname{Spec} A[x_1, \ldots, x_n]$.

Up to isomorphism, there is no dependence on the choice of basis of $\mathscr{E}_U$. Indeed, each we had chosen a different basis $e'_1, \ldots, e'_n$ for $\mathscr{E}_U$, we could always define a map $e_i \mapsto e'_i$ and do this over each open affine U. This would then induce a map of sheaves $\mathscr{E} \to \mathscr{E}$ that is an isomorphism. Hence, we get an isomorphism $S(\mathscr{E}) \to S(\mathscr{E})$ and therefore an isomorphism on the underlying schemes $g : \mathbf{Spec}\, S(\mathscr{E}) \to \mathbf{Spec}\, S(\mathscr{E})$.

Of course, $f = f' \circ g$ under these choices because our isomorphisms were Y-scheme isomorphisms and the universal property in Hartshorne II.5.17(c). Furthermore, covering of Y is still the same and the isomorphisms ψ_i and $\psi'_i \circ g$ serve as trivialization's of the open sets.

(b) We provide more details on how to glue and restrict sections to smaller open sets. Let $s : U \to X$. We can then restrict to an open set $V \subseteq U$ by in the usual way $s|_V : V \to X$ and then $f \circ s|_V = (f \circ s)|_V = \mathrm{id}_U|_V = \mathrm{id}_V$.

For gluing, suppose $s_i : U_i \to X$ were sections of f and assume they agreed on overlaps. Then we can glue the s_i's together a morphism $s : U \to X$ with $s|_{U_i} = s_i$. Hence, $f \circ s|_{U_i} = \mathrm{id}_{(\bigcup_i U_i)}|_{U_i} = \mathrm{id}_{U_i}$. Thus, we have the following lemma.

Lemma 5.26. The presheaf $U \mapsto \mathscr{S}(X/Y)(U) := \{\text{sections of } f \text{ over } U\}$ is a sheaf of sets on Y which is denoted $\mathscr{S}(X/Y)$.

Now let $f : X \to Y$ be a geometric vector bundle of rank n.

Lemma 5.27. $\mathscr{S}(X/Y)$ is naturally a $\mathcal{O}_Y$-module and in particular, it is a free $\mathcal{O}_Y$-module of rank n.

Proof. Follow the hint. It suffices to define the module structure locally and in particular, we may do it affine locally.

Assume WLOG that $Y = \operatorname{Spec} A$ and the construction in (a) and Hartshorne II.5.17 mean we can take $X = \mathbb{A}^n_Y$. Then a section $s : Y \to X$ is precisely a ring homomorphism

$$\theta : A[x_1, \ldots, x_n] \to A.$$

Every such ring homomorphism is determined by where it maps each x_i. Thus, it determines an n-tuple $\langle \theta(x_1), \ldots, \theta(x_n) \rangle$ of elements of A. We define the the module structure by have $a \in \mathcal{O}_Y(Y) = A$ act on the n-tuple by multiplication.

The fact that $\mathscr{S}(X/Y) \cong \mathcal{O}_Y^n$ is a locally free sheaf of rank n follows from the existence of generators $\theta(x_i)$ for $1 \leq i \leq n$. □

(c) Let $\mathbf{V}(\mathscr{E})$ be the geometric vector bundle associated to $\mathscr{E}$. Let $\mathscr{S}(X/Y)$ be the sheaf of sections of X over Y associated to the map $f : \mathbf{V}(\mathscr{E}) \to Y$.

Lemma 5.28. $\mathscr{S} \cong \mathscr{E}^\smile$ is an isomorphism of sheaves of $\mathcal{O}_Y$-modules.

Proof. Let $s \in \Gamma(V, \mathscr{E}^\smile)$ be a section over an open set V. We may view s as an element of $\operatorname{Hom}(\mathscr{E}|_V, \mathcal{O}_Y|_V)$. Thus, s determines an $\mathcal{O}_V := \mathcal{O}_Y|_V$-module homomorphism $\mathscr{E}|_V \to \mathcal{O}_Y|_V$. Of course, this then determines a map $S(\mathscr{E}|_V) \to \mathcal{O}_V$. As these sheaves are quasi-coherent $\mathcal{O}_V$-modules, we obtain a morphism of spectras

$$s : V = \mathbf{Spec}(\mathcal{O}_V) \to \mathbf{Spec}\, S(\mathscr{E}|_V) = f^{-1}(V).$$

We show s is a section of $f : X \to Y$ by showing $f \circ s = \mathrm{id}_V$. Indeed, $f : f^{-1}(V) \to V$ post-composed to s is the identity.

Now we show that this defines an isomorphism of $\mathscr{E}^\smile \to \mathscr{S}$. It suffices to show that this gives local isomorphisms and that they behave w.r.t. restrictions.

Firstly, this construction is well-defined so we check that it is surjective i.e. every section $s : V \to X$ arises in this way. To specify a morphism $s : V \to X$, it is sufficient to specify a map $S(\mathscr{E}(V)) \to A$. We need $\operatorname{Spec} A \to \operatorname{Spec} S(\mathscr{E}(V)) \to \operatorname{Spec} A$ to be the identity or equivalently, $A \to S(\mathscr{E}(V)) \to A$ is the identity. This is always possible since $\mathcal{O}_V(V)$ is a free $\mathcal{O}_V(V)$-module.

Next, everything done here is canonical so we get the indicated sheaf isomorphism $\mathscr{S} \cong \mathscr{E}^\smile$. □

(d) We now show the indicated one-to-one correspondence between isomorphisms class:

$$\frac{\left\{\begin{array}{c}\text{locally free sheaves of}\\ \text{rank } n \text{ on } Y\end{array}\right\}}{\{\text{isomorphisms of } \mathcal{O}_Y\text{-modules}\}} \xrightarrow{\Phi} \frac{\left\{\begin{array}{c}\text{geometric vector bundles}\\ \text{of rank } n \text{ over } Y\end{array}\right\}}{\{\text{isomorphism classes of vector bundles over } Y\}}.$$

Let Ψ be the arrow going the other direction. Given a locally free sheaf $\mathscr{E}$ of rank n on Y, we associate the geometric vector bundle $\mathbf{V}(\mathscr{E}) =: \Phi(\mathscr{E})$. Given a geometric vector bundle of rank n over Y, say $\mathbf{V}(\mathscr{E})$, we associate to it the sheaf of sections $\mathscr{S}(X/Y)$ which was shown to be a locally free sheaf of rank n on Y in (b) and (c).

We start by showing $\Phi \circ \Psi$ is the identity. Let $\mathbf{V}(\mathscr{E})$ be a geometric vector bundle. Then Ψ gives the sheaf $\mathscr{S}(X/Y) \cong \mathscr{E}^\smile$. Under the isomorphism class of $\mathcal{O}_Y$-modules, $\mathscr{E}^\smile \cong \mathscr{E}$. So Φ sends $\mathscr{E} \cong \mathscr{E}^\smile$ to $\mathbf{V}(\mathscr{E})$.

On the other end, let $\mathscr{E}$ be a given locally free sheaf of rank n. Then Φ sends $\mathscr{E}$ to $\mathbf{V}(\mathscr{E})$. Then Ψ sends this vector bundle to $\mathscr{S} \cong \mathscr{E}^\smile \cong \mathscr{E}$.

At last, we have the stated one-to-one correspondence. □

6. Divisors

Hartshorne II.6.1 Let X be a scheme satisfying $(*)$. Then $X \times \mathbb{P}^n$ also satisfies $(*)$, and $\mathrm{Cl}(X \times \mathbb{P}^n) \cong (\mathrm{Cl}X) \times \mathbb{Z}$.

Proof. *Verify* $(*)$ *for* $X \times \mathbb{P}^n$: Let $Y := X \times \mathbb{P}^n$. By Corollary II.4.6d, since $h : X \to \operatorname{Spec}\mathbb{Z}$ and $h' : \mathbb{P}^n \to \operatorname{Spec}\mathbb{Z}$ are separated, $h \times h' : Y \to \operatorname{Spec}\mathbb{Z}$ is also separated.

Next, Y is integral because first, it is reduced since the rings $\mathcal{O}_Y(U)$ over any open affine set is an integral domain, and second, it is irreducible because fibre products preserve irreducibility when both schemes are irreducible.

Furthermore, Y is noetherian because there is a cover by open affine sets $\operatorname{Spec} A$ where A is the tensor product of a noetherian ring with and a finite k-algebra which means the ring is still noetherian and the underlying topological space of Y is quasi-compact because we take the inverse images of the associated projection maps p_1, p_2.

Finally, we show Y is regular in codimension one. Suppose $\mathcal{O}_{Y,y}$ is a local ring of Y of dimension one. Since Y is noetherian, $\mathcal{O}_{Y,y}$ is a noetherian local ring and it suffices[49] to show that $\mathcal{O}_{Y,y}$ is a DVR. There are two points[50] to consider, (1) the point y whose image x in Y in X is a point of codimension one, and (2) the point y whose image in X is the generic point of X. For (1), this means y is the generic point of $p_1^{-1}(x)$. So, $\mathcal{O}_{Y,y} \cong \left(\mathcal{O}_{X,x}\left[\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}\right]\right)_{\mathfrak{m}_y}$. This is a DVR. For (2), this means $\mathcal{O}_{Y,y}$ is the localization of $K[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$ where K is the function field of X. This is also a DVR.

Alternatively, observe that being regular in codimension one is a local condition. Of course, $\mathbb{P}^n$ can be covered by $D(x_i)$ which are all affine and so Hartshorne Proposition II.6.6 completes the proof.

Lemma 6.1. There is an isomorphims $\mathrm{Cl}(X \times \mathbb{P}^n) \cong (\mathrm{Cl}\, X) \times \mathbb{Z}$.

Proof. Let $i : X \times \mathbb{P}^{n-1} \to X \times \mathbb{P}^n$ be the closed immersion whose image is $X \times V(X_0)$ where $X_0, X_1, \dots, X_n$ are the coordinates of $\mathbb{P}^n$.

[49] See Lecture 7 Winter 2021 of Math 203B.

[50] This might be wrong? I am following Hartshorne Proposition II.6.6's analysis here and I might be mistaken to think I can generalize in this way. Here, I'm assuming the points are over an open affine subset of $\mathbb{P}^n$.

Let $j : X \times \mathbb{A}^n \to X \times \mathbb{P}^n$ be the open immersion whose image is $X \times D_+(X_0)$.

By Hartshorne Proposition II.6.5 there is an exact sequence

$$\mathbb{Z} \cdot (X \times \mathbb{P}^{n-1}) \to \mathrm{Cl}(X \times \mathbb{P}^n) \to \mathrm{Cl}(X \times \mathbb{A}^n) \to 0$$

and by Hartshorne Proposition II.6.6, $\mathrm{Cl}(X \times \mathbb{A}^n) \cong \mathrm{Cl}(X)$. We can obtain a section $\mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{P}^n)$ by mapping any divisor prime divisor D to $D \times \mathbb{P}^n$.

We show that the LHS map is injective so set $Z := i(X \times \mathbb{P}^{n-1})$. We show mZ is never a principal divisor for any nonzero integer m. Suppose not and mZ is a principal divisor of $X \times \mathbb{P}^n$. Then it is cut out by some rational function $K(X \times \mathbb{P}^n)$ which we call f.

Then, f cuts out $aZ \cap (X \times \mathbb{P}^n) = aZ \cap \mathbb{P}^n_{K(X)}$ as an element of $K(\mathbb{P}^n_{K(X)})$.

Lemma 6.2. The function field $K(X \times \mathbb{P}^n)$ is the same as the function field $K(\mathbb{P}^n_{K(X)})$.

Proof. The function field is computed by taking the localization at the generic point. If $\xi \in X \times \mathbb{P}^n$ is the generic point, there is an affine open subset $\operatorname{Spec} A \subseteq X$ and an affine chart $D_+(X_0)$ (WLOG assuming it is X_0) s.t. $\xi \in \operatorname{Spec}(A) \times D_+(X_0) \cong \operatorname{Spec} A[y_1, \ldots, y_n]$. Localization at ξ (which is then precisely the ideal generated by zero) gives the function field $K(X \times \mathbb{P}^n) = K(A)(y_1, \ldots, y_n)$.

On the LHS, we know $K(X) \cong K(A)$ by uniquness of functions fields. Then $\mathbb{P}^n_{K(X)} \cong \operatorname{Proj} K(X)[X_0, X_1, \ldots, X_n]$. The generic point is then locally the zero ideal in the chart $D_+(X_0)$ so $K(\mathbb{P}^n_{K(X)}) \cong K(A)(y_1, \ldots, y_n)$. □

As $Z \subseteq X \times V(X_0)$, it has valuation equal to zero on $K(X) \times D_+(X_0) \cong K(X) \times \mathbb{A}^n = \operatorname{Spec} K(X)[y_1, \ldots, y_n]$. It follows that f and $1/f$ lays inside every codimension 1 point. That is, $f, 1/f$ lay inside every prime ideal $\mathfrak{p} \subseteq K(X)[y_1, \ldots, y_n]$ of height 1. By Hartshorne Proposition II.6.3A, we deduce that $f, 1/f \in K(X)[y_1, \ldots, y_n]$. Hence, $f \in (K(X)[y_1, \ldots, y_n])^\times = K(X)^\times$. This means f has valuation zero in the local ring of $V(X_0)$ (as the $y_1, \ldots, y_n$ are $X_1/X_0, \ldots, X_n/X_0$). We conclude that f has valuation zero on $V(X_0)$. But then f has valuation zero on all of $Z = X \times V(X_0)$ and so cannot cut out $aZ \cap \mathbb{P}^n_{K(X)}$ unless $a = 0$. □

□

Remark 30. A reference for another proof of Hartshorne II.6.1 can be found here written by James McKernan. Another proof can be found here by KReiser.

Hartshorne II.6.2 *Varieties in Projective Space.* Let k be an algebraically closed field, and let X be a closed subvariety of $\mathbb{P}^n_k$ which is nonsingular in codimension one (hence satisfies (*)). For any divisor $D = \sum n_i Y_i$ on X, we define the degree of D to be $\sum n_i \deg Y_i$, where $\deg Y_i$ is the degree of Y_i, considered as a projective variety itself (I, §7).

(a) Let V be an irreducible hypersurface in $\mathbb{P}^n$ which does not contain X, and let Y_i be the irreducible components of $V \cap X$. They all have codimension 1 by (I, Ex. 1.8). For each i, let f_i be a local equation for V on some open set U_i of $\mathbb{P}^n$ for which $Y_i \cap U_i \neq \varnothing$, and let $n_i = v_{Y_i}(\bar{f}_i)$, where $\bar{f}_i$ is the restriction of f_i to $U_i \cap X$. Then we define the divisor $V.X$ to be $\sum n_i Y_i$. Extend by linearity, and show that this gives a well-defined homomorphism from the subgroup of $\operatorname{Div} \mathbb{P}^n$ consisting of divisors, none of whose components contain X, to $\operatorname{Div} X$.

(b) If D is a principal divisor on $\mathbb{P}^n$, for which $D.X$ is defined as in (a), show that $D.X$ is principal on X. Thus we get a homomorphism $\operatorname{Cl} \mathbb{P}^n \to \operatorname{Cl} X$.

(c) Show that the integer n_i defined in (a) is the same as the intersection multiplicity $i(X, V; Y_i)$ defined in (I, §7). Then use the generalized Bézout theorem (I, 7.7) to show that for any divisor D on $\mathbb{P}^n$, none of whose components contain X,

$$\deg(D.X) = (\deg D) \cdot (\deg X).$$

(d) If D is a principal divisor on X, show that there is a rational function f on $\mathbb{P}^n$ such that $D = (f).X$. Conclude that $\deg D = 0$. Thus the degree function defines a homomorphism $\deg\colon \operatorname{Cl} X \to \mathbb{Z}$. (This gives another proof of (6.10), since any complete nonsingular curve is projective.) Finally, there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Cl}\mathbb{P}^n & \longrightarrow & \operatorname{Cl} X \\ {\scriptstyle\cong}\downarrow{\scriptstyle\deg} & & \downarrow{\scriptstyle\deg} \\ \mathbb{Z} & \xrightarrow{\cdot(\deg X)} & \mathbb{Z} \end{array}$$

and in particular, we see that the map $\operatorname{Cl}\mathbb{P}^n \to \operatorname{Cl} X$ is injective.

Proof. For this exercise, one should consider reading this question about the problem.
(a) Let $D = \sum_i n_i W_i$ be a divisor on $\mathbb{P}^n$. Then the W_i are irreducible hypersurfaces from Hartshorne I.2.8. It follows that $W_i.X$ is welldefined and from the construction, if we set $W_i.X = \sum_j n_{ij} Y_j$, then Y_j being codimension 1 subvarieties of X gives the desired map $\operatorname{Div}\mathbb{P}^n \to \operatorname{Div} X$.

(b) Let D be a principal divisor on $\mathbb{P}^n$ i.e. $D = \div(f) = \sum n_i W_i$ for $f \in K(\mathbb{P}^n)^*$. It follows from Hartshorne Proposition II.6.4 that $\deg(f) = 0$.

(c)

(d) □

Proof. [**Incomplete**]I do not have all of the details figured out.

(a) I extend by linearity to show that we have a well-defined homomorphism from the subgroup of $\operatorname{Div}\mathbb{P}^n$ consisting of divisors none of whose components contain X to $\operatorname{Div} X$.

I think the trick is to view each prime divisor as an irreducible hypersurface in some way. Then, extending by linearity isn't bad. Something like

$$D = \sum m_i Y_i' \mapsto D.X = \sum_j v_{Y_j'}(\overline{f_j}) Y_j.$$

[**Incomplete**]

(b) [**Incomplete**]

The verification here looks straightforward. I just need to follow the mapping through.

(c) [**Incomplete**]

I have zero ideas on how to check that n_i is equal to the intersection multiplicity that Hartshorne defines earlier.

Suppose that I did have the verification done and that $n_i = i(X, V; Y_i)$. I then have

$$\deg(D.X) = \sum n_i \deg Y_i = \deg(X)\deg(V)$$

(d) [Incomplete]

The choice of f comes from considering the function fields. I want $f \in K(X)$ somehow. So, I want $\mathcal{O}_{\mathbb{P}^n_k,\eta} \to \mathcal{O}_{X,\eta} = K(X)$ where η is the generic point. I'm still not sure...

□

Hartshorne II.6.3 *Cones.* In this exercise we compare the class group of a projective variety V to the class group of its cone (I, Ex. 2.10). So let V be a projective variety in $\mathbb{P}^n$, which is of dimension $\geqslant 1$ and nonsingular in codimension 1. Let $X = C(V)$ be the affine cone over V in $\mathbb{A}^{n+1}$, and let $\bar{X}$ be its projective closure in $\mathbb{P}^{n+1}$. Let $P \in X$ be the vertex of the cone.

(a) Let $\pi : \overline{X} - P \to V$ be the projection map. Show that V can be covered by open subsets U_i such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^1$ for each i, and then show as in (6.6) that $\pi^* : \operatorname{Cl} V \to \operatorname{Cl}(\bar{X} - P)$ is an isomorphism. Since $\operatorname{Cl} \bar{X} \cong \operatorname{Cl}(\bar{X} - P)$, we have also $\operatorname{Cl} V \cong \operatorname{Cl} \bar{X}$.

(b) We have $V \subseteq \bar{X}$ as the hyperplane section at infinity. Show that the class of the divisor V in $\operatorname{Cl} \bar{X}$ is equal to π^* (class of $V.H$) where H is any hyperplane of $\mathbb{P}^n$ not containing V. Thus conclude using (6.5) that there is an exact sequence

$$0 \to \mathbb{Z} \to \operatorname{Cl} V \to \operatorname{Cl} X \to 0,$$

where the first arrow sends $1 \mapsto V.H$, and the second is π^* followed by the restriction to $X - P$ and inclusion in X. (The injectivity of the first arrow follows from the previous exercise.)

(c) Let $S(V)$ be the homogeneous coordinate ring of V (which is also the affine coordinate ring of X). Show that $S(V)$ is a unique factorization domain if and only if (1) V is projectively normal (Ex. 5.14), and (2) $\operatorname{Cl} V \cong \mathrm{Z}$ and is generated by the class of $V.H$.

(d) Let $\mathcal{O}_P$ be the local ring of P on X. Show that the natural restriction map induces an isomorphism $\operatorname{Cl} X \to \operatorname{Cl}(\operatorname{Spec} \mathcal{O}_P)$.

Proof. □

Hartshorne II.6.4 Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \ldots, x_n]$ be a square-free nonconstant polynomial, i.e., in the unique factorization of f into irreducible polynomials, there are no repeated factors. Let $A = k[x_1, \ldots, x_n, z] / (z^2 - f)$. Show that A is an integrally closed ring.

[Hint: The quotient field K of A is just $k(x_1, \ldots, x_n)[z]/(z^2 - f)$. It is a Galois extension of $k(x_1, \ldots, x_n)$ with Galois group $\mathbb{Z}/2\mathbb{Z}$ generated by $z \mapsto -z$. If $\alpha = g + hz \in K$, where $g, h \in k(x_1, \ldots, x_n)$, then the minimal polynomial of α is $X^2 - 2gX + (g^2 - h^2 f)$. Now show that α is integral over $k[x_1, \ldots, x_n]$ if and only if $g, h \in k[x_1, \ldots, x_n]$. Conclude that A is the integral closure of $k[x_1, \ldots, x_n]$ in K.]

Proof. Note that $(z^2 - f)$ is a prime ideal in $k[x_1, \ldots, x_n, z]$ because it does not factor. The field of fractions of A is then the localization at the multiplicative subset $S := A \setminus (z^2 - f)$. That is,

$$K = (A - \backslash (z^2 - f))^2 k[x_1, \ldots, x_n, z]/(z^2 - f) = k(x_1, \ldots, x_n, z)/(z^2 - f) \cong k(x_1, \ldots, x_n)[z]/(z^2 - f).$$

The RHS shows that this is a Galois extension of $k(x_1, \ldots, x_n)$ of degree two. The Galois group G is generated by the automorphism $z \mapsto -z$. Now let $\alpha \in K$ and factor $\alpha := g + hz \in$

K with $g, h \in k(x_1, \ldots, x_n)$. Then the minimal polynomial for α is

$$\begin{aligned} F(X) &= (X - (g + hz))(X - (g - hz)) \\ &= X^2 - Xg - Xhz - Xg + Xhz + (g + hz)(g - hz) \\ &= X^2 - 2gX + [g^2 - h^2 z^2] \\ &= X^2 - 2gX + g^2 - h^2 f. \end{aligned}$$

If $g, h \in k[x_1, \ldots, x_n]$, then we immediately have α integral over $k[x_1, \ldots, x_n]$. OTOH, if follows that $g, h \in k[x_1, \ldots, x_n]$ by taking $\mathfrak{a} = k[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_n] \subseteq K$ and applying Proposition 5.15 of [2].

Now we know $\alpha \in K$ is integral over $k[x_1, \ldots, x_n]$ iff $\alpha \in k[x_1, \ldots, x_n][z]/(z^2 - f)$. Therefore, A is the integral closure of $k[x_1, \ldots, x_n]$ in K. Applying Corollary 5.5 in [2] tells us that A is integrally closed in $K \supseteq k[x_1, \ldots, x_n]$. □

Hartshorne II.6.5 *Quadric Hypersurfaces.* Let char $k \neq 2$, and let X be the affine quadric hypersurface Spec $k[x_0, \ldots, x_n] / (x_0^2 + x_1^2 + \ldots + x_r^2)$ − cf. (I, Ex. 5.12).
(a) Show that X is normal if $r \geqslant 2$ (use (Ex. 6.4)).
(b) Show by a suitable linear change of coordinates that the equation of X could be written as $x_0 x_1 = x_2^2 + \ldots + x_r^2$. Now imitate the method of (6.5.2) to show that:
(1) If $r = 2$, then $\mathrm{Cl}X \cong \mathbb{Z}/2\mathbb{Z}$
(2) If $r = 3$, then $\mathrm{Cl}X \cong \mathbb{Z}$ (use (6.6.1) and (Ex.6.3) above)
(3) If $r \geqslant 4$ then $\mathrm{Cl}X = 0$
(c) Now let Q be the projective quadric hypersurface in $\mathbb{P}^n$ defined by the same equation. Show that:
(1) If $r = 2, \mathrm{Cl}Q \cong \mathbb{Z}$, and the class of a hyperplane section $Q.H$ is twice the generator;
(2) If $r = 3, \mathrm{Cl}Q \cong \mathbb{Z} \oplus \mathbb{Z}$
(3) If $r \geqslant 4, \mathrm{Cl}Q \cong \mathbb{Z}$, generated by $Q.H$.
(d) Prove Klein's theorem, which says that if $r \geqslant 4$, and if Y is an irreducible subvariety of codimension 1 on Q, then there is an irreducible hypersurface $V \subseteq \mathbb{P}^n$ such that $V \cap Q = Y$, with multiplicity one. In other words, Y is a complete intersection. (First show that for $r \geqslant 4$, the homogeneous coordinate ring $S(Q) = k[x_0, \ldots, x_n] / (x_0^2 + \ldots + x_r^2)$ is a UFD.)

Proof. **(a)** In the case of $r \geq 2$, we may apply Hartshorne II.6.4 with

$$z := x_0 \qquad \& \qquad f = -(x_1^2 + x_2^2 + \cdots + x_r^2).$$

Note that $r = 2$ gives $f = -(x_1 + ix_2)(x_1 - ix_2)$ which is permissible in the hypothesis. So $k[x_0, \ldots, x_r]/(x_0^2 + \cdots + x_r^2)$ is integrally closed for $r \geq 2$. Then Atiyah-Macdonald Proposition 5.13 implies X is a normal scheme.
(b) Define the linear change of coordinates by $x_0 \mapsto \frac{y_0 + y_1}{2i}$ and $x_1 \mapsto \frac{y_1 - y_0}{2}$. The new equation is

$$-\frac{y_0^2 + 2y_0 y_1 + y_1^2}{4} + \frac{y_1^2 - 2y_1 y_0 + y_0^2}{4} + x_2^2 + x_3^2 + \cdots + x_r^2 = 0 \;\Rightarrow\; x_2^2 + x_3^2 + \cdots + x_r^2 = y_0 y_1.$$

Replace y_0, y_1 by x_0, x_1 for the new coordinates. This induces an isomorphism of affine schemes so we can compute $\mathrm{Cl}\, X$ this equation.

(1) If $r = 0$, we know $X \cong \mathrm{Spec}\, k[x_0, x_1, x_2]/(x_0 x_1 - x_2^2)$ which is precisely the scheme in Hartshorne Example II.6.5.2. Hence, $\mathrm{Cl}\, X \cong \mathbb{Z}/2\mathbb{Z}$.

(2) Fix $r = 3$ and the equation becomes $x_0x_1 = x_2^2+x_3^2$. By taking a change of coordinates $x_2 \to \frac{y_0+y_1}{2}$ and $x_1 \to \frac{y_0-y_1}{2i}$, we see that $x_2^2 + x_3^2 \to y_0y_1$. So our new variety is cut out by the equation $x_0x_1 = y_0y_1$. For ease, we relabel and work with the scheme cute out by $x_0x_1 = x_2x_3$. Thus far, we are viewing $X := \operatorname{Spec} k[x_0, x_1, x_2, x_3]/(x_0x_1 - x_2x_3)$ as an affine scheme. Let Q be the projective variety cut out by the same equation as in Hartshorne Example II.6.6.1. Then X is the cone of $Q \subseteq \mathbb{P}^3$ inside $\mathbb{A}^4$. By Hartshorne II.6.3 there is a SES

$$0 \to \mathbb{Z} \to \operatorname{Cl}(Q) = \mathbb{Z} \oplus \mathbb{Z} \to \operatorname{Cl}(X) \to 0.$$

where the first map is $1 \mapsto 1 \cdot Q.H$.

To finish the proof, it suffices to show that $Q.H$ is of type $(1, 1)$ inside $\operatorname{Cl}(Q)$. If one traces the construction of $Q.H$ in Hartshorne II.6.2(a) we see that $Q.H$ is the divisor of $Q \cap H$ and if we take H to be cut out by $x_0 = 0$, Hartshorne Example II.6.6.2 tells us $Q.H$ is of type $(1, 1)$ inside $\operatorname{Cl}(Q)$. Therefore,

$$\operatorname{Cl}(X) \cong \mathbb{Z} \oplus \mathbb{Z}/\langle(1, 1)\rangle$$

Of course, $\mathbb{Z} \oplus \mathbb{Z}/\langle(1, 1)\rangle \cong \mathbb{Z}$ is via the group homomorphism $(a, b) \mapsto a - b$. We conclude that $\operatorname{Cl}(X) \cong \mathbb{Z}$.

(3) For this proof, we follow the method of Hartshorne Example II.6.5.2. Fix

$$A := k[x_0, x_1, \ldots, x_r]/(x_0x_1 - x_2^2 - x_3^2 - \cdots - x_r^2)$$

for $r \geq 4$. Consider the subscheme cut out by $Y : x_1 = x_2^2 + x_3^2 + \cdots + x_r^2 = 0$. This gives a codimension 1 subscheme. Indeed, Y can be cut out by x_1 alone and hence,

$$A_{(x_1)} \cong k[x_1](x_0, x_2, \ldots, x_r)/(x_0x_1 - x_2^2 - \cdots - x_r^2) \cong k[x_1](x_1, \ldots, x_r) \cong k(x_2, \ldots, x_r)[x_1]$$

where used the relation $x_0 = \frac{x_1}{x_2^2+\cdots+x_r^2}$ to eliminate x_0. Next, $X - Y = \operatorname{Spec} Ax_1$ and hence,

$$A_{x_1} \cong k[x_0, x_1, x_1^{-1}, x_2, \ldots, x_r]/(x_0x_1 = x_2^2 + \cdots + x_r^2) \cong k[x_1, x_1^{-1}, x_2, \ldots, x_r]$$

which is still a UFD. It follows $\operatorname{Cl}(X - Y) = 0$ and the exact sequence of Hartshorne Proposition II.6.5 is

$$\mathbb{Z} \to \operatorname{Cl} X \to 0 \to 0$$

where $\mathbb{Z} \to \operatorname{Cl} X$ is given by $n \mapsto n \cdot Y$. By exactness and $\operatorname{Cl} X \to 0$ surjective, $\operatorname{im}(\mathbb{Z}) = \operatorname{Cl} X$. To show $\operatorname{Cl} X = 0$, it suffices to show that Y is a principal Weil divisor. A priori, we know that

$$\mathfrak{m}_Y = (x_1, x_2^2 + x_2^2 + \cdots + x_r^2) = (x_1)$$

because $x_0x_1 = x_2^2 + \cdots + x_r^2$. Here,the condition that $r \geq 4$ is necessary because it implies $x_2^2 + x_3^2 + \cdots + x_r^2$ is an irreducible polynomial. Because of this, x_1 is a prime element of $\mathfrak{m}_Y$. This means x_1 is the uniformizer and generates $\mathfrak{m}_Y A_{\mathfrak{m}_Y}$. Since $\div(x_1) = 1 \cdot Y$, it follows that Y is a principal divisor.

(c) Fix $Q := \operatorname{Proj} A$ where A is as before. Let $X := \operatorname{Spec} A$ as before.

(1) If we use Hartshorne II.6.3(b) and part (b) above, we obtain a SES

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto Q.H} \operatorname{Cl} Q \to \operatorname{Cl} X = \mathbb{Z}/2\mathbb{Z} \to 0.$$

Any[51] SES of abelian groups where the middle group is squeezed between two f.g. abelian groups is also finitely generated.

[51] See this post or prove it yourself.

The generator $Q.H$ of the image $\mathrm{im}(\mathbb{Z} \to \mathrm{Cl}\, Q)$ is twice a generator of $\mathrm{Cl}\, Q$ since that generator $Q.H$ must generate the kernel of $\mathrm{Cl}\, X \to \mathbb{Z}/2\mathbb{Z}$.

There are two possibilities for $\mathrm{Cl}\, Q$ which are $\mathrm{Cl}\, Q \cong \mathbb{Z}$ or $\mathrm{Cl}\, Q \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Lift D of $\overline{1} \in \mathbb{Z}/2\mathbb{Z}$ to $\mathrm{Cl}\, Q$ and with element $Q.H$, they generate $\mathrm{Cl}\, Q$. That is, $\mathrm{Cl}\, X = \langle Q.H, D\rangle$. It suffices to show that $Q.H = 2(D)$ to rule out the possibility that $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ because that says

$$\mathbb{Z}(Q.H) \subseteq \ker(\mathrm{Cl}\, Q \to \mathrm{Cl}\, X) \subseteq \mathrm{Cl}\, Q.$$

Lemma 6.3. One has $Q.H = 2(D)$. Note that Q is cut out by $x_0x_1 = x_2^2$ and we take $Q.H$ to be cut out by $\{y = 0\} \cap Q \subseteq \mathbb{P}^2$.

Proof. It is psychologically easier if we write x, y, z for x_0, x_1, x_2.

From part (b) and Hartshorne Example II.6.5.2, we know that $\mathrm{Cl}\, X$ was generated by the ruling $\{y = z = 0\} \cap X$.

The divisor class $\{y = z = 0\} \cap Q \in \mathrm{Cl}\, Q$ gets sent to this ruling. We take this divisor class to be the choice of D.

We are reduced to showing that $\{y = 0\} \cap Q = 2(\{y = z = 0\} \cap Q)$. To prove this, we examine the local ring of Q at the generic point ξ of $\{y = z = 0\} \cap Q$. Indeed, it is

$$\mathcal{O}_{Q,\xi} = (k[x, y, z]/(xy - z^2))_{(y,z)} \cong k[x, y, \sqrt{xy}]_{(y,\sqrt{xy})}$$

which is a DVR with maximal ideal

$$(y, \sqrt{xy})k[x, y, \sqrt{xy}]_{(y,\sqrt{xy})}.$$

Since we have a DVR, we know that there should be a choice of uniformizer. Indeed, one observes that $y = (\sqrt{xy})^2/x$ and x is inverted because $x \notin (y, \sqrt{xy})$. So we conclude that the uniformizer is $\sqrt{xy} =: \pi$. Let $v(-)$ be the discrete valuation of this ring. Then

$$v(y) = 2 \text{ because } y = \frac{(\sqrt{xy})^2}{x} = \frac{1}{x}\sqrt{xy}^2 = \frac{1}{x}\pi^2 = u\pi^2.$$

This implies that $\{y = 0\} \cap Q$ vanishes at order 2 on $\{y = z = 0\} \cap Q$. □

(2) Using Hartshorne II.6.3(b), there is a SES

$$0 \to \mathbb{Z} \to \mathrm{Cl}\, Q \to \mathrm{Cl}\, X \to 0 \;\Rightarrow\; 0 \to \mathbb{Z} \to \mathrm{Cl}\, Q \to \mathbb{Z} \to 0$$

from part (b) and $r = 3$. The divisor class group is always an abelian group which means we have a splitting $\mathrm{Cl}\, Q \cong \mathbb{Z} \oplus \mathbb{Z}$.

(3) Following exactly the same method as above,

$$0 \to \mathbb{Z} \to \mathrm{Cl}\, Q \to \mathrm{Cl}\, X \to 0 \;\Rightarrow\; 0 \to \mathbb{Z} \to \mathrm{Cl}\, Q \to 0 \to 0.$$

It follows $\mathrm{Cl}\, Q \to \mathbb{Z}$ is an isomorphism and the morphism being given by $1 \mapsto Q.H$ implies $Q.H$ generates $\mathrm{Cl}\, X$.

(d) From (a) and (b), we know that $\mathrm{Cl}(\mathrm{Spec}\, k[x_0, \ldots, x_r]/(x_0^2 + \cdots + x_r^2)) = 0$ and that this affine scheme is normal. Hence, $S(Q)$ is a UFD.

Fix Y to be an irreducible subvariety of codimension 1 on Q. Then Y is cut out by a prime ideal of $S(Q)$ that is of height 1. By Hartshorne Proposition I.1.12A, the prime ideal is principal $\mathfrak{p} = (f)$. From primality, f is an irreducible polynomial.

Now we have a map $\mathrm{Cl}\, P^n \to \mathrm{Cl}\, Q \cong \mathbb{Z}$ and it follows that Y itself is a principal divisor of $\mathrm{Cl}\, Q$. But we know that $\mathrm{Cl}\, \mathbb{P}^n \to \mathrm{Cl}\, Q$ is an injective map from Hartshorne II.6.2(d). It

follows that $\operatorname{div}(f)$ is mapped to by some $\operatorname{div}(g) = 0 \in \operatorname{Cl}(\mathbb{P}^n)$. Let V be the hypersurface cut out by g. Then $V \cap Q = Y$ it is clear from $\operatorname{div}(f) = 1 \cdot Y$ that the intersection multiplicity n_i between V and Y is 1 as in Hartshorne II.6.2. □

Hartshorne II.6.6 Let X be the nonsingular plane cubic curve $y^2z = x^3 - xz^2$ of (6.10.2).

(a) Show that three points P, Q, R of X are collinear if and only if $P + Q + R = 0$ in the group law on X. (Note that the point $P_0 = (0, 1, 0)$ is the zero element in the group structure on X.)

(b) A point $P \in X$ has order 2 in the group law on X if and only if the tangent line at P passes through P_0.

(c) A point $P \in X$ has order 3 in the group law on X if and only if P is an inflection point. (An inflection point of a plane curve is a nonsingular point P of the curve, whose tangent line (I, Ex. 7.3) has intersection multiplicity $\geqslant 3$ with the curve at P.)

(d) Let $k = \mathbb{C}$. Show that the points of X with coordinates in $\mathbb{Q}$ form a subgroup of the group X. Can you determine the structure of this subgroup explicitly?

Proof. **[Incomplete]** □

Hartshorne II.6.7 Let X be the nodal cubic curve $y^2z = x^3 + x^2z$ in $\mathbb{P}^2$. Imitate (6.11.4) and show that the group of Cartier divisors of degree 0, $\operatorname{CaCl}^\circ X$, is naturally isomorphic to the multiplicative group $\mathbb{G}_m$.

Proof. **[Incomplete]** □

Hartshorne II.6.8

(a) Let $f : X \to Y$ be a morphism of schemes. Show that $\mathscr{L} \mapsto f^*\mathscr{L}$ induces a homomorphism of Picard groups, $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$.

(b) If f is a finite morphism of nonsingular curves, show that this homomorphism corresponds to the homomorphism $f^* : \operatorname{Cl} Y \to \operatorname{Cl} X$ defined in the text, via the isomorphisms of (6.16) .

(c) If X is a locally factorial integral closed subscheme of $\mathbb{P}^n_k$, and if $f : X \to \mathbb{P}^n$ is the inclusion map, then f^* on Pic agrees with the homomorphism on divisor class groups defined in (Ex. 6.2) via the isomorphisms of (6.16).

Proof. **(a)** To show that we have a homomorphism of Picard groups, we must show that f^* is a well-defined map between Picard groups and that is preserves the group structure.

Showing well-definedness: Suppose $\mathscr{L}$ is an invertible sheaf over Y. Let $\{U_i\}$ cover Y and $\mathscr{L}|_{U_i} \cong \mathcal{O}_Y|_U$ be the trivializations. Then, for open affine $V_i = f^{-1}(U_i)$, we have

$$f^*\mathscr{L}|_{V_i} \cong f^{-1}\mathscr{L}|_{V_i} \otimes_{f^{-1}\mathcal{O}_Y|_{V_i}} \mathcal{O}_X|_{f^{-1}(V_i)} \cong f^{-1}\mathcal{O}_Y|_{V_i} \otimes_{f^{-1}\mathcal{O}_Y|_{V_i}} \mathcal{O}_X|_{f^{-1}(V_i)} \cong \mathcal{O}_X|_{f^{-1}(V_i)}.$$

Since $\{f^{-1}(V_i)\}$ covers X and our isomorphisms were canonical, we have a well-defined map.

Group structure preserved: Let $\mathscr{L}, \mathscr{M} \in \operatorname{Pic} Y$. We know $\mathscr{L} \otimes \mathscr{M} \in \operatorname{Pic} Y$. Next[52], we want to prove

$$f^*(\mathscr{L} \otimes \mathscr{M}) \cong f^*\mathscr{L} \otimes f^*\mathscr{M}.$$

[52] A quicker proof is detailed here.

We have isomorphisms

$$\begin{aligned}
\operatorname{Hom}(f^*(\mathscr{L}\otimes\mathscr{M}),\mathscr{P}) &\cong \operatorname{Hom}(\mathscr{L}\otimes\mathscr{M}, f_*\mathscr{P}) && \text{(adjointness of } f^* \text{ \& } f_*)\\
&\cong \operatorname{Hom}(\mathscr{L}, \mathscr{H}om(\mathscr{M}, f_*\mathscr{P}) && \text{(Hartshorne II.5.1c)}\\
&\cong \operatorname{Hom}(\mathscr{L}, f_*\mathscr{H}om(f^*\mathscr{M}, \mathscr{P}))\\
&\cong \operatorname{Hom}(f^*\mathscr{L}, \mathscr{H}om(f^*\mathscr{M}, \mathscr{P}))\\
&\cong \operatorname{Hom}(f^*\mathscr{L}\otimes f^*\mathscr{M}, \mathscr{P}) && \text{(Hartshorne II.5.1c).}
\end{aligned}$$

This shows that $f^*(\mathscr{L}\otimes\mathscr{M}) \cong f^*\mathscr{L}\otimes f^*\mathscr{M}$. So we have a homomorphism $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$.

(b) Fix $f : X \to Y$ a finite morphism of nonsingular curves. We want to show

$$\begin{array}{ccc}
\operatorname{Cl} Y & \xrightarrow{f^*} & \operatorname{Cl} X\\
\downarrow & & \downarrow\\
\operatorname{Pic} Y & \xrightarrow{f^*} & \operatorname{Pic} X
\end{array}$$

commutes.

We being by proving the result for a prime divisor $(Q) \in \operatorname{Cl} Y$. From the vertical map, we associate to Q the invertible sheaf $\mathscr{L}(D)$ where D a Cartier divisor.

Define $D := \{(U_Q, t), (Y - Q), 1)\}$ as follows. Let U_Q be a neighborhood of Q where $t = 0$ at Q only. That means $t \in \Gamma(U_Q, \mathscr{K}_Y^*)$ has $t|_Q \neq 0$ only at Q. In our setting of curves, we know $\mathscr{K}_Y^*$ is the constant sheaf at with $K(Y)$ ring.

To see that D defines a Cartier divisor, we check that $\frac{1/t}{1} = 1/t$ is invertible on $\Gamma(U_Q \cap (Y - Q), \mathscr{K}^*)$. It is invertible since t has nonzero germ on all points of $U_Q \cap (Y - Q)$.

Now following the map to $\operatorname{Pic} X$, we study $f^*\mathscr{L} := f^*\mathscr{L}(D)$. We shall show that $f^*\mathscr{L} = \mathscr{L}(E)$ for a certain Cartier divisor E.

Define E as follows. Let (U_{P_i}, t_Q) be the pair where t_Q is the t above, U_{P_i} is the largest open set around P_i in which t_Q vanishes only at P_i. Let $(X - f^{-1}(Q), 1)$ be another pair. Then take $E := \{(X - f^{-1}(Q), 1), (U_{P_1}, t_Q), \ldots, (U_{P_r}, t_Q)\}$ for $P_1, P_2, \ldots, P_r \in f^{-1}(Q)$.

We check E defines a divisor. It is clear that $1/t_Q$ is invertible on $U_{P_i} \cap (X - f^{-1}(Q))$. Additionally, $\frac{1/t_Q}{1/t_Q} = 1$ is always invertible on $U_{P_i} \cap U_{P_j}$. So this gives a well-defined Cartier divisor.

Now, after setting up everything, we prove the following lemma.

Lemma 6.4. $f^*\mathscr{L}(D) \cong \mathscr{L}(E)$.

Proof. This itself is relatively unenlightening. The key thing to understand is the proof of Hartshorne Proposition II.5.8(a). We do things locally.

First off, let ξ be the generic point of Y and η the generic point of X. It is helpful to note that a dominant morphism of integral schemes will necessarily send generic point to generic point. Note that we use $1 \cdot -$ to write the sub-$\mathcal{O}_Y$-module generated by 1.

$$f^*\mathscr{L}|_{X-f^{-1}(Q)} = f^{-1}\mathscr{L}|_{X-f^{-1}(Q)} \otimes_{f^{-1}\mathcal{O}_Y|_{X-f^{-1}(Q)}} \mathcal{O}_X|_{X-f^{-1}(Q)} \tag{12}$$

$$= f^{-1}\mathscr{L}|_{X-f^{-1}(Q)} \otimes_{f^{-1}\mathcal{O}_Y|_{X-f^{-1}(Q)}} \mathcal{O}_X|_{X-f^{-1}(Q)} \tag{13}$$

$$= 1 \cdot f^{-1}\mathcal{O}_Y|_{X-f^{-1}(Q)} \otimes_{f^{-1}\mathcal{O}_Y|_{X-f^{-1}(Q)}} \mathcal{O}_X|_{X-f^{-1}(Q)} \tag{14}$$

$$= 1 \cdot \mathcal{O}_X|_{X-f^{-1}(Q)}. \tag{15}$$

Next, for each i there is an isomorphism

$$f^*\mathscr{L}|_{U_{P_i}} = f^{-1}\mathscr{L}|_{U_{P_i}} \otimes_{f^{-1}\mathcal{O}_Y|_{U_{P_i}}} \mathcal{O}_X|_{U_{P_i}} \tag{16}$$

$$= \frac{1}{t_Q} \cdot f^{-1}\mathcal{O}_Y|_{U_{P_i}} \otimes_{f^{-1}\mathcal{O}_Y|_{U_{P_i}}} \mathcal{O}_X|_{U_{P_i}} \tag{17}$$

$$= \frac{1}{t_Q}\mathcal{O}_X|_{U_{P_i}}. \tag{18}$$

Putting these sheaf isomorphisms together and observing that they agree on overlaps, we have the isomorphism. □

Now suppose we start with (Q) and obtain a Weil divisor $f^*(Q) = \sum_{f(P_i)=Q} v_{P_i}(t_Q) \cdot (P_i) =: W \in \operatorname{Cl} X$

Taking $P_i \in X$, we know that W induces a divisor W_{P_i} on $\operatorname{Spec} \mathcal{O}_{P_i} \cong \{P_i\}$. On $\operatorname{Spec} \mathcal{O}_{P_i}$, it is cut out by a single equation called f_{P_i}. Of course, we can take $f_{P_i} = t_Q$ for all i and the corresponding open neighborhood of P_i to be U_{P_i} as above. It follows that we must associate to $f^*(Q)$ the invertible sheaf $\mathscr{L}(E)$. We conclude that the diagram above commutes.

Now we explain how to extend the result to any divisor in $\operatorname{Cl} Y$ and not just prime divisors. From the construction of Hartshorne Proposition II.6.11, to go from a Weil divisor to a Cartier divisor then becomes a matter of taking the product of $f_x g_x \in K$ for f_x, g_x corresponding to D, E Weil divisors respectively.

Of course, this then coincides with taking the tensor product of two invertible sheaves and from (a), we know that f^* commutes with $\otimes$.

(c) Let $f : X \to \mathbb{P}^n$ be the inclusion map. The goal is to show that the diagram is commutative:

$$\begin{array}{ccc} \operatorname{Cl}(\mathbb{P}^n) & \xrightarrow{H \mapsto H.X} & \operatorname{Cl}(X) \\ \downarrow & & \downarrow \\ \operatorname{Pic}(\mathbb{P}^n) & \xrightarrow{f^*} & \operatorname{Pic}(X) \end{array}$$

We know from Hartshorne Proposition II.6.4 that the generator of $\operatorname{Cl}(\mathbb{P}^n)$ is given by $H = \{x_0 = 0\}$.

Following the vertical and using the fact that $\operatorname{Pic}(\mathbb{P}^n) \cong (\mathcal{O}(l), \otimes)_{l \in \mathbb{Z}} \cong \mathbb{Z}$, we shall associate to H the sheaf $\mathcal{O}(1)$.

Lemma 6.5. The isomorphism of Hartshorne Corollary II.6.16 identifies H with the invertible sheaf $\mathcal{O}(1)$.

Proof. Define the Cartier divisor D as follows. Take the pair $(D_+(x_i), x_0/x_i)$ for $i = 0, 1, \ldots, n$. This clearly defines a Cartier divisor because on $D_+(x_i) \cap D_+(x_j)$, we have $\frac{x_0/x_i}{x_0/x_j} = \frac{x_j}{x_i}$ which is an invertible section.

The associated invertible sheaf $\mathscr{L}$ is then determined by

$$\mathscr{L}|_{D_+(x_i)} = \frac{x_i}{x_0}\mathcal{O}|_{D_+(x_i)}.$$

We claim $\mathscr{L} \cong \mathcal{O}(1)$. This is discussed in Hartshorne Corollary II.6.17, we provide justification.

From definition, $\mathcal{O}(1) \cong \widetilde{S(1)} = \widetilde{(x_0, x_1, \dots, x_n)}$ where $(x_0, x_1, \dots, x_n)$ is the subring generated by these elements inside $S = k[x_0, x_1, \dots, x_n]$. Now, locally on each $D_+(x_i)$, we have

$$\begin{aligned}\mathcal{O}(1)|_{D_+(x_i)} &\cong \widetilde{(x_0, x_1, \dots, x_n)}_{(x_i)} \cong \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)^{\sim} \\ &\cong \left(\frac{x_i}{x_0} S_{(x_i)}\right)^{\sim} \cong \frac{x_i}{x_0}\mathcal{O}|_{D_+(x_i)} = \mathscr{L}|_{D_+(x_i)}\end{aligned}$$

where the subscript (x_i) indicates we localize at the set $\{1, x_i, x_i^2, \dots\}$ and take degree zero. Since everything here works well w.r.t. restrictions, we may glue and obtain the claimed sheaf isomorphism. □

Now taking $\mathcal{O}(1)$, we want to consider $f^*\mathcal{O}(1)$ which is a sheaf on X. Observe,

$$f^*\mathcal{O}(1)|_{X\cap D_+(x_i)} \cong f^{-1}\mathcal{O}(1)|_{X\cap D_+(x_i)} \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^n}|_{X\cap D_+(x_i)}} \mathcal{O}_X|_{X\cap D_+(x_i)} \tag{19}$$

$$= \frac{x_i}{x_0} f^{-1}\mathcal{O}_{\mathbb{P}^n}|_{X\cap D_+(x_i)} \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^n}|_{X\cap D_+(x_i)}} \mathcal{O}_X|_{X\cap D_+(x_i)} \tag{20}$$

$$= \frac{x_i}{x_0}\mathcal{O}_X|_{X\cap D_+(x_i)} \tag{21}$$

is the indicated invertible subsheaf $\mathscr{K}_X$ and an element of $\operatorname{Pic} X$. Such a sheaf has an associated Cartier divisor $D := \{(X \cap D_+(x_i), \frac{x_0}{x_i})\}_{i=0,1,\dots,n}$ which satisfies $f^*\mathcal{O}(1) = \mathscr{L}(D)$ on X.

Now we follow along the horizontal map. From Hartshorne II.6.2, we may WLOG write

$$H.X = \sum_{j=0,i}^{n} n_j H_{ij}$$

where $n_j = v_{H_{ij}}(\overline{x_0/x_j})$ and H_{ij} is the intersection of an irreducible component of $H \cap X$ with $D_+(x_j)$. Here, we are using i as a separate index since $H \cap X$ may have more than one irreducible component that intersects $D_+(x_j)$. The reason we take the valuation of $\overline{x_0/x_j}$ is because the equation that cuts out H on $D_+(x_j)$ is x_0/x_j and the bar indicates that we restrict it to $D_+(x_i) \cap X$.

To show the diagram commutes, it suffices to show that $H.X$ as above has associated invertible sheaf equal to $\mathscr{L}(D)$ above. It is vastly more difficult to go from $H.X$ to an invertible sheaf follow the proof of Hartshorne Proposition II.6.11 so we shall begin with $\mathscr{L}(D)$ and show that it gives rise to $H.X$.

First off, $\{(D_+(x_i) \cap X, x_0/x_i)\}$ has $D_+(x_i) \cap X$ serving as an open covering of X. Furthermore, $x_0/x_i \in \Gamma(D_+(x_i)\cap X, \mathscr{K}_X^*)$ is clear when we view it as its restriction to $D_+(x_i)\cap X$ from $D_+(x_i)$.

Next, for each prime divisor Y of X, we take the coefficient of Y to be $v_Y(x_0/x_i)$ where i is the index s.t. $D_+(x_i) \cap X \cap Y \neq \emptyset$. Then, the associated Weil divisor is $\sum v_Y(x_0/x_i)Y$.

Of course, one must note that $v_Y(x_0/x_i)$ can only have nonzero valuation iff Y intersects H. It then follows that $\sum v_Y(x_0/x_i)Y$ is precisely the same as that of $H.X$. □

Hartshorne II.6.9 *Singular Curves.* Here we give another method of calculating the Picard group of a singular curve. Let X be a projective curve over k, let $\widetilde{X}$ be its normalization, and let $\pi : \widetilde{X} \to X$ be the projection map (Ex. 3.8). For each point $P \in X$, let $\mathcal{O}_P$ be its local ring, and let $\widetilde{\mathcal{O}}_P$ be the integral closure of $\mathcal{O}_P$. We use a * to denote the group of units in a ring.

(a) Show there is an exact sequence

$$0 \to \bigoplus_{P \in X} \widetilde{\mathcal{O}}_P^*/\mathcal{O}_P^* \to \text{ Pic } X \xrightarrow{\pi_*} \text{ Pic } \widetilde{X} \to 0.$$

[Hint: Represent Pic X and Pic $\widetilde{X}$ as the groups of Cartier divisors modulo principal divisors, and use the exact sequence of sheaves on X

$$0 \to \pi_*\mathcal{O}_{\widetilde{X}}^*/\mathcal{O}_X^* \to \mathscr{K}^*/\mathcal{O}_X^* \to \mathscr{K}^*/\pi_*\mathcal{O}_{\widetilde{X}}^* \to 0.\big] .$$

(b) Use (a) to give another proof of the fact that if X is a plane cuspidal cubic curve, then there is an exact sequence

$$0 \to \mathbb{G}_a \to \operatorname{Pic} X \to \mathbb{Z} \to 0$$

and if X is a plane nodal cubic curve, there is an exact sequence

$$0 \to \mathbb{G}_m \to \operatorname{Pic} X \to \mathbb{Z} \to 0.$$

Remark 31. There is a typo in Hartshorne. The map $\operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$ should π_*.

Proof. **(a)** By Hartshorne Proposition II.5.8(c) and Hartshorne Caution II.5.8.1, the push-forward map $\pi_* : \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$ makes sense. Also, $\operatorname{CaCl} X \cong \operatorname{Pic} X$ via $D \mapsto \mathscr{L}(D)$.

Lemma 6.6. There is a SES $0 \to \pi_*\mathcal{O}_{\widetilde{X}}^*/\mathcal{O}_X^* \to \mathscr{K}^*/\mathcal{O}_X^* \to \mathscr{K}^*/\pi_*\mathcal{O}_{\widetilde{X}}^* \to 0$.

Proof. This SES of sheaves exists because there is an inclusion $\mathcal{O}_X^* \hookrightarrow \pi_*\mathcal{O}_{\widetilde{X}}^*$. Indeed, this is induced from $\mathcal{O}_X^*(D(f)) \hookrightarrow \mathcal{O}_{\widetilde{X}}^*(\pi^{-1}(D(f))$ which was the normalization map. Additionally, we get an exact sequence

$$0 \to \Gamma(X, \pi_*\mathcal{O}_{\widetilde{X}}^*/\mathcal{O}_X^*) \to \Gamma(X, \mathscr{K}^*/\mathcal{O}_X^*) \to \Gamma(X, \mathscr{K}^*/\pi_*\mathcal{O}_{\widetilde{X}}^*)$$

□

Now we study the map $\pi_* : \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$. Let (U_i, f_i) be a Cartier divisor of X and (V_i, g_i) be the image under this map. That is, $g_i = \pi_*(f_i)$ and $V_i = \pi^{-1}(U_i)$.

Lemma 6.7. The map $\pi_* : \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$ is surjective.

Proof. □

Lemma 6.8. The kernel of $\pi_* : \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$ is $\bigoplus_{P \in X} \widetilde{\mathcal{O}_P}^*/\mathcal{O}_P^*$ and there is an injective map Φ from this group to $\operatorname{Pic} X$.

Proof. First off, one may observe that for our hypotheses, $\bigoplus_{P \in X} \widetilde{\mathcal{O}_P}^*/\mathcal{O}_P^*$ is isomorphic to $\Gamma(X, \pi_*\mathcal{O}_{\widetilde{X}}^*/\mathcal{O}_X^*)$ by virtue of how taking stalks of a pushforward works and how $\mathcal{O}_{\widetilde{X}}$ is defined. □

From the exact sequence above, we get

$$0 \to \bigoplus_{P \in X} \widetilde{\mathcal{O}_P}^*/\mathcal{O}_P^* \to \Gamma(X, \mathscr{K}^*/\mathcal{O}_X^*) \to \Gamma(X, \mathscr{K}^*/\pi_*\mathcal{O}_{\widetilde{X}}^*).$$

It suffices to show that $\Gamma(X, \mathscr{K}^*/\pi_*\mathcal{O}_{\widetilde{X}}^*)$ can be identified with $\mathrm{CaCl}(\widetilde{X})$ and that the RHS map is surjective. These are done in the next two lemmas.

Lemma 6.9. $\operatorname{Pic} \widetilde{X} \cong \Gamma(X, \mathscr{K}^*/\pi_*\mathcal{O}_{\widetilde{X}}^*)$

Proof. This boils down to showing that the Cartier divisors given on the LHS can be identified with the RHS.

Suppose (V_i, g_i) represents a section on the RHS. Then $g_i \in \Gamma(V_i, \mathscr{K}^*)$ with $g_i/g_j \in \Gamma(V_i \cap V_j, \pi_*\mathcal{O}_{\widetilde{X}}^*)$. Of course, we identify $g_i \in \Gamma(V_i, \mathscr{K}^*)$ with $g_i \in \Gamma(\pi^{-1}(V_i), \mathscr{K}^*)$ based on how $\widetilde{X}$ was constructed and

$$g_i/g_j \in\in \Gamma(V_i \cap V_j, \pi_*\mathcal{O}_{\widetilde{X}}^*) = \Gamma(\pi^{-1}(V_i) \cap \pi^{-1}(V_j), \mathcal{O}_{\widetilde{X}}^*).$$

It follows that $(\pi^{-1}(V_i), g_i)$ represents a Cartier divisor in $\operatorname{Pic} \widetilde{X}$. There is an obvious map going the other direction so we get the indicated isomorphism. □

Lemma 6.10. The RHS map $\Gamma(X, \mathscr{K}^*/\mathcal{O}_X^*) \to \Gamma(X, \mathscr{K}^*/\pi_*\mathcal{O}_{\widetilde{X}}^*)$ is surjective.

Proof. One could attempt to use cohomology to show surjectivity since $\pi_*\mathcal{O}_{\widetilde{X}}^*/\mathcal{O}_X^*$ can be observed to be a direct sum of skyscrapper sheaves.

A more elementary approach to proving this surjectivity can be found in this post. □

(b) Plane Cuspidal Cubic. Recall that the plane cuspidal cubic X is cut out by $y^2z = x^3$ in $\mathbb{P}^2_k$. The notation $\mathbb{G}_a$ is for the additive group given by $\mathbb{A}^1$.

Let U_0, U_1, U_2 denote the open affines where we dehomogenize w.r.t x, y, z respectively. So for instance on U_0, the scheme is $\operatorname{Spec} k[y/x, z/x]/\left(\frac{y^2z}{x^3} - 1\right)$. Observe

$$U_0 \cap X = \operatorname{Spec} \frac{k[y/x, z/x]}{(\frac{y^2z}{x^3} - 1)},\ U_1 \cap X = \operatorname{Spec} \frac{k[x/y, z/y]}{(\frac{z}{y} - (\frac{x}{y})^3)},\ U_2 \cap X = \operatorname{Spec} \frac{k[x/z, y/z]}{(\frac{y}{z})^2 - (\frac{x}{z})^3}.$$

Observe that $U_0 \cap X$ need not be considered because if x is nonzero, then $y^2z = x^3$ implies that neither y or z is zero. So we may omit the consderation of $U_0 \cap X$.

Observe the rings for $U_1 \cap X$ and $U_2 \cap X$ are isomorphic to

$$\frac{k[s_1, t_1]}{(t_1 - s_1^3)},\ \frac{k[s_2, t_2]}{(t_2^2 - s_2^3)}.$$

We will need to normalize each of these rings and then remember how the gluing is done. The normalizations only affect the RHS ring so we know the normalization is isomorphic to

$$k[s_1] = k[\frac{x}{y}] \qquad \& \qquad k[\frac{t_2}{s_2}] = k[\frac{y/z}{x/z}] = k[\frac{y}{z}].$$

Since we understand the gluing of these two open affine subsets, one observes that $\operatorname{Spec} \widetilde{X} \cong \mathbb{P}^1_k$. Hence, $\operatorname{Pic} \widetilde{X} \cong \operatorname{Pic} \mathbb{P}^1_k \cong \mathbb{Z}$. So we get the SES

$$0 \to \bigoplus_{P \in X} \widetilde{\mathcal{O}_P}^*/\mathcal{O}_P^* \to \operatorname{Pic} X \to \mathbb{Z} \to 0.$$

The proof is complete if we prove the following lemma.

Lemma 6.11. Let $\mathcal{O}_P$ be the stalk of $\mathcal{O}_X$ at $P \in X$. Then,

$$\bigoplus_{P \in X} \widetilde{\mathcal{O}_P}^* / \mathcal{O}_P^* \cong \mathbb{G}_a$$

in a canonical manner.

PROOF. The normalization only affects a single point in the affine chart. Indeed, that point is the origin of the affine chart

$$\operatorname{Spec} \frac{k[s_2, t_2]}{(t_2^2 - s_2^3)}$$

and so, we focus our attention on here and drop the subscripts. Let Q be the origin in this affine chart. A priori, we see that

$$\mathcal{O}_{X,Q}^* = \left(\frac{k[s,t]}{(s^2 - t^3)} \right)^*_{(s,t)} \cong \left(k[\left(\frac{s}{t}\right)^3, \left(\frac{s}{t}\right)^2] \right)^*_{\left(\left(\frac{s}{t}\right)^3, \left(\frac{s}{t}\right)^2\right)} = \left\{ \frac{\{\text{polynomials in } (s/t)^2, (s/t)^3\} + k^*}{\{\text{polynomials in } (s/t)^2, (s/t)^3\} + k^*} \right\}$$

Now we compute $\widetilde{\mathcal{O}_{X,Q}}$. Since normalization commutes with localization, we will localize after we take the integral closure. Indeed, the integral closure of $\frac{k[s,t]}{(s^2-t^3)}$ is $k[\frac{s}{t}]$. Now localize this ring at the multiplicative system $k[\left(\frac{s}{t}\right)^3, \left(\frac{s}{t}\right)^2] - \left(\left(\frac{s}{t}\right)^3, \left(\frac{s}{t}\right)^2\right)$ and call it R. One can deduce that the units are

$$R^* = \left\{ \frac{\{\text{polynomials in } (s/t)\} + k^*}{\{\text{polynomials in } (s/t)\} + k^*} \right\}$$

because $\alpha \frac{s}{t} + \beta$ for $\beta \in k^*$ can be multiplied to a unit by

$$(\alpha \frac{s}{t} + \beta)(\alpha \frac{s}{t} - \beta) = \alpha^2 \left(\frac{s}{t}\right)^2 - \beta^2 \in \{\text{polynomials in } (s/t)^2, (s/t)^3\} + k^*.$$

It follows that the cokernel can be determined

$$\widetilde{\mathcal{O}_{X,Q}}^* / \mathcal{O}_{X,Q} \cong \{\alpha : \alpha \in k \ \& \ \alpha \text{ is the coefficient of } \frac{s}{t}\}. \tag{22}$$

Indeed, the RHS has an additive structure because the LHS is a cokernel and we know the cokernel is precisely the $\alpha \frac{s}{t} + 1$ which gives

$$\alpha \cdot \alpha' \equiv \left(\alpha \frac{s}{t} + 1\right)\left(\alpha' \frac{s}{t} + 1\right) = \alpha\alpha' \left(\frac{s}{t}\right)^2 + (\alpha + \alpha') \frac{s}{t} + 1 \equiv \alpha + \alpha'.$$

Hence, the isomorphism in (22) is $\widetilde{\mathcal{O}_{X,P}}^* / \mathcal{O}_{X,P} \cong \mathbb{G}_a$. We concdlude that

$$\bigoplus_{P \in X} \widetilde{\mathcal{O}_P}^* / \mathcal{O}_P^* \cong \widetilde{\mathcal{O}_Q}^* / \mathcal{O}_Q^* \cong \mathbb{G}_a.$$

□

Plane Nodal Cubic Curve. Recall that if X is the nodal cubic curve, then X is cut out by $y^2 z = x^3 + x^2 z$ in $\mathbb{P}^2$. The notation $\mathbb{G}_m$ is for the multiplicative group given by $\mathbb{A}^1 - \{0\}$.

Note that the normalization of a curve always yields a nonsingular curve. In particular, the normalization of X is a complete nonsingular curve. If we take chart $U_0 \cap X$ where $z \neq 0$, we get

$$U_0 \cap X = \operatorname{Spec} \frac{k[\frac{x}{z}, \frac{y}{z}]}{((y/z)^2 - (x/z)^3 - (x/z)^2)} \overleftarrow{\text{normalize}} \operatorname{Spec} k[\frac{y/z}{x/z}] \cong \mathbb{A}^1_k.$$

This shows that the normalization is birational to $\mathbb{P}^1_k$. A general fact is that a complete nonsingular rational curve is isomorphic to $\mathbb{P}^1_k$. Hence, $\operatorname{Pic}\widetilde{X} \cong \mathbb{Z}$.

For the direct sum $\bigoplus_{P\in X} \widetilde{\mathcal{O}_{X,P}}^*/\mathcal{O}^*_{X,P}$, note that the only time where the summand $\widetilde{\mathcal{O}_{X,P}}^*/\mathcal{O}^*_{X,P}$ is nonzero is when P is the origin of $U_0 \cap X$. Observe that

$$\mathcal{O}^*_{X,P} = \frac{\{\text{polynomials in } s\} + kt + k^*}{\{\text{polynomials in } s + kt + k^*}$$

and because taking integral closures commutes with localization, one has (where $s = \frac{x}{z}$ and $t = \frac{y}{z}$)

$$\widetilde{\mathcal{O}_{X,P}}^* = (\{\text{polynomials in } s\} + kt + k^*)^{-1}k\left[\frac{s}{t}\right]$$

which contains $\mathcal{O}^*_{X,P}$. Furthermore,

$$\left(\frac{s}{t}\right)^2 = \frac{s^2}{s^3+s^2} = \frac{1}{s+1}$$

which is a unit. So

$$\widetilde{\mathcal{O}_{X,P}}^*/\mathcal{O}^*_{X,P} \cong \left\{m\frac{s}{t} : m \in k^*\right\} \cong \mathbb{G}_m.$$

□

Hartshorne II.6.10 *The Grothendieck Group $K(X)$.* Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on X, by the subgroup generated by all expressions $\mathscr{F} - \mathscr{F}' - \mathscr{F}''$, whenever there is an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves on X. If $\mathscr{F}$ is a coherent sheaf, we denote by $\gamma(\mathscr{F})$ its image in $K(X)$.

(a) If $X = \mathbb{A}^1_k$, then $K(X) \cong \mathbb{Z}$.

(b) If X is any integral scheme, and $\mathscr{F}$ a coherent sheaf, we define the rank of $\mathscr{F}$ to be $\dim_K \mathscr{F}_\xi$, where ξ is the generic point of X, and $K = \mathcal{O}_\xi$ is the function field of X. Show that the rank function defines a surjective homomorphism $\operatorname{rank} : K(X) \to \mathbb{Z}$.

(c) If Y is a closed subscheme of X, there is an exact sequence

$$K(Y) \to K(X) \to K(X - Y) \to 0$$

where the first map is extension by zero, and the second map is restriction. [Hint: For exactness in the middle, show that if $\mathscr{F}$ is a coherent sheaf on X, whose support is contained in Y, then there is a finite filtration $\mathscr{F} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \ldots \supseteq \mathscr{F}_n = 0$, such that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is an $\mathcal{O}_Y$ -module. To show surjectivity on the right, use (Ex. 5.15). For further information about $K(X)$, and its applications to the generalized Riemann-Roch theorem, see Borel-Serre [1], Manin [1], and Appendix A.]

Proof. **(a)** Every coherent sheaf over X is of form $\widetilde{M}$ for M a f.g. $k[x]$-module. But $k[x]$ is a PID so we need only use the fact that $\sim$ is an exact functor which yields an equivalence of category. Use[53] Exercise 7.26iii) of Atiyah-Macdonald.

[53]We shall not include the proof of this exercise here as it is a relatively straightforward exercise.

(b) We begin by reducing to the affine case. Suppose we knew that the rank map $K(\operatorname{Spec} A) \to \mathbb{Z}$ were true for any integral affine scheme $\operatorname{Spec} A$. Now let X be an arbitrary integral noetherian scheme. Let $U = \operatorname{Spec} A$ be an open affine neighborhood of the generic point ξ. Choose a coherent sheaf $\mathscr{G}$ on $\operatorname{Spec} A$ that has rank $n \in \mathbb{Z}_{\geq 0}$. Now Hartshorne II.5.15(d) to extend $\mathscr{G}$ to a coherent sheaf $\mathscr{F}$ on X s.t. $\mathscr{F}|_U \cong \mathscr{G}$. Of course,

$$\operatorname{rank}(\mathscr{G}) = \dim_{\mathcal{O}_{X,\xi}} \mathscr{G}_\xi \cong \dim_{\mathcal{O}_{\operatorname{Spec} A,\xi}} \mathscr{F}_\xi = n.$$

As $n \geq 0$ was arbitrary, we have the result (to get negative values, take the negative of the sheaf as an element inside $K(X)$).

Now we prove the result in the affine case. Fix $X = \operatorname{Spec} A$. Again, let $n \geq 0$ be an arbitrary integer.Then the sheaf $\mathscr{F} := \widetilde{A^n}$ satisfies $\operatorname{rank}(\mathscr{F}) = n$.

(c) It is clear that the sequence is exact on the right. Indeed, any sheaf in $K(X - Y)$ is mapped to a sheaf in $K(X)$ which we can take to be the extension of zero.

We assume the hint for now. Let $\mathscr{F}$ be given with the filtration. From the relation and existence of SESs of $\mathcal{O}_X$-modules

$$0 \to \mathscr{F}_{i+1} \to \mathscr{F}_i \to \mathscr{F}_i/\mathscr{F}_{i+1} \to 0,$$

we have the formula $(\mathscr{F}) = \sum_{j=0}^{n-1} (\mathscr{F}_j/\mathscr{F}_{j+1})$. But we may view $\mathscr{F}_j/\mathscr{F}_{j+1}$ as $\mathcal{O}_Y$-modules so the RHS is an element of $K(Y)$. Since $\mathscr{F}$ is supported on Y, if $P \in X - Y$, the stalks gives a sequence

$$0 \to \mathscr{F}_{i+1,P} = 0 \to \mathscr{F}_{i,P} = 0 \to (\mathscr{F}_i/\mathscr{F}_{i+1})_P \to 0.$$

This means $(\mathscr{F}_i/\mathscr{F}_{i+1})_P = 0$ for $P \in X - Y$. Taking the sum $\sum_{j=0}^{n-1} (\mathscr{F}_j/\mathscr{F}_{j+1})$ as an element in $K(Y)$ and applying extension by zero $j_!$ gives a sum of $\mathcal{O}_X$-modules $\sum_{j=0}^{n-1} (j_!\mathscr{F}_j/\mathscr{F}_{j+1})$. From the observation above, the map $j_!\mathscr{F}_j/\mathscr{F}_{j+1} \to \mathscr{F}_j/\mathscr{F}_{j+1}$, with the RHS viewed as an $\mathcal{O}_X$-module, is a sheaf isomorphism if one checks the stalks. Thus, we have shown that if $(\mathscr{F}) \in \ker(K(X) \to K(X - Y))$, then $(\mathscr{F}) \in \operatorname{im}(K(Y) \to K(X))$.

It suffices to have proven this result with $(\mathscr{F}) \in \ker(K(X) \to K(X - Y))$ rather than a finite sum. If we were given a finite sum, the restriction of scalars would given a finite sum of sheaves that would be congruent to zero in $K(X - Y)$. To be congruent to zero would entail being able to rewrite the sum using the relation. Of course, we could then look at the element $(\mathscr{F}) - (\mathscr{F}') - (\mathscr{F}'')$ fitting inside the $K(X - Y)$. Then repeat the above proof on each term.

At the start, we had asusmed the hint. Now, we prove it in the next lemma.

Lemma 6.12. There exists a finite filtration of $\mathcal{O}_X$-modules

$$\mathscr{F} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{F}_{n-1} \supseteq \mathscr{F}_n = 0$$

s.t. each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is an $\mathcal{O}_Y$-module when $\mathscr{F}$ is a coherent sheaf with $\operatorname{supp} \mathscr{F} \subseteq Y$. Here, Y is a closed subscheme of X.

Proof. Fix $i : Y \to X$ to be the closed immersion determined by Y. From the adjoint property of the inverse-image and push-forward functors, we always have a natural map $\eta : \mathscr{F} \to i_* i^* \mathscr{F}$.

Let $\mathscr{F}_0 := \mathscr{F}$ and define inductively $\mathscr{F}_j = \ker(\mathscr{F}_{j-1} \to i_* i^* \mathscr{F}_{j-1})$.

Now, if $\operatorname{Spec} A \subseteq X$ is s.t. $\mathscr{F}|_{\operatorname{Spec} A} = \widetilde{M}$, then the map $\eta|_{\operatorname{Spec} A} : \widetilde{M} \to \widetilde{M/IM}$. Indeed, the inverse image functor i^* would pull $\widetilde{M}$ back to the the subset $\operatorname{Spec} A \cap Y$ which is a

closed subscheme of $\operatorname{Spec} A$. Therefore, $\operatorname{Spec} A \cap Y = \operatorname{Spec} A/I$ for some ideal I. Of course, $i_*\widetilde{M/IM} = \widetilde{M/IM}$ since it is viewed as an A-module.

Next, we note that $\mathscr{F}_j/\mathscr{F}_{j+1}$ are all $\mathcal{O}_Y$-modules because on $\operatorname{Spec} A \cap Y$ they are isomorphic to $I^j\widetilde{M/I^{j+1}M}$ and each $I^jM/I^{j+1}M$ is an $\mathcal{O}_Y$-module by our construction.

From our construction, we have a possibly infinite filtration $\mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \mathscr{F}_2 \supseteq \dots$ and we must show that this process terminates. On an open affine $\operatorname{Spec} A \cap Y$, we see that

$$V(\operatorname{Ann}(M)) = \operatorname{Supp}(\widetilde{M}) \subseteq Y \cap \operatorname{Spec} A \cap Y = \operatorname{Spec} A/I = V(I)$$

where the LHS is by Hartshorne II.5.6(b). Since $V(\operatorname{Ann}(M)) \subseteq V(I)$, we know $\sqrt{I} \subseteq \operatorname{Ann}(M)$. Since A is necessarily Noetherian and M is f.g., $\operatorname{Ann}(M)$ is f.g.. This means $I^n = 0$ for sufficiently large n. Since X was a Noetherian scheme, there are only finitely many open affines to consider. Take the maximum N among the integers n and we have the filtration

$$\mathscr{F} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \dots \mathscr{F}_{N-1} \supseteq \mathscr{F}_N = 0.$$

□

□

Remark 32. One may be interested if the approach of "do it in the affine case and generalize to the arbitrary case" would work. It is easy to see from a result in Bourbaki's commutative algebra text book that it is true in the affine case. However, the general case cannot follow due to the arbitrary nature of our filtration. The proof of the lemma is motivated by the proof inside that question's refernced solutions manual by Cutrone and Marshburn.

Lemma 6.13 (Bourbaki's "Algèbre Commutative", Chap. IV n°4 Théorème 2)**.** Fix A a noetherian ring and M a f.g.A-module. Then there is a filtration as above with $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ where

$$\operatorname{Ass}(M) \subseteq \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\} \subseteq \operatorname{Supp}(M).$$

Hartshorne II.6.11 *The Grothendieck Group of a Nonsingular Curve.* Let X be a nonsingular curve over an algebraically closed field k. We will show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$, in several steps.

(a) For any divisor $D = \sum n_i P_i$ on X, let $\psi(D) = \sum n_i \gamma(k(P_i)) \in K(X)$, where $k(P_i)$ is the skyscraper sheaf k at P_i and 0 elsewhere. If D is an effective divisor, let $\mathcal{O}_D$ be the structure sheaf of the associated subscheme of codimension 1 , and show that $\psi(D) = \gamma(\mathcal{O}_D)$. Then use (6.18) to show that for any $D, \psi(D)$ depends only on the linear equivalence class of D, so ψ defines a homomorphism $\psi : \operatorname{Cl} X \to K(X)$.

(b) For any coherent sheaf $\mathscr{F}$ on X, show that there exist locally free sheaves $\mathscr{E}_0$ and $\mathscr{E}_1$ and an exact sequence $0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$. Let $r_0 = \operatorname{rank} \mathscr{E}_0$, $r_1 = \operatorname{rank} \mathscr{E}_1$, and define $\det \mathscr{F} = (\bigwedge^{r_0} \mathscr{E}_0) \otimes (\bigwedge^{r_1} \mathscr{E}_1)^{-1} \in \operatorname{Pic} X$. Here $\bigwedge$ denotes the exterior power (Ex. 5.16). Show that $\det \mathscr{F}$ is independent of the resolution chosen, and that it gives a homomorphism $\det : K(X) \to \operatorname{Pic} X$. Finally show that if D is a divisor, then $\det(\psi(D)) = \mathscr{L}(D)$.

(c) If $\mathscr{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence $0 \to \mathscr{L}(D)^{\oplus r} \to \mathscr{F} \to \mathscr{T} \to 0$, where $\mathscr{T}$ is a torsion sheaf. Conclude that if $\mathscr{F}$ is a sheaf of rank r, then $\gamma(\mathscr{F}) - r\gamma(\mathcal{O}_X) \in \operatorname{Im} \psi$.

(d) Using the maps ψ, det, rank, and $1 \mapsto \gamma(\mathcal{O}_X)$ from $\mathbb{Z} \to K(X)$, show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

PROOF. **(a)** To prove $\psi(D) = \gamma(\mathcal{O}_D)$, we prove a few lemmas and put them together. The only nontrivial lemma is the third.

Lemma 6.14. If $D = P$, then $\psi(D) = \gamma(k(P)) = \gamma(\mathcal{O}_D)$.

PROOF. This is clear since $k(P) = \mathcal{O}_P = \mathcal{O}_D$. □

Lemma 6.15. If $D = P + Q$ where $P \neq Q$, then $\psi(D) = \gamma(\mathcal{O}_D)$.

PROOF. In this case, $\psi(D) = \gamma(k(P)) + \gamma(k(Q))$. However, $\mathcal{O}_D = \mathcal{O}_P \oplus \mathcal{O}_Q = k(P) \oplus k(Q)$ so the equality $\psi(D) = \gamma(\mathcal{O}_D)$ clearly holds. □

Lemma 6.16. If $D = nP$, then $\psi(D) = \gamma(\mathcal{O}_D)$.

PROOF. We get $\psi(D) = n\gamma(k(P))$ while $\mathcal{O}_D \cong \mathcal{O}_{\mathrm{Spec}(A)}$ where $A \cong k[x]/(x^n)$. Of course, $A \cong k \oplus k\langle x\rangle \oplus k\langle x^2\rangle \oplus \cdots \oplus k\langle x^{n-1}\rangle$ as a k-module so as an element of the Grothendieck group, $\gamma(\mathcal{O}_D) = n\gamma(k(P))$. □

Lemma 6.17. Let D and D' be two divisors. Then $\psi(D + D') = \gamma(\mathcal{O}_{D+D'})$.

PROOF. Let $D = \sum_{i\in I} n_i P_i$ and $D' = \sum_{j\in J} n_j P_j$. Then $\psi(D + D') = \sum_{\ell\in I\cup J} n_i \gamma(k(P_\ell))$. Meanwhile, $\mathcal{O}_{D+D'} = \left(\bigoplus_{\ell\in I\cup J, n_\ell = 1} k(P_\ell)\right) \oplus \left(\bigoplus_{\ell\in I\cup J, n_\ell > 1} \mathcal{O}_{\mathrm{Spec}\, k[x]/(x^{n_\ell})}\right)$. Applying γ to this, it is easy to see that it equals the RHS of the previous equation (via the relations in the Grothendieck group). □

Lemma 6.18. If $D \sim D'$, then $\psi(D) = \psi(D')$.

PROOF. From the previous lemma, $D - D'$ is a principal divisor and so

$$\psi(D) - \psi(D') = \psi(D - D') = \gamma(\mathcal{O}_{D-D'}).$$

Since $D - D'$ is a principal divisor, then $\mathcal{O}_{D-D'}$ is cut out by a single equation. In which case,

$$0 \to \mathscr{I}_{D-D'} \to \mathcal{O}_X \to \mathcal{O}_{D-D'} \to 0$$

which means

$$\gamma(\mathcal{O}_{D-D'}) = \gamma(\mathcal{O}_X) - \gamma(\mathscr{I}_{D-D'}) = \gamma(\mathcal{O}_X) - \gamma(\mathscr{L}(D - D')) = \gamma(\mathcal{O}_X) - \gamma(\mathcal{O}_X) = 0$$

where the second to last equality follows from Hartshorne Proposition II.6.13(c):

$$\mathscr{L}(D - D') \cong \mathscr{L}(D) \otimes \mathscr{L}(D')^{-1} \cong \mathscr{L}(D) \otimes \mathscr{L}(D)^{-1} \cong \mathcal{O}_X.$$

□

(b) As usual, we split the problem into parts. They shall each be in the form of a lemma.

Lemma 6.19. The resolution via locally free sheaves $0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$ for any coherent sheaf $\mathscr{F}$ on the curve X exists.

PROOF. [54] This result follows from Hartshorne III.6.9(a) so we shall leave the proof to that section. Note that the proof relied on facts about homological dimension and cohomology which we do not have at the moment. □

Lemma 6.20. $\det \mathscr{F}$ is well-defined i.e. it is independent of the choice of resolution.

[54] **[Incomplete]** This is a to-do for the future – give a proof that does not rely on cohomology.

PROOF. First off, we shall show that there exists a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}_1' & \longrightarrow & \mathscr{E}_0' & \longrightarrow & \mathscr{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\psi & \longrightarrow & \mathscr{E}_0 \oplus \mathscr{E}_0' & \xrightarrow{\psi} & \mathscr{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\psi/\mathscr{E}_1' & \longrightarrow & \mathscr{E}_0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which the top two rows are exact and the columns are exact. The top row is just the two step resolution of $\mathscr{F}$. The second row is obtained by taking the addition map $\psi : \mathscr{E}_0 \oplus \mathscr{E}_0' \xrightarrow{\psi} \mathscr{F}$ which is surjective.

The fact that $\ker\psi/\mathscr{E}_1' \to \mathscr{E}_0$ exists is justified as follows. There is a map $\ker\psi \to \mathscr{E}_0 \oplus \mathscr{E}_0' \to \mathscr{E}_0$ where the second map is projection. Under this composition, $\mathscr{E}_1' \mapsto \mathscr{E}_0' \mapsto 0$. So we factor through the composition to get a map $\ker\psi/\mathscr{E}_1' \to \mathscr{E}_0$. This is injective because $[a] \in \ker\psi/\mathscr{E}_1'$ goes to zero in $\mathscr{E}_0$ iff a goes to $\mathscr{E}_0'$ in $\mathscr{E}_0 \oplus \mathscr{E}_0'$ which is iff $a \in \mathscr{E}_1'$ because of the commutativity of the top left diagram.

Apply the 9-lemma to deduce that the bottom row is exact and so, $\frac{\ker\psi}{\mathscr{E}_1'} \cong \mathscr{E}_0$. Apply this with the indices interchanged to get $\frac{\ker\psi'}{\mathscr{E}_1} \cong \mathscr{E}_0$.

Independence of resolution is equivalent to

$$\left(\bigwedge^{r_0}\mathscr{E}_0\right)\otimes\left(\bigwedge^{r_1}\mathscr{E}_1\right)^{-1} \cong \left(\bigwedge^{r_0'}\mathscr{E}_0'\right)\otimes\left(\bigwedge^{r_1'}\mathscr{E}_1'\right)^{-1} \iff$$

$$\left(\bigwedge^{r_0}\frac{\ker\psi}{\mathscr{E}_1'}\right)\otimes\left(\bigwedge^{r_1}\mathscr{E}_1\right)^{-1} \cong \left(\bigwedge^{r_0'}\frac{\ker\psi'}{\mathscr{E}_1'}\right)\otimes\left(\bigwedge^{r_1'}\mathscr{E}_1'\right)^{-1}.$$

The RHS equation is equivalent to

$$\left(\bigwedge^{r_0}\frac{\ker\psi}{\mathscr{E}_1'}\right)\otimes\left(\bigwedge^{r_1}\mathscr{E}_1'\right) \cong \left(\bigwedge^{r_0'}\frac{\ker\psi'}{\mathscr{E}_1}\right)\otimes\left(\bigwedge^{r_1'}\mathscr{E}_1\right) \tag{23}$$

But we can apply Hartshorne II.5.16(d) to the SES

$$0 \to \mathscr{E}_1' \to \ker\phi \to \ker\phi/\mathscr{E}_1' \to 0$$

to get that $\left(\bigwedge^{r_0}\frac{\ker\psi}{\mathscr{E}_1'}\right)\otimes\left(\bigwedge^{r_1}\mathscr{E}_1'\right) \cong \bigwedge^{r}\ker\psi$ where $r = \operatorname{rank}\ker\psi$. Of course, this posits that $\ker\psi$ is a locally free sheaf.

Taking stalks in the above SES, we get a SES

$$0 \to (\mathscr{E}_1')_P \to (\ker\psi)_P \to (\mathscr{E}_0)_P \to 0$$

and because $(\mathscr{E}_0)_P$ is free and therefore projective, the sequence splits $(\ker\psi)_P \cong (\mathscr{E}_1')_P \oplus (\mathscr{E}_0)_P$. This implies $(\ker\psi)_P$ is free and so $\ker\psi$ is locally free by Hartshorne II.5.7. Now do this with the SES $0 \to \mathscr{E}_1 \to \ker\psi' \to \mathscr{E}_0' \to 0$ to get $\left(\bigwedge^{r_0'} \frac{\ker\psi'}{\mathscr{E}_1}\right) \otimes \left(\bigwedge^{r_1'} \mathscr{E}_1\right) \cong \bigwedge^r \ker\psi$. With these two tensor product formulas for $\bigwedge^r \ker\psi$, we have (23). □

Lemma 6.21. The determinant det extends to a homomorphism $\det : K(X) \to \mathrm{Pic}(X)$ where $K(X)$ is an additive abelian group and $\mathrm{Pic}(X)$ is an abelian group under $\otimes$.

Proof. Take a SES $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$. Then obtain the two step filtrations of $\mathscr{F}$ and $\mathscr{H}$ and use the Horseshoe Lemma of homological algebra to get the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}_1' & \longrightarrow & \mathscr{E}_1 & \longrightarrow & \mathscr{E}_1'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}_0' & \longrightarrow & \mathscr{E}_0 & \longrightarrow & \mathscr{E}_0'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{H} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and the sheaves $\mathscr{E}_1, \mathscr{E}_0$ are locally free since they are a direct sum of locally free sheaves. Now the fact that $\gamma(\mathscr{F}) - \gamma(\mathscr{G}) + \gamma(\mathscr{H})$ goes to zero in $\mathrm{Pic}(X)$ follows from the formulas in Hartshorne II.5.16 and a basic computation. □

Lemma 6.22. Let D be a divisor. Then $\det(\psi(D)) = \mathscr{L}(D)$.

Proof. Let $D = \sum n_i P_i$ be a divisor. Then $\psi(D) = \sum_{i\in I} n_i \gamma(k(P_i))$. Consider the representative $\bigoplus_{i\in I} k(P_i)^{\oplus n_i}$ of $\psi(D)$. This is the class in $K(X)$ represented by $\mathcal{O}_D$ and if D is effective, there is a SES

$$0 \to \mathscr{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

The determinant of this is then $\mathscr{L}(D)$.

Now consider the case where D is not effective. Then write $D = D' - D''$ where D' is effective and D'' is also effective. Apply the result to get

$$\begin{aligned}
\det(\psi(D)) &= \det(\psi(D') + \psi(-D'')) \\
&= \det(\psi(D') - \psi(D'')) \\
&= \det(\psi(D')) \otimes (\det(\psi(D''))^{-1} \\
&= \mathscr{L}(D') \otimes \mathscr{L}(D'')^{-1} \\
&= \mathscr{L}(D' - D'') \\
&= \mathscr{L}(D).
\end{aligned}$$

□

(c)[55]

Since $\mathscr{F}_\xi \cong \mathcal{O}_{X,\xi}^r$, we can upgrade this via Hartsrhone II.5.7(a) to a local isomorphism $\mathscr{F}|_U \cong \mathcal{O}_X|_U^r$. The argument here gives the morphism

$$0 \to \mathscr{L}(D) \to \mathscr{F} \to \mathscr{T} \to 0.$$

To prove the second part, we begin with a lemma.

Lemma 6.23. If $\mathscr{T}$ is a torsion sheaf, then $\gamma(\mathscr{T}) \in \operatorname{im}\psi$.

Proof. [Incomplete] □

Lemma 6.24. $\gamma(\mathscr{F}) - r\gamma(\mathcal{O}_X) \in \operatorname{Im}\psi$.

Proof. From the sequence and applying γ,

$$\gamma(\mathscr{F}) = r\gamma(\mathscr{L}(D)) + \gamma(\mathscr{T}) \;\Rightarrow\; \gamma(\mathscr{T}) = \gamma(\mathscr{F}) - \gamma(\mathscr{L}(D)).$$

However, the sequence $0 \to \mathscr{L}(D) \to \mathcal{O}_X \to \mathcal{O}_{-D} \to 0$ whenever $-D$ is effective gives

$$\gamma(\mathscr{T}) = \gamma(\mathscr{F}) - r\gamma(\mathcal{O}_X) + r\gamma(\mathcal{O}_{-D}) = \gamma(\mathscr{F}) - r\gamma(\mathcal{O}_X) + r\psi(-D).$$

Since $\gamma(\mathscr{T}) \in \operatorname{im}\psi$, we get the claim. □

(d) The Grothendieck group $K_0(X) \cong \mathbb{Z} \oplus \operatorname{Pic}(X)$ is given by rank and determinant. Here is how it is done. [Incomplete] □

Hartshorne II.6.12 Let X be a complete nonsingular curve. Show that there is a unique way to define the *degree* of any coherent sheaf on X, $\deg \mathscr{F} \in \mathbb{Z}$, such that:

(1) If D is a divisor, $\deg \mathscr{L}(D) = \deg D$.

(2) If $\mathscr{F}$ is a *torsion sheaf* (meaning a sheaf whose stalk at the generic point is zero), then $\deg \mathscr{F} = \sum_{P \in X} \operatorname{length}(\mathscr{F}_P)$; and

(3) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, then $\deg \mathscr{F} = \deg \mathscr{F}' + \deg \mathscr{F}''$.

Proof 1. We present a proof that does not use sheaf cohomology. We use Hartshorne II.6.11 without comment.

There is a projection $K(X) \to \operatorname{Pic}(X)$ from the isomorphism $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$ which we denote by p_1. Identify any invertible sheaf $\mathscr{L} \in \operatorname{Pic} X$ to a Weil divisor $\sum n_i P_i$ and map to $\mathbb{Z}$ by $\mathscr{L} \to \sum n_i$. Let $\deg : K(X) \to \operatorname{Pic}(X) \to \mathbb{Z}$ be the composition.

Clearly, condition (3) is satisfied since taking associated Weil divisors and then taking the degree is additive.

To check condition (1), we note that $\mathscr{L}(D) \in \operatorname{Pic} X$ and corresponds to the divisor D under the Weil divisor and invertible sheaf correspondence. Hence, $\deg \mathscr{L}(D) = \deg D$.

The nontrivial part is to check that $\mathscr{F}$ is a torsion sheaf. By similar methods below, we deduce that $\gamma(\mathscr{F}) = \gamma(\mathcal{O}_D)$ for some Weil divisor $D = \sum n_i P_i$ which is effective. Since the degree function satisfies (3), $\deg \mathscr{F} = \deg \mathcal{O}_D$. So we reduce to computing $\deg D = \sum n_i$ and showing it equals $\sum_{P \in X} \operatorname{length}(\mathscr{F}_P)$. This reduces down to proving $n_i = \operatorname{length} \mathcal{O}_{P_i}$. The proof in our second proof can replace what is below.

First off, the stalk of $\mathcal{O}_D$ at P_i is isomorphic to k^{n_i}. Now $\mathscr{F}_P$ is a priori just an $\mathcal{O}_P$-module. Take a filtration of $\mathscr{F}_P$ as an $\mathcal{O}_P$-module. Then this extends to a filtration as k-modules since $k \hookrightarrow \mathcal{O}_P$ is an embedding (due to k-algebraically closed). Now a k-filtration

[55] See the post here and this post here

of k^{n_i} has length n_i and if it is maximal, it has simple quotients. The quotients remain[56] simple even over $\mathcal{O}_P$. It follows that $n_i = \operatorname{length} \mathcal{O}_{P_i}$. □

Proof 2. **Disclaimer.** The proof here assigns some familiarity with sheaf cohomology. The choice

$$\deg \mathscr{F} = \chi(X, \mathscr{F}) - (\operatorname{rank} \mathscr{F})\chi(X, \mathcal{O}_X)$$

works according to Vakil's notes and is usually the one taken as definition. We sketch out the reason why.

First off, it is clearly additive since the Euler characteristic is additive (see Hartshorne III.5.1). To see that (1) is satisfied by this choice, one needs

$$\chi(X, \mathscr{L}(D)) - \chi(X, \mathcal{O}_X) = \deg D$$

since the rank of an invertible sheaf is 1. Of course, this is proven in Hartshorne Theorem IV.1.13.

To check (2), suppose $\mathscr{F}$ were a torsion sheaf. We want

$$\chi(X, \mathscr{F}) - (\operatorname{rank} \mathscr{F})\chi(X, \mathcal{O}_X) = \sum_{P \in X} \operatorname{length} \mathscr{F}_P.$$

The fact that $\mathscr{F}$ is torsion and the curve is a complete nonsingular curve implies that $\mathscr{F}$ has finite support. So the sum on the RHS is well-defined.

Because $\mathscr{F}$ has finite support, it follows that

$$\mathscr{F}(U) = \prod_{P \in U} \mathscr{F}_P \qquad \text{for any open subset } U \subseteq X \text{ and the product is over closed points}$$

(for a reference, see this post), and so let us attempt to study the situation where $\mathscr{F}$ was supported at a single point.

Lemma 6.25. If $\mathscr{F}$ is supported at a single point Q, then $\chi(X, \mathscr{F}) - (\operatorname{rank} \mathscr{F})\chi(X, \mathcal{O}_X) = \operatorname{length} \mathscr{F}_P$.

Proof. First off,

$$\chi(X, \mathscr{F}) = \dim_k H^0(X, \mathscr{F}) - \dim_k H^1(X, \mathscr{F}) = \Gamma(X, \mathscr{F}) - H^1(X, \mathscr{F}) = \dim_k \mathscr{F}_Q - \dim_k H^1(X, \mathscr{F}).$$

But dimension and length coincides for k-vector spaces which means $\dim_k \mathscr{F}_Q = \operatorname{length} \mathscr{F}_Q$. OTOH, $\mathscr{F}$ being a torsion sheaf implies that $\operatorname{rank} \mathscr{F} = \dim_K \mathscr{F}_\xi = \dim_{\mathcal{O}_{X,\xi}} 0 = 0$ so that

$$\chi(X, \mathscr{F}) - (\operatorname{rank} \mathscr{F})\chi(X, \mathcal{O}_X) = \chi(X, \mathscr{F}) = \operatorname{length} \mathscr{F}_Q - \dim_k H^1(X, \mathscr{F}).$$

It suffices to show $\dim_k H^1(X, \mathscr{F}) = 0$. Of course, $\mathscr{F}$ is clearly flasque from the above line so $H^1(X, \mathscr{F}) = 0$. □

The proof of the lemma generalizes to the case where $\mathscr{F}$ is supported on more than a single point. □

[56] If $M = \langle a \rangle$ is a simple k-module, then it is a simple $\mathcal{O}_{P_i}$-module. The idea is that $\mathcal{O}_{P_i} \cong \operatorname{Ann}\langle a \rangle$. Then $M \cong \mathfrak{m}_{P_i} / \operatorname{Ann}(a) \cong k$.

7. Projective Morphisms

Hartshorne II.7.1 Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $f : \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism. [Hint: Reduce to a question of modules over a local ring by looking at the stalks.

Proof. Since $f : \mathscr{L} \to \mathscr{M}$ is surjective map of sheaves, $\operatorname{im} f \cong \mathscr{M}$. It suffices to show that the induced maps on stalks are injective by Hartshorne II.1.2. If $x \in X$, then there is an open neighborhood $U \ni x$ on which $\mathscr{L}$ trivializes and by shrinking U, $\mathscr{M}$ also trivializes. Therefore, $\mathscr{L}|_U \cong \mathcal{O}_X|_U \cong \mathscr{M}|_U$ and over the stalks, $\mathscr{L}_x \cong \mathcal{O}_x \cong \mathscr{M}_x$ which means

$$f_x : \mathscr{L}_x \to \mathscr{M}_x$$

is a surjective map of rank one free $\mathcal{O}_x$-modules. Every surjective map of rank one free $\mathcal{O}_x$-modules is injective as well.

Lemma 7.1. Let $f : A \to A$ be a surjective map of rank one free A-modules. Then $f : A \to A$ is also injective.

Proof. Suppose $f(a) = 0$ and that the map was surjective. Then $f(a) = a \cdot f(1) = 0$. For the map to be surjective, $f(1)$ is a unit in A. Multiplying both sides by $f(1)^{-1}$ gives $a = 0$. This proves injectivity. □

□

Hartshorne II.7.2 Let X be a scheme over a field k. Let $\mathscr{L}$ be an invertible sheaf on X, and let $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ be two sets of sections of $\mathscr{L}$, which generate the same subspace $V \subseteq \Gamma(X, \mathscr{L})$, and which generate the sheaf $\mathscr{L}$ at every point. Suppose $n \leqslant m$. Show that the corresponding morphisms $\varphi : X \to \mathbb{P}^n_k$ and $\psi : X \to \mathbb{P}^m_k$ differ by a suitable linear projection $\mathbb{P}^m - L \to \mathbb{P}^n$ and an automorphism of $\mathbb{P}^n$, where L is a linear subspace of $\mathbb{P}^m$ of dimension $m - n - 1$.

Proof. Hartshorne Theorem II.7.1 allows us to identify $\{s_i\}$ and $\{t_j\}$ with morphisms $\varphi : X \to \mathbb{P}^n_k$ and $\psi : X \to \mathbb{P}^m_k$.

For each s_i, we may write $s_i = \sum_{j=0}^m a_{ij} t_j$. By a suitable change of coordinates of the s_i, we may assume $s_0 = t_0, s_1 = t_1, \ldots, s_n = t_n$. There is a map $\mathbb{P}^m \to \mathbb{P}^n$ which sends $[t_0, t_1, \ldots, t_m] \mapsto [t_0, t_1, \ldots, t_n]$ which us defined away from the hyperplane cut out by $t_0 = t_1 = \cdots = t_n = 0$. This hyperplane has dimension $m - (n + 1) = m - n - 1$.

Now we put together everything we did above. First off, the map $\psi : X \to \mathbb{P}^m_k$ is given by mapping to $[t_0, t_1, \ldots, t_m]$. Then we compose with $\mathbb{P}^m - L \to \mathbb{P}^n$ which is a map $[t_0, t_1, \ldots, t_m] \to [t_0, t_1, \ldots, t_n]$. Our change of coordinates in the last paragraph then induces an automorphism $\mathbb{P}^n_k \to \mathbb{P}^n_k$ which maps $[t_0, t_1, \ldots, t_n] \to [s_0, s_1, \ldots, s_n]$. Of course, the map $\varphi : X \to \mathbb{P}^m_k$ is defined precisely by mapping to the $[s_0, s_1, \ldots, s_n]$. □

Hartshorne II.7.3 Let $\varphi : \mathbb{P}^n_k \to \mathbb{P}^m_k$ be a morphism. Then:

(a) either $\varphi(\mathbb{P}^n) = pt$ or $m \geqslant n$ and $\dim \varphi(\mathbb{P}^n) = n$.

(b) in the second case, φ can be obtained as the composition of (1) a d-uple embedding $\mathbb{P}^n \to \mathbb{P}^N$ for a uniquely determined $d \geqslant 1$, (2) a linear projection $\mathbb{P}^N - L \to \mathbb{P}^m$, and (3) an automorphism of $\mathbb{P}^m$. Also, φ has finite fibres.

PROOF. **(a)** Either $m < n$ or $m \geq n$. Assume first that $m < n$. The morphism $\varphi : \mathbb{P}^n_k \to \mathbb{P}^m_k$ is determined by $\mathcal{O}_{\mathbb{P}^n_k}(\ell) = \varphi^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ for $\ell \in \mathbb{Z}$ and specifying the sections $\varphi^*(x_i)$ with $0 \leq i \leq m$. To ensure the sections $\varphi^*(x_i)$ generate the sheaf $\mathcal{O}_{\mathbb{P}^n_k}(\ell)$, we must have $\ell = 0$.

Lemma 7.2. If $\ell \neq 0$, then $\mathcal{O}_{\mathbb{P}^n_k}(\ell)$ is not generated by the $\varphi^*(x_i)$ for $x_0, \ldots, x_m \in \mathcal{O}_{\mathbb{P}^m_k}(1)$.

PROOF. Clearly, we cannot have $\ell < 0$ because there are no global sections in this case. Assume $\ell > 0$. Then the sections $\varphi^*(x_i) =: s_i \in \mathcal{O}_{\mathbb{P}^n_k}(\ell)$ cut out hypersurfaces inside $\mathbb{P}^n_k$. In particular, each s_i is the equation of a degree ℓ hypersurface inside $\mathbb{P}^n_k$.

Now we use Hartshorne Theorem I.7.2. Let H_1, H_2 be two of these hypersurfaces. Then the irreducible components of $H_1 \cap H_2$ have dimension $\geq (n-1) + (n-1) - n = n - 2$. Proceeding inductively, the irreducible components of the intersections of all of these hypersurfaces has dimension $\geq n - m > 0$. □

Using the lemma, we find sections inside $\mathcal{O}_{\mathbb{P}^n_k}$. Let $a_i := \varphi^*(x_i) \in \mathcal{O}_{\mathbb{P}^n_k}$ be a set of sections that generates $\mathcal{O}_{\mathbb{P}^n_k}$. For these to generate, not all of the a_i are zero. Then $\mathbb{P}^n_k \to \mathbb{P}^m_k$ is given by sending $[x_0, x_1, \ldots, x_n] \mapsto [a_0, a_1, \ldots, a_n]$ where not all of the a_i are zero. So the map is constant.

Now assume $m \geq n$. Let $e := \dim \mathbb{P}^n - \dim \varphi(\mathbb{P}^n)$. Then Hartshorne II.3.22(e) asserts that C_e contains an open dense subset of $\varphi(\mathbb{P}^n)$. Of course, $\varphi(\mathbb{P}^n)$ is closed so C_e is all of $\varphi(\mathbb{P}^n)$. But because C_e is the set of points $y \in \varphi(\mathbb{P}^n)$ with $\dim \varphi^{-1}(y) = e$, finiteness of the fibres in part (b) shows this is zero. So $e = 0$ which means $\dim \varphi\mathbb{P}^n = \dim \varphi\mathbb{P}^n = n$.

(b) Let $\phi : \mathbb{P}^n \to \mathbb{P}^m$ be a morphism. Then there is an invertible sheaf $\mathscr{L}$ s.t. $\mathscr{L} \cong \phi^*\mathcal{O}_{\mathbb{P}^m}(1)$ that is generated by the corresponding global sections $s_i = \varphi^*(x_i)$ for $i = 0, 1, \ldots, m$. Now $\mathscr{L} \cong \mathcal{O}(d)$ for some d and the only ample ones are for $d > 0$.

Assuming $\mathscr{L} \cong \mathcal{O}(d)$, we get an associated d-uple embedding $\mathbb{P}^n \xrightarrow{\phi_d} \mathbb{P}^N$. Since the sections which define the d-uple embedding map also define the morphism into $\mathbb{P}^m$, Hartshorne II.7.2 says that ϕ and ϕ_d differ only by a linear projection $\mathbb{P}^N - L \to \mathbb{P}^m$ and an automorphism of $\mathbb{P}^m$. That is, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\quad\phi\quad} & \mathbb{P}^m \\ \downarrow{\scriptstyle \varphi_d} & & \uparrow{\scriptstyle \text{automorphism}} \\ \varphi_d(\mathbb{P}^n) \subseteq \mathbb{P}^N - L & \xrightarrow{\text{linear projection}} & \mathbb{P}^m \\ \downarrow & & \\ \mathbb{P}^N & & \end{array}$$

Note that we must have $N \geq m$ because the m sections must define a map into $\mathbb{P}^N$ and if $N < m$, we could not pullback the N sections of $\mathcal{O}_{\mathbb{P}^N}(1)$.

Observe that φ_d, the linear projection, and the potential automorphism all define finite morphisms. First, φ_d is a closed immersion and these are known to be finite [10, Tag 035C]. One can also show using an argument similar to the one here that the linear projection is separated. Indeed, the linear projection is defined $\mathbb{P}^N \to \mathbb{P}^m$ by forgetting coordinates and $\mathbb{P}^N - L, \mathbb{P}^m$ are both proper over k. It follows by Hartshorne Corollary II.4.8. that the linear projection is proper and hence, finite by Hartshorne II.4.1.

The preceding paragraph showed that ϕ must be a finite map. Hartshorne II.3.5. asserts that the fibres must be finite. □

Remark 33. A helpful read for the previous two exercise can be found here: Vakil's Math 216 Class 32 Notes. Alternatively, see Section 16.4 of Vakil's Notes.

Remark 34. For Hartshorne II.7.3(b), one needs to assume that φ is nondengenerate i.e. the image of φ is not contained in a hyperplane. Otherwise (b) fails by taking $\mathbb{P}^n \to \mathbb{P}^m$ which sends $[x_0, \ldots, x_n] \to [x_0, \ldots, x_n, 0, 0, \ldots, 0]$ for $m \geq n$. We used this implicitly when getting our choice of φ_d.

Hartshorne II.7.4

(a) Use (7.6) to show that if X is a scheme of finite type over a noetherian ring A, and if X admits an ample invertible sheaf, then X is separated.

(b) Let X be the affine line over a field k with the origin doubled (4.0.1). Calculate Pic X, determine which invertible sheaves are generated by global sections, and then show directly (without using (a)) that there is no ample invertible sheaf on X.

Proof. (a) From Hartshorne Theorem II.7.6, the ample invertible sheaf $\mathscr{L}$ has a tensor power $\mathscr{L}^m$ for $m > 0$ that is very ample. So there is an immersion $i : X \to \mathbb{P}^n_A$ s.t. $i^*\mathcal{O}(1) \cong \mathscr{L}^m$. The immersion i factors as an open immersion followed by a closed immersion. These are both separated and so $i : X \to \mathbb{P}^n_A$ is separated. The structure morphism $\mathbb{P}^n_A \to \operatorname{Spec} A$ is always separated. So the composition $X \to \mathbb{P}^n_A \to \operatorname{Spec} A$ is separated. But this just means X is separated.

(b) Given X and we determine $\operatorname{Pic} X$ by computing the invertible sheaves and verifying which invertible sheaves are generated by global sections. The scheme X is integral because the sections over the open sets are integral domains. If we let $\mathbb{A}_1$ and $\mathbb{A}_2$ denote the two affine open subsets, then $\mathcal{O}_X(\mathbb{A}_i) \cong k[z]$ which is an integral domain. Furthermore, $\mathcal{O}_X := \{(s_1, s_2) : s_i \in \mathcal{O}_X(\mathbb{A}_i) \ \& \ s_1|_{\mathbb{A}_1 \cap \mathbb{A}_2} = s_2|_{\mathbb{A}_1 \cap \mathbb{A}_2}\}$ which also has no zero divisors. Since X is integral, we use Hartshorne Proposition II.6.15 to deduce that $\operatorname{Pic} X \cong \operatorname{CaCl} X$.

Computing $\operatorname{CaCl} X$. We have an open cover $U_1 = \mathbb{A}_1$ and $U_2 = \mathbb{A}_2$ and for each i, an element $f_i \in \Gamma(U_i, \mathscr{K}^*)$ is just an element of $k(x)$ (and we write $k(x)$ when $i = 1$ and $k(y)$ when $i = 2$ to distinguish them). We also require that $f_1/f_2 \in \Gamma(U_1 \cap U_2, \mathcal{O}^*)$ and $f_2/f_1 \in \Gamma(U_1 \cap U_2, \mathcal{O}^*)$. In lieu of this, we deduce that f_1 and f_2 must differ by a constant. Therefore, the Cartier divisor is given by elements $f_1 = x^n$ and $f_2 = cx^n$ for some $c \in k^\times$ and $n \in \mathbb{N}$.

Since $\operatorname{CaCl} X$ is the group of Cartier divisors modulo linear equivalence, and two Cartier divisors are linearly equivalent iff their difference is principal, we deduce that elements of $\operatorname{CaCl} X$ are uniquely given by Cartier divisors of the form $f_1 = cx^n$ and $f_2 = y^n$ for some $c \in k^\times$. Indeed, any two Cartier divisors $\{(cx^n, U_1), (y^n, U_2)\}$ and $\{(c'x^n, U_1), (y^n, U_2)\}$ are linearly equivalent where $c, c' \in k^\times$. So up to linear equivalence, the only Cartier divisors have form $D_n := \{(x^n, U_1), (y^n, U_2)\}$. The group operation for Cartier divisors is given by multiplication. Defining the map $\operatorname{CaCl} X \to \mathbb{Z}$ by $D_n \mapsto n$, we get the isomorphism $\operatorname{CaCl} X \cong \mathbb{Z}$. So,

$$\operatorname{Pic} X \cong \operatorname{CaCl} X \cong \mathbb{Z}.$$

Next, we identify the sheaves which are generated by global sections. Let $\mathscr{L}_n := \mathcal{O}(D_n)$.

Lemma 7.3. For $n \neq 0$, there are no global sections of $\mathscr{L}_n$. If $n = 0$, then $\mathscr{L}_n \cong \mathcal{O}(D_0) \cong \mathcal{O}_X$.

PROOF. First off, $\mathscr{K}_X$ is given by $\mathscr{K}_X(U_1) = k(x)$ and $\mathscr{K}_X(U_2) = k(y)$. A global section of $\mathscr{L}_n$ is an s that restricts to an element of $k(x)$ and an element of $k(y)$. The only elements that can do this are the constants $s \in k$. But $s \notin \mathcal{O}(D_n)(X)$ if $n \neq 0$.

Now assume $n = 0$. Then $\mathscr{L} := \mathscr{L}$ can be given by the Cartier divisor $\{(1, U_1), (1, U_2)\}$. It follows that $\mathscr{L} \cong \mathcal{O}_X$. □

Finally, we show that there is no ample invertible sheaf on X.

Lemma 7.4. There are no ample invertible sheaves on X. This proof will not use part (a) because this is trivial: an ample invertible sheaf on X would imply X is separated which it clearly is not.

PROOF. An ample invertible sheaf on X would be generated by global sections if raised to a high enough tensor power. In this case, $\mathscr{L}_n$ cannot be ample unless $n = 0$. Of course, the sheaf $\mathcal{O}_X$ is not generated by global sections and any higher tensor power has the same global sections. Fix $m > 1$ and the coherent sheaf $\mathcal{O}_X$. Consider the sheaf $\mathcal{O}_X \otimes \mathcal{O}_X^m$ and the stalk at one of the origins is isomorphic to $k(x_1) \otimes_k k(x_2) \otimes \cdots \otimes_k k(x_m) \otimes_k k(x_{m+1})$. This stalk cannot be generated by k as an $\mathcal{O}_{X,x}^m$-module. □

Remark 35. Part (b) is trivial if one applies (a). Indeed, if an ample invertible sheaf on X exists, then X is separated which is absurd by Hartshorne Example II.4.0.1.

□

Hartshorne II.7.5 Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathscr{L}, \mathscr{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a noetherian ring A.

(a) If $\mathscr{L}$ is ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is ample.

(b) If $\mathscr{L}$ is ample and $\mathscr{M}$ is arbitrary, then $\mathscr{M} \otimes \mathscr{L}^n$ is ample for sufficiently large n.

(c) If $\mathscr{L}, \mathscr{M}$ are both ample, so is $\mathscr{L} \otimes \mathscr{M}$.

(d) If $\mathscr{L}$ is very ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is very ample.

(e) If $\mathscr{L}$ is ample, then there is an $n_0 > 0$ such that $\mathscr{L}^n$ is very ample for all $n \geqslant n_0$.

PROOF. For (a), (b), and (c), we let $\mathscr{F}$ be a coherent sheaf.

(a) Fix $n_0 > 0$ s.t. $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections for $n \geq n_0$. The tensor product of two globally generated sheaves is also globally generated and hence, $\mathscr{F} \otimes (\mathscr{L} \otimes \mathscr{M})^n \cong (\mathscr{F} \otimes \mathscr{L}^n) \otimes \mathscr{M}^n$ is globally generated.

(b) Fix $\mathscr{L}$ ample. Since $\mathscr{M}$ is invertible, choose k_0 s.t. $\mathscr{M} \otimes \mathscr{L}^k$ is globally generated for $k \geq k_0$. From (a), $\mathscr{M} \otimes \mathscr{L}^{k+1} \cong \mathscr{L} \otimes (\mathscr{M} \otimes \mathscr{L}^k)$ is ample. So it suffices to take $n \geq k_0$.

(c) Let $\mathscr{F}$ be coherent. Let $n \gg 0$. Then, $\mathscr{F} \otimes (\mathscr{L} \otimes \mathscr{M})^n \cong (\mathscr{F} \otimes \mathscr{M}^n) \otimes \mathscr{L}^n$ is generated by global sections by (a) and (b).

From (b), for n sufficiently large, we know $\mathscr{M} \otimes \mathscr{L}^n$ is ample and then, for m sufficiently large, we know $\mathscr{M}^n \otimes \mathscr{L}^n$ is ample. Therefore, for values l larger than n_0, m_0, we know $(\mathscr{L} \otimes \mathscr{M})^l$ is ample. Therefore, $\mathscr{L} \otimes \mathscr{M}$ is ample by Hartshorne Proposition II.7.5.

(d) First off, $\mathscr{L}$ being very ample gives an immersion $i : X \to \mathbb{P}^n_A$ s.t. $i^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \mathscr{L}$. Since $\mathscr{M}$ is generated by global sections, it defines a morphism $j : X \to \mathbb{P}^m_A$ s.t. $j^*\mathcal{O}_{\mathbb{P}^m}(1) \cong \mathscr{M}$. Now, we have a composition of maps

$$X \overset{\Delta}{\to} X \times_A X \overset{i\times j}{\to} \mathbb{P}^n_A \times_A \mathbb{P}^m_A \overset{\sigma\times \mathrm{id}_A}{\to} \mathbb{P}^N_{\mathbb{Z}} \times A \cong \mathbb{P}^N_A.$$

Since X is separated by Hartshorne II.7.4(a) Δ is a closed immersion. OTOH, $i \times j$ is the fibre product between an immersion and a morphism which is always an immersion. Similarly, $\sigma \times \mathrm{id}_A$ is an immersion since σ is a closed immersion. It follows that the composite $h := (\sigma \times \mathrm{id}_A) \circ (i \times j) \circ \Delta$ is an immersion.

We now compute $h^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathscr{L} \otimes \mathscr{M}$. Note that

$$X \overset{\Delta}{\to} X \times_A X \overset{\pi_X}{\to} X \overset{i}{\to} \mathbb{P}^n_A$$

is just the map $i : X \to \mathbb{P}^n_A$ and similarly for j. Using the sheaf computation for the Segre embedding and the contravariance of $(-)^*$,

$$h^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \Delta^*(i \times j)^*[\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^m}(1)] \cong i^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes j^*\mathcal{O}_{\mathbb{P}^m}(1) \cong \mathscr{L} \otimes \mathscr{M}.$$

One should note here that $\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^m}(1)$ is the tensor product of two sheaves on $\mathbb{P}^n_A \times_A \mathbb{P}^m_A$.

(e) Since $\mathscr{L}$ is ample, Hartshorne Theorem II.7.6 tells us $\mathscr{L}^m$ is very ample for an $m > 0$. Since $\mathscr{L}$ is coherent, choose $k_0 > 0$ s.t. $\mathscr{L} \otimes \mathscr{L}^n$ is generated by global sections for all $k \geq k_0$. Then part (d) tells us that $\mathscr{L}^m \otimes \mathscr{L} \otimes \mathscr{L}^n$ is very ample for all $k \geq k_0$. That is, $\mathscr{L}^l$ is globally generated for all $l \geq k_0 + m + 1 =: n_0$. □

Remark 36. For this exercise, some basic facts about globally generated sheaves were used i.e. the tensor product of two globally generated sheaves globally generated. For more general results about ampleness (only requiring the scheme be quasi-compact), see [10, Section 01PR]. More details for part (d) can be found in this discussion.

Hartshorne II.7.6 *The Riemann-Roch Problem.*

Let X be a nonsingular projective variety over an algebraically closed field, and let D be a divisor on X. For any $n > 0$ we consider the complete linear system $|nD|$. Then the Riemann-Roch problem is to determine $\dim |nD|$ as a function of n, and, in particular, its behavior for large n. If $\mathscr{L}$ is the corresponding invertible sheaf, then $\dim |nD| = \dim \Gamma(X, \mathscr{L}^n) - 1$, so an equivalent problem is to determine $\dim \Gamma(X, \mathscr{L}^n)$ as a function of n.

(a) Show that if D is very ample, and if $X \hookrightarrow \mathbf{P}^n_k$ is the corresponding embedding in projective space, then for all n sufficiently large, $\dim |nD| = P_X(n) - 1$, where P_X is the Hilbert polynomial of X(I, §7). Thus in this case $\dim |nD|$ is a polynomial function of n, for n large.

(b) If D corresponds to a torsion element of Pic X, of order r, then $\dim |nD| = 0$ if $r \mid n, -1$ otherwise. In this case the function is periodic of period r. It follows from the general Riemann-Roch theorem that $\dim |nD|$ is a polynomial function for n large, whenever D is an ample divisor. See (IV, 1.3.2), (V, 1.6), and Appendix A. In the case of algebraic surfaces, Zariski [7] has shown for any effective divisor D, that there is a finite set of polynomials $P_1, \ldots, P_r$, such that for all n sufficiently large, $\dim |nD| = P_{i(n)}(n)$, where $i(n) \in \{1, 2, \ldots, r\}$ is a function of n.

Proof. **[Incomplete]** □

Hartshorne II.7.7 *Some Rational Surfaces.* Let $X = \mathbb{P}^2_k$, and let $|D|$ be the complete linear system of all divisors of degree 2 on X (conics). D corresponds to the invertible sheaf $\mathcal{O}(2)$, whose space of global sections has a basis $x^2, y^2, z^2, xy, xz, yz$, where x, y, z are the homogeneous coordinates of X.

(a) The complete linear system $|D|$ gives an embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$, whose image is the Veronese surface (I, Ex. 2.13).

(b) Show that the subsystem defined by $x^2, y^2, z^2, y(x-z), (x-y)z$ gives a closed immersion of X into $\mathbb{P}^4$. The image is called the Veronese surface in $\mathbb{P}^4$. Cf. (IV, Ex.3.11)

(c) Let $\mathfrak{d} \subseteq |D|$ be the linear system of all conics passing through a fixed point P. Then $\mathfrak{d}$ gives an immersion of $U = X - P$ into $\mathbb{P}^4$. Furthermore, if we blow up P, to get a surface $\widetilde{X}$, then this map extends to give a closed immersion of $\widetilde{X}$ in $\mathbb{P}^4$. Show that $\widetilde{X}$ is a surface of degree 3 in $\mathbb{P}^4$, and that the lines in X through P are transformed into straight lines in $\widetilde{X}$ which do not meet. $\widetilde{X}$ is the union of all these lines, so we say $\widetilde{X}$ is a ruled surface (V, 2.19.1).

Proof. (a) It is clear that the morphism $|D|$ defines is the 2-uple embedding $\mathbb{P}^2 \to \mathbb{P}^5$. The 2-uple embedding is also a closed immersion and so an immersion. By definition, the image of the 2-uple embedding is the Veronese surface.

We can check this explicitly. The sheaf $\mathcal{O}(2)$ is very ample and so the 2-uple map determines an embedding (immersion). Now since $\mathbb{P}^2$ is proper, the immersion will be a closed immersion. To check that the sections determine the 2-uple embedding, fix the conic D being the zero locus of x^2. Then $\Gamma(\mathbb{P}^2, \mathscr{L}(D))$ is the span of $\frac{x^2}{x^2}, \frac{y^2}{x^2}, \frac{xy}{x^2}, \frac{xz}{x^2}, \frac{yz}{x^2}$. So the embedding corresponding to D is $(x, y, z) \to \left(\frac{x^2}{x^2}, \frac{y^2}{x^2}, \frac{z^2}{x^2}, \frac{xy}{x^2}, \frac{xz}{x^2}, \frac{yz}{x^2}\right)$ and clearing denominators gives the 2-uple embedding.

Remark 37. It is beneficial to describe how we determine $|D|$ gives this map. First off, $|D|$ is the set of all effective divisors linearly equivalent to the divisor given by D. By definition, $\mathscr{L}(D) \cong \mathcal{O}_{\mathbb{P}^2}(2)$. Then Hartshorne Proposition II.7.7 implies that $|D|$ is in 1-1 correspondence with $(\Gamma(X, \mathcal{O}(2)) - \{0\})/k^*$. So the morphism determined by $|D|$ is going to correspond to a morphism determined by $\mathscr{L}(D)$ on X. Everything said here works in the other direction. Explicitly D is given by $\{U_i, \frac{1}{x_i^2}\}$ for U_i the standard basis open set.

(b) It suffices to use Hartshorne Proposition II.7.2. It is clear that the sections define a morphism since the system itself is basepoint free (any base point P would have to have each coordinate vanishing).

We begin by checking that each $(\mathbb{P}^2_k)_{s_i}$ is affine. Ordering $s_0 = x^2, s_1 = y^2, s_2 = z^2, s_3 = x(y-z), s_4 = (x-y)z$, it is clear it is always affine.

Now we can identify $(\mathbb{P}_k)^2_{s_0}$ with $\operatorname{Spec} k[\frac{y}{x}, \frac{z}{x}]$. We get surjectivity

$$A[\frac{y_1}{y_0}, \frac{y_2}{y_0}, \frac{y_3}{y_0}, \frac{y_4}{y_0}] \to k[\frac{y}{x}, \frac{z}{x}]$$

as follows. First, $\frac{y_3 - y_4}{y_0} = \frac{y-z}{x}$. Squaring this gives $\frac{y^2+z^2-2yz}{x^2}$ and so $\frac{-2yz}{x^2}$ is certainly in the image. Now the image of $\frac{y_3}{y_0}$ is $\frac{y(x-z)}{x^2} = \frac{xy-yz}{x^2}$ and so $\frac{xy}{x^2}$ is in the image. That means $\frac{y}{x}$ is in the image as well. A similar argument shows that $\frac{z^2}{x^2}$ is in the image. i.e. consider $\frac{y_4}{y_0}$ in place of $\frac{y_3}{y_0}$ whose image is $\frac{xz-yz}{x^2}$.

A similar method of proof shows that surjectivity of the corresponding maps in the case of X_{y^2}, X_{z^2}. We now deal with the sections $x(y-z)$ and $(x-y)z$. One observes that

$$(\mathbb{P}^2_k)_{x(y-z)} \cong \operatorname{Spec} k[\frac{y}{x}, \frac{z}{x}]_{(\frac{y}{x}-\frac{z}{x})} = \operatorname{Spec} k[\frac{y}{x}, \frac{z}{x}]_{\frac{y-z}{x}}.$$

We have $\frac{s_0}{s_3} \to \frac{x^2}{x(y-z)} = \frac{x}{y-z} = \frac{1}{\frac{y-z}{x}}$. So all that is left is to check we can map onto $\frac{y}{x}$ and $\frac{z}{x}$. If we look at the images of $\frac{s_0}{s_3}, \frac{s_1}{s_3}, \frac{s_2}{s_3}, \frac{s_4}{s_3}$, we know we have

$$\frac{x^2}{x(y-z)} = \frac{x}{(y-z)}, \frac{y^2}{x(y-z)}, \frac{z^2}{x(y-z)}, \frac{(x-y)z}{x(y-z)}.$$

From subtracting the second and third elements from each other, we know $\frac{y+z}{x}$ is in the image. Now $\frac{y+z}{x}$ in the image means its square $\frac{y^2+2yz+z^2}{x^2}$ is in the image. Multiply by the first element to get $\frac{y^2+2yz+z^2}{x(y-z)}$ in the image. It follows that $\frac{2yz}{x(y-z)}$ is in the image and so $\frac{yz}{x(y-z)}$ is in the image.

Subtracting $\frac{yz}{x(y-z)}$ by the third element gives $\frac{z(y-z)}{x(y-z)}$ and so obtaining $\frac{z}{x}$. Taking the second element and subtracting by $\frac{yz}{x(y-z)}$ gives $\frac{y^2-yz}{x(y-z)} = \frac{y(y-z)}{x(y-z)} = \frac{y}{x}$. So we have all of the generators needed to generate the target ring (note $x/(y-z)$ is one of the unit generators we needed). A similar approach can be used for the case of $X_{(x-y)z}$ and so we omit it.

Remark 38. It is also possible to do this part of the exercise via Hartshorne Proposition II.7.3.

(c)[57] If $\mathfrak{d}$ consists of those conics passing through a point, then it is clearly a linear system. Then $U \to \mathbb{P}^4$ is an immersion since the base-point of the system was P. Blow up X at P to get $\widetilde{X}$.

Let us assume WLOG that $P = (0,0,1)$ so we take the sections x^2, y^2, xy, xz, yz. This gives an immersion $U := X - P \to \mathbb{P}^4$.

Lemma 7.5. The map $U \to \mathbb{P}^4$ extends to a closed immersion $\widetilde{X} \to \mathbb{P}^4$.

Proof. See Hartshorne Proposition V.4.1, Remark V.4.0.1, and Remark V.4.0.3 which shows that the linear system has no assigned base point and there are no base-points (on U).

A more direct proof is as follows. First, the blow-up $\widetilde{X}$ can be described as the zero-locus of the equations $X_1Y_1 - X_2Y_2, X_1Y_2 - X_0Y_1$ where $([X_0, X_1, X_2], [Y_1, Y_2])$ are the coordinates of $\mathbb{P}^2 \times \mathbb{P}^1$. Then we can compose with the Segre mapping to get a map $\widetilde{X} \to \mathbb{P}^5$ given by

$$([X_0, X_1, X_2], [Y_1, Y_2]) \to [X_0Y_1, X_0Y_2, X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2] = [X_0Y_1, X_1Y_1, X_1Y_1, X_0Y_1, X_2Y_1, X_2Y_2]$$

and the equations imply the equality. We can factor $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^4$ by mapping to

$$[X_0Y_1, X_0Y_2, X_1Y_1, X_2Y_1, X_2Y_2]$$

[57] Some discussion of this exercise can be found here.

and then mapping $\mathbb{P}^4 \to \mathbb{P}^5$ by $[A, B, C, D, E] \to [A, B, C, C, D, E]$. One observes this map gives a commutative diagram

$$\begin{array}{ccccc} \widetilde{X} & \hookrightarrow & \mathbb{P}^2 \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^5 \\ & \searrow & \vdots \\ & & \phi \\ & & \downarrow \\ & & \mathbb{P}^4 & \nearrow & \end{array}$$

and the map $\mathbb{P}^4 \to \mathbb{P}^5$ is separated. Apply Hartshorne II.4.8. to deduce that the first map $\widetilde{X} \to \mathbb{P}^4$ is a closed immersion. Apply an automorphism of $\mathbb{P}^4$ to get a map extending $U \to \mathbb{P}^4$.

□

Lemma 7.6. $\widetilde{X}$ is a surface of degree 3 in $\mathbb{P}^4$ and the lines in X through P are transformed into lines in $\widetilde{X}$ which do not meet.

Proof. The computation that $\widetilde{X}$ is a surface of degree 3 is in Hartshorne Example V.2.19.1 so we omit this.

The process of blowing-up a point always separates distinct lines through the point. Indeed, a conic and a line will intersect at two points by Bezout's Theorem. After blowing-up, they intersect at only one point.

This explains how the lines through P get transformed to lines that do not meet. □

Lemma 7.7. $\widetilde{X}$ is the union of all of the lines in the preceding lemma.

Proof. $\mathbb{P}^2$ can be covered by lines crossing the point P. We had assumed $P = (0, 0, 1)$ so a line through P has form $ax + by = 0$. Let $Q = (q_0, q_1, q_2)$ be an arbitrary point. Then we would like $aq_0 + bq_1 = 0$. Of course, there is always a choice of a and b s.t. this is solvable. Blowing up and working in an affine chart, the strict transform of the line is given by $a + bt = 0$. A simple analysis shows that the two lines $ax + by$, $cx + dy$ (assuming WLOG that $b, d \neq 0$) intersecting transversely at P have strict transform which no longer intersect.

Indeed, if the two lines intersected $a + bt_0 = c + dt_0$, then $\frac{a-c}{d-b} = t_0$. That means $t_0 = \frac{a-c}{d-b} = \frac{y}{x}$ which is absurd if $ax + by$ and $cx + dy$ intersected at P. Indeed. two lines can intersect only at a single point by Bezout's theorem so this second point of intersect cannot exist.

Blowing up separates all of these lines and they also give all of $\widetilde{X} = \mathbb{P}^4$ by Hartshorne Proposition II.7.16(b). Indeed, the map is surjective and we can cover $\mathbb{P}^2$ by the lines crossing P. So their strict transforms cover $\widetilde{X}$. □

□

Remark 39. The term "embedding" is not defined anywhere in Hartshorne's textbook, but from context it appears he uses it synonymously with "closed immersion". We took this interpretation here.

For the problem, we have implicitly assumed that $\operatorname{char} k \neq 2$. Otherwise, it hopelessly fails.

A useful post about linear systems can be found here.

Hartshorne II.7.8 Let X be a noetherian scheme, let $\mathscr{E}$ be a coherent locally free sheaf on X, and let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the corresponding projective space bundle. Show that there

is a natural 1-1 correspondence between sections of π (i.e., morphisms $\sigma : X \to \mathbb{P}(\mathscr{E})$ such that $\pi \circ \sigma = \mathrm{id}_X$) and quotient invertible sheaves $\mathscr{E} \to \mathscr{L} \to 0$ of $\mathscr{E}$.

PROOF. This "proof" should be viewed as an attempt to thoroughly understand the projective space bundle. We show a correspondence

$$\Theta : \left\{ \begin{array}{c} \text{sections of} \\ \pi : \mathbb{P}(\mathscr{E}) \to X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{quotient invertible} \\ \text{sheaves} \\ \mathscr{E} \to \mathscr{L} \to 0 \\ \text{of } \mathscr{E} \end{array} \right\}$$

Lemma 7.8. Define Θ as follows.

PROOF. Apply Hartshorne Proposition II.7.12. Given a section $\sigma : X \to \mathbb{P}(\mathscr{E})$, the surjective map $\pi^*\mathscr{E} \to \mathcal{O}(1)$ pulls back to give a surjective map

$$\mathscr{E} = (\pi \circ \sigma)^*\mathscr{E} = \sigma^*\pi^*\mathscr{E} \to \sigma^*\mathcal{O}(1) \to 0$$

and the invertible sheaf $\mathscr{L}$ is $\sigma^*\mathcal{O}(1)$. □

Lemma 7.9. From Hartshorne Proposition II.7.12, a surjective map of sheaves $\mathrm{id}^*\mathscr{E} = \mathscr{E} \to \mathscr{L} \to 0$ gives rise to a morphism $s : X \to \mathbb{P}(\mathscr{E})$ over X. In other words, s is a morphism s.t. $\pi \circ s = \mathrm{id}_X$.

$$\begin{array}{ccc} X & \xrightarrow{s} & \mathbb{P}(\mathscr{E}) \\ & \searrow^{\mathrm{id}_X} & \downarrow \pi \\ & & X \end{array}$$

The two lemmas with the proof of Hartshorne Proposition II.7.12 proves the correspondence. □

Hartshorne II.7.9 Let X be a regular noetherian scheme, and $\mathscr{E}$ a locally free coherent sheaf of rank ≥ 2 on X.

(a) Show that $\operatorname{Pic}\mathbb{P}(\mathscr{E}) \cong \operatorname{Pic} X \times \mathbb{Z}$.

(b) If $\mathscr{E}'$ is another locally free coherent sheaf on X, show that $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$ (over X) if and only if there is an invertible sheaf $\mathscr{L}$ on X such that $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$.

PROOF. For this exercise, we shall need to assume X is irreducible. Otherwise, take X to be the disjoint union of two copies of $\mathbb{A}^1_k$. Let $\mathscr{E}$ be $\mathcal{O}_{\mathbb{A}^1_k} \times \mathcal{O}_{\mathbb{A}^1_k}$. Then $\mathbb{P}(\mathscr{E})$ is a disjoint union of two copies of $\mathbb{P}^1_k$. The Picard group of this space would be $\mathbb{Z} \times \mathbb{Z}$, but $\operatorname{Pic} X \times \mathbb{Z} = \mathbb{Z}$ since $\operatorname{Pic} X = \operatorname{Pic} \mathbb{A}^1_k \times \operatorname{Pic} \mathbb{A}^1_k = 0$.

(a) The projection map $\pi : \mathbb{P}(\mathscr{E}) \to X$ defines an injective map $\pi^* : \operatorname{Pic}(X) \to \operatorname{Pic}(\mathbb{P}(\mathscr{E}))$ (injectivity is via a simply stalk computation). Now we have a sequence

$$0 \to \operatorname{Pic} X \xrightarrow{\pi^*} \operatorname{Pic}\mathbb{P}(\mathscr{E})$$

and we define a map $\operatorname{Pic}\mathbb{P}(\mathscr{E}) \to \mathbb{Z}$ as follows. An invertible sheaf $\mathscr{L}$ on $\mathbb{P}(\mathscr{E})$ restricts to an invertible sheaf $\mathscr{L}|_{\mathbb{P}^n_U}$ for any open affine $U \subseteq X$.

Lemma 7.10. Any regular noetherian scheme is reduced and locally factorial.

PROOF. A famous result known as the Auslander-Buchsbaum-Nagata theorem (see Theorem 2.2.19 of Bruns and Herzog's "Cohen-Macaulay Rings") asserts that regular local rings are UFDs. This proves locally factorial. This also implies that the scheme is reduced. □

We shall assume from here that X was assumed to be irreducible. This gives us the rest of the hypotheses needed to apply our results on divisors.

Lemma 7.11. If $U \subseteq X$ is an open affine subscheme, then it satisfies (*) in Hartshorne Section II.5.

PROOF. The previous lemma did most of the work. A regular affine scheme is clearly regular in codimension one. Our assumption of irreduciblity implies U is irreducible and hence, integral. Any affine scheme is separated. □

Under these hypotheses, $\operatorname{Pic} \mathbb{P}^n_U \cong \operatorname{Pic} U \times \mathbb{Z}$ from Hartshorne II.6.1 and Hartshorne Corollary II.6.16.

We define a map $\Phi : \operatorname{Pic} X \times \mathbb{Z} \to \operatorname{Pic} \mathbb{P}(\mathscr{E})$ by sending $(\mathscr{L}, d) \to \pi^*\mathscr{L} \otimes \mathcal{O}_{\mathbb{P}(\mathscr{E})}(d)$.

Lemma 7.12. The map is a homomorphism of abelian groups.

PROOF. Recall that $\operatorname{Pic} X$ is endowed with the group operation $\otimes$ which is gives it an abelian group structure. It follows that

$$\begin{aligned}
\Phi((\mathscr{L}_1, d))\Phi((\mathscr{L}_2, e)) &= (\pi^*\mathscr{L}_1 \otimes \mathcal{O}(d)) \otimes (\pi^*\mathscr{L}_2 \otimes \mathcal{O}(e)) && \text{(Definition)} \\
&= \pi^*(\mathscr{L}_1 \otimes \mathscr{L}_2) \otimes \mathcal{O}(d+e) && ((-)^* \text{ commutes with } \otimes) \\
&= \Phi((\mathscr{L}_1 \otimes \mathscr{L}_2, d+e)) && \text{(Definition)}
\end{aligned}$$

□

Lemma 7.13. Φ is a monomorphism of abelian groups.

PROOF. Suppose $\pi^*\mathscr{L} \otimes \mathcal{O}(d) \cong \mathcal{O}_{\mathbb{P}(\mathscr{E})}$. Apply π_* to get

$$\pi_*(\mathcal{O}(d) \otimes \pi^*\mathscr{L}) = \pi_*(\pi^*\mathscr{L} \otimes \mathcal{O}(d)) = \pi_*\mathcal{O}_{\mathbb{P}(\mathscr{E})} = \mathcal{O}_X.$$

Apply the projection formula (Hartshorne II.5.1(d)) on the LHS to get

$$\pi_*\mathcal{O}(d) \otimes_{\mathcal{O}_X} \mathscr{L} = \mathcal{O}_X.$$

Now $\pi_*\mathcal{O}(d)$ is the dth rank part of $S(\mathscr{E})$ from Hartshorne Proposition II.7.11. Comparing the ranks of the LHS with the RHS, we must have $d = 0$ and so $\mathscr{L} \cong \mathcal{O}_X$. This proves injectivity. □

Lemma 7.14. Φ is an epimorphism of abelian groups.

PROOF. Cover X by open affine subschemes U_i. Then $\pi^{-1}(U_i) = \mathbb{P}^n_{U_i}$ and so $\operatorname{Pic} \pi^{-1}(U_i) \cong \operatorname{Pic} \mathbb{P}^n_{U_i} \cong \operatorname{Pic} U \times \mathbb{Z}$.

Let $\mathscr{M} \in \operatorname{Pic} \mathbb{P}(\mathscr{E})$ be an invertible sheaf on $\mathbb{P}(\mathscr{E})$. We can restrict $\mathscr{M}$ to $\pi^{-1}(U_i)$. It can be identified as an element of $\operatorname{Pic} U_i \times \mathbb{Z}$. But this is identified with an element of $\operatorname{Pic} \mathbb{P}^n_{U_i}$. In particular, it is identified with the element $(\mathscr{M}_i, d_i) \in \operatorname{Pic} U_i \times \mathbb{Z}$ where $\pi_i : \pi^{-1}(U_i) \to U_i$ and d_i as in

$$\pi_i^*\mathscr{M}_i \otimes \mathcal{O}_{\mathbb{P}^n_{U_i}}(d_i) \in \operatorname{Pic} \mathbb{P}^n_{U_i}.$$

Additionally, there are transition functions

$$\alpha_{ij} : \pi_i^*\mathscr{M}_i \otimes \mathcal{O}_{\mathbb{P}^n_{U_i}}(d_i)|_{\pi^{-1}(U_i)\cap\pi^{-1}(U_j)} \to \pi_j^*\mathscr{M}_j \otimes \mathcal{O}_{\mathbb{P}^n_{U_j}}(d_j)|_{\pi^{-1}(U_j)\cap\pi^{-1}(U_j)}$$

which agree on triple intersections (as they are determined from the restriction of $\mathscr{M}$ to $\mathbb{P}^n_{U_i}$). We show these transition functions are isomorphism. Apply $(\pi_i)_*$ to each α_{ij} and use the projection formula. We get an isomorphism

$$(\pi_i)_*\alpha_{ij} : \mathscr{M}_i \otimes (\pi_i)_*\mathcal{O}_{\mathbb{P}^n_{U_i}}(d_i)|_{\pi^{-1}(U_i)\cap\pi^{-1}(U_j)} \to \mathscr{M}_j \otimes (\pi_j)_*\mathcal{O}_{\mathbb{P}^n_{U_j}}(d_j)|_{\pi^{-1}(U_i)\cap\pi^{-1}(U_j)}$$

From rank considerations (as before, applying Hartshorne Proposition II.7.11), $d_i = d_j$. Fix $d := d_i$. If we examine α_{ij} above once more, it is an isomorphism (setting $W_{ij} := \pi^{-1}(U_i) \cap \pi^{-1}(U_j)$)

$$\alpha_{ij} : \pi_i^*\mathscr{M}_i|_{W_{ij}} \otimes \mathcal{O}_{\mathbb{P}^n_{U_i\cap U_j}}(d) \to \pi_j^*\mathscr{M}_j|_{W_{ij}} \otimes \mathcal{O}_{\mathbb{P}^n_{U_i\cap U_j}}(d).$$

Tensor by $\mathcal{O}_{\mathbb{P}^n_{U_i\cap U_j}}(-d)$ to get

$$\beta_{ij} : \pi_i^*\mathscr{M}_i|_{W_{ij}} \otimes \mathcal{O}_{W_{ij}} \to \pi_j^*\mathscr{M}_j|_{W_{ij}} \otimes \mathcal{O}_{W_{ij}}.$$

Apply the projection formula (with the map being the restriction of π_i to W_{ij}) to get another isomorphism

$$\gamma_{ij} : (\pi_i)_*\mathcal{O}_{\mathbb{P}^n_{U_i}} \otimes \mathscr{M}_i|_{W_{ij}} \to (\pi_j)_*\mathcal{O}_{\mathbb{P}^n_{U_j}} \otimes \mathscr{M}_j|_{W_{ij}}$$

but this is just an isomorphism

$$\gamma_{ij} : \mathcal{O}_{U_i\cap U_j} \otimes \mathscr{M}_i|_{W_{ij}} \cong \mathscr{M}_i|_{W_{ij}} \to \mathcal{O}_{U_i\cap U_j} \otimes \mathscr{M}_j|_{W_{ij}} \cong \mathscr{M}_j|_{W_{ij}}$$

which means $\mathscr{M}_i \cong \mathscr{M}_j$ on W_{ij}. The checks show that we can glue as in Hartshorne II.1.22. Glue the $\mathscr{M}_i$ to an invertible sheaf $\mathscr{N} \in \operatorname{Pic}(X)$ on X. By construction, $\pi^*\mathscr{N} \otimes \mathcal{O}(d)$ agrees with $\mathscr{M}$. □

Remark 40. How would one guess this result should hold? Heuristically, an invertible sheaf on $\mathbb{P}(\mathscr{E})$ should restrict to an invertible sheaf on each $\mathbb{P}^k_U$ and as $\operatorname{Pic} P^k_U = \mathbb{Z}$, we get the fact of $\mathbb{Z}$ and gluing these invertible sheaves give the factor of $\operatorname{Pic} X$.

(b) ($\Rightarrow$). Assume $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$. Let $\mathcal{O}(1)$ and $\mathcal{O}'(1)$ be respective sheaves constructed in the Definition on p. 162 of Hartshorne.

Let $f : \mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}')$ be a fixed isomorphism. Then $f^*\mathcal{O}'(1) \cong \mathcal{O}(1) \otimes \pi^*\mathscr{L}$ for some $\mathscr{L} \in \operatorname{Pic}(X)$ from (a) and $\pi : \mathbb{P}(\mathscr{E}) \to X$ and rank considerations. Applying the projection formula,

$$\pi_*\mathcal{O}(1) \otimes \mathscr{L} \cong \pi_* f^*\mathcal{O}'(1).$$

Apply Hartshorne Proposition II.7.11 to get $\pi_*\mathcal{O}(1) \cong \mathscr{E}$ and so,

$$\mathscr{E} \otimes \mathscr{L} \cong \pi_* f^*\mathcal{O}'(1).$$

Observe that there is a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\mathscr{E}) & \xrightarrow{\ \pi\ } & X \\ & \searrow^{f} & \uparrow{\scriptstyle \pi'} \\ & & \mathbb{P}(\mathscr{E}') \end{array}$$

By Hartshorne Proposition II.7.11

$$\mathscr{E}' \cong \pi'_*(\mathcal{O}'(1)) \cong \pi_* f^*(\mathcal{O}'(1)) \cong \mathscr{E} \otimes \mathscr{L}$$

as desired.

($\Leftarrow$). Let $\mathscr{L}$ be an invertible sheaf s.t. $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$. By Hartshorne II.7.11(b), there is a natural surjection $\pi^*\mathscr{E}' \cong \pi^*\mathscr{E} \otimes \pi^*\mathscr{L} \twoheadrightarrow \mathcal{O}(1) \otimes \pi^*\mathscr{L}$. By Hartshorne Proposition II.7.12, this gives a morphism $f : \mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}')$ over X. Tensoring by $\mathscr{L}^{-1}$ and doing things the other way around, we have two commutative diagrams from two maps f, f'

$$\begin{array}{ccc} \mathbb{P}(\mathscr{E}) & \xrightarrow{\pi} & X \\ & \searrow^{f} & \uparrow{\scriptstyle \pi'} \\ & & \mathbb{P}(\mathscr{E}') \end{array} \qquad \begin{array}{ccc} \mathbb{P}(\mathscr{E}') & \xrightarrow{\pi} & X \\ & \searrow^{f'} & \uparrow{\scriptstyle \pi'} \\ & & \mathbb{P}(\mathscr{E}) \end{array}$$

From our choice, f' is the inverse of f. □

Hartshorne II.7.10 $\mathbb{P}^n$*-Bundles Over a Scheme.* Let X be a noetherian scheme.

(a) By analogy with the definition of a vector bundle (Ex. 5.18), define the notion of a **projective n-space bundle over** X, as a scheme P with a morphism $\pi : P \to X$ such that P is locally isomorphic to $U \times \mathbb{P}^n, U \subseteq X$ open, and the transition automorphisms on $\operatorname{Spec} A \times \mathbb{P}^n$ are given by A-linear automorphisms of the homogeneous coordinate ring $A[x_0, \ldots, x_n]$ (e.g., $x_i' = \sum a_{ij}x_j, a_{ij} \in A$).

(b) If $\mathscr{E}$ is a locally free sheaf of rank $n+1$ on X, then $\mathbb{P}(\mathscr{E})$ is a $\mathbb{P}^n$-bundle over X.

(c) Assume that X is regular, and show that every $\mathbb{P}^n$-bundle P over X is isomorphic to $\mathbb{P}(\mathscr{E})$ for some locally free sheaf $\mathscr{E}$ on X.
[Hint: Let $U \subseteq X$ be an open set such that $\pi^{-1}(U) \cong U \times \mathbb{P}^n$, and let $\mathscr{L}_0$ be the invertible sheaf $\mathcal{O}(1)$ on $U \times \mathbb{P}^n$. Show that $\mathscr{L}_0$ extends to an invertible sheaf $\mathscr{L}$ on P. Then show that $\pi_*\mathscr{L} = \mathscr{E}$ is a locally free sheaf on X and that $P \cong \mathbb{P}(\mathscr{E})$.]

Can you weaken the hypothesis "X regular"?

(d) Conclude (in the case X regular) that we have a 1-1 correspondence between $\mathbb{P}^n$-bundles over X, and equivalence classes of locally free sheaves $\mathscr{E}$ of rank $n+1$ under the equivalence relation $\mathscr{E}' \sim \mathscr{E}$ if and only if $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{M}$ for some invertible sheaf, $\mathscr{M}$ on X.

Proof. **(a)** There is nothing to prove – a projective n-space bundle is just a bundle in which the local isomorphism is with $U \times \mathbb{P}^n$ and whose transition automorphisms are A-linear automorphisms of homogeneous coordinate rings.

(b) The local isomorphisms follow from $\pi^{-1}(U) \cong \mathbb{P}^n_U$ for all $U \subseteq X$ open and $\pi : \mathbb{P}^n(\mathscr{E}) \to X$ being the projection.

Let $U_i, U_j \subseteq X$ be two open subsets and $\operatorname{Spec} A \subseteq U_i \cap U_j$ be an open affine subset. Let $\psi_i : \pi^{-1}(U_i) \cong \mathbb{P}_{U_i}$ be the isomorphism determined by $\mathbb{P}\diagdown\!\!\rtimes\!\!\rfloor$. We want to show that $\psi_j \circ \psi_i^{-1} : \mathbb{P}_{U_i} \to \mathbb{P}_{U_j}$ an automorphism on $\mathbb{P}_A$ defined by an A-linear automorphism of $A[x_0, \ldots, x_n]$ (which is due to $\mathscr{E}$ being of rank $n+1$).

First, ψ_i^{-1} identifies $A[x_0, \ldots, x_n]$ with a choice of trivialization of $S(\mathscr{E})$ and then ψ_j identifies $S(\mathscr{E}|_U)$'s basis elements $e_0, \ldots, e_n$ with a choice of elements $y_0, \ldots y_n$ to get a map to $A[y_0, \ldots, y_n]$. That is, we A-linear isomorphisms

$$A[y_0, \ldots, y_n] \to S(\mathscr{E}|_U) \to A[x_0, \ldots, x_n]$$

which induce the map on Proj. In this way $\mathbb{P}(\mathscr{E})$ is a $\mathbb{P}^n$-bundle.

(c) First off, since X is Noetherian and regular, we may assume that X is irreducible. Indeed, one knows from [10, Tag 033N] that X is a disjoint union of integral regular Noetherian schemes and the proof runs by taking the direct sum of the $\pi_*\mathscr{L}$ for each irreducible component.

Choose a set of open affines $U_i = \operatorname{Spec}(A_i)$'s covering X and so $\pi^{-1}(U_i) = \mathbb{P}^n_{U_i}$. Let $\mathscr{L}_0$ be the sheaf $\mathcal{O}(1)$ on $U \times \mathbb{P}^n$. Next, $\operatorname{Cl}(X) \cong \operatorname{Pic}(X)$ because[58] X is locally factorial and Noetherian (as the Auslander-Buchsbaum-Nagata Theorem says regular local rings are UFDs). Therefore $\mathcal{O}(1)$ corresponds to a Weil divisor of $\pi^{-1}(U_i) \cong \mathbb{P}^n_{U_i}$ Take the closure inside P of the codimension one subscheme W_0 it corresponds to and call it W. Over each $\mathbb{P}^n_{U'}$, $\operatorname{Pic}(\mathbb{P}^n_{U'}) \cong \operatorname{Pic}(U') \times \mathbb{Z}$ and our solution to Hartshorne II.7.9 implies

$$\pi_*(\mathscr{L})|_{U'} \cong \pi_*(\pi^*(\mathscr{M}) \otimes \mathcal{O}_{\mathbb{P}^n_{U'}}(m))) \cong \pi_*\mathcal{O}_{\mathbb{P}^n_{U'}}(m) \otimes \mathscr{M}$$

where $\mathscr{M} \in \operatorname{Pic}(U')$ and we applied the projection formula. When $U' = U$ from before,

$$\pi_*(\mathscr{L}) \cong \pi_*\mathcal{O}_{U\times\mathbb{P}^n}(1) \;\Rightarrow\; m = 1 \text{ and } \mathscr{M} \cong \mathcal{O}_U.$$

Over each other U_i open affine subset from the cover of X, we can use the transition maps and the isomorphism

$$\pi_*(\mathscr{L})|_{U_i} \cong \pi_*\mathcal{O}_{\mathbb{P}^n_{U_i}}(m) \otimes \mathscr{M}_i$$

and restricting it further to U shows that we must have $m = 1$ and $\mathscr{M}_i \cong \mathcal{O}_{U_i\cap U}$. So over each U_i, we find $\pi_*(\mathscr{L})|_{U_i} \cong \pi_*\mathcal{O}_{\mathbb{P}^n_{U_i}}(1)$. Let $U_i = \operatorname{Spec} A_i$ and consider the diagram

$$\begin{array}{ccc} \operatorname{Spec} A \times \mathbb{P}^n_{\mathbb{Z}} & \xrightarrow{p} & \mathbb{P}^n_{\mathbb{Z}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \phi} \\ \operatorname{Spec} A & \xrightarrow{\psi} & \operatorname{Spec} \mathbb{Z} \end{array}$$

By Hartshorne Proposition II.5.8 $(-)_*$ preserves quasi-coherence. So we get

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^n_A}(1) = \widetilde{A[x_0,\ldots,x_n]_{(1)}} & \xleftarrow{p_*} & \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) \\ \downarrow{\scriptstyle \pi_*} & & \\ \widetilde{A[x_0,\ldots,x_n]_{(1)}} & & \end{array}$$

i.e. $\pi_*\mathcal{O}_{\mathbb{P}^n_{U_i}}(1)|_{\operatorname{Spec} A} \cong \widetilde{A[x_0,\ldots,x_n]_{(1)}}$ which is clearly a free A-module. Hence, $\pi_*\mathcal{O}_{\mathbb{P}^n_{U_i}}(1)$ is locally which means $\pi_*(\mathscr{L})$ is locally free.

Set $\mathscr{E} := \pi_*\mathscr{L}$. We define the isomorphism $P \cong \mathbb{P}(\mathscr{E})$ locally. Over each U_i, we know $P \cong \mathbb{P}^n_{U_i}$ and $\mathscr{E} \cong \pi_*\mathscr{L}|_{U_i} \cong \mathcal{O}_{\mathbb{P}^n_{U_i}}(1)$ which means

$$\mathbb{P}(\mathscr{E})|_{U_i} \cong \operatorname{Proj} \bigoplus_{d\geq 0} S^d(\mathscr{E}) \cong \operatorname{Proj} \bigoplus_{d\geq 0} S^d(\mathcal{O}_{\mathbb{P}^n_{U_i}}(1)) \cong \mathbb{P}^n_{U_i} \cong \pi^{-1}(U_i) \subseteq P$$

and since these isomorphisms are canonical and agree on intersections, they glue to an isomorphism of locally ringed spaces. Hence, $\mathbb{P}(\mathscr{E}) \cong P$.

[58] See Vakil's FOAG.

(d) Assume X is regular. Every $\mathbb{P}^n$-bundle over X is isomorphic to $\mathbb{P}(\mathscr{E})$ for some locally free sheaf $\mathscr{E}$ and then this corresponds to the equivalence classes of free sheaves of rank $n+1$ by Hartshorne II.7.9. □

Hartshorne II.7.11 On a noetherian scheme X, different sheaves of ideals can give rise to isomorphic blown up schemes.

(a) If $\mathscr{I}$ is any coherent sheaf of ideals on X, show that blowing up $\mathscr{I}^d$ for any $d \geq 1$ gives a scheme isomorphic to the blowing up of $\mathscr{I}$ (cf. Ex. 5.13).

(b) If $\mathscr{I}$ is any coherent sheaf of ideals, and if $\mathscr{J}$ is an invertible sheaf of ideals, then $\mathscr{I}$ and $\mathscr{I} \cdot \mathscr{J}$ give isomorphic blowings-up.

(c) If X is regular, show that (7.17) can be strengthened as follows. Let $U \subseteq X$ be the largest open set such that $f : f^{-1}U \to U$ is an isomorphism. Then $\mathscr{I}$ can be chosen such that the corresponding closed subscheme Y has support equal to $X - U$.

Proof. (a) By definition, $\mathbb{P}_\diagdown \rtimes \bigoplus_{k\geq 0} \mathscr{I}^k = \mathbb{P}_\diagdown \rtimes \mathscr{S}$ is the blow-up along $\mathscr{I}$. The exercise wants us to prove an isomorphism with $\mathbb{P}_\diagdown \rtimes \bigoplus_{k\geq 0} \mathscr{I}^{kd} = \mathbb{P}_\diagdown \rtimes \mathscr{S}^d$ for any $d \geq 1$.

Using Hartshorne II.5.13, we see that we can define an isomorphism is locally which agree on overlaps. Indeed, over each open open affine subset $U \subseteq X$, we get a diagram

$$\begin{array}{ccc} \operatorname{Proj}(\mathscr{S}(U)) & \xrightarrow{\cong} & \operatorname{Proj}(\mathscr{S}^d(U)) \\ & \searrow^{\pi_1} & \downarrow{\scriptstyle \pi_2} \\ & & U \end{array}$$

and since the isomorphisms on top are canonical (from Hartshorne II.5.13), we are done.

(b) By definition, $\mathscr{I} \cdot \mathscr{J}$ is a the sheaf of ideals generated by products of $\mathscr{I}$ and $\mathscr{J}$ inside $\mathcal{O}_X$. That is, over each open subset U, $\mathscr{I} \cdot \mathscr{J}(U) = \langle \mathscr{I}(U)\mathscr{J}(U) \rangle \subseteq \mathcal{O}_X(U)$.

Since $\mathscr{J}$ is an invertible sheaf of ideals, choose trivializations $(U_i)_{i=1,2,\ldots,\ell}$ s.t. $\mathscr{J}(U_i)$ is generated by a single section s_i as an $\mathcal{O}_X(U_i)$-module. Taking the U_i to be affine, we can also trivialize $\mathscr{I}$ on these open subsets.

The blow-up along $\mathscr{I}$ is $\operatorname{Bl}_{\mathscr{I}} X = \mathbb{P}_\diagdown \rtimes \bigoplus_{k\geq 0} \mathscr{I}^k$ and $\operatorname{Bl}_{\mathscr{I}} X = \mathbb{P}_\diagdown \rtimes \bigoplus_{k\geq 0} (\mathscr{I} \cdot \mathscr{J})^k$. We define the isomorphisms locally and show they glue.

For a given $U_i \in (U_i)_{i=1,2,\ldots,\ell}$, we want a map

$$\operatorname{Bl}_{\mathscr{I}}(X)|_U = \mathbb{P}_\diagdown \rtimes \bigoplus_{k\geq 0} \mathscr{I}^k|_U \to \operatorname{Bl}_{\mathscr{I}\cdot\mathscr{J}}(X)|_U = \operatorname{Bl}_{\mathscr{I}\cdot\mathscr{J}}(X)|_U = \bigoplus_{k\geq 0} (\mathscr{I} \cdot \mathscr{J})^k|_U.$$

We first define the map globally. Take a section $t \in \mathscr{I}^k(U)$ and map it to $ts \in (\mathscr{I}\mathscr{J})^k(U)$ where s was from before. Do this for each $k \geq 0$. This is an $\mathcal{O}_X(U)$-module isomorphism. Over each open $D(f) \subseteq U$, we understand how the restriction maps behave so the map agrees with restriction. Over each intersection $U_i \cap U_j$, the map will still be multiplication by the choice of generators. Thus, we get a map $\operatorname{Bl}_{\mathscr{I}}(X)|_U \to \operatorname{Bl}_{\mathscr{I}\cdot\mathscr{J}}(X)|_Y$ which is an isomorphism (this is easy to check via stalks). The maps are canonical so they glue together to give an isomorphism $\operatorname{Bl}_{\mathscr{I}}(X) \to \operatorname{Bl}_{\mathscr{I}\cdot\mathscr{J}}(X)$.

(c) **[Incomplete]** □

Hartshorne II.7.12 Let X be a noetherian scheme, and let Y, Z be two closed subschemes, neither one containing the other. Let $\widetilde{X}$ be obtained by blowing up $Y \cap Z$ (defined by the ideal sheaf $\mathscr{I}_Y + \mathscr{I}_Z$). Show that the strict transforms $\widetilde{Y}$ and $\widetilde{Z}$ of Y and Z in $\widetilde{X}$ do not meet.

PROOF. [Incomplete] □

Hartshorne II.7.13 *A Complete Nonprojective Variety.* Let k be an algebraically closed field of char $\neq 2$. Let $C \subseteq \mathbf{P}^2_k$ be the nodal cubic curve $y^2z = x^3 + x^2z$. If $P_0 = (0,0,1)$ is the singular point, then $C - P_0$ is isomorphic to the multiplicative group $\mathbf{G}_m = \operatorname{Spec} k[t, t^{-1}]$ (Ex. 6.7). For each $a \in k, a \neq 0$, consider the translation of $\mathbf{G}_m$ given by $t \mapsto at$. This induces an automorphism of C which we denote by φ_a. Now consider $C \times (\mathbf{P}^1 - \{0\})$ and $C \times (\mathbf{P}^1 - \{\infty\})$. We glue their open subsets $C \times (\mathbf{P}^1 - \{0, \infty\})$ by the isomorphism $\varphi : \langle P, u\rangle \mapsto \langle \varphi_u(P), u\rangle$ for $P \in C, u \in \mathbf{G}_m = \mathbf{P}^1 - \{0, \infty\}$. Thus we obtain a scheme X, which is our example. The projections to the second factor are compatible with φ, so there is a natural morphism $\pi : X \to \mathbf{P}^1$.
(a) Show that π is a proper morphism, and hence that X is a complete variety over k.
(b) Use the method of (Ex. 6.9) to show that $\operatorname{Pic}(C \times \mathbf{A}^1) \cong \mathbf{G}_m \times \mathbf{Z}$ and $\operatorname{Pic}(C \times (\mathbf{A}^1 - \{0\})) \cong \mathbf{G}_m \times \mathbf{Z} \times \mathbf{Z}$. [Hint: If A is a domain and if $*$ denotes the group of units, then $(A[u])^* \cong A^*$ and $(A[u, u^{-1}])^* \cong A^* \times \mathbf{Z}$.]
(c) Now show that the restriction map $\operatorname{Pic}(C \times \mathbf{A}^1) \to \operatorname{Pic}(C \times (\mathbf{A}^1 - \{0\}))$ is of the form $\langle t, n\rangle \mapsto \langle t, 0, n\rangle$, and that the automorphism φ of $C \times (\mathbf{A}^1 - \{0\})$ induces a map of the form $\langle t, d, n\rangle \mapsto \langle t, d + n, n\rangle$ on its Picard group.
(d) Conclude that the image of the restriction map $\operatorname{Pic} X \to \operatorname{Pic}(C \times \{0\})$ consists entirely of divisors of degree 0 on C. Hence X is not projective over k and π is not a projective morphism.

PROOF. [Incomplete] □

Hartshorne II.7.14 (a) Give an example of a noetherian scheme X and a locally free coherent sheaf $\mathscr{E}$, such that the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathscr{E})$ is not very ample relative to X.

(b) Let $f : X \to Y$ be a morphism of finite type, let $\mathscr{L}$ be an ample invertible sheaf on X, and let $\mathscr{S}$ be a sheaf of graded $\mathcal{O}_X$-algebras satisfying (†). Let $P = \mathbf{Proj}\,\mathscr{S}$, let $\pi : P \to X$ be the projection, and let $\mathcal{O}_P(1)$ be the associated invertible sheaf. Show that for all $n \gg 0$, the sheaf $\mathcal{O}_P(1) \otimes \pi^*\mathscr{L}^n$ is very ample on P relative to Y.
[Hint: Use (7.10) and (Ex. 5.12).]

PROOF. **(a)** Let $X = \mathbb{P}^1_k$ and $\mathscr{E} = \mathcal{O}_X(-1)^2$. Now $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}^1_k \times_k \mathbb{P}^1_k$ with $\pi : \mathbb{P}^1_k \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$ being the projection onto the second coordinate. We also know that $\pi_*\mathcal{O}_Y(1) \cong \mathcal{O}^2_X(-1)$ by Hartshorne Proposition II.7.11. Taking global sections, we know that $\mathcal{O}_Y(1)(\mathbb{P}^1_k \times \mathbb{P}^1_k) = \mathcal{O}^2_X(-1)(\mathbb{P}^1_k) = 0$.

If $\mathcal{O}_Y(1)$ were very ample, then we would get an immersion $i : Y \to \mathbb{P}^m_{\mathbb{P}^1_k}$. The class group of $\mathbb{P}^m_{\mathbb{P}^1_k}$ would be $\mathbb{Z} \oplus \mathbb{Z}$ and $i^*\mathcal{O}_{\mathbb{P}^m_{\mathbb{P}^1_k}}(1) = \mathcal{O}_Y(1)$. But to define a morphism into projective space, we would need to pullback sections of $\mathcal{O}_{\mathbb{P}^m_{\mathbb{P}^1_k}}(1)$. But $\mathcal{O}_Y(1)$ has no global sections. Contradiction.

(b) By [10, Tag 01VS], $\mathscr{L}$ being an ample invertible sheaf on X and $f : X \to Y$ being a finite morphism implies that $\mathscr{L}^n$ is very ample relative to Y for some n. From Hartshorne Proposition II.7.10(b), we know that $\mathcal{O}_P(1) \otimes \pi^*\mathscr{L}^m$ is very ample on P relative to Y for some $m > 0$. By Hartshorne II.5.12(b),

$$\mathcal{O}_P(1) \otimes \pi^*\mathscr{L}^m \otimes \pi^*\mathscr{L}^n \cong \mathcal{O}_P(1) \otimes \pi^*\mathscr{L}^{m+n}$$

is very ample on P relative to Y. □

8. Differentials

Hartshorne II.8.1 Here we will strengthen the results of the text to include information about the sheaf of differentials at a not necessarily closed point of a scheme X.

(a) Generalize (8.7) as follows. Let B be a local ring containing a field k, and assume that the residue field $k(B) = B/\mathfrak{m}$ of B is a separably generated extension of k. Then the exact sequence of (8.4A)

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

is exact on the left also. [Hint: In copying the proof of (8.7), first pass to $B/\mathfrak{m}^2$, which is a complete local ring, and then use (8.25A) to choose a field of representatives for $B/\mathfrak{m}^2$.]

(b) Generalize (8.8) as follows. With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $= \dim B + \text{tr. deg}\, k(B)/k$.

(c) Strengthen (8.15) as follows. Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $x \in X$, not necessarily closed, show that the local ring $\mathcal{O}_{x,X}$ is a regular local ring if and only if the stalk $\left(\Omega_{X/k}\right)_x$ of the sheaf of differentials at x is free of rank n

(d) Strengthen (8.16) as follows. If X is a variety over an algebraically closed field k, then $U = \{x \in X \mid \mathcal{O}_x$ is a regular local ring $\}$ is an open dense subset of X.

Proof. **(a)** From Hartshorne Proposition II.8.4A, there is an exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes_B k(B) \to \Omega_{k(B)/k} \to 0$$

by taking $A := k \subseteq B$ and $C := k(B)$. The statement we prove is weaker than in Hartshorne Proposition II.8.7 since we lose surjectivity due to $k(B)/k$ being a separably generated extension.

To show left exactness, prove that δ is injective. It is sufficient to show

$$\delta' : \text{Hom}(\Omega_{B/k} \otimes k(B), k(B)) \to \text{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k(B))$$

is surjective since we are working with vector spaces.

Lemma 8.1. The ring $B/\mathfrak{m}^2$ is a complete local ring. Applying Hartshorne Theorem II.8.25A, we find a subfield $L \subseteq B/\mathfrak{m}^2$ s.t. $k \subseteq L$ and $L \to (B/\mathfrak{m}^2)/(\mathfrak{m}/\mathfrak{m}^2) \cong B/\mathfrak{m}$ is an isomorphism.

Lemma 8.2. Let L be as in the previous lemma. Then Hartshorne Proposition II.8.7 is applicable and gives an isomorphism $\delta : \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{(B/\mathfrak{m}^2)/L} \otimes_{B/\mathfrak{m}^2} L$.

PROOF. All of the hypotheses of Hartshorne Proposition II.8.7 are satisfied. With this isomorphism,

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{(B/\mathfrak{m}^2)/L} \otimes_{B/\mathfrak{m}^2} L \to \Omega_{L/L} \to 0$$

being a short exact sequence implies that the second map is an isomorphism. □

From our lemma, we reduce to showing

$$\delta' : \mathrm{Hom}_{k(B)}(\Omega_{B/k} \otimes_B k(B), k(B)) \to \mathrm{Hom}_{k(B)}(\Omega_{(B/\mathfrak{m}^2)/L} \otimes_{B/\mathfrak{m}^2} L, k(B))$$

is surjective. Using Hom-tensor adjunction, the map δ' maps

$$\delta' : \mathrm{Der}_{k(B)}(B, k(B)) \to \mathrm{Der}_{k(B)}(B/\mathfrak{m}^2, k(B)).$$

This map is clearly surjective by an argument similar to that in Hartshorne Proposition II.8.7.

Remark 41. Another method of proof can be found in [10, Tag 00TU]. The only nontrivial step in that proof is the reduction of B to $B/\mathfrak{m}^2$. This relies on [10, Tag 02HQ] which proves that the tensored differential does not notice quotients by higher powers.

(b) ($\Leftarrow$). If $\Omega_{B/k}$ is free of rank $\dim B + \mathrm{tr.deg}\, k(B)/k$, then

$$\dim_{k(B)} \mathfrak{m}/\mathfrak{m}^2 + \dim_{k(B)} \Omega_{k(b)/k} = \dim_{k(B)} \Omega_{B/k} \otimes k(B) \tag{24}$$

from the SES in part (a). Using Hartshorne Theorem II.8.6A and the fact that $k(B)/k$ is separable,

$$\dim_{k(B)} \mathfrak{m}/\mathfrak{m}^2 + \mathrm{tr.deg}\, k(B)/k = \dim_{k(B)} \Omega_{B/k} \otimes k(B) = \dim B + \mathrm{tr.deg}\, k(B)/k.$$

Hence, $\dim_{k(B)} \mathfrak{m}/\mathfrak{m}^2 = \dim B$ which means B is a regular local ring.

($\Rightarrow$). This direct requires some work. From equation (24) and Harsthorne Lemma II.8.9, it suffices to show

$$\dim_F \Omega_{B/k} \otimes F = \dim B + \mathrm{tr.deg}\, k(B)/k$$

for $F := \mathrm{Frac}(B)$. By hypothesis, $B \cong A_{\mathfrak{p}}$ where A is a finite type k-algebra and $\mathfrak{p}$ is a prime ideal (recall $S \subseteq A$ is a multiplicatively closed subset iff $A - S$ is a prime ideal). In this way, B is a k-algebra and hence, F is also a k-algebra. By k being perfect, F/k is separably generated. Applying Hartshorne Proposition II.8.2A and Hartshorne Theorem II.8.6A,

$$\dim_F \Omega_{B/k} \otimes F = \dim_F \Omega_{F/k} = \mathrm{tr.deg}\, F/k.$$

Next, Hartshorne Theorem I.1.8A(a) says that $\mathrm{tr.deg}\, F/k = \dim A$ because F is a k-algebra and

$$F = \mathrm{Frac}(B) = \mathrm{Frac}(A_{\mathfrak{p}}) = \mathrm{Frac}(A).$$

Then Hartshorne Theorem I.1.8A(b) implies $\dim A = \mathrm{height}\, \mathfrak{p} + \dim A/\mathfrak{p}$. But $\mathrm{height}\, \mathfrak{p} = \dim B_{\mathfrak{p}}$ which now gives

$$\mathrm{tr.deg}\, F/k = \dim B + \dim A/\mathfrak{p}.$$

Now $A/\mathfrak{p}$ is a f.g. k-algebra since A was. By Hartshorne Theorem I.1.8A and

$$k(B) = k(A_{\mathfrak{p}}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = (A/\mathfrak{p})_{\mathfrak{p}} = (A/\mathfrak{p} - \overline{0})^{-1}(A/\mathfrak{p}) = \mathrm{Frac}(A/\mathfrak{p}),$$

we get $\dim A/\mathfrak{p} = \mathrm{tr.deg}\, k(B)/k$. So

$$\dim_F \Omega_{B/k} \otimes F = \dim B + \mathrm{tr.deg}\, k(B)/k.$$

Remark 42. A better written proof can be found in this post. The tricky part of the proof is the repeated use of Hartshorne Proposition I.1.8A.

(c) ($\Rightarrow$). Let $\mathcal{O}_{x,X}$ be a regular local ring and let $\mathfrak{p}$ be the prime ideal corresponding to x. Choose an open neighborhood $\operatorname{Spec} A \ni x$ s.t. A is a finite type k-algebra. Then $\mathcal{O}_{x,X} = A_{\mathfrak{p}}$ and $\dim \operatorname{Spec} A = \dim X = n$ by Hartshorne I.1.6. From (b),

$$\operatorname{rank}(\Omega_{X/k})_x = \operatorname{rank} \Omega_{\mathcal{O}_{x,X}/k} = \dim \mathcal{O}_{x,X} + \operatorname{tr.deg} k(\mathcal{O}_{x,X})/k.$$

But this RHS is

$$\dim \mathcal{O}_{x,X} + \operatorname{tr.deg} k(\mathcal{O}_{x,X})/k = \dim A_{\mathfrak{p}} + \operatorname{tr.deg}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})/k = \operatorname{height} \mathfrak{p} + \dim A/\mathfrak{p} = \dim A$$

by repeated use of Hartshorne Theorem I.1.8A. Indeed, our work in part (b) and the theorem proves the second equality while the third equality is by the theorem.
($\Leftarrow$). Everything we did for ($\Rightarrow$) works in the other direction.

(d) If $\mathcal{O}_x$ is a regular local ring, then $(\Omega_{X/k})_x$ is of rank n and there is a neighborhood $V \ni x$ s.t. $\Omega_{X/k}$ is free with $\operatorname{rank}(\Omega_{X/k})_x = n$. Since the space is connected, the ranks are constant on all of X. Every point $y \in V$ gives rise to a $\mathcal{O}_{y,X}$ that is also regular and local. Letting $x \in X$ vary over those $\mathcal{O}_x$ which are regular and local, it follows that $U \subseteq X$ is open.

For density, it suffices to show that $\mathcal{O}_{\xi,X}$ is a regular local ring where $\xi \in X$ is the generic point. In this case,

$$\big(\Omega_{X/k}\big)_{\xi} = \Omega_{\mathcal{O}_{\xi,X}/k}.$$

We use (b) to show $\mathcal{O}_{\xi,X} =: K(X)$ is regular. Since $K(X)$ is a field, $\dim K(X) = 0$. Meanwhile, $\operatorname{tr.deg} K(X)/k = \dim X$ from Hartshorne II.3.20(b) and so, $\dim K(X) + \operatorname{tr.deg} K(X)/k = \dim X$. So, we have reduced to proving $\Omega_{K(X)/k}$ is free of rank $\dim X$.

□

Hartshorne II.8.2 Let X be a variety of dimension n over k. Let $\mathscr{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subseteq \Gamma(X, \mathscr{E})$ be a vector space of global sections which generate $\mathscr{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x \mathscr{E}_x$. Conclude that there is a morphism $\mathcal{O}_X \to \mathscr{E}$ giving rise to an exact sequence

$$0 \to \mathcal{O}_X \to \mathscr{E} \to \mathscr{E}' \to 0$$

where $\mathscr{E}'$ is also locally free. [Hint: Use a method similar to the proof of Bertini's theorem (8.18).]

Proof. The useful techniques are in the last pargraph of the proof of Bertini's theorem. Define $Z := \{(x, s) : s \in \mathfrak{m}_x \mathscr{E}_x\} \subseteq X \times V$. Let $p_1 : X \times V \to X$ and $p_2 : X \times V \to V$ be the projections and restrict these to Z. Call the restrictions q_1, q_2 respectively.

For any $w \in X$, one has $q_1^{-1}(w) = \{(w, s) : s \in \mathfrak{m}_w \mathscr{E}_w\}$. This is precisely the kernel of the map

$$\phi : k(w) \otimes_k V \to k(w) \otimes_{\mathcal{O}_{X,w}} \mathscr{E}_w \cong \mathscr{E}_w/\mathfrak{m}_w \mathscr{E}_w.$$

Here, if we assume that k is algebraically closed, then $k(w) = k$ (which we must assume for this to work). We know $\mathscr{E}_w$ is generated by the global sections in V so that means the map is surjective. Recall that we had assumed that $\mathscr{E}$ is free of rank r. The above equation gives

$$\dim_k V - \dim \ker \phi = \operatorname{rank} \mathscr{E}_{x_0} = r \;\Rightarrow\; \dim \ker \phi = \dim V - r = n - r.$$

Because $Z = \bigcup_{w\in X} q^{-1}(w)$, and $q^{-1}(w) = \ker\phi$, and $r > \dim X$,

$$\dim Z = \dim X + \dim\ker(\phi) = \dim X + \dim V - r < \dim V.$$

Because $\dim Z < \dim V$, the map $q_2 : X \to V$ is not surjective. Any element not in the image would suffice for choice of s.

Define $\mathcal{O}_X \to \mathscr{E}$ as multiplication by s. Then the cokernel is $\mathscr{E}'$ and it gives a SES

$$0 \to \mathcal{O}_X \to \mathscr{E} \to \mathscr{E}' \to 0.$$

Now $\mathscr{E}'$ is locally free if we take the stalks and apply Hartshorne II.5.7b with our choice of s. The fact that $s \notin \mathfrak{m}_x\mathscr{E}_x$ for any s ensures that $\mathcal{O}_{X,x} \to \mathscr{E}_x$ has a locally free cokernel.

Lemma 8.3. The cokernel is locally free.

Proof. Consider the SES $0 \to \mathcal{O}_{X,x} \to \mathscr{E}_x \to \mathscr{E}'_x \to 0$ of $\mathcal{O}_{X,x}$-modules. The first map is given by multiplication by the germ of s. The SES is given by mapping $\mathcal{O}_{X,x}$ via multiplication by x into the first factor of $\mathscr{E}_{X,x} = \bigoplus_{k=1}^n \mathcal{O}_{X,x}$,

$$0 \to \mathcal{O}_{X,x} \overset{\times s}{\to} \bigoplus_{k=1}^{n} \mathcal{O}_{X,x} \to \mathscr{E}'_x \to 0.$$

Since $s \notin \mathfrak{m}_x\mathscr{E}_x$, the map is an isomorphism onto the first factor and hence, $\mathscr{E}'_x \cong \bigoplus_{k=2}^n \mathcal{O}_{X,x}$. This means $\mathscr{E}'_x$ is free and Hartshorne II.5.7b does the job. □

□

Hartshorne II.8.3 *Product Schemes.*

(a) Let X and Y be schemes over another scheme S. Use (8.10) and (8.11) to show that $\Omega_{X\times_S Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$

(b) If X and Y are nonsingular varieties over a field k, show that $\omega_{X\times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y$

(c) Let Y be a nonsingular plane cubic curve, and let X be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$ (I, Ex. 7.2). This shows that the arithmetic genus and the geometric genus of a nonsingular projective variety may be different.

Proof. **(a)** Let $f : X \to S$ and $g : Y \to S$. Base extend to get $f' : X \times_S Y \to Y$. Then Hartshorne Proposition II.8.10 implies $\Omega_{X\times_S Y/Y} \cong p_1^*\Omega_{X/S}$ for $p_1 : X \times_S Y \to X$ the first projection. We could also base extend the other way with $g' : X \times_S Y \to X$ to get $\Omega_{X\times_S Y/X} \cong p_2^*\Omega_{Y/S}$ for $\pi_2 : X \times_S Y \to Y$ the second projection. The first isomorphism with Hartshorne Proposition II.8.11 gives an exact sequence

$$\Omega_{X\times_S Y/Y} \cong p_1^*\Omega_{X/S} \to \Omega_{X\times_S Y/S} \to \Omega_{X\times_S Y/X} \to 0 \tag{25}$$

Similarly, for the second isomorphism, there is an exact sequence

$$\Omega_{X\times_S Y/X} \cong p_2^*\Omega_{Y/S} \to \Omega_{X\times_S Y/S} \to \Omega_{X\times_S Y/Y} \to 0$$

From the sequence (25), we shall show that the first map is a split monomorphism. It suffices to this affine locally so replace $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $S = \operatorname{Spec} C$. Then we have

ring maps $C \to A \to A \otimes_C B$ and $C \to B \to A \otimes_C B$ which give the associated sequences of modules

$$\Omega_{A/C} \otimes_C B \cong \Omega_{A/C} \otimes_A (A \otimes_C B) \xrightarrow{f_1} \Omega_{A\otimes_C B/C} \xrightarrow{g_1} \Omega_{A\otimes_C B/A} \to 0 \tag{26}$$

$$\Omega_{B/C} \otimes_C B \cong \Omega_{B/C} \otimes_B (A \otimes_C B) \xrightarrow{f_2} \Omega_{A\otimes_C B/C} \xrightarrow{g_2} \Omega_{A\otimes_C B/B} \to 0. \tag{27}$$

By definition, $f_1(da \otimes b) = bd(a \otimes 1)$ and $g_1(d(a \otimes b)) = d(a \otimes b)$. Similarly, the maps f_2, g_2 are defined. Because $\Omega_{A/C} \otimes_C B \cong \Omega_{A\otimes_C B/B}$ by Hartshorne Proposition II.8.2A, $g_2 : \Omega_{A\otimes_C B/C} \to \Omega_{A\otimes_C B/B}$ in the first sequence. For this to be a sequence, we need $g_2 \circ f_1 = \mathrm{id}$. Following the isomorphism $\Omega_{A/C} \otimes_C B \cong \Omega_{A\otimes_C B/B}$, for any derivation $d(a \otimes b)$,

$$g_2 \circ f_1(d(a \otimes b)) = g_2(bd(a \otimes 1)) = bd(a \otimes 1) = d(a \otimes b) \in \Omega_{A\otimes_C B/B}$$

The last equality is as follows. By the Leibniz rule and $d(1 \otimes b) = 0$,

$$\begin{aligned} d(a \otimes b) &= d((a \otimes 1)(1 \otimes b)) = d(a \otimes 1)(1 \otimes b) + d(1 \otimes b)(a \otimes 1) \\ &= d(a \otimes 1)b + db(a \otimes 1) = d(a \otimes 1)b = bd(a \otimes 1). \end{aligned}$$

(b) Assume X and Y are nonsingular varieties of dimension n' and n'' while $X \times_k Y$ is of dimension n. From (a), there is a SES

$$0 \to p_1^*\Omega_{X\times Y/k} \to \Omega_{X\times Y/k} \to p_2^*\Omega_{X\times Y/k} \to 0$$

and an application of Hartshorne II.5.16 gives an isomorphisms

$$\begin{aligned} \omega_{X\times Y} &= \bigwedge^n \Omega_{X\times Y/k} = \bigwedge^n (p_1^*\Omega_{X/k} \oplus p_2^*\Omega_{Y/k}) = \bigwedge^{n'} p_1^*\Omega_{X/k} \otimes \bigwedge^{n''} p_2^*\Omega_{Y/k} \\ &= p_1^* \bigwedge^{n'} \Omega_{X/k} \otimes p_2^* \bigwedge^{n''} \Omega_{Y/k} = p_1^*\Omega_{X/k} \otimes p_2^*\Omega_{Y/k}. \end{aligned}$$

The only nontrivial part hypothesis is that $p_1^*\Omega_{X/k}$ is of rank n' (and similarly for $p_2^*\Omega_{Y/k}$). Of course, this follows from Proposition II.5.2(e), Hartshorne II.5.7 and $\Omega_{X/k}$ being free of rank n'.

(c) Since $\omega_X \cong p_1^*\omega_Y \otimes p_2^*\omega_Y$ and $p_i^*\omega_Y$ are free of rank 1,

$$p_g(X) = \dim_k \Gamma(X, p_1^*\omega_1 \otimes p_2^*\omega_X) = 1$$

OTOH, Hartshorne I.7.2 gives $p_a(Y) = \frac{1}{2}(3-1)(3-2) = 1$ and as $\dim Y = 1$, $p_a(X) = p_a(Y)^2 - p_a(Y) - p_a(Y) = -1$. □

Hartshorne II.8.4 *Complete Intersections in* $\mathbb{P}^n$. A closed subscheme Y of $\mathbb{P}_k^n$ is called a **(strict, global) complete intersection** if the homogeneous ideal I of Y in $S = k[x_0, \ldots, x_n]$ can be generated by $r = \mathrm{codim}(Y, \mathbb{P}^n)$ elements (I, Ex. 2.17).

(a) Let Y be a closed subscheme of codimension r in $\mathbb{P}^n$. Then Y is a complete intersection if and only if there are hypersurfaces (i.e., locally principal subschemes of codimension 1) $H_1, \ldots, H_r$, such that $Y = H_1 \cap \ldots \cap H_r$ as schemes, i.e., $\mathscr{I}_Y = \mathscr{I}_{H_1} + \ldots + \mathscr{I}_{H_r}$. [Hint: Use the fact that the unmixedness theorem holds in S (Matsumura [2, p.107]).]

(b) If Y is a complete intersection of dimension $\geqslant 1$ in $\mathbb{P}^n$, and if Y is normal, then Y is projectively normal (Ex. 5.14). [Hint: Apply (8.23) to the affine cone over Y.]

(c) With the same hypotheses as (b), conclude that for all $l \geqslant 0$, the natural map $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) \to \Gamma(Y, \mathcal{O}_Y(l))$ is surjective. In particular, taking $l = 0$, show that Y is connected.

(d) Now suppose given integers $d_1, \ldots, d_r \geqslant 1$, with $r < n$. Use Bertini's theorem (8.18) to show that there exist nonsingular hypersurfaces $H_1, \ldots, H_r$ in $\mathbb{P}^n$, with $\deg H_i = d_i$, such that the scheme $Y = H_1 \cap \ldots \cap H_r$ is irreducible and nonsingular of codimension r in $\mathbb{P}^n$.

(e) If Y is a nonsingular complete intersection as in (d), show that $\omega_Y \cong \mathcal{O}_Y(\sum d_i - n - 1)$.

(f) If Y is a nonsingular hypersurface of degree d in $\mathbb{P}^n$, use (c) and (e) above to show that $p_g(Y) = \binom{d-1}{n}$. Thus $p_g(Y) = p_a(Y)$ (I, Ex.7.2). In particular, if Y is a nonsingular plane curve of degree d, then $p_g(Y) = \frac{1}{2}(d-1)(d-2)$.

(g) If Y is a nonsingular curve in $\mathbb{P}^3$, which is a complete intersection of nonsingular surfaces of degrees d, e, then $p_g(Y) = \frac{1}{2}de(e+d-4)+1$. Again the geometric genus is the same as the arithmetic genus (I, Ex. 7.2).

PROOF. **(a)** For reference, we define the relevant terminology for the unmixedness theorem and the theorem that shows S satisfies the condition.

Definition 8.4. A ring R is said to satisfy the **unmixedness theorem** if for any ideal $I = (a_1, \ldots, a_r)$ generated by r elements has height r.

Theorem 8.5. If S is a Noetherian ring, then the unmixedness theorem holds for S iff $A_{\mathfrak{p}}$ is Cohen-Macaulay for all primes $\mathfrak{p}$ iff $A_{\mathfrak{m}}$ is free

($\Rightarrow$). If Y is a complete intersection, then $I = (f_1, f_2, \ldots, f_r)$. By the unmixedness theorem, height $I = r$. Furthermore, (f_i) has height 1 for all i which means that each f_i cuts out a hypersurface. As $\mathscr{I}_Y = \widetilde{I}$ (see the proof of Hartshorne Corollary II.5.16), the fact that $I = (f_1) + (f_2) + \cdots + (f_r)$ implies that $\mathscr{I}_Y = \mathscr{I}_{H_1} + \cdots + \mathscr{I}_{H_r}$ where H_i is the hypersurface cut out by f_i.
($\Leftarrow$). We shall show that I is generated by r elements. By unmixedness, the ideal I_{H_i} associated to each hypersurface H_i can be generated by a single polynomial f_i. Then $Y = H_1 \cap \cdots \cap H_r$ says that Y is determined by the simultaneous vanishing of each f_i. Hence, $I = (f_1, \ldots, f_r)$.

(b) Let $C(Y)$ be the affine cone over Y. Then Hartshorne I.2.20 says that $S(Y) = k[x_0, x_1, \ldots, x_n]/I = A/I$ where I associated to Y except viewed as an ordinary ideal in A.

Lemma 8.6. Since Y is a complete intersection and normal, $C(Y)$ is a locally complete intersection of $\mathbb{A}^{n+1}$.

PROOF. First off, Y is cut out by I which is generated by r elements due to unmixedness. It follows that each of the local rings are as well. That means, $\mathscr{I}_Y$ is locally generated by $r = \operatorname{codim}(Y, \mathbb{A}^{n+1})$ elements and so $C(Y)$ is a local complete intersection. □

Applying Hartshorne Proposition II.8.23, Y is a normal scheme iff it is regular in codimension 1.

Lemma 8.7. $C(Y)$ is regular in codimension 1, and hence, $C(Y)$ is normal.

PROOF. First off, $A(C(Y)) = k[x_0, \ldots, x_n]/I(Y)$ with $I(Y)$ viewed as a "usual" ideal of $k[x_0, \ldots, x_n]$. However, the fact that Y is the intersection of hypersurfaces means that itself was also regular in codimension 1. This implies that $C(Y)$ is as well. □

Because $C(Y)$ is normal, we know that all of its local rings are integrally closed domains. By Atiyah-MacDonald Proposition 5.13, the ring A/I is normal. But this implies that $S(Y)$ itself was normal meaning Y is projective normal by Hartshorne II.5.14.

(c) With the same hypotheses in (b), we know that Y is projectively normal. By Hartshorne II.5.14(d), the maps $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) \to \Gamma(X, \mathcal{O}_X(l))$ are surjective for all $l \geq 0$.

Take $l = 0$. Then $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = k$ which means $k \to \Gamma(Y, \mathcal{O}_Y)$ is a surjective map. Now if Y were not connected, we could write $Y = W_1 \cup W_2$ for open sets W_i with $W_1 \cap W_2 = \emptyset$. In that case, $\Gamma(Y, \mathcal{O}_Y) \cong \Gamma(W_1, \mathcal{O}_Y) \times \Gamma(W_2, \mathcal{O}_Y)$ with both component rings being nonzero (see Hartshorne II.2.19 for the case of affine schemes). There is no such surjective map in this case.

(d) The idea of proof is to construct them inductively. In the case of $r = 1$, we can just take the hypersurface cut out by $x_0^{d_1}$. Apply Bertini's Theorem to find H_2 s.t. $H_1 \cap H_2$ is irreducible and nonsingular (a consequence of being regular see Hartshorne Remark III.7.9.1). Furthermore, H_2 can be taken to have degree d_2 by raising it to a higher power.

The intersection of irreducible varieties is also irreducible and so Y is irreducible. For nonsingular in codimension r, let $\mathcal{O}_{Y,y}$ be a local ring of codimension r.

Alternatively, apply Hartshorne Example II.8.20.2 r-times.

(e) Let $Y := \bigcap_{i=1}^r H_i$ from (d). Then consider $Y \subseteq X := \mathbb{P}^n_k$.

Consider the case of just one hypersurface of degree d_1. Then,

$$\begin{aligned}\omega_{H_1} &\cong \omega_Y \cong \omega_{\mathbb{P}^n_k} \otimes \mathcal{O}_{\mathbb{P}^n_k}(d_1) \otimes \mathcal{O}_Y \\ &\cong \mathcal{O}_{\mathbb{P}^n_k} \otimes \mathcal{O}_{\mathbb{P}^n_k}(d_1) \otimes \mathcal{O}_Y \\ &\cong \mathcal{O}_Y(d_1 - n - 1).\end{aligned}$$

Then in the case of $H_1 \cap H_2$, one has

$$\begin{aligned}\omega_{H_1 \cap H_2} &\cong \omega_{H_1} \otimes \mathcal{O}_{H_1}(H_1 \cap H_2) \otimes \mathcal{O}_{H_1 \cap H_2} \\ &\cong \mathcal{O}_{H_1}(d_1 - n - 1) \otimes \mathcal{O}_{H_1}(d_2) \otimes \mathcal{O}_{H_1 \cap H_2} \\ &\cong \mathcal{O}_{H_1 \cap H_2}(d_1 + d_2 - n - 1)\end{aligned}$$

and if one continues inductively, we get the desired isomorphism.

(f) Hartshorne I.7.2 gives $p_a(Y) = \binom{d-1}{n}$ so we show $p_g(Y) = \binom{d-1}{n}$. From (e), we know $\omega_Y \cong \mathcal{O}_Y(d-n-1)$. The result is clear if $\ell = d-n-1 < 0$. If $\ell \geq 0$, we know that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)) \to \Gamma(Y, \mathcal{O}_Y(\ell))$ is surjective. In this case, we know $p_g(Y) \leq \dim_k \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) = \binom{n+l}{l} = \binom{d-1}{n}$. We are done if we prove the following lemma.

Lemma 8.8. The map $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) \to \Gamma(X, \mathcal{O}_Y(l))$ is injective for $l = d - n - 1$.

PROOF. Tracing how we defined the map in Hartshorne II.5.14 and the fact that $l \geq 0$ implies $d \geq n + 1$. When $d = n + 1$, $p_g(Y) \leq 1$ and $\Gamma(Y, \mathcal{O}_Y)$ having global sections shows $p_g(Y) = 1$. Assuming $d > n + 1$, the ideal I in the proof of Hartshorne II.5.14 contains only elements of degree d. So quotienting by I does not give a kernel and so, the map is injective. We conclude that $p_g(Y) = \binom{d-1}{n}$. □

For the statement about nonsingular plane curves Y, we take $Y \subseteq \mathbb{P}^2$ and apply the first part. This gives $p_g(Y) = \binom{d-1}{2} = \frac{(d-1)!}{2!(d-3)!} = \frac{1}{2}(d-1)(d-2)$.

(g) The arithmetic genus computation is in Hartshorne I.7.2 so we prove the statement about geometric genus. From (e), we know that

$$\omega_Y = \mathcal{O}_Y(e+d-4).$$

Now the ideal of Y is cut out by two equations $I_Y = (f, g)$. There is a SES

$$0 \to \mathscr{I}_Y \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Y \to 0.$$

Twist by $e+d-4$ to get

$$0 \to \mathscr{I}_Y(e+d-4) \to \mathcal{O}_{\mathbb{P}^3}(e+d-4) \to \omega_Y \to 0.$$

Taking global sections,

$$p_g(Y) = \dim_k \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(e+d-4)) + \dim_k \Gamma(\mathbb{P}^3, \mathscr{I}_Y(e+d-4)).$$

Now we must compute the dimensions. We know that $\dim_k \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(e+d-4)) = \binom{e+d-4+3}{3} = \binom{e+d-1}{3}$. OTOH, note that $I_Y = (f, g)$ means any element has form $h_1 f + h_2 g$ for $h_1, h_2 \in k[x_0, \ldots, x_2]$. Then to get an element of $I_Y(e+d-4)$, note that we will have $\deg(f) = d$ and $\deg(g) = e$. so that means we must have $\deg(h_1) = e-4$ and $\deg(h_2) = d-4$.

Now we must count the number of possible basis elements. In particular, we have a total of $\binom{d-4+3}{3} + \binom{e-4+3}{3}$ possibilities. Writing this out,

$$\begin{aligned} p_g(Y) &= \binom{e+d-1}{3} + \binom{d-1}{3} + \binom{e-1}{3} \\ &= \frac{(e+d-1)(e+d-2)(e+d-3)}{6} - \frac{(e-1)(e-2)(e-3)}{6} - \frac{(d-1)(d-2)(d-3)}{6} \\ &= \frac{3d^2e + ede^2 - 12de + 6}{6} \\ &= \frac{1}{2}de(e+d-4) + 1 \end{aligned}$$

□

Hartshorne II.8.5 *Blowing up a Nonsingular Subvariety.* As in (8.24), let X be a nonsingular variety, let Y be a nonsingular subvariety of codimension $r \geqslant 2$, let $\pi : \widetilde{X} \to X$ be the blowing-up of X along Y, and let $Y' = \pi^{-1}(Y)$.

(a) Show that the maps $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$, and $\mathbb{Z} \to \operatorname{Pic} X$ defined by $n \mapsto$ class of nY', give rise to an isomorphism $\operatorname{Pic} \widetilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

(b) Show that $\omega_{\widetilde{X}} \cong f^*\omega_X \otimes \mathscr{L}((r-1)Y')$.

[Hint: By (a) we can write in any case $\omega_{\widetilde{X}} \cong f^*\mathscr{M} \otimes \mathscr{L}(qY')$ for some invertible sheaf $\mathscr{M}$ on X, and some integer q. By restricting to $\widetilde{X} - Y' \cong X - Y$, show that $\mathscr{M} \cong \omega_X$. To determine q, proceed as follows. First show that $\omega_{Y'} \cong f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)$. Then take a closed point $y \in Y$ and let Z be the fibre of Y' over y. Then show that $\omega_Z \cong \mathcal{O}_Z(-q-1)$. But since $Z \cong \mathbb{P}^{r-1}$, we have $\omega_Z \cong \mathcal{O}_Z(-r)$, so $q = r-1$.]

PROOF. **(a)** Apply Hartshorne Proposition II.6.5(c) to get an ES

$$\mathbb{Z} \to \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} \widetilde{X} - Y' \to 0.$$

because Y' itself is an irreducible subset of codimension 1 (since it corresponds to a divisor). But note that $\operatorname{Pic} \widetilde{X} - Y'$ is isomorphic to $\operatorname{Pic} X - Y$ and to $\operatorname{Pic} X$. Indeed, $\operatorname{Pic} X - Y \cong \operatorname{Cl} X - Y \cong \operatorname{Cl} X$ because Y is of codimension ≥ 2. So the sequence becomes

$$\mathbb{Z} \to \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} X \to 0.$$

At this point, it is easier to just work with divisors and show that the sequence splits

$$\mathbb{Z} \to \operatorname{Cl} \widetilde{X} \to \operatorname{Cl} X \to 0$$

and here, $\mathbb{Z} \cdot 1 \to \mathbb{Z} \cdot \pi^{-1}(Y)$. Define the map $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$ which gives a section of the map $\operatorname{Cl} \widetilde{X} \to \operatorname{Cl} X$.

We want to show that $\mathbb{Z} \to \operatorname{Cl} \widetilde{X}$ is injective. Assume $nY' \sim 0$ for some $n > 0$. Suppose it is linearly equivalent to the divisor of $f' \in k(\widetilde{X})^\times$. Now $\widetilde{X} \to X$ is a birational surjective map, so f' corresponds to a regular function on X with only zeros on Y. But of course, Y is codimension 2 which is absurd. Therefore, f' is zero which is absurd.

(b)[59] First off, we shall write π instead of f since that is the notation used in the section.

By (a), let $\omega_{\widetilde{X}} \cong \pi^* \mathscr{M} \otimes \mathscr{L}(qY')$ for some $q \in \mathbb{Z}$ and invertible sheaf $\mathscr{M} \in \operatorname{Pic} X$.

We can then restrict $\mathscr{M} \in \operatorname{Pic} X$ to $X - Y \cong X' - Y$. Then $\omega_{\widetilde{X}}|_{\widetilde{X}-Y} \cong \omega_X|_{X-Y}$ and $\operatorname{Pic} X \cong \operatorname{Pic} X - Y$ implies that $\mathscr{M} = \omega_X$.

Next,

$$\begin{aligned}
\omega_{Y'} &\cong \omega_{\widetilde{X}} \otimes \mathscr{L}(Y') \otimes \mathcal{O}_{Y'} && \text{(Hartshorne Proposition II.8.20)} \\
&\cong \pi^*\omega_X \otimes \mathscr{L}((q+1)Y') \otimes \mathcal{O}_{Y'} && \text{(from above)} \\
&\cong \pi^*\omega_X \otimes \mathscr{I}_Y^{-q-1} \otimes \mathcal{O}_{Y'} && \text{(from Proposition II.6.18)} \\
&\cong \pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(1)^{-q-1} \otimes \mathcal{O}_{Y'} && \text{(proof of Hartshorne Proposition II.7.13)} \\
&\cong \pi^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)
\end{aligned}$$

Note that everything above is viewed as a sheaf on Y' so $\pi^*\omega_X$ is being viewed as a sheaf on Y'.

OTOH, let $y \in Y$ be a closed point and $Z := \pi^{-1}(y)$ be the fibre of Y' over y. Let p_1, p_2 be the respective projections of Z. Then,

$$\begin{aligned}
\omega_Z &\cong p_1^*\omega_y \otimes p_2^*\omega_{Y'} && \text{(Hartshorne II.8.3(b))} \\
&\cong \mathcal{O}_Z \otimes p_2^*(\pi^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)) && (p_1^*\omega_y \cong \mathcal{O}_y \cong \mathcal{O}_Z) \\
&\cong p_2^*(\pi^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)) \\
&\cong p_2^*\pi^*\omega_X \otimes p_2^*\mathcal{O}_{Y'}(-q-1) && ((-)^* \text{ distributes over tensor products}) \\
&\cong (\pi \circ p_2)^*\omega_X \otimes p_2^*\mathcal{O}_{Y'}(-q-1) && ((-)^* \text{ is contravariant}) \\
&\cong \mathcal{O}_Z \otimes p_2^*\mathcal{O}_{Y'}(-q-1) && (p_2^*\pi^*\omega_{X/k} \cong \mathcal{O}_Z) \\
&\cong \mathcal{O}_Z \otimes \mathcal{O}_Z(-q-1) \\
&\cong \mathcal{O}_Z(-q-1).
\end{aligned}$$

[59] A much more thorough discussion can be found in this post. The link is inside the proof as well.

The third to last isomorphism is justified as follows. First, by definition, Z fits into a fibre product diagram

$$\begin{array}{ccc} Z & \xrightarrow{p_2} & Y' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ y & \xrightarrow{i} & Y \end{array}$$

It follows that $\pi \circ p_2 = i \circ p_1$ and so, $(\pi \circ p_2)^* \omega_X = p_1^* i^* \omega_X = p_1^* \mathcal{O}_y = \mathcal{O}_Z$. The fact that $i^* \omega_X = \mathcal{O}_y$ follows form the fact that the only line bundles on $\operatorname{Spec} k(y)$ is just $\mathcal{O}_y$ (note that ω_X is a linear bundle by Hartshorne II.5.16(a)).

But now one observes[60] that $Z \cong \operatorname{Spec} k(y) \times_Y Y' \cong \mathbb{P}^{r-1}$ so that Hartshorne Example II.8.20.1 says $\omega_Z \cong \mathcal{O}_Z(-r)$.

Hence, $-q - 1 = -r$ which means $q = r - 1$. At last,

$$\omega_{\widetilde{X}} \cong \pi^* \omega_X \otimes \mathscr{L}((r-1)Y').$$

□

Hartshorne II.8.6 ***The Infinitesimal Lifting Property.*** The following result is very important in studying deformations of nonsingular varieties. Let k be an algebraically closed field, let A be a finitely generated k-algebra such that $\operatorname{Spec} A$ is a nonsingular variety over k. Let $0 \to I \to B' \to B \to 0$ be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f : A \to B$. Then there exists a k-algebra homomorphism $g : A \to B'$ making a commutative diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & I & & \\ & & \downarrow & & \\ & & B' & & \\ & \overset{g}{\nearrow} & \downarrow & & \\ A & \xrightarrow{f} & B & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

We call this result the *infinitesimal lifting property* for A. We prove this result in several steps.

(a) First suppose that $g : A \to B'$ is a given homomorphism lifting f. If $g' : A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, which we can consider as an element of $\operatorname{Hom}_A\left(\Omega_{A/k}, I\right)$. Note that since $I^2 = 0, I$ has a natural structure of B-module and hence also of A-module. Conversely, for any $\theta \in \operatorname{Hom}_A\left(\Omega_{A/k}, I\right), g' = g + \theta$ is another homomorphism lifting f. (For this step, you do not need the hypothesis about $\operatorname{Spec} A$ being nonsingular.)

[60] This is sort of just stated, but see Eisenbud-Harris Section IV.2 on blow-ups. In the situation of a local complete intersection, the exceptional divisor is a projective space (or more specifically a projective bundle). This also follows from Hartshorne Theorem II.8.24.

(b) Now let $P = k[x_1, \ldots, x_n]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there does exist a homomorphism $h : P \to B'$ making a commutative diagram,

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ J & & I \\ \downarrow & & \downarrow \\ P & \xrightarrow{h} & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

and show that h induces an A-linear map $h : J/J^2 \to I$.

(c) Now use the hypothesis $\operatorname{Spec} A$ nonsingular and (8.17) to obtain an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0.$$

Show furthermore that applying the functor $\operatorname{Hom}_A(\cdot, I)$ gives an exact sequence

$$0 \to \operatorname{Hom}_A\left(\Omega_{A/k}, I\right) \to \operatorname{Hom}_P\left(\Omega_{P/k}, I\right) \to \operatorname{Hom}_A\left(J/J^2, I\right) \to 0.$$

Let $\theta \in \operatorname{Hom}_P\left(\Omega_{P/k}, I\right)$ be an element whose image gives $\bar{h} \in \operatorname{Hom}_A\left(J/J^2, I\right)$. Consider θ as a derivation of P to B'. Then let $h' = h - \theta$, and show that h' is a homomorphism of $P \to B'$ such that $h'(J) = 0$. Thus h' induces the desired homomorphism $g : A \to B'$.

PROOF. **(a)** ($\Rightarrow$). Let $\phi : B' \to B$ the map in the SES and $\psi : I \hookrightarrow B'$ the inclusion. For any $a \in A$,

$$f(\theta(a)) = \phi(a) - \phi(a) = 0 \;\Rightarrow\; \theta(a) \in \ker(\psi) = \operatorname{im}(\phi) = I.$$

For any $\ell \in k$, we have $\phi\theta(\ell) = \ell\phi\theta(1) = 0$ which means $\phi\theta(1) = 0$. Hence, $\theta(1) \in I$ and since these are k-algebra homomorphisms, $\theta(1) = 0$. It follows that $\theta(\ell) = 0$ for all $\ell \in k$.

The fact that θ is additive clear from additivity of g, g'.

For the Leibniz rule, we check $\theta(aa') = \theta(a)a' + a\theta(a')$. We have (based on the A-module structure on B' given by g)

$$\theta(aa') = g(aa') - g(aa')$$

and

$$\theta(a)a' + a\theta(a') = g(aa') - g'(a)g(a') + g(aa') - g(a)g'(a') \tag{28}$$

$$= 2g(aa') - g'(a)g(a') - g(a)g'(a'). \tag{29}$$

Because $I^2 = 0$ and $\theta : A \to I$,

$$\begin{aligned} 0 = \theta(a)\theta(a') &= g(aa') + g'(aa') - g(a)g'(a') - g'(a)g(a') \\ \Rightarrow -g(aa') - g'(aa') &= -g(a)g'(a') - g'(a')g(a'). \end{aligned}$$

From this line and the RHS of Equation (29),

$$\begin{aligned}\theta(a)a' + a\theta(a') &= g(a)g(a') - g'(a)g(a') + g(a)g(a') - g(a)g'(a') \\ &= 2g(aa') - g'(a)g(a') - g(a)g'(a') \\ &= 2g(aa') - g(aa') - g'(aa') \\ &= g(aa') - g'(aa') \\ &= \theta(aa').\end{aligned}$$

Note that the proof would be the same had we assumed the A-module structure on I was given by g' since g, g' lift f.

Clearly $\theta(s) = g(s) - g'(s) = 0$ for all $s \in k$.
($\Leftarrow$). Let $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I)$. Then we can identify θ with an A-derivation $\theta : A \to I$. Define $g' := g + \theta : A \to I$. So,

$$\phi(g'(a)) = \phi(g(a) + \theta(a)) = f(a) + \phi\theta(a).$$

This equals $f(a)$ iff $\phi\theta(a) = 0$ which we now show. Of course, $\theta(a) \in I$ and since

$$0 \to I \to B' \xrightarrow{\phi} B \to 0$$

is exact, $\phi\theta(a) = 0$.

(b) Construct h as follows. Another k-algebra homomorphism out of P is determined by where we send each x_i. Map x_i into A via $\psi' : P \to A$ and then $f(\psi(x_i)) \in B$. We can choose lifts $h_i \in B'$ of $f(\psi(x_i))$. Then define $h(x_i) = h_i$ and this is certainly a k-linear homomorphism.

Write $\overline{h} : J/J^2 \to I$ for the induced map. By commutativity of the diagram, $h(J) \in I$. OTOH, it si a homomorphism which means $h(J^2) = h(J)^2 \in I^2 = 0$. So $\overline{h} : J/J^2 \to I$ exists and is well-defined.

To check A-linearity, observe that $J/J^2 \cong J \otimes P/J \cong J \otimes A$ so that $\overline{h} : J \otimes P/J \to I$ is defined by $h(j \otimes p) = h(j) \otimes h(p)$. This is clearly A-linear since the action of A on I is determined by the action of B on $I/I^2 = I$.

(c) Because $\mathrm{Spec}\, A$ is nonsingular, the functor $\widetilde{(-)}$ is fully faithful by Hartshorne Corollary II.5.5, $\widetilde{(-)}$ distributes over tensor products, and Theorem II.8.17 holds due to nonsingularity of $\mathrm{Spec}\, A$, the sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0$$

is exact.

The functor $\mathrm{Hom}_A(-, J)$ is left exact and contravariant. To prove that

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, I) \to \mathrm{Hom}_A(\Omega_{P/k} \otimes A, I) \to \mathrm{Hom}_A(J/J^2, I) \to 0$$

is exact, we establish surjectivity on the RHS map. For the middle term, Hom-Tensor adjunction gives

$$\mathrm{Hom}_A(\Omega_{P/k} \otimes_P A, I) \cong \mathrm{Hom}_P(\Omega_{P/k}, \mathrm{Hom}_A(A, I)) \cong \mathrm{Hom}_P(\Omega_{P/k}, I)$$

and so we have the sequence in the problem statement. Take the LES of Ext-modules

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, I) \to \mathrm{Hom}_P(\Omega_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I) \to \mathrm{Ext}^1_A(\Omega_{A/k}, I) \to \dots$$

We know that $\Omega_{A/k}$ is free of rank $\dim \operatorname{Spec} A$ so $\Omega_{A/k}$ is projective. So, $\operatorname{Ext}^i_A(\Omega_{A/k}, I) = 0$ for all $i > 0$ and in particular, $\operatorname{Ext}^1_A(\Omega_{A/k}, I) = 0$.

Having established surjectivity, let $\theta \in \operatorname{Hom}_P(\Omega_{P/k}, I)$ map to $\overline{h} \in \operatorname{Hom}_A(J/J^2, I)$. Consider θ as a derivation of P to $B' \supseteq I$. Let $h' = h - \theta : P \to B'$. The fact that h' is a homomorphism follows from h, θ being homomorphisms. The fact that θ, h both induce the map $\overline{h} : J/J^2 \to I \to B$ means that their difference induces a map $J/J^2 \to B$ that is zero. Hence, the difference maps J to zero i.e. $h'(J) = 0$.

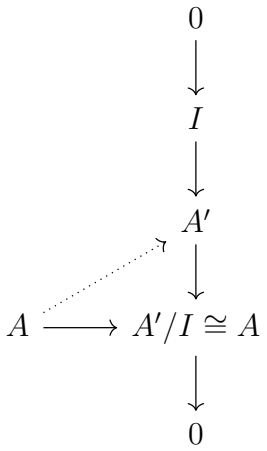
Now we explain how h' induces the desired $g : A \to B'$. First off, $h' : P \to B'$ with $h'(J) = 0$ means that h' induces a map $h' : A = P/J \to B'$. Furthermore, $h' : A \to B'$ makes the first diagram commute. □

Hartshorne II.8.7 As an application of the infinitesimal lifting property, we consider the following general problem. Let X be a scheme of finite type over k, and let $\mathscr{F}$ be a coherent sheaf on X. We seek to classify schemes X' over k, which have a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$ and $(X', \mathcal{O}_{X'}/\mathscr{I}) \cong (X, \mathcal{O}_X)$, and such that $\mathscr{I}$ with its resulting structure of $\mathcal{O}_X$-module is isomorphic to the given sheaf $\mathscr{F}$. Such a pair $X', \mathscr{I}$ we call an **infinitesimal extension** of the scheme X by the sheaf $\mathscr{F}$. One such extension, the **trivial** one, is obtained as follows. Take $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathscr{F}$ as sheaves of abelian groups, and define multiplication by $(a \oplus f) \cdot (a' \oplus f') = aa' \oplus (af' + a'f)$. Then the topological space X with the sheaf of rings $\mathcal{O}_{X'}$ is an infinitesimal extension of X by $\mathscr{F}$.

The general problem of classifying extensions of X by $\mathscr{F}$ can be quite complicated. So for now, just prove the following special case: if X is affine and nonsingular, then any extension of X by a coherent sheaf $\mathscr{F}$ is isomorphic to the trivial one. See (III, Ex. 4.10) for another case.

Proof. Let $X = \operatorname{Spec} A$ be nonsingular. The problem reduces to proving that if A' is a ring with ideal I s.t. $A'/I \cong A$, $I \cong M$, and $I^2 = 0$, then $A' \cong A \oplus M$ where the RHS has multiplication $(a, f)(a', f') = (aa', af' + a'f)$.

In terms of an actuall diagram, we have



a lift from Hartshorne II.8.6 which lifts the isomorphism. The kernel of $A' \to A$ was $I \cong M$. So the lift induces a splitting of the SES as A-modules (but a priori only as abelian groups) i.e. $A' \cong A \oplus I \cong A \oplus M$. Now we check that the multiplication on the RHS is defined as in the trivial extension. Indeed, $(a, f)(a', f') \in A \oplus I$ and we must use the A-module structure

on I defined from $M \cong I \cong I/I^2$. For any $a \in A$, $(a,0)(a',f') = (aa', af' + a'f)$ because

$$\begin{aligned}
(a,f)(a',f') &= (a,0)(a',f') + (0,f)(a',f') \\
&= (aa', af') + (0,f)((a',0) + (0,f')) \\
&= (aa', af') + (0,f)(a',0) + (0,f)(0,f') \\
&= (aa', af') + (a',0)(0,f) + (0,ff') && \text{(By commutativity)} \\
&= (aa', af') + (0,a'f) + (0,ff') && \text{(by definition of the action)} \\
&= (aa', af' + a'f) + (0,ff') \\
&= (aa', af' + a'f) && \text{(Since } ff' \in I^2 = 0\text{).}
\end{aligned}$$

□

Hartshorne II.8.8 Let X be a projective nonsingular variety over k. For any $n > 0$ we define the n**th plurigenus** of X to be $P_n = \dim_k \Gamma\left(X, \omega_X^{\otimes n}\right)$. Thus in particular $P_1 = p_g$. Also, for any $q, 0 \leqslant q \leqslant \dim X$ we define an integer $h^{q,0} = \dim_k \Gamma\left(X, \Omega^q_{X/k}\right)$ where $\Omega^q_{\bar{X}/k} = \bigwedge^q \Omega_{X/k}$ is the sheaf of regular q-forms on X. In particular, for $q = \dim X$, we recover the geometric genus again. The integers $h^{q,0}$ are called **Hodge numbers**.

Using the method of (8.19), show that P_n and $h^{q,0}$ are **birational invariants** of X, i.e., if X and X' are birationally equivalent nonsingular projective varieties, then $P_n(X) = P_n(X')$ and $h^{q,0}(X) = h^{q,0}(X')$.

Proof. We shall just observe that the proof of Hartshorne Theorem II.8.19 follows with only few modification.

First, in the first paragraph, we get an induced map by taking exterior powers. This was possible because Hartshorne II.5.16(e) showed that f^* commutes with tensor operations. So of course, this part of the proof is okay since we tensor powers of the canonical sheaf or direct sums of the sheaf of differential forms.

Second, the last paragraph required that we show $\Gamma(X, \omega_X) \to \Gamma(V, \omega_V)$ was bijective. This was reduced to showing $\Gamma(U, \mathcal{O}_U) \to \Gamma(U \cap V, \mathcal{O}_{U\cap V})$ is bijective. Of course, this latter part implies the bijectivity of $\Gamma(X, \omega_X^{\otimes n}) \to \Gamma(V, \omega_V^{\otimes n})$ and $\Gamma(X, \Omega^q_{X/k}) \to \Gamma(V, \Omega^q_{V/k})$.

Putting these two parts together and interchanging the role of X and X', we know P_n and $h^{q,0}$ are birational invariants. □

Remark 43. Why are the Hodge numbers $h^{0,q}$ not necessarily birational invariants? One heuristic reason is that these would involve higher cohomology groups and so the argumnet in Hartshorne Theorem II.8.19 has no hope of passing through.

9. Formal Scheme

Hartshorne II.9.1 Let X be a noetherian scheme, Y a closed subscheme, and $\widehat{X}$ the completion of X along Y. We call the ring $\Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}})$ the ring of formal-regular functions on X along Y.

In this exercise we show that if Y is a connected, nonsingular, positive dimensional subvariety of $X = \mathbb{P}^n_k$ over an algebraically closed field k, then $\Gamma\left(\widehat{X}, \mathcal{O}_{\widehat{X}}\right) = k$.

(a) Let $\mathscr{I}$ be the ideal sheaf of Y. Use (8.13) and (8.17) to show that there is an inclusion of sheaves on Y, $\mathscr{I}/\mathscr{I}^2 \hookrightarrow \mathcal{O}_Y(-1)^{n+1}$.

(b) Show that for any $r \geqslant 1, \Gamma\left(Y, \mathscr{I}^r/\mathscr{I}^{r+1}\right) = 0$.

(c) Use the exact sequences

$$0 \to \mathscr{I}^r/\mathscr{I}^{r+1} \to \mathcal{O}_X/\mathscr{I}^{r+1} \to \mathcal{O}_X/\mathscr{I}^r \to 0$$

and induction on r to show that $\Gamma\left(Y, \mathcal{O}_X/\mathscr{I}^r\right) = k$ for all $r \geqslant 1$. (Use (8.21Ae).)

(d) Conclude that $\Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}}) = k$. (Actually, the same result holds without the hypothesis Y nonsingular, but the proof is more difficult-see Hartshorne [3, (7.3)].)

Proof. (a) From Hartshorne Theorem II.8.17, there is a SES

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0.$$

Now take the sequence in Hartshorne Theorem II.8.13 (and then tensor by $\mathcal{O}_Y$) to get

$$0 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \mathcal{O}_Y(-1)^{n+1} \to \mathcal{O}_Y \to 0$$

and so there is an inclusion $\mathscr{I}/\mathscr{I}^2 \hookrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \hookrightarrow \mathcal{O}_Y(-1)^{n+1}$.
(b) The result for $r = 1$ is clear from above since

$$\Gamma(X, \mathscr{I}/\mathscr{I}^2) \hookrightarrow \Gamma(X, \mathcal{O}_X(-1)^{n+1}|_Y) = 0$$

since $\mathcal{O}_X(-1)^{n+1}|_Y = \mathcal{O}_X(-1)^{n+1} \otimes \mathcal{O}_Y$. To get the result for $\mathscr{I}^2/\mathscr{I}^3$, we tensor by $\mathcal{O}_X/\mathscr{I} = \mathcal{O}_Y$ since

$$0 \to \mathscr{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

where we really mean $\mathcal{O}_Y = i_*\mathcal{O}_Y$ for $i : Y \hookrightarrow X$. Tensor repeatedly to get the result – provided we show tensoring by $\mathcal{O}_Y := \mathcal{O}_X/\mathscr{I}_Y$ does not affect injectivity. This follow from some facts in Hartshorne Theorem II.8.17 which we formulate in a lemma.

Lemma 9.1. Tensoring by $\mathcal{O}_X/\mathscr{I}_Y =: \mathcal{O}_Y$ preserves injectivity i.e. the following is injective

$$\mathscr{I}/\mathscr{I}^2 \otimes \mathcal{O}_Y \to \mathcal{O}_Y(-1)^{n+1}.$$

Proof. [Incomplete] □

(c) The existence of the sequence is clear. The claim for $r = 1$ is clear since $\Gamma(Y, \mathcal{O}_X/\mathscr{I})$ is the global sections of a structure sheaf of a projective variety. Now consider the sequence with $r = 1$,

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \mathcal{O}_X/\mathscr{I}^2 \to \mathcal{O}_X/\mathscr{I} \to 0.$$

Then we get

$$0 \to 0 \to \Gamma(X, \mathcal{O}_X/\mathscr{I}^2) \to \Gamma(X, \mathcal{O}_X/\mathscr{I}) \to H^1(X, \mathscr{I}/\mathscr{I}^2) \to \dots$$

Now $\Gamma(\mathcal{O}_X/\mathscr{I}^2)$ injects into $\Gamma(\mathcal{O}_X/\mathscr{I})$ so we need only show that it is nonzero. But it being nonzero is just a globalized version of Hartshorne Theorem II.8.21.A(e) i.e. $\Gamma(\mathcal{O}_X/\mathscr{I})$ is computing the degree zero part of the associated graded ring.
(d) Taking cohomology commutes in the following way;

$$\Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}}) = \varprojlim_i \Gamma(X_i, \mathcal{O}_X/\mathscr{I}^i) = k$$

since the individual global sections is isomorphic to k. □

Remark 44. It is crucial here that Y is positive dimensional. If Y were a point, then the completion along Y would give Spec $\widehat{\mathcal{O}_Y}$ and the global sections do not need to be a field (it can be bigger).

Hartshorne II.9.2 Use the result of (Ex. 9.1) to prove the following geometric result. Let $Y \subseteq X = \mathbb{P}^n_k$ be as above, and let $f : X \to Z$ be a morphism of k-varieties. Suppose that $f(Y)$ is a single closed point $P \in Z$. Then $f(X) = P$ also.

PROOF. Let $\widehat{X}$ be the formal completion of X along Y. Now

$$\Gamma(P, f_*\mathcal{O}_{\widehat{X}}) = \Gamma(f^{-1}(P), \mathcal{O}_{\widehat{X}}) \to \Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}}) = k$$

where the map is the restriction map. On the other hand,

$$\mathcal{O}_{\widehat{X}} = \varprojlim \mathcal{O}_X/\mathscr{I}^n \;\Rightarrow\; f_*\mathcal{O}_{\widehat{X}} = \varprojlim_n f_*\mathcal{O}_X/\mathscr{I}^n$$

by RAPL. Taking global sections,

$$\Gamma(P, f_*\mathcal{O}_{\widehat{X}}) = \varprojlim_n \Gamma(P, f_*\mathcal{O}_X/\mathscr{I}^n) = \varprojlim_n \Gamma(Y, (\mathcal{O}_X/\mathscr{I})|_Y) = \Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}}) = k.$$

[Incomplete] □

PROOF. The proof hinges on computing $\Gamma(P, f_*\mathcal{O}_X)$. □

Hartshorne II.9.3 Prove the analogue of (5.6) for formal schemes, which says, if $\mathfrak{X}$ is an affine formal scheme, and if

$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$

is an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules, and if $\mathfrak{F}'$ is coherent, then the sequence of global sections

$$0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0$$

is exact. For the proof, proceed in the following steps.

(a) Let $\mathfrak{I}$ be an ideal of definition for $\mathfrak{X}$, and for each $n > 0$ consider the exact sequence

$$0 \to \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}'' \to 0$$

Use (5.6), slightly modified, to show that for every open affine subset $\mathfrak{U} \subseteq \mathfrak{X}$, the sequence

$$0 \to \Gamma(\mathfrak{U}, \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}'') \to 0$$

is exact. and that the sequence of global sections above is exact.

(b) Now pass to the limit, using (9.1),(9.2), and (9.6). Conclude that $\mathfrak{F} \cong \lim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}'$ and that the sequence of global sections above is exact.

PROOF. (a) The exact sequence clearly exists by sheaf theory. We first modify Hartshorne Proposition II.5.6 in the form of the following lemma.

Lemma 9.2. Let $\mathfrak{X}$ be a Noetherian affine scheme and $\mathfrak{F}$ a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Then the exact sequence

$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$

gives rise to an exact sequence

$$0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0$$

(b) □

Hartshorne II.9.4 Use (Ex. 9.3) to prove that if

$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$

is an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules on a noetherian formal scheme $\mathfrak{X}$, and if $\mathfrak{F}', \mathfrak{F}''$ are coherent, then $\mathfrak{F}$ is coherent also.

Proof. The question is local so assume $\mathfrak{X}$ is an affine Noetherian formal scheme. Then by Hartshorne II.9.3 and Hartshorne Theorem II.9.7,

$$0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0$$

is a SES in which the first and third terms are finitely generated A-modules (provided A was I-adically complete). SO the middle term is finitely generated and we get the conclusion.

The fact that coherence does not matter on if A was I-adically complete follows from taking base change $A \to \widehat{A}$ and recalling this is a faithfully flat extension (see Matsumura's Commutative Ring Theory). □

Hartshorne II.9.5 If $\mathfrak{F}$ is a coherent sheaf on a noetherian formal scheme $\mathfrak{X}$, which can be generated by global sections, show in fact that it can be generated by a finite number of its global sections.

Proof. One takes a finite cover of $\mathfrak{X}$ by open affine sets, then uses the equivalence of categories of Hartshorne Theorem II.9.6, and then observes that we can then glue to a finite number of global sections which generate everywhere. □

Hartshorne II.9.6 Let $\mathfrak{X}$ be a noetherian formal scheme, let $\mathfrak{J}$ be an ideal of definition, and for each n, let Y_n be the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{J}^n)_n$. Assume that the inverse system of groups $(\Gamma(Y_n, \mathcal{O}_{Y_n}))$ satisfies the Mittag-Leffier condition. Then prove that $\operatorname{Pic} \mathfrak{X} = \varinjlim \operatorname{Pic} Y_n$. As in the case of a scheme, we define $\operatorname{Pic} \mathfrak{X}$ to be the group of locally free $\mathcal{O}_{\mathfrak{X}}$-modules of rank 1 under the operation $\otimes$. Proceed in the following steps.

(a) Use the fact that $\ker\left(\Gamma(Y_{n+1}, \mathcal{O}_{Y_{n+1}}) \to \Gamma(Y_n, \mathcal{O}_{Y_n})\right)$ is a nilpotent ideal to show that the inverse system $\left(\Gamma\left(Y_n, \mathcal{O}_{Y_n}^*\right)\right)$ of units in the respective rings also satisfies (ML).

(b) Let $\mathfrak{F}$ be a coherent sheaf of $\mathcal{O}_{\mathbb{X}}$-modules, and assume that for each n, there is some isomorphism $\varphi_n : \widetilde{F}/\mathfrak{J}^n\mathfrak{F} \cong \mathcal{O}_{Y_n}$. Then show that there is an isomorphism $\mathfrak{F} \cong \mathcal{O}_x$. Be careful, because the φ_n may not be compatible with the maps in the two inverse systems $(\mathfrak{F}/\mathfrak{J}^n\mathfrak{F})$ and $(\mathcal{O}_{Y_n})$. Conclude that the natural map $\operatorname{Pic} \mathfrak{X} \to \varprojlim_n \operatorname{Pic} Y_n$ is injective.

(c) Given an invertible sheaf $\mathscr{L}_n$ on Y_n for each n, and given isomorphisms $\mathscr{L}_{n+1} \otimes \mathcal{O}_{Y_n} \cong \mathscr{L}_n$, construct maps $\mathscr{L}_{n'} \to \mathscr{L}_n$ for each $n' \geqslant n$ so as to make an inverse system, and show that $\mathfrak{L} = \varprojlim_n \mathscr{L}_n$ is a coherent sheaf on $\mathfrak{X}$. Then show that $\mathfrak{L}$ is locally free of rank 1, and thus conclude that the map $\operatorname{Pic} \mathfrak{X} \to \varprojlim_n \operatorname{Pic} Y_n$ is surjective. Again be careful, because even though each $\mathscr{L}_n$ is locally free of rank 1 , the open sets needed to make them free might get smaller and smaller with n.

(d) Show that the hypothesis "$(\Gamma(Y_n, \mathcal{O}_{Y_n}))$ satisfies (ML)" is satisfied if either $\mathfrak{X}$ is affine, or each Y_n is projective over a field k. Note: See (III, Ex. 11.5-11.7) for further examples and applications.

PROOF. (a) By the Noetherian hypothesis, we can do this affine locally i.e. assume $Y := V(I)$, and $\mathcal{O}_{Y_{n+1}} = \widetilde{A/I^{n+1}}$ then the kernel of $A/I^{n+1} \to A/I^n$ is just I^n/I^{n+1} which is certainly nilpotent for $n \geq 1$.
(b)
(c)
(d) □

CHAPTER 3

Chapter III Exercises

1. Derived Functors

Hartshorne III.1.0 Take any book on homological algebra, and prove all the theorems without looking at the proofs given in that book. See [12]. More seriously, look at your notes on homological methods.

2. Cohomology of Sheaves

Hartshorne III.2.1 (a) Let $X = \mathbb{A}^1_k$ be the affine line over an infinite field k. Let P, Q be distinct closed points of X, and let $U = X - \{P, Q\}$. Show that $H^1(X, \mathbb{Z}_U) \neq 0$.

*(b) More generally, let $Y \subseteq X = \mathbb{A}^n_k$ be the union of $n+1$ hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X, \mathbb{Z}_U) \neq 0$. Thus the result of (2.7) is the best possible.

Proof. **(a)** Assume $H^1(X, \mathbb{Z}_U) = 0$. There is a SES of sheaves

$$0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_{P,Q} \to 0$$

which induce a LES in sheaf cohomology

$$0 \to \Gamma(X, \mathbb{Z}_U) \to \Gamma(X, \mathbb{Z}_X) \to \Gamma(X, \mathbb{Z}_{P,Q}) \to H^1(X, \mathbb{Z}_U) \to \dots$$

and if $H^1(X, \mathbb{Z}_U) = 0$ the sequence gives rise to a SES

$$0 \to \Gamma(X, \mathbb{Z}_U) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to 0.$$

This is impossible since there cannot be any surjection $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. Contradiction.

(b) Consider the SES of sheaves

$$0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_Y \to 0$$

and take the LES in sheaf cohomology

$$0 \to \Gamma(X, \mathbb{Z}_U) \to \Gamma(X, \mathbb{Z}_X) \to \Gamma(X, \mathbb{Z}_Y) \to H^1(X, \mathbb{Z}_U) \to \dots$$

and we examine in particular the part

$$\dots \to H^{n-1}(X, \mathbb{Z}_X) \to H^{n-1}(X, \mathbb{Z}_Y) \to H^n(X, \mathbb{Z}_U) \to H^n(X, \mathbb{Z}_X) \to \dots$$

Now $H^{n-1}(X, \mathbb{Z}_X) = H^n(X, \mathbb{Z}_X) = 0$ since $\mathbb{Z}_X$ is a flasque sheaf (we assume $n \geq 2$ since (a) dealt with the case of $n = 1$). This gives

$$0 \to H^{n-1}(X, \mathbb{Z}_Y) \to H^n(X, \mathbb{Z}_U) \to 0.$$

It suffices to show that $H^{n-1}(X, \mathbb{Z}_Y) \neq 0$. This computation is done in this post, but we present a solution not utilizing Čech cohomology. Indeed, we just construct the Čech complex directly and prove it is indeed a flasque resolution.

Let $Y = \bigcap_{i=1}^{n+1} H_i$.

First off, construct a flasque resolution for $\mathbb{Z}_Y$ as follows with $i : Y \hookrightarrow X$ the inclusion map. Now construct a resolution

$$0 \to \mathbb{Z}_Y \to \prod_{i_1<i_2} \mathbb{Z}_{H_{i_1}\cap H_{i_2}} \to \prod_{i_1<i_2<i_3} \mathbb{Z}_{H_{i_1}\cap H_{i_2}\cap H_{i_3}} \to \ldots \to \prod_{i_1<\cdots<i_n} \mathbb{Z}_{H_{i_1}\cap\cdots\cap H_{i_n}} \to 0$$

where the differentials is precisely the formula from Čech cohomology. In particular, one sends a section s in $\Gamma(U, \mathbb{Z}_{H_{i_1}\cap\cdots\cap H_{i_k}})$ to

$$(ds)_{H_{i_1}\cap\cdots\cap H_{i_{k+1}}} = \sum_{l=0}^{k+1} (-1)^k s_{i_0,\ldots,\widehat{i_l},\ldots,i_{k+1}}|_{H_{i_1}\cap\cdots\cap H_{i_{p+1}}}.$$

One can check at the level of stalks that this is always exact. Furthermore, the direct image of a flasque sheaf is flasque so this is a flasque resolution (also each of the intersections are irreducible).

Afterwards, we take global sections and drop $\mathbb{Z}_Y$ to get

$$0 \to \prod_{i_1<i_2} \mathbb{Z} \to \prod_{i_1<i_2<i_3} \mathbb{Z} \to \ldots \to \prod_{i_1<\cdots<i_n} \mathbb{Z} \to 0.$$

The $(n-1)$th cohomology is the cohomology at $\prod_{i_1<\cdots<i_n} \mathbb{Z}$. The cohomology taken here gives the simplicial homology of S^{n-1}. So the $(n-1)$th cohomology is nonzero i.e. $H^{n-1}(X, \mathbb{Z}_Y) \cong \mathbb{Z}$. Contradiction. □

Hartshorne III.2.2 Let $X = \mathbb{P}^1_k$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathscr{K} \to \mathscr{K}/\mathcal{O} \to 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

Proof. We break up the proof into multiple parts and recall that Hartshorne I.21(e) says $0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}) \to 0$ is exact.

Lemma 2.1. $\mathscr{K}$ is a flasque sheaf on X.

Proof. We know that $\mathbb{P}^1_k$ is irreducible and so Hartshorne I.1.16(a) applies to show $\mathscr{K}$ is flasque. □

Lemma 2.2. $\mathscr{K}/\mathcal{O}$ is a flasque sheaf on X.

Proof. We show $\mathscr{K}(U)/\mathcal{O}(U) \to \mathscr{K}(V)/\mathcal{O}(V)$ is surjective for all $V \subseteq U$. First, $\mathscr{K}(U) \to \mathscr{K}(V) \to \mathscr{K}(V)/\mathcal{O}(V)$. It suffices to show that $\mathcal{O}(U)$ is in the kernel of this composition.

First off, there is a map $\mathcal{O}(U) \to \mathcal{O}_\xi$ by definition of the stalk and ξ is the generic point. So there is a map $\mathcal{O}(U) \to \mathscr{K}(U) = \mathcal{O}_\xi$.

Second, $\mathscr{K}(U) \to \mathscr{K}(V)/\mathcal{O}(V)$ sends a section s to zero iff s was in the image of $\mathcal{O}(V) \to \mathscr{K}(V) = \mathcal{O}_\xi$. But there is a composition

$$(\mathcal{O}(U) \to \mathcal{O}_\xi) = (\mathcal{O}(U) \to \mathcal{O}(V) \to \mathcal{O}_\xi)$$

which shows that $s \in \mathcal{O}(U)$ lands in the image of $\mathcal{O}(V) \to \mathcal{O}_\xi$. So so quotienting by $\mathcal{O}(U)$ does not affect surjectivity. Therefore, $\mathscr{K}/\mathcal{O}$ is a flasque sheaf. □

Lemma 2.3. $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

PROOF. Using the LES of cohomology,

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathscr{K}) \to H^1(X, \mathscr{K}/\mathcal{O}) \to \dots$$

and combined with Hartshorne I.1.21(e), shows $H^i(X, \mathcal{O}) = 0$ for all $i > 0$. Indeed, we get a sequence (and using flasqueness),

$$0 \to H^1(X, \mathcal{O}) \to 0 \to 0 \to H^2(X, \mathcal{O}) \to 0 \to 0 \to \dots.$$

□

□

Hartshorne III.2.3 Cohomology with Supports (Grothendieck *Local Cohomology Lecture Notes by Robin Hartshorne*). Let X be a topological space, let Y be a closed subset, and let $\mathscr{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathscr{F})$ denote the group of sections of $\mathscr{F}$ with support[1] in Y (II, Ex. 1.20).

(a) Show that $\Gamma_Y(X, \cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$.

We denote the right derived functors of $\Gamma_Y(X, \cdot)$ by $H^i_Y(X, \cdot)$. They are the **cohomology groups of X with supports in Y**, and coefficients in a given sheaf.

(b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, with $\mathscr{F}'$ flasque, show that

$$0 \to \Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'') \to 0$$

is exact.

(c) Show that if $\mathscr{F}$ is flasque, then $H^i_Y(X, \mathscr{F}) = 0$ for all $i > 0$.

(d) If $\mathscr{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X - Y, \mathscr{F}) \to 0$$

is exact.

(e) Let $U = X - Y$. Show that for any $\mathscr{F}$, there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 \to H^0_Y(X, \mathscr{F}) \to H^0(X, \mathscr{F}) \to H^0(U, \mathscr{F}|_U) \to \\ \to H^1_Y(X, \mathscr{F}) \to H^1(X, \mathscr{F}) \to H^1(U, \mathscr{F}|_U) \to \\ \to H^2_Y(X, \mathscr{F}) \to \dots \end{aligned}$$

(f) *Excision.* Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathscr{F}$,

$$H^i_Y(X, \mathscr{F}) \cong H^i_Y(V, \mathscr{F}|_V)$$

[1]The support $\operatorname{Supp} s$ of a section is the set of points P s.t. $s_P \neq 0$.

PROOF. **(a)** Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$ be an exact sequence of sheaves. Then take global sections with support $\Gamma_Y(X,\mathscr{F}) \xrightarrow{\phi} \Gamma_Y(X,\mathscr{G}) \to \Gamma_Y(X,\mathscr{H})$. We claim that ϕ is injective.

Let $s \in \Gamma_Y(X,\mathscr{F})$ and suppose $\phi(s) = 0$. It follows that the germs $\phi(s)_P \in \mathscr{F}_P$ are all zero and hence, there are open sets U_P and sections t_P that are zero on U_P s.t. $\phi_P(s_P) = t_P$. But ϕ_P is necessarily injective so that means $s_P = 0$ on U_P. As collection $\{U_P\}$ covers all of X, it follows that $s = 0$.

Alternatively, one can use left exactness of $\Gamma(X,-)$ in the following way. First, if $\phi(s) = 0$, we may view s as a section of $\Gamma(X,\mathscr{F})$ to deduce that $s = 0$. Then we verify that s has support in Y. If $P \notin Y$, we know that $0 = \phi(s)_P = \phi_P(s_P)$ and by injectivity of $\phi_P : \mathscr{F}_P \to \mathscr{G}_P$, we deduce that $s_P = 0$. So $\operatorname{Supp} s \subseteq Y$.

(b) The inclusion $\Gamma_Y(X,\mathscr{F}') \to \Gamma_Y(X,\mathscr{F})$ is clear since $\mathscr{F}' \to \mathscr{F}$ is injective. Indeed, a section $s \in \Gamma_Y(X,\mathscr{F}')$ goes to zero in $\Gamma(X,\mathscr{F})$ only if it went to zero under the map $\Gamma(X,\mathscr{F}') \to \Gamma(X,\mathscr{F})$.

Now we check surjectivity on the right. By Hartshorne II.1.6(b), we know that $\Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{F}'')$ is surjective. So if $s \in \Gamma_Y(X,\mathscr{F}'')$, we can lift to a $t \in \Gamma(X,\mathscr{F})$ and we need only show that the support of t is in Y.

Let $P \in X$ s.t. $t_P \neq 0$. Then t_P maps to s_P and $s_P \neq 0$ since s has support in Y. So t's support is in Y.

Now we prove exactness in the middle. Assume $t \in \Gamma_Y(X,\mathscr{F})$ goes to $0 \in \Gamma_Y(X,\mathscr{F}'')$. Then Hartshorne II.1.16(c) says there is an $r \in \Gamma(X,\mathscr{F}')$ mapping to t. Now $r_P \neq 0$ iff $t_P \neq 0$ by injectivity. So the support of r is contained in the support of t which is contained in Y.

(c) We can repeat the proof of Hartshorne Proposition II.2.5 and then use part (b) above in place of Hartshorne II.1.16(b).

(d) [Incomplete]

(e) Let $\mathscr{I}^\bullet$ be an flasque resolution of $\mathscr{F}$. Then we obtain a SES of chain cochain complexes

$$0 \to \Gamma_Y(X,\mathscr{I}^\bullet) \to \Gamma(X,\mathscr{I}^\bullet) \to \Gamma(X-Y,\mathscr{I}^\bullet) \to 0$$

and taking the LES of cohomology gives the desired LES of cohomology groups.

(f) For any sheaf $\mathscr{F}$, observe that $\Gamma_Y(X,\mathscr{F}) = \Gamma_Y(V,\mathscr{F}|_V)$ for any open subset $V \subseteq Y$. So any injective resolution $\mathscr{I}^\bullet$ of $\mathscr{F}$ will give rise to two equal chain cochain complexes

$$\Gamma_Y(X,\mathscr{I}^\bullet) = \Gamma_Y(V,\mathscr{I}^\bullet|_V).$$

Hence, the cohomology groups are isomorphism and as everything here is functorial, the natural functorial isomorphisms indicated exist. □

Hartshorne III.2.4 ***Mayer-Vietoris Sequence.*** Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\begin{aligned}\ldots \to H^i_{Y_1\cap Y_2}(X,\mathscr{F}) \to H^i_{Y_1}(X,\mathscr{F}) \oplus H^i_{Y_2}(X,\mathscr{F}) \to H^i_{Y_1\cup Y_2}(X,\mathscr{F}) \to \\ \to H^{i+1}_{Y_1\cap Y_2}(X,\mathscr{F}) \to \ldots\end{aligned}$$

PROOF. Let $\mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution for $\mathscr{F}$. We show there is an exact sequence of chain complexes constructed from sequences of form

$$0 \to \Gamma_{Y_1 \cap Y_2}(X, \mathscr{I}^i) \to \Gamma_{Y_1}(X, I^i) \oplus \Gamma_{Y_2}(X, \mathscr{I}^i) \to \Gamma_{Y_1 \cup Y_2}(X, \mathscr{I}^i) \to 0.$$

The LES then exists from taking the LES via the Snake Lemma. We reduce to showing the above sequence is exact with maps $s \mapsto s \oplus s$ and $(s,t) \mapsto s - t$ respectively. Injectivity is clear by consider the germs inside the stalks.

Now we check surjectivity. Let $s \in \Gamma_{Y_1 \cup Y_2}(X, \mathscr{I})$ (dropping the superscript i). Now extend s by zero outside of $Y_1 \cup Y_2$ to get an $s \in \Gamma(X, \mathscr{I})$.

Consider $s|_{Y_1} \in \Gamma(Y_1, \mathscr{I})$. Extend this section outside by zero to get a section $t_1 \in \Gamma_{Y_1}(X, \mathscr{I})$. Similarly, let t_2 be the extension of $(-s)|_{Y_2} \in \Gamma(Y_2, \mathscr{I})$ by zero outside of Y_2. Then $t_1 \oplus t_2$ is in the middle term, but $t_1 - t_2 = s$ as elements in $\Gamma_{Y_1 \cup Y_2}(X, \mathscr{I}^i)$. □

Hartshorne III.2.5 Let X be a Zariski space (II, Ex. 3.17). Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \{Q\}^-$. We call X_P the local space of X at P, and give it the induced topology. Let $j : X_P \to X$ be the inclusion, and for any sheaf $\mathscr{F}$ on X, let $\mathscr{F}_P = j^*\mathscr{F}$. Show that for all $i, \mathscr{F}$, we have

$$H^i_P(X, \mathscr{F}) = H^i_P\left(X_P, \mathscr{F}_P\right).$$

PROOF. Consider the global sections $\Gamma_P(X, \mathscr{F})$ and $\Gamma_P(X_P, \mathscr{F}_P)$.

Any open set $U \supseteq P$ contains X_P. Indeed, $P \in \overline{\{Q\}}$ and as open sets are stable under generization, $Q \in U$.

So there is an isomorphism of global sections

$$\Gamma(X_P, \mathscr{F}_P) \cong \varinjlim_{U \ni P} \Gamma(U, \mathscr{F}).$$

Given a section $s \in \Gamma_P(X_P, \mathscr{F}_P)$. We can view s as a section represented by a collection $\{(U_i, s_i)\}$ where $U_i \subseteq X_P$ is open and $s_i \in \Gamma_P(U_i, \mathscr{F}_P) \subseteq \mathscr{F}_P(U_i) = \mathscr{F}(j^{-1}(U_i)) = \mathscr{F}(U_i)$. Indeed, we are simply viewing s as an element of $\Gamma(U_i, \mathscr{F})$ for some U_i and $s|_{U_i} = 0$ if $P \notin U_i$. Now by definition, s is supported on P. So glue s with $0 \in \Gamma(X - P, \mathscr{F})$ to get a new section $s' \in \Gamma(X, \mathscr{F})$. Because s' is still supported on P, $s' \in \Gamma_P(X, \mathscr{F})$.

Clearly, each section of $\Gamma_P(X_P, \mathscr{F})$ gives rise to a section $\Gamma_P(X_P, \mathscr{F}_P)$.

Lemma 2.4. We have a natural bijection

$$\Gamma_P(X_P, \mathscr{F}) \leftrightarrow \Gamma_P(X_P, \mathscr{F}_P).$$

PROOF. Clearly, the bijection we define can be upgraded to an isomorphism of abelian groups.

It suffices to show that the map $\Phi : \Gamma_P(X_P, \mathscr{F}_P) \to \Gamma_P(X, \mathscr{F})$ is inverse to the restriction $\Psi : \Gamma_P(X, \mathscr{F}) \to \Gamma_P(X_P, \mathscr{F}_P)$. In particular, Ψ is defined by taking a section s on the RHS and restricting it to X_P.

Given a section $s \in \Gamma_P(X_P, \mathscr{F}_P)$. Take $\Phi(s) = s'$ is as above. □

Now any flasque resolution

$$0 \to \mathscr{F} \to \mathscr{J}_0 \to \mathscr{J}_1 \to \dots$$

corresponds to an flasque resolution

$$0 \to \mathscr{F}_P \to \mathscr{J}_{0,P} \to \mathscr{J}_{1,P} \to \dots$$

as in the proof of Hartshorne Lemma II.2.10. Taking global sections are using the argument above to show there is a bijection $\Gamma_P(X, I_i) \leftrightarrow \Gamma_P(X_P, \mathscr{I}_{i,P})$ and hence, the cohomology are the same. □

Hartshorne III.2.6 Let X be a noetherian topological space, and let $\{\mathscr{I}_\alpha\}_{\alpha\in A}$ be a direct system of injective sheaves of abelian groups on X. Then $\varinjlim \mathscr{I}_\alpha$ is also injective.

[*Hints*: First show that a sheaf $\mathscr{I}$ is injective if and only if for every open set $U \subseteq X$, and for every subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$, and for every map $f : \mathscr{R} \to \mathscr{I}$, there exists an extension of f to a map of $\mathbb{Z}_U \to \mathscr{I}$. Secondly, show that any such sheaf $\mathscr{R}$ is finitely generated, so any map $\mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ factors through one of the $\mathscr{I}_\alpha$.]

Proof.

Lemma 2.5. $\mathscr{I}$ is injective iff for every open set $U \subseteq X$, for every subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$, for every map $f : \mathscr{R} \to \mathscr{I}$, there exists an extension of f to a map $\mathbb{Z}_U \to \mathscr{I}$.

Proof 1. **[Incomplete]** □

Proof 2. ($\Rightarrow$): Assume $\mathscr{I}$ is injective. Let $U \subseteq X$ and $\mathscr{R} \subseteq \mathbb{Z}_U$. There is always a surjective map $\mathbb{Z}_U \to \mathscr{R}$ and applying $\mathrm{Hom}(-, \mathscr{I})$, we get a surjection

$$\mathrm{Hom}(\mathscr{R}, \mathscr{I}) \twoheadrightarrow \mathrm{Hom}(\mathbb{Z}_U, \mathscr{I}).$$

This proves statement about extensions.

($\Leftarrow$): **[Incomplete]** □

Lemma 2.6. Any such sheaf $\mathscr{R}$ is f.g., so any map $\mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ factors through one of the $\mathscr{I}_\alpha$.

Proof. We shall take the definition that $\mathscr{R}$ is finitely generated iff it is a direct sum of copies of $\mathbb{Z}_U$. In this case, this is just a consequence of the fact that $\mathbb{Z}$ is a Noetherian $\mathbb{Z}$-module and say $\bigoplus_{i=1}^r \mathbb{Z}_U \cong \mathscr{R}$.

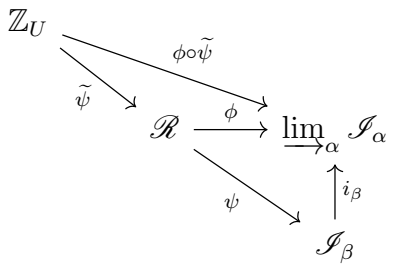
Now any map $\phi : \mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ is determined by where it sends each generator $s_i \in \Gamma(V, \mathbb{Z}_U)$ to an element in $\varinjlim \mathscr{I}_\alpha(V)$ for $V \subseteq X$ open. Of course, there will be some $\mathscr{I}_\beta$ in which each $s_1, \ldots, s_r$ maps into and that determines the map. It follows that ϕ factors through $\mathscr{I}_\beta$ to get a commutative diagram

$$\begin{array}{ccc} \mathscr{R} & \xrightarrow{\phi} & \varinjlim_\alpha \mathscr{I}_\alpha \\ & \searrow^{\psi} & \uparrow{\scriptstyle i_\beta} \\ & & \mathscr{I}_\beta \end{array}$$

□

Now we can finish the proof. Return to the situation in the proof above. Then we can apply Lemma 2.5 to get the extension

Apply the previous lemma to get an extension



and also by Lemma 2.5, $\varinjlim \mathscr{I}_\alpha$ is an injective sheaf.

Remark 45. Note this discussion here.

□

Hartshorne III.2.7 Let S^1 be the circle (with its usual topology), and let $\mathbb{Z}$ be the constant sheaf $\mathbb{Z}$.

(a) Show that $H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$, using our definition of cohomology.

(b) Now let $\mathscr{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1, \mathscr{R}) = 0$.

Proof. **(a)** We will construct an injective resolution

$$0 \to \mathbb{Z} \to I_0 \to I_1 \to I_2 \to I_3 \to \dots$$

and specify I_0, I_1, I_2 as follows. Define

$$I_0 := \prod_{x \in S^1} (j_x)_*(\mathbb{Q})$$

where $\mathbb{Q}$ is the constant sheaf on S^1. Then, define the injective sheaf

$$I_1 := \prod_{x \in S^1} (j_x)_*(\mathbb{Q}/\mathbb{Z})$$

where $\mathbb{Q}$ is the stalk $I_{0,P}$. Then set

$$I_2 := \prod_{x \in S^1} (j_x)_*(I_{1,x}/I_{0,x}).$$

Observe that we do not obtain an exact sequence. In particular, we only obtain a chain cocomplex because **[Incomplete]**

Now consider the complex

$$0 \to \Gamma(S^1, I_0) \xrightarrow{d_0} \Gamma(S^1, I_1) \xrightarrow{d_1} \Gamma(S^1, I_2).$$

We can describe the image and kernels

$$\ker d_1 := \{s : S^1 \to \coprod_{x \in S^1} \mathbb{Q}/\mathbb{Z} : \mathbb{Q}\text{-valued functions modulo which are locally constant}\}$$

$$\operatorname{im} d_0 := \{s : S^1 \to \coprod_{x \in S^1} \mathbb{Q}/\mathbb{Z} : \mathbb{Q}\text{-valued functions modulo constant } \mathbb{Z}\text{-valued functions}\} = 0.$$

It follows that the quotient is $\mathbb{Z}$ since the classes of maps can differ by an integer.

Remark 46. If $j_x : \{x\} \hookrightarrow X$ is the inclusion and $\mathbb{Q}$ is the constant sheaf on X, recall from Hartshorne II.1.17 that the skyscrapper sheaf $(j_x)_*(\mathbb{Q})$ has nonzero stalk only at x and the stalk there is $\mathbb{Q}$.

The global sections $\Gamma(X, (j_x)_*(\mathbb{Q})) = \{0, s_x\}$ where s_x is the function $s_x : X \to \coprod_{x\in X}((j_x)_*(\mathbb{Q}))_x$. So the function is $s_x(y) = 0$ if $y \neq x$ (assuming here that all points of X are closed) and $s_x(x) \in \mathbb{Q}$ for some choice.

A useful reference for reading can be found here.

[Incomplete]

(b) Let $\mathscr{R}$ be the sheaf of continuous real-valued functions on S^1 and $\mathscr{F}$ be the sheaf of all real-valued functions on S^1. There is a SES

$$0 \to \mathscr{R} \to \mathscr{F} \to \mathscr{F}/\mathscr{R} \to 0$$

which induces a LES

$$0 \to \Gamma(S^1, \mathscr{R}) \to \Gamma(S^1, \mathscr{F}) \xrightarrow{\phi} \Gamma(S^1, \mathscr{F}/\mathscr{R}) \to H^1(S^1, \mathscr{R}) \to 0$$

since the sheaf $\mathscr{F}$ is flasque. To prove that $H^1(S^1, \mathscr{R})$, we must show that ϕ is surjective.

Let $s \in \Gamma(S^1, \mathscr{F}/\mathscr{R})$. Then s can be represented by $\{(U_i, s_i)\}$ where $s_i \in \Gamma(U_i, \mathscr{F})$ and $s_i - s_j \in \Gamma(S^1, \mathscr{R})$. Since S^1 is compact, we may assume there are finitely many $s_1, \dots, s_N$ and that $U_i \cap U_j \cap U_k = \emptyset$ for any triple i, j, k.

Let $r_i := s_{i+1} - s_i \in \Gamma(U_{i+1} \cap U_i, \mathscr{R})$ and continue it by zero to all of U_i to get an element $r_i \in \Gamma(U_i, \mathscr{F})$. So we have sections $t_1, t_2, \dots, t_n$ where $s_{n+1} = s_1$.

Now define a new collection $\{(U_i, r_i)\}$ where $t_i = s_i + r_i$ for $1 \leq i \leq n$ and $r_i \in \Gamma(U_i, \mathscr{F})$. Observe that

$$t_i|_{U_{i+1}\cap U_i} = (s_i + s_{i+1} - s_i)|_{U_{i+1}\cap U_i} = s_{i+1}|_{U_{i+1}\cap U_i} = t_{i+1}|_{U_{i+1}\cap U_i}.$$

The last equality is justified as follows. Write

$$t_{i+1} = s_{i+1} + r_{i+1} = (s_{i+1} + s_{i+2} - s_{i+1})|_{U_{i+1}\cap U_i} = (s_{i+1} - s_{i+1})|_{U_{i+1}\cap U_i} = s_{i+1}|_{U_{i+1}\cap U_i}$$

because the second s_{i+1} was a priori just defined on $U_{i+2} \cap U_{i+1}$ and then continued by zero to all of U_{i+1}. But $(U_{i+2} \cap U_{i+1}) \cap (U_{i+1} \cap U_i) = \emptyset$ which means this second s_{i+1} is evaluated to zero.

Gluing together these r_i's, there is a $r \in \Gamma(S^1, \mathscr{F})$ and it maps to itself because $r_{i+1} - r_i = (s_{i+1} - s_i) + (t_{i+1} - s_i)$ which is a sum of two continuous functions on U_i.

Now define a new function

$$r' := \begin{cases} r_i & U_{i+1} \cap U_i \\ 0 & \end{cases}.$$

Now the desired function $t - r' \in \Gamma(S^1, \mathscr{F})$ maps to s under ϕ. Indeed,

$$\phi(t - r')|_{U_i} = t|_{U_i} - r'|_{U_i} = s_i|_{U_i} + r_i|_{U_i} - r'|_{U_i} = s_i|_{U_i} + 0|_{U_i} + 0|_{U_i} = s_i|_{U_i}.$$

Apply identity axiom of sheaves to deduce that $\phi(t - r') = s$.

Remark 47. Let $\mathscr{R}$ be the sheaf of germs of continuous real-valued functions on X. If one examines our proof, the only property we used was that X is compact and that we can make the cover consist of sets that have empty triple intersection. Hence,

$$H^1(X, \mathscr{R}) = 0$$

for said spaces. In particular, we can do this for $X = S^1$ and $X = D^1$.

A more general fact is that $\mathscr{R}$ is a soft sheaf on S^n and D^n for $n \geq 1$. So their cohomology vanish.

□

3. Cohomology of Noetherian Affine Schemes

Hartshorne III.3.1 Let X be a noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine.

[*Hint*: Use (3.7), and for any coherent sheaf $\mathscr{F}$ on X, consider the filtration $\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \ldots$, where $\mathscr{N}$ is the sheaf of nilpotent elements on X.]

Proof. The fact that X is affine implies X_{red} is affine is clear (if $X = \operatorname{Spec} A$, then $X_{red} = \operatorname{Spec} A/N$ where N is the nilradical).

Now suppose X_{red} is affine. Let $\mathscr{F}$ be a coherent sheaf of ideals on X. We start with proving the hint.

Lemma 3.1. If $\mathscr{F}$ just a coherent sheaf on X, there exists a finite filtration

$$\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \ldots \supseteq \mathscr{N}^{k+1} \cdot \mathscr{F} = 0.$$

Proof. This follows from the fact that X is a Noetherian scheme. Indeed, if $\mathscr{N}$ is the sheaf of nilpotents, then we know $\mathscr{N}^{k+1} \cdot \mathscr{F} = 0$ iff this occurs when restricted over each open affine subset $U_i = \operatorname{Spec} A_i$ (which we may assume there are only finitely many by Noetherian) and this is iff $\widetilde{N^{k+1}M} = 0$ for N the ideal of nilpotents and M an A-module where A is a Noetherian ring. Of course, this is certainly true when k is sufficiently large since M and N are finitely generated. □

Now we prove the result via induction. Our goal is to show that $H^1(X, \mathscr{F}) = 0$ so that Serre's Affine Criterion applies. Let us climb the filtration. Take the SES

$$0 \to \mathscr{N}^{i+1}\mathscr{F} \to \mathscr{N}^i\mathscr{F} \to \mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F} \to 0$$

and get a LES

$$\begin{aligned} 0 \to H^0(X, \mathscr{N}^{i+1}\mathscr{F}) \to H^0(X, \mathscr{N}^i\mathscr{F}) \to H^0(X, \mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F}) \\ \to H^1(X, \mathscr{N}^{i+1}) \to H^1(X, \mathscr{N}^i\mathscr{F}) \to H^1(X, \mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F}) \\ \to \ldots \end{aligned}$$

Now if $i = k$, then $H^j(X, \mathscr{N}^{i+1}\mathscr{F}) = 0$ for all $j > 0$. From which, we can extract the sequence

$$0 \to H^1(X, \mathscr{N}^i\mathscr{F}) \to H^1(X, \mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F}) \to 0.$$

If we can show that $H^1(X, \mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F}) = 0$ for any i, then we have shown the vanishing of $H^1(X, \mathscr{N}^{i+1}\mathscr{F})$ implies the vanishing of $H^1(X, \mathscr{N}^i\mathscr{F})$. Afterwards, we use induction to deduce that $H^1(X, \mathscr{F}) = 0$.

Lemma 3.2. $H^1(X, \mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F}) = 0$ for any i.

Proof. The sheaf $\mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F}$ is a well-defined coherent sheaf of ideals on X_{red}. Apply Serre's Affine Criterion to X_{red} to deduce that the first cohomology is trivial (here, we use the fact that X_{red} and X are topologically the same and cohomology only observes the topological space structure). □

□

Hartshorne III.3.2 Let X be a reduced noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

PROOF. ($\Rightarrow$). Let $X = \operatorname{Spec}(A)$. Serre's Affine Criterion (Hartshorne Theorem III.3.7) says that $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent sheaves and $i > 0$. If $Z \subseteq X$ is an irreducible component, then by[2] Hartshorne Lemma II.2.10,

$$H^i(Y, \mathscr{F}) = H^i(X, j_*\mathscr{F}) = 0 \text{ for all } i > 0 \text{ and all quasi-coherent sheaves } \mathscr{F}.$$

Another application of Serre's Affine Criterion shows Z is affine.

($\Leftarrow$). Let $X_1, \dots, X_m$ be the irreducible components and set $X_i = \operatorname{Spec}(A_i)$. Let $j_i : X_i \hookrightarrow X$ be the inclusions.

Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Serre's Affine Criterion says $H^k(X_i, \mathscr{F}) = 0$ for all $k > 0$.

Let $\mathscr{I}_1, \dots, \mathscr{I}_m$ be the ideal sheaves of the irreducible components $X_1, \dots, X_m$. Then we can filter

$$\mathscr{F} \supseteq \mathscr{I}_1\mathscr{F} \supseteq \mathscr{I}_1\mathscr{I}_2\mathscr{F} \supseteq \cdots \supseteq \mathscr{I}_1\mathscr{I}_2 \dots \mathscr{I}_m\mathscr{F} \supseteq 0.$$

Now we do reverse induction. The base case is

$$H^k(X, \mathscr{I}_1\mathscr{I}_2 \dots \mathscr{I}_m\mathscr{F}) = H^k(\emptyset, j_*\mathscr{I}_1\mathscr{I}_2 \dots \mathscr{I}_m\mathscr{F}) = 0$$

for all $k \geq 0$ where $j : \emptyset \hookrightarrow X$ is the inclusion. Now for every $0 \leq i \leq m-1$

$$0 \to \mathscr{I}_1 \dots \mathscr{I}_i\mathscr{F} \to \mathscr{I}_1 \dots \mathscr{I}_{i+1}\mathscr{F} \to \frac{\mathscr{I}_1 \dots \mathscr{I}_{i+1}\mathscr{F}}{\mathscr{I}_1 \dots \mathscr{I}_i\mathscr{F}} \to 0.$$

This induces a LES of cohomology

$$\begin{aligned}
0 \to & H^0(X, \mathscr{I}_1 \dots \mathscr{I}_i\mathscr{F}) \to H^0(X, \mathscr{I}_1 \dots \mathscr{I}_{i+1}\mathscr{F}) \to H^0\left(X, \frac{\mathscr{I}_1 \dots \mathscr{I}_{i+1}}{\mathscr{I}_1 \dots \mathscr{I}_i}\mathscr{F}\right) \\
\to & H^1(X, \mathscr{I}_1 \dots \mathscr{I}_i\mathscr{F}) \to H^1(X, \mathscr{I}_1 \dots \mathscr{I}_{i+1}\mathscr{F}) \to H^1\left(X, \frac{\mathscr{I}_1 \dots \mathscr{I}_{i+1}}{\mathscr{I}_1 \dots \mathscr{I}_i}\mathscr{F}\right) \\
\to & H^2(X, \mathscr{I}_1 \dots \mathscr{I}_i\mathscr{F}) \to H^2(X, \mathscr{I}_1 \dots \mathscr{I}_{i+1}\mathscr{F}) \to H^2\left(X, \frac{\mathscr{I}_1 \dots \mathscr{I}_{i+1}}{\mathscr{I}_1 \dots \mathscr{I}_i}\mathscr{F}\right) \\
\to & \dots
\end{aligned}$$

and the inductive hypotheses says there are isomorphisms

$$H^k(X, \mathscr{I}_1 \dots \mathscr{I}_{i+1}\mathscr{F}) \cong H^{k+1}(X, \frac{\mathscr{I}_1 \dots \mathscr{I}_{i+1}}{\mathscr{I}_1 \dots \mathscr{I}_i}\mathscr{F}).$$

But the sheaf on the RHS is a sheaf X_{i+1} so this is isomorphic to

$$H^{k+1}(X, \frac{\mathscr{I}_1 \dots \mathscr{I}_{i+1}}{\mathscr{I}_1 \dots \mathscr{I}_i}\mathscr{F}) = H^{k+1}(X_i, j_*\frac{\mathscr{I}_1 \dots \mathscr{I}_{i+1}}{\mathscr{I}_1 \dots \mathscr{I}_i}\mathscr{F})$$

where $j : X_i \hookrightarrow X$. This allows us to climb the tower of filtration's to get that $H^k(X, \mathscr{F}) = 0$ for all $k > 0$. Serre's Affine Criterion applies. □

[2]As a reminder, Hartshorne Proposition II.5.8 implies $j_*\mathscr{F}$ is also quasi-coherent whenever $\mathscr{F}$ is.

Hartshorne III.3.3 Let A be a noetherian ring, and let $\mathfrak{a}$ be an ideal of A.

(a) Show that $\Gamma_{\mathfrak{a}}(\cdot)$ (II, Ex. 5.6) is a left-exact functor from the category of A-modules to itself. We denote its right derived functors, calculated in $\mathfrak{Mod}(A)$, by $H^i_{\mathfrak{a}}(\cdot)$.

(b) Now let $X = \operatorname{Spec} A, Y = V(\mathfrak{a})$. Show that for any A-module M,

$$H^i_{\mathfrak{a}}(M) = H^i_Y(X, \widetilde{M}),$$

where $H^i_Y(X, \cdot)$ denotes cohomology with supports in Y (Ex. 2.3).

(c) For any i, show that $\Gamma_{\mathfrak{a}}\left(H^i_{\mathfrak{a}}(M)\right) = H^i_{\mathfrak{a}}(M)$.

Proof. **(a)** Let $0 \to N \to^{\phi} M \to^{\psi} L \to 0$ be a SES of A-modules. Apply $\Gamma_{\mathfrak{a}}(-)$ to get

$$0 \to \Gamma_{\mathfrak{a}}(N) \xrightarrow{\phi} \Gamma_{\mathfrak{a}}(M) \xrightarrow{\psi} \Gamma_{\mathfrak{a}}(L).$$

Clearly the sequence is a complex and ϕ is injective. All that remains is showing $\ker(\psi) \subseteq \operatorname{im}(\phi)$. Suppose $\psi(m) = 0$. Then the original sequence says there is an $n \in N$ s.t. $\phi(n) = m$. Choose k s.t. $\mathfrak{a}^k m = 0$ and then $\mathfrak{a}^k \phi(n) = 0$ and injectivity of ϕ says $\mathfrak{a}^k n = 0$ meaning $n \in \Gamma_{\mathfrak{a}}(N)$.

(b) From the hint in Hartshorne II.5.6(d), we know that $\Gamma_{\mathfrak{a}}(M) \cong \Gamma_Y(X, \widetilde{M})$. So they agree for $i = 0$.

To compute $H^i_{\mathfrak{a}}(M)$, we take an injective resolution

$$0 \to M \to I^0 \to I^1 \to \dots$$

and apply $\Gamma_{\mathfrak{a}}(-)$ to get a sequence

$$0 \to \Gamma_{\mathfrak{a}}(I^0) \to \Gamma_{\mathfrak{a}}(I^1) \to \Gamma_{\mathfrak{a}}(I^2) \to \dots. \tag{30}$$

OTOH, if we apply the exact functor to the injective resolution, then there is a flasque resolution

$$0 \to \widetilde{I^0} \to \widetilde{I^1} \to \widetilde{I^2} \to \dots$$

of $\widehat{M}$ and if we apply $\Gamma_Y(X, -)$, we get

$$0 \to \Gamma_Y(X, \widetilde{I^0}) \to \Gamma_Y(X, \widetilde{I^1}) \to \Gamma_Y(X, \widetilde{I^2}) \to \dots \tag{31}$$

which is the same sequence as that in (30). Indeed, we would only need to show that

$$\Gamma_{\mathfrak{a}}(M) = \Gamma_Y(X, \widetilde{M})$$

for any A-module M. Writing out the definition of the two sides, we only need to show

$$\{m \in M : \mathfrak{a}^n m = 0 \text{ for some } n > 0\} = \{m \in \Gamma(X, \widetilde{M}) : \operatorname{Supp} m \subseteq Y\}$$

For this to hold, we will use the fact that A is Noetherian. Let $\mathfrak{a} = (f_1, \dots, f_r)$. Suppose $m \in M$ s.t. $\mathfrak{a}^n m = 0$ for some n. Then for all i, there is an $n_i > 0$ s.t. $f_i^{n_i} m = 0$. This says that $m \in A_{\mathfrak{p}}$ is zero for all primes $\mathfrak{p}$ s.t. $f_i \in A - \mathfrak{p}$. This equivalently means that $m \in A_{\mathfrak{p}}$ is zero for all primes $\mathfrak{p} \notin V(\mathfrak{a})$. So $\operatorname{Supp} m \subseteq Y$. The other containment is also clear. Taking cohomology, we get

$$H^i_{\mathfrak{a}}(M) = H^i_Y(X, \widetilde{M}).$$

(c) We study the case of $i = 0$ to understand the proof for arbitrary i. In this case, $H^0_{\mathfrak{a}}(M) = H^0_Y(X, \widetilde{M}) = \Gamma_Y(X, \widetilde{M}) = \Gamma_{\mathfrak{a}}(M)$. Of course, $\Gamma_{\mathfrak{a}}(-)$ of this is equal to itself. In the general case, we look at the cochain complex

$$0 \to \Gamma_{\mathfrak{a}}(I^0) \to \ldots \to \Gamma_{\mathfrak{a}}(I^{i-1}) \overset{\phi_{i-1}}{\to} \Gamma_{\mathfrak{a}}(I^i) \overset{\phi_i}{\to} \Gamma_{\mathfrak{a}}(I^{i+1}) \overset{\phi_{i+1}}{\to} \ldots$$

Then $H^i_{\mathfrak{a}}(M) = \ker(\phi_i)/\operatorname{im}(\phi_{i-1})$. Take $\Gamma_{\mathfrak{a}}(-)$ of this. The elements are then those $m + \operatorname{im}(\phi_{i-1})$ s.t. $\mathfrak{a}^n m \in \operatorname{im}(\phi_{i-1})$ for some $n > 0$. Of course, we know each $m \in \Gamma_{\mathfrak{a}}(I^i)$ so $\mathfrak{a}^l m = 0$ for some $l > 0$. So $\Gamma_{\mathfrak{a}}(\ker(\phi_i)/\operatorname{im}(\phi_{i-1})) = \ker(\phi_i)/\operatorname{im}(\phi_{i-1})$ as desired. □

Hartshorne III.3.4 *Cohomological Interpretation of Depth.* If A is a ring, $\mathfrak{a}$ an ideal, and M an A module, then depth M is the maximum length of an M-regular sequence $x_1, \ldots, x_r$, with all $x_i \in \mathfrak{a}$. This generalizes the notion of depth introduced in (II,§8).

(a) Assume that A is noetherian. Show that if $\operatorname{depth}_{\mathfrak{a}} M \geq 1$, then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated.

[*Hint*: When M is finitely generated, both conditions are equivalent to saying that $\mathfrak{a}$ is not contained in any associated prime of M.]

(b) Show inductively, for M finitely generated, that for any $n \geq 0$, the following conditions are equivalent:

(i) $\operatorname{depth}_{\mathfrak{a}} M \geq n$;

(ii) $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$.

For more details, and related results, see Grothendieck [7].

Proof. **(a) First Statement.** Suppose not and $m \in \Gamma_{\mathfrak{a}}(M) \subseteq M$ is a nonzero element. Then $\mathfrak{a}^l m = 0$ for some $l > 0$. If $x_1, \ldots, x_r$ is an M-regular sequence with $x_i \in \mathfrak{a}$, then that means x_1 is not a zero-divisor inside M. But $\mathfrak{a}^l m = 0$ implies $x_1^l m = 0$. Of course, that means $x_1(x_1)^{l-1} m = 0$. In either situation, x_1 is a zero-divisor inside M. So the M-regular sequence cannot exist to being with. Contradiction.

Second Statement. Since M is f.g., then M is Noetherian.

We shall prove the contrapositive i.e. $\operatorname{depth}_{\mathfrak{a}} M < 1$ implies $\Gamma_{\mathfrak{a}}(M) \neq 0$. The assumption means every element of $\mathfrak{a}$ is a zerodivisor of M. So if $\mathfrak{a} = (y_1, \ldots, y_r)$, choose $m_1, \ldots, m_r$ s.t. $y_i m_i = 0$. Consider the element $m_1 + m_2 + \cdots + m_r$.

Suppose $\Gamma_{\mathfrak{a}}(M) = 0$. Suppose not and $\operatorname{depth}_{\mathfrak{a}}(M) < 1$. Then every element of $\mathfrak{a}$ is a zerodivisor in M. Choose $y_1, \ldots, y_r$ generators of $\mathfrak{a}$ and then $y_i m_i = 0$ for some $m_i \in M$.

Suppose $\Gamma_{\mathfrak{a}}(M) = 0$ and $M = (m_1, \ldots, m_r)$. Choose $n_i > 0$ large enough so that $\mathfrak{a}^{n_i} m_i = 0$. Then set n to be the maximum of these n_i so that $\mathfrak{a}^n m = 0$ for all $m \in M$.

First off, $\mathfrak{a} m_i \neq 0$ for each m_i. So choose an $x_i \in \mathfrak{a}$ s.t. $x_i m_i \neq 0$.

(b) We do induction for both directions. Let $M = (m_1, \ldots, m_r)$ and the base case of $n = 1$ was dealt with in (a) while the case of $n = 0$ is a vacuous base case.

($\Rightarrow$). Assume we knew the result for an $n \geq 1$. We wish to prove the result for $n + 1$. Assume $\operatorname{depth}_{\mathfrak{a}} M \geq n + 1$. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$. We want to show $H^n_{\mathfrak{a}}(M) = 0$. Choose a maximal regular sequence $x_1, \ldots, x_r$ in $\mathfrak{a}$ where $r = \operatorname{depth}_{\mathfrak{a}} M$. Then there is a SES $0 \to M \overset{\cdot x_1}{\to} M \to M/x_1 M \to 0$. From definition $\operatorname{depth}_{\mathfrak{a}} M/x_1 M \geq n$. Taking the LES of cohomology, we can extract the sequence

$$0 \to H^{n-1}_{\mathfrak{a}}(M/x_1 M) \to H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M/x_1 M) \to \ldots$$

The first term vanishes by the inductive hypothesis. So the $\cdot x_1 : H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M)$ must be injective. This cannot be injective if the cohomologies are nonzero by Hartshorne III.3.3(c) since $H^n_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(H^n_{\mathfrak{a}}(M))$. So $H^n_{\mathfrak{a}}(M) = 0$.
($\Leftarrow$). Suppose we knew the result for some $n \geq 1$. Suppose $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n+1$. Look at the SES from before and there is a sequence

$$0 \to 0 \to \ldots \to 0 \to H^n_{\mathfrak{a}}(M/x_1M) \to H^{n+1}_{\mathfrak{a}}(M) \to H^{n+1}_{\mathfrak{a}}(M) \to H^{n+1}_{\mathfrak{a}}(M/x_1M) \to \ldots$$

which establishes that $H^i_{\mathfrak{a}}(M/x_1M) = 0$ for all $i < n$. So $\operatorname{depth}_{\mathfrak{a}}(M/x_1M) \geq n$ and this means $\operatorname{depth}_{\mathfrak{a}}(M) \geq n+1$ by x_1 an M-regular element. □

Hartshorne III.3.5 Let X be a noetherian scheme, and let P be a closed point of X. Show that the following conditions are equivalent:
(i) $\operatorname{depth} \mathcal{O}_P \geq 2$;
(ii) if U is any open neighborhood of P, then every section of $\mathcal{O}_X$ over $U - P$ extends uniquely to a section of $\mathcal{O}_X$ over U.

This generalizes (I, Ex. 3.20), in view of (II, 8.22A).

Proof. We can prove this result via a sequence of equivalences. Since P is a closed point, let $\operatorname{Spec} A \subseteq X$ be an open affine neighborhood of P. Then,

$$\begin{aligned}
\text{every section } s \in \mathcal{O}_X(U-P) \text{ extends uniquely to } U &\iff \Gamma(U-P, \mathcal{O}_X|_U) \to \Gamma(U, \mathcal{O}_X|_U) \text{ is bijective} \\
&\iff H^0_P(U, \mathcal{O}_X|_U) = H^1_P(U, \mathcal{O}_X|_U) \text{ by Hartshorne III.2.3(e)} \\
&\iff H^0_P(X, \mathcal{O}_X) = H^1_P(X, \mathcal{O}_X) = 0 \text{ by Hartshorne III.2.3(f)} \\
&\iff H^0_P(X_P, \mathcal{O}_{X,P}) = H^1_P(X_P, \mathcal{O}_{X,P}) \text{ by Hartshorne III.2.5} \\
&\iff H^0_P((\operatorname{Spec} A)_P, \mathcal{O}_{X,P})) = H^1_P((\operatorname{Spec} A)_P, \mathcal{O}_{X,P})) = 0 \text{ from above} \\
&\iff H^0_{\mathfrak{m}}(\mathcal{O}_{X,P}) = H^1_{\mathfrak{m}}(\mathcal{O}_{X,P}) = 0 \text{ by Hartshorne III.3.3(b)} \\
&\iff H^i(\mathcal{O}_P) = 0 \text{ for } i < 2 \\
&\iff \operatorname{depth} \mathcal{O}_P \geq 2 \text{ by Hartshorne III.3.4.}
\end{aligned}$$

C.f. this MSE thread. □

Hartshorne III.3.6 Let X be a noetherian scheme.

(a) Show that the sheaf $\mathscr{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X. Thus $\mathfrak{Qco}(X)$ has enough injectives.

(b)* Show that any injective object of $\mathfrak{Qco}(X)$ is flasque. [*Hints*: The method of proof of (2.4) will not work, because $\mathcal{O}_U$ is not quasi-coherent on X in general. Instead, use (II, Ex. 5.15) to show that if $\mathscr{I} \in \mathfrak{Qco}(X)$ is injective, and if $U \subseteq X$ is an open subset, then $\mathscr{I}|_U$ is an injective object of $\mathfrak{Qco}(U)$. Then cover X with open affines...]

(c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.

Proof. **[Incomplete]** □

Hartshorne III.3.7 Let A be a noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

(a) For any A-module M, establish the following formula of Deligne:

$$\Gamma(U, \widetilde{M}) \cong \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M).$$

(b) Apply this in the case of an injective A-module I, to give another proof of (3.4).

Proof. **(a).** It is instructive to deal with the case where $\mathfrak{a} = (f)$ is principal. Then $U = X - V((f)) = D(f)$ and so the isomorphism we want is

$$M_f \cong \varinjlim_n \operatorname{Hom}_A((f^n), M).$$

Determining an A-module morphism $(f^n) \to M$ is the same as determining the image of f^n in M and extending A-linearly. Of course, $f \in A$ and so, we need to only determine the image of $1 \in A$ under the morphism. The colimit is taken over the system

$$\operatorname{Hom}_A((f), M) \to \operatorname{Hom}_A((f^2), M) \to \operatorname{Hom}_A((f^3), M) \to \dots$$

by precomposing with maps $\psi_i : (f^{i+1}) \to (f^i)$. Now define a map $\operatorname{Hom}_A((f^k), M) \to M_f$ by taking an $\phi : (f^k) \to M$ and sending it to $\frac{\phi(f^k)}{f^k}$. This gives a diagram

$$\begin{array}{ccccccc} \operatorname{Hom}((f), M) & \longrightarrow & \operatorname{Hom}((f^2), M) & \longrightarrow & \operatorname{Hom}((f^3), M) & \longrightarrow & \dots \\ & \searrow & \downarrow & \swarrow & \swarrow & & \\ & & M_f & & & & \end{array}$$

which commutes. By the universal property, we get an induced map $\varinjlim_n \operatorname{Hom}_A((f), M) \to M_f$. It is easy to see that this is neccesarily an isomorphism so we are done.

From the definition of the structure sheaf (via inverse limits), we know

$$\Gamma(U, \widetilde{M}) \cong \varprojlim_{D(f) \subseteq U} \Gamma(D(f), \widetilde{M}) = \varprojlim_{D(f) \subseteq U} M_f.$$

Since A is a Noetherian ring, $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = (g_1, \dots, g_k)$. The inverse system in the problem is given by

$$\operatorname{Hom}(\mathfrak{a}, M) \to \operatorname{Hom}(\mathfrak{a}^2, M) \to \operatorname{Hom}(\mathfrak{a}^3, M) \to \dots$$

where the maps are given by precomposing by the inclusion $\mathfrak{a}^{i+1} \to \mathfrak{a}^i$. Now we define a map

$$\psi : \operatorname{Hom}((g_1, \dots, g_k), M) \to \Gamma(U, \widetilde{M})$$

as follows. Let ϕ be a morphism on the LHS. Then $\phi(g_1) = m_1$ and we map this to the section $\frac{m_1}{g_1} \in \Gamma(D(g_1), \widetilde{M})$. Define a similar map for the other g_i's and so we get a collection of sections $\frac{m_i}{g_i} \in \Gamma(D(g_i), \widetilde{M})$. These sections agree on intersections i.e.

$$\frac{m_i}{g_i}|_{D(g_i g_j)} = \frac{m_i g_j}{g_i g_j}|_{D(g_i g_j)} = \frac{m_j g_i}{g_i g_j}|_{D(g_i g_j)} = \frac{m_j}{g_j}|_{D(g_i g_j)}$$

because $\phi(g_ig_j) = g_jm_i = m_jg_i$ and so they glue to a section say s. So define $\Psi(\phi) = s$. A similar approach can be used to define a morphism

$$\Psi_l : \mathrm{Hom}(\mathfrak{a}^l, M) \to \Gamma(U, \widetilde{M}).$$

The l-th power of $\mathfrak{a}$ is generated by elements $g_1^{i_1} \dots g_k^{i_k}$ s.t. $\sum_{j=1}^k i_j = l$. Then one defines a map by sending a $\phi : \mathfrak{a}^l \to M$ to a section we glue up as follows. First, $\phi(g_i^l) = m_1$ gives us a section $\Gamma(D(g_i), \widetilde{M})$ which is $\frac{m_i}{g_i^l}$. Meanwhile if $\phi(g_1^{i_1} \dots g_n^{i_n}) = m$, then send ϕ to

$$\frac{m}{g_1^{i_1} \dots g_n^{i_n}} \in \Gamma(D(g_1 \dots g_n), \widetilde{M}).$$

Of course, these choices of sections are compatible and glue up to a section $s \in \Gamma(U, \widetilde{M})$ which we take as $\Psi_l(\phi) = s$. After this, one needs only check that $\Gamma(U, \widetilde{M})$ satisfies the universal property of the colimit (which is an easy check).

(b). Let I be an injective module over a Noetherian ring A. We show $\widetilde{I}$ is flasque on $\mathrm{Spec}\, A$. It suffices to show that for any open set $U \subseteq X$,

$$\Gamma(X, \widetilde{I}) \to \Gamma(U, \widetilde{I})$$

is surjective. Using the isomorphism in (a), we want to prove a surjection

$$\varinjlim_n \mathrm{Hom}_A(A^n, I) \to \varinjlim_n \mathrm{Hom}_A(\mathfrak{a}^n, I)$$

where $\mathfrak{a}$ is the ideal s.t. $X - V(\mathfrak{a}) \cong U$. Now $\mathrm{Hom}_A(-, I)$ is right exact and taking colimits is exact in this situation ([10, Tag 04B0]). □

Hartshorne III.3.8 Without the noetherian hypothesis, (3.3) and (3.4) are false. Let $A = k[x_0, x_1, x_2, \dots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2, \dots$. Let I be an injective A-module containing A. Show that $I \to I_{x_0}$ is not surjective.

Proof. Clearly A is not Noetherian. Let I be an arbitrary injective A-module s.t. $A \subseteq I$.

Lemma 3.3. The morphism $\psi : I \to I_{x_0}$ given by $f \mapsto \frac{f}{x_0}$ is not surjective.

Proof. We do not actually need to use the fact that I is injective in any nontrivial manner outside of saying there exists an $I \supseteq A$ due to there being enough injectives.

Suppose $\psi : I \to I_{x_0}$ is surjective. Choose $a \in I$ s.t. $\frac{a}{1} = \frac{1}{x_0}$ which means there is an $n \geq 0$ s.t.

$$x_0^n(x_0 a - 1) = 0 \;\Rightarrow\; x_0^{n+1} a = x_0^n.$$

Now consider the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & (x_0) & \hookrightarrow & A \\ & & & \searrow^{x_0 \mapsto a x_0} & \\ & & & & I \end{array}$$

and from this diagram, we can choose x_m for $m > n$ and look at $x_0^m x_m \in (x_0)$. As an element of A, this is zero. But

$$0 = a(0) = a(x_0^m x_m) = a x_0^m x_m = x_0^{m-1} x_m.$$

But that means $x_0^{m-1} x_m \in A$ is zero which is ridiculous. Contradiction. □

Having proven the lemma, Hartshorne Lemma III.3.3 does not hold if A is not assumed to be Noetherian. For Hartshorne Proposition III.3.4, $\widetilde{I}$ is not a flasque Spec A-module since the restriction map $\Gamma(\operatorname{Spec} A, \widetilde{I}) = I \to \Gamma(D(x_0), \widetilde{I}) = I_{x_0}$ is not injective. □

4. Čech Cohomology

Hartshorne III.4.1 Let $f : X \to Y$ be an affine morphism of noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \geq 0$,

$$H^i(X, \mathscr{F}) \cong H^i\left(Y, f_*\mathscr{F}\right).$$

[Hint: Use (II, 5.8)]

Proof. Let $\{V_i\}_{i=1,\dots,n}$ be an open affine cover of Y and then $\{f^{-1}(V_i)\}_{i=1,\dots,n}$ is an open affine cover of f. Since the schemes are separated, we know the the intersections between open affines are also open affine.

Since X is Noetherian, we know $f_*\mathscr{F}$ is also quasicoherent. By Hartshorne Theorem III.4.5,

$$H^i(X, \mathscr{F}) = \check{H}(\{f^{-1}(V_i)\}, \mathscr{F})$$

is a natural isomorphism. Recall Čech cohomology is defined using the Čech cohomology and

$$\begin{aligned}\mathscr{C}^p(\{f^{-1}(V_i)\}_{i=1,\dots,n}, \mathscr{F}) &= \prod_{i_0<\cdots<i_p} g_*(\mathscr{F}|_{f^{-1}(V_{i_0})\cap\cdots\cap f^{-1}(V_{i_p})}) \\ &\cong \prod_{i_0<\cdots<i_p} g_*(f_*\mathscr{F}|_{V_{i_0}\cap\cdots\cap V_{i_p}}) \\ &= \mathscr{C}^p(\{V_i\}_{i=1,\dots,n}, f_*\mathscr{F}).\end{aligned}$$

This shows that there is a natural isomorphism

$$\check{H}(\{f^{-1}(V_i)\}, \mathscr{F}) \cong \check{H}(\{V_i\}, f_*\mathscr{F}).$$

Applying Hartshorne Theorem III.4.5,

$$H^i(X, \mathscr{F}) \cong \check{H}^i(\{f^{-1}(V_i)\}, \mathscr{F}) \cong \check{H}^i(\{V_i\}, \mathscr{F}) = H^i(Y, f_*\mathscr{F}).$$

□

Hartshorne III.4.2 Prove Chevalley's theorem: Let $f : X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.

(a) Let $f : X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathscr{M}$ on X, and a morphism of sheaves $\alpha : \mathcal{O}_Y^r \to f_*\mathscr{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

(b) For any coherent sheaf $\mathscr{F}$ on Y, show that there is a coherent sheaf $\mathscr{G}$ on X, and a morphism $\beta : f_*\mathscr{G} \to \mathscr{F}^r$ which is an isomorphism at the generic point of Y.
[*Hint*: Apply $\mathscr{H}om(\cdot, \mathscr{F})$ to α and use (II, Ex. 5.17e).]

(c) Now prove Chevalley's theorem. First use (Ex. 3.1) and (Ex. 3.2) to reduce to the case X and Y integral. Then use (3.7), (Ex. 4.1), consider $\ker\beta$ and $\operatorname{coker}\beta$, and use noetherian induction on Y.

PROOF. **(a)** See this post [14] for a detailed discussion on the problem.

(b) Applying $\mathscr{H}om(\alpha, \mathscr{F}) : \mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) \to \mathscr{H}om(\mathcal{O}_Y^r, \mathscr{F}) = \mathscr{F}^r$. By Hartshorne II.5.17(e), $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F})$ is isomorphic to $f_*\mathscr{G}$ for some coherent sheaf $\mathscr{G}$. Since α was an isomorphism at the generic point of Y, the map β is also an isomorphism at the generic point.

(c) By Hartshorne III.3.1 and III.3.2, we may assume X, Y are integral. We use Hartshorne Theorem III.3.7 to show Y is affine. Consider the SESs

$$0 \to \ker\beta \to f_*\mathscr{G} \to \operatorname{im}\beta \to 0$$
$$0 \to \operatorname{im}\beta \to \mathscr{F}^r \to \operatorname{coker}\beta \to 0.$$

We use Noetherian induction (Hartshorne II.3.16) on Y. Let $Z \subsetneq Y$ be a proper closed subset and assume all proper closed subsets of Z are affine. We show Z is affine.

Let $\mathscr{I}$ be a coherent sheaf of ideals on Z. The LES of cohomology

$$0 \to H^0(Z, \ker\beta) \to H^0(Z, f_*\mathscr{G}) \to H^0(Z, \operatorname{im}\beta) \to H^1(Z, \ker\beta) \to$$

gives the isomorphisms $H^i(Z, \operatorname{im}\beta) \cong H^{i+1}(Z, \ker\beta)$ for all $i > 0$ if we can show that $H^i(Z, f_*\mathscr{G})$ vanishes for all $i > 0$. Indeed, by Hartshorne III.4.1, $H^i(Z, f_*\mathscr{G}) = H^i(f^{-1}(Z), \mathscr{G})$ and because $f^{-1}(Z) \subseteq X$ is a closed subset of an affine scheme, it is affine. By Serre's Criterion, $H^i(f^{-1}(Z), \mathscr{G})$ always vanishes. The second LES of cohomology with this isomorphism gives

$$\ldots \to H^{i+1}(Z, \ker\beta) \to H^i(Z, \mathscr{F}^r) \to H^i(Z, \operatorname{coker}\beta) \to H^{i+2}(Z, \ker\beta) \to \ldots$$

Using Čech cohomology and Hartshorne Theorem III.4.5, $H^{i+1}(Z, \ker\beta) = H^i(Z, \operatorname{coker}\beta) = 0$ for all i. Take $\mathscr{F} := \mathscr{I}$ and $i = 1$. It follows that $H^1(Z, \mathscr{I}^r) = 0$. Therefore, $H^1(Z, \mathscr{I}) = 0$ and by Serre's Criterion, Z is affine. The Noetherian induction is complete. □

Hartshorne III.4.3 Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, and let $U = X - \{(0, 0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j \mid i, j < 0\}$. In particular, it is infinite-dimensional. (Using (3.5), this provides another proof that U is not affine–cf. (I, Ex. 3.6).)

PROOF. Cover U by $D(x) = \operatorname{Spec} A[x, y]_x$ and $D(y) = \operatorname{Spec} A[x, y]_y$ and let $\mathscr{F} := \mathcal{O}_U$ and $\mathfrak{U} = \{D(x), D(y)\}$. We have

$$C^0(\mathfrak{U}, \mathscr{F}) = k[x, y]_x \times k[x, y]_y \qquad \& \qquad C^1(\mathfrak{U}, \mathscr{F}) = k[x, y]_{xy} \qquad \& \qquad C^k(\mathfrak{U}, \mathscr{F}) = 0 \text{ for } k \geq 2.$$

Then $d : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F})$ maps $\langle \frac{f}{x^n}, \frac{g}{y^m} \rangle \mapsto \frac{f}{x^n} - \frac{g}{y^m} = \frac{fy^m - gx^n}{x^n y^m}$ with $f, g \in k[x, y]$. Since the Čech complex is

$$0 \to C^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{d} C^1(\mathfrak{U}, \mathscr{F}) \to 0.$$

Now we show that $H^1(U, \mathcal{O}_U) = \check{H}^1(\mathfrak{U}, \mathscr{F}) = \frac{C^1(\mathfrak{U}, \mathscr{F})}{d(C^0(\mathfrak{U}, \mathscr{F}))} = \frac{k[x,y]_{xy}}{d(C^1(\mathfrak{U}, \mathscr{F}))}$ is a k-vector space with basis $\{x^i y^j : i, j < 0\}$. Now

$$d(C^1(\mathfrak{U}, \mathscr{F})) = \left\langle \frac{fy^m - gx^n}{x^n y^m} : f, g \in k[x, y] \right\rangle.$$

From these relations, we know the quotient has

$$0 = \frac{x^n f y^m - y^m x^n}{x^n y^m} = f - \frac{1}{y}$$

so we can identify all polynomials in $f \in k[x,y]$ with $\frac{1}{y}$ We also have

$$0 = \frac{f y^m - 0 \cdot x^n}{x^n y^m} = \frac{f}{x^n}$$

which means we can kill any ratios with only powers of x in the denominator. We do a similar thing with y. Next, we show that we can relate any ratio $\frac{f}{x^n y^m}$ to one of form $x^i y^j$ with $i < 0, j < 0$. Using the fact that we can kill any polynomials, we can assume that all monomials $kx^l x^k$ of f have $l < n, k < m$. In this way, we have written $\frac{f}{x^n y^m}$ as a k-linear combination of the $x^i y^j$ where $i < 0, j < 0$. So we get the statement that

$$H^1(U, \mathcal{O}_U) \cong k\text{-vector space spanned by } \{x^i y^j : i, j < 0\}.$$

Applying Serre's Affine Criterion, we get another proof that U is not affine. □

Hartshorne III.4.4 On an arbitrary topological space X with an arbitrary abelian sheaf, $\mathscr{F}$ Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if one takes the limit over all coverings.

(a) Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{B} = (V_j)_{j \in J}$, together with a map $\lambda : J \to I$ of the index sets, such that for each $j \in J, V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{B}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology, for any abelian sheaf $\mathscr{F}$, and for each i,

$$\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{B}, \mathscr{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathscr{F}).$$

(b) For any abelian sheaf $\mathscr{F}$ on X, show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$$

are compatible with the refinement maps above.

(c) Now prove the following theorem. Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$

is an isomorphism.

[*Hint*: Embed $\mathscr{F}$ in a flasque sheaf $\mathscr{G}$, and let $\mathscr{R} = \mathscr{G}/\mathscr{F}$, so that we have an exact sequence

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0.$$

Define a complex $D^\bullet(\mathfrak{U})$ by

$$0 \to C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{U}, \mathscr{G}) \to D^\bullet(\mathfrak{U}) \to 0.$$

Then use the exact cohomology sequence of this sequence of complexes, and the natural map of complexes

$$D^\bullet(\mathfrak{U}) \to C^\bullet(\mathfrak{U}, \mathscr{R}),$$

and see what happens under refinement.]

PROOF. **(a)** First, we define a morphism

$$\phi_p : \prod_{i_0<\cdots<i_p} \mathscr{F}(U_{i_0,\ldots,i_p}) \to \prod_{i_0<\cdots<i_p} \mathscr{F}(V_{i_0,\ldots,i_p})$$

by

$$\phi_p(\alpha)_{j_0,\ldots,j_p} = \alpha_{\lambda(j_0),\ldots,\lambda(j_p)}|_{V_{j_0,\ldots,j_p}}.$$

So we get a diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \prod_{i_0<\cdots<i_p} \mathscr{F}(U_{i_0,\ldots,i_p}) & \longrightarrow & \prod_{i_0<\cdots<i_{p+1}} \mathscr{F}(U_{i_0,\ldots,i_{p+1}}) & \longrightarrow & \prod_{i_0<\cdots<i_{p+2}} \mathscr{F}(U_{i_0,\ldots,i_{p+2}}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \phi_p} & & \downarrow{\scriptstyle \phi_{p+1}} & & \downarrow{\scriptstyle \phi_{p+2}} & \\
\cdots \longrightarrow & \prod_{j_0<\cdots<j_p} \mathscr{F}(V_{j_0,\ldots,j_p}) & \longrightarrow & \prod_{i_0<\cdots<i_{p+1}} \mathscr{F}(V_{i_0,\ldots,i_{p+1}}) & \longrightarrow & \prod_{i_0<\cdots<i_{p+2}} \mathscr{F}(V_{i_0,\ldots,i_{p+2}}) & \longrightarrow \cdots
\end{array}$$

We claim that ϕ itself is a chain map. Indeed,

$$\begin{aligned}
\phi(d\alpha)_{j_0,\ldots,j_{p+1}} &= d\alpha_{\lambda(j_0),\ldots,\lambda(j_{p+1})}|_{V_{j_0,\ldots,j_p,j_{p+1}}} \\
&= \sum_{k=0}^{p+1} (-1)^k \alpha_{\lambda(j_0),\ldots,\widehat{\lambda(j_k)},\ldots\lambda(j_{p+1}}|_{V_{j_0,\ldots,j_{p+1}}}
\end{aligned}$$

and

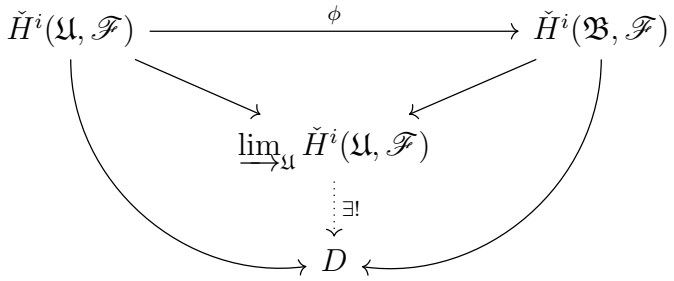
$$\begin{aligned}
d(\phi\alpha)_{j_0,\ldots,j_{p+1}} &= \sum_{k=0}^{p+1} (-1)^k (\phi\alpha)_{j_0,\ldots,\widehat{j_i},\ldots,j_{p+1}}|_{V_{j_0,\ldots,j_{p+1}}} \\
&= \sum_{k=0}^{p+1} (-1)^k \alpha_{\lambda(j_0),\ldots,\widehat{\lambda(j_i)},\ldots,\lambda_{(j_{p+1})}}|_{V_{j_0,\ldots,j_{p+1}}}.
\end{aligned}$$

Therefore we get a chain map $\phi : C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{V}, \mathscr{F})$ and this induces a morphism $\phi : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F})$ of Čech cohomology groups. Since the collection of open covers partially ordered[3] by refinement (i.e. $\mathfrak{U} \leq \mathfrak{V}$ if $\mathfrak{V}$ is a refinement of $\mathfrak{U}$), we can take the direct limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathscr{F}).$$

[3]For antisymmetry to hold, we say $\mathfrak{U}$ is equivalent to $\mathfrak{V}$ iff $\mathfrak{U} \leq \mathfrak{V}$ and $\mathfrak{V} \leq \mathfrak{U}$. That is, two refinements are equivalent if they can be "fitted" into the other. This does not imply that the two refinements are precisely the same open cover.

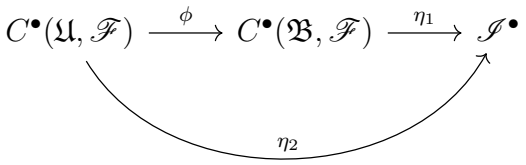
That is, we it has the universal property described by the diagram



(b) To show that the refinement map is compatible, we essentially show the following diagram commutes

$$\begin{array}{ccc} \check{H}^i(\mathfrak{U},\mathscr{F}) & \longrightarrow & H^i(X,\mathscr{F}) \\ \downarrow{\scriptstyle \phi} & & \Downarrow \\ \check{H}^i(\mathfrak{B},\mathscr{F}) & \longrightarrow & H^i(X,\mathscr{F}) \end{array}$$

Consider the diagram



where η_1, η_2 are defined via the map in the proof of Hartshorne Lemma III.4.4. But the composite $\phi \circ \eta_1$ induces the identity so $\phi \circ \eta_1$ and η_2 both give the same maps on cohomology. So we get the desired commutative diagram.

(c) Let $\mathscr{G}$ be a flasque sheaf s.t. $\mathscr{F} \hookrightarrow \mathscr{G}$ and look at the SES

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0.$$

Then we can look at the SES

$$0 \to C^\bullet(\mathfrak{U},\mathscr{F}) \to C^\bullet(\mathfrak{U},\mathscr{G}) \to D^\bullet(\mathfrak{U}) \to 0.$$

In this way, we can take a LES in cohomology. There is a natural map $D^\bullet(\mathfrak{U}) \to C^\bullet(\mathfrak{U},\mathscr{R})$ which is given by the obvious inclusion. That is,

$$\begin{array}{ccccccc} 0 & \longrightarrow & D^0(\mathfrak{U}) & \longrightarrow & D^1(\mathfrak{U}) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C^0(\mathfrak{U},\mathscr{R}) & \longrightarrow & C^1(\mathfrak{U},\mathscr{R}) & \longrightarrow & \dots \end{array}$$

has the vertical maps obtained from the sheafification functor $\prod_{i_0<\dots<i_p} \mathscr{F}(U_{i_0,\dots,i_p})/\mathscr{G}(U_{i_0,\dots,i_p}) \to \prod_{i_0<\dots<i_p} \mathscr{R}(U_{i_0,\dots,i_p})$.

Lemma 4.1. From here, we get a natural map $H^0(D^\bullet(\mathfrak{U})) \to \check{H}^0(\mathfrak{U},\mathscr{R})$ which we claim is always an inclusion.

PROOF. The elements are just the kernels of the first map in each of the cochain complexes. Commutativity of the diagram ensures that an element $\alpha \in D^0(\mathfrak{U})$ going to zero maps to zero in $C^1(\mathfrak{U}, \mathscr{R})$. That means the image inside $C^0(\mathfrak{U}, \mathscr{R})$ is in the kernel of $C^0(\mathfrak{U}, \mathscr{R}) \to C^1(\mathfrak{U}, \mathscr{R})$.

If an element of $H^0(D^\bullet(\mathfrak{U}))$ goes to zero, that means each coordinate is zero in $\mathscr{R}(U_i)$ for each $U_i \in \mathfrak{U}$. This occurs iff $t = (t_i)_{U_i \in \mathfrak{U}}$ has $t_i = 0$ for all i. □

Passing to the direct limit (which is exact), we get an inclusion $\varinjlim_{\mathfrak{U}} H^0(D^\bullet(\mathfrak{U})) \to \check{H}^0(\mathfrak{U}, \mathscr{R})$.

Lemma 4.2. The map $\varinjlim_{\mathfrak{U}} H^0(D^\bullet(\mathfrak{U})) \to \check{H}^0(\mathfrak{U}, \mathscr{R}) = \Gamma(X, \mathscr{R})$ is surjective.

PROOF. Let $s \in \Gamma(X, \mathscr{R})$ be a global section. So by sheafification, we get

- for any $P \in X$, $s(P) \in (\mathscr{G}/\mathscr{F})_P$,
- for each $P \in X$, there is a neighborhood $P \in V \subseteq X$ and a $t \in \mathscr{G}/\mathscr{F}(V)$ s.t. for all $Q \in V$, the germ $t_Q = s(Q)$.

It is not clear how to specify an element on the LHS of the map. For each $P \in X$, choose the corresponding V_P open subset and a section $t \in \mathscr{G}/\mathscr{F}(V_P)$. This gives an element in $H^0(D^\bullet(\mathfrak{U}))$ where $\mathfrak{U} = (V_P)_{P \in X}$. The condition that $t_Q = s(Q)$ ensures that we can extend this to a coherent sequence in $\varinjlim_{\mathfrak{U}} H^0(D^\bullet(\mathfrak{U})$ and this coherent sequence maps to s. □

Lemma 4.3. There is an isomorphism $\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$.

PROOF. From part (b), we know there are natural maps from Čech cohomology to sheaf cohomology. So taking LESs and consider the map to sheaf cohomology, we get a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{H}^0(\mathfrak{U}, \mathscr{F}) & \longrightarrow & \check{H}^0(\mathfrak{U}, \mathscr{G}) & \longrightarrow & H^0(D^\bullet(\mathfrak{U}) & \longrightarrow & \check{H}^1(\mathfrak{U}, \mathscr{F}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(X, \mathscr{F}) & \longrightarrow & H^0(X, \mathscr{G}) & \longrightarrow & H^0(X, \mathscr{R}) & \longrightarrow & H^1(X, \mathscr{F}) & \longrightarrow & 0
\end{array}$$

The zero's on the RHS appear because $\mathscr{G}$ was flasque. Naturality ensures that this is a commutative diagram. The first two vertical maps are isomorphisms because we simply get the global sections of the sheaves. Passing to direct limits which is an exact functor and applying our previous lemmas, we get

$$\begin{array}{ccccccccccc}
\Gamma(X, \mathscr{F}) & \longrightarrow & \Gamma(X, \mathscr{G}) & \longrightarrow & \varinjlim H^0(D^\bullet(\mathfrak{U}) & \longrightarrow & \varinjlim \check{H}^1(\mathfrak{U}, \mathscr{F}) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
\Gamma(X, \mathscr{F}) & \longrightarrow & \Gamma(X, \mathscr{G}) & \longrightarrow & \varinjlim H^0(X, \mathscr{R}) & \longrightarrow & H^1(X, \mathscr{F}) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

□

A 5-lemma application means $\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$ is an isomorphism. □

Hartshorne III.4.5 For any ringed space $(X, \mathcal{O}_X)$, let Pic X be the group of isomorphism classes of invertible sheaves (II, §6). Show that Pic $X \cong H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X)$, with multiplication as the group operation.

[*Hint*: For any invertible sheaf $\mathscr{L}$ on X, cover X by open sets U_i on which $\mathscr{L}$ is free, and fix isomorphisms $\varphi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathscr{L}\Big|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_i \cap U_j}$ with itself. These isomorphisms give an element of $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$. Now use (Ex. 4.4).]

PROOF. Let $\mathscr{L} \in \operatorname{Pic}(X)$ and fix isomorphisms φ_i as in the hint. Since $\varphi_i \circ \varphi_j^{-1}$ are isomorphisms (as they are transition functions for locally free sheaves), we get induced ring isomorphisms

$$\varphi_i^{-1} \circ \varphi_j : \Gamma(U_i \cap U_j, \mathcal{O}_X) \to \Gamma(U_i \cap U_j, \mathcal{O}_X)$$

which is determined by where it maps 1. Let $\alpha_{i,j} := \varphi_i^{-1} \circ \varphi_j(U_i \cap U_j)(1)$. The $\alpha_{i,j}$ are all units so $\alpha_{i,j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X)^\times$.

Putting these elements together, we get an element

$$(\alpha_{i,j}) \in \prod_{i,j} \Gamma(U_i \cap U_j, \mathcal{O}_X).$$

Given $\alpha = (\alpha_{i,j})$, we have

$$(d\alpha)_{i,j,k} = (\alpha_{j,k}|_V)(\alpha_{i,k}|_V)^{-1}(\alpha_{i,j}|_V) = 1 \tag{32}$$

because $\varphi_i^{-1} \circ \varphi_j \circ \varphi_j^{-1} \circ \varphi_k = \varphi_i^{-1} \circ \varphi_k$. Note we wrote Equation 32 out using multiplicative notation since we are working with $\mathcal{O}_X^*$. OTOH, every $\beta = (\beta_i) \in \prod_i \mathcal{O}_X^*(U_i)$ gets mapped to

$$(d\beta)_{i,j} = (\beta_i|_{U_i \cap U_j})(\beta_j|_{U_i \cap U_j})^{-1}.$$

So inside $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$, the $(\beta_i|_{U_i \cap U_j})(\beta_j|_{U_i \cap U_j})^{-1}$ are all zero. This is equivalent to the fact that $\alpha_{i,i}|_{U_i}\alpha_{i,i}|_{U_i}^{-1} = 1$.

Lemma 4.4. With the above work, there is an isomorphism between $\operatorname{Pic}(X)$ and $H^1(X, \mathcal{O}_X^*)$.

PROOF. Given the fact that $(X, \mathcal{O}_X)$ has a basis of open sets by distinguished open affine subsets and the trivializations can be taken to be over these distinguish open affine subsets, we get a map

$$\operatorname{Pic}(X) \to \varinjlim_{\mathfrak{U}} \check{H}^1(X, \mathcal{O}_X^*).$$

This is a group homomorphism since tensoring two invertible sheaves corresponds to the trivializations being multiplied.

Now we check that this is an isomorphism.

First off, it is surjective since any coherent sequence in the RHS gives rise to gluing maps $\varphi_{i,j} : \mathcal{O}_X|_{U_i \cap U_j} \to \mathcal{O}_X|_{U_i \cap U_j}$ that are compatible on triple intersections (we get these at the level of sheaves since restrictions to finer open subsets is equivalent to restricting our sections to refinements which are compatible by the direct limit). Therefore, we get a gluing of the sheaves $\mathcal{O}_X|_{U_i}$ to an invertible sheaf on $\mathscr{L}$. Since any invertible sheaf $\mathscr{L}$ is determined precisely by its trivializations and its transition functions, the map is surjective.

Second, the morphism is injective. If an invertible sheaf $\mathscr{L}$ corresponds to the "trivial" coherent sequence $((1)_{i,j})_{\mathfrak{U}}$, then the gluing is just the identity $\varphi_i^{-1}\varphi_j = \mathrm{id} : \mathcal{O}_X(U_i \cap U_j) \to \mathcal{O}_X(U_i \cap U_j)$. So the sheaf we get when gluing is $\mathcal{O}_X$.

Now by an application of Hartshorne III.4.5,

$$\operatorname{Pic}(X) \cong \varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathcal{O}_X^*) \cong H^1(X, \mathcal{O}_X^*).$$

□

□

Hartshorne III.4.6 Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathscr{I}$ be a sheaf of ideals with $\mathscr{I}^2 = 0$, and let X_0 be the ringed space $(X, \mathcal{O}_X/\mathscr{I})$. Show that there is an exact sequence of sheaves of abelian groups on X,

$$0 \to \mathscr{I} \to \mathcal{O}_X^* \to \mathcal{O}_{X_0}^* \to 0,$$

where $\mathcal{O}_X^*$ (respectively, $\mathcal{O}_{X_0}^*$) denotes the sheaf of (multiplicative) groups of units in the sheaf of rings $\mathcal{O}_X$ (respectively, $\mathcal{O}_{X_0}$); the map $\mathscr{I} \to \mathcal{O}_X^*$ is defined by $a \mapsto 1 + a$, and $\mathscr{I}$ has its usual (additive) group structure. Conclude there is an exact sequence of abelian groups

$$\ldots \to H^1(X, \mathscr{I}) \to \operatorname{Pic} X \to \operatorname{Pic} X_0 \to H^2(X, \mathscr{I}) \to \ldots$$

Proof. The map $\mathcal{O}_X^* \to \mathcal{O}_{X_0}^*$ is simply the quotient map since taking units commutes with taking quotients. To check exactness, we take stalks

$$0 \to \mathscr{I}_x \to \mathcal{O}_{X,x}^* \to \mathcal{O}_{X_0,x}^* \to 0.$$

We know $\mathcal{O}_{X,x_0}^* = (\mathcal{O}_{X,x}/\mathscr{I}_x)^*$. The map $\mathscr{I}_x \to \mathcal{O}_{X,x}^*$ is clearly injective since $s_x \to 1 + s_x$ is trivial iff $s_x = 0$. The map $\mathcal{O}_{X,x}^* \to \mathcal{O}_{X_0,x}^*$ is surjective as follows. The map sends $s_x \mapsto s_x + (1 + \mathscr{I}_x)$. Given $s_x + (1 + \mathscr{I}_x)$ invertible. So there is a $t_x + (1 + \mathscr{I}_x)$ s.t.

$$(s_x + (1 + \mathscr{I}_x))\,(t_x + (1 + \mathscr{I}_x)) = t_x s_x + t_x + s_x + \mathscr{I}_x = 1 + \mathscr{I}_x.$$

This is equivalent to saying $t_x s_x + t_x + s_x \in 1 + \mathscr{I}_x$. Then $t_x s_x + t_x + s_x = 1 + r_x$ for $r_x \in \mathscr{I}_x$ and we deduce that

$$s_x(t_x + 1) + t_x = 1 + r_x \;\Rightarrow\; s_x = \frac{1 + r_x - t_x}{t_x + 1} \in \mathcal{O}_{X,x_0}^*.$$

So the RHS element maps to the class $s_x + (1 + \mathscr{I}_x)$ which proves surjecitivty. For exactness in the middle, suppose $s_x \mapsto s_x + 1 + \mathscr{I}_x = 1 + \mathscr{I}_x$. Then $s_x \in 1 + \mathscr{I}_x$ and we write $s_x = 1 + r_x$ for $r_x \in \mathscr{I}_x$. Then the element $r_x \in \mathscr{I}_x$ mapsto s_x under $\mathscr{I}_x \to \mathcal{O}_{X,x}^*$.

The existence of the LES of abelian groups follows from taking the LES of cohomology and using Hartshorne III.4.5. □

Remark 48. We remark that the hypothesis $\mathscr{I}^2 = 0$ was made to ensure that $1 + \mathscr{I} \subseteq \mathcal{O}_X^*$ made sense.

Hartshorne III.4.7 Let X be a subscheme of $\mathbb{P}_k^2$ defined by a single homogeneous equation $f(x_0, x_1, x_2) = 0$ of degree d. (Do not assume f is irreducible.) Assume that $(1, 0, 0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$$

explicitly, and thus show that

$$\dim H^0(X, \mathcal{O}_X) = 1$$
$$\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2).$$

Proof. First, we observe that $f(1, 0, 0) \neq 0$ implies that $f(x_0, x_1, x_2) = a x_0^d + \text{other terms}$ for $a \in k^*$.

Lemma 4.5. The subscheme X can be covered by U and V as indicated.

PROOF. Certainly U, V are open affine subsets. To show that they cover X, let $y \in X$ be an arbitrary point. Writing $y = (y_0, y_1, y_2)$, we know that $f(y_0, y_1, y_2) = 0$. If both $y_1 = y_2 = 0$, then $f(y_0, y_1, y_2) \neq 0$ which is absurd. So we get $y \in U$ if $y_1 \neq 0$ and $y \in V$ if $y_2 \neq 0$. □

The Čech complex is

$$0 \to \Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \xrightarrow{\phi} \Gamma(U \cap V, \mathcal{O}_X) \to 0$$

with the middle map defined by $g \oplus h \mapsto g|_{U\cap V} - h|_{U\cap V}$.

Writing it out, we know that

$$\begin{aligned} U &= \operatorname{Spec} k[\frac{x_0}{x_1}, \frac{x_2}{x_1}] / \left(f(\frac{x_0}{x_1}, \frac{x_2}{x_1} \right) \\ U &= \operatorname{Spec} k[\frac{x_0}{x_2}, \frac{x_1}{x_2}] / \left(f(\frac{x_0}{x_2}, \frac{x_0}{x_2} \right) \\ U \cap V &= \operatorname{Spec} k[\frac{x_0}{x_1}, \frac{x_1}{x_2}, \frac{x_0}{x_2}, \frac{x_2}{x_1}] / \left(f(x_0, x_1, x_2) \right). \end{aligned}$$

The corresponding global sections are just the rings on the RHS.

Lemma 4.6. $\dim H^0(X, \mathcal{O}_X) = 1$ because $\check{H}^0(\{U, V\}, \mathcal{O}_X) = k$ and so $\Gamma(X, \mathcal{O}_X) \cong k$.

PROOF. By definition, the zeroth Čech cohomology is the kernel of ϕ. An element $g \oplus h$ goes to zero iff the difference is equal to zero. The polynomial g is allowed to have x_1's in its denominator while h is only allowed to have x_2's. The difference being zero indicates that the power of x_1 and x_2 are zero. The only elements in $k[\frac{x_0}{x_1}, \frac{x_2}{x_1}]$ and $k[\frac{x_0}{x_2}, \frac{x_1}{x_2}]$ with that property are those in k. So $\check{H}(\{U, V\}, \mathcal{O}_X) = k$. □

Lemma 4.7. $\dim H^1(X, \mathcal{O}_X) = \binom{d-1}{2} = \frac{1}{2}(d-1)(d-2)$.

PROOF. Because of the form $f(x_0, x_1, x_2) = x_0^d + \text{other terms}$, we can kill off any monomials in $k[\frac{x_0}{x_1}, \frac{x_1}{x_2}, \frac{x_0}{x_2}, \frac{x_2}{x_1}]$ with denominators x_0^n, x_1^n, or x_2^n for $n \geq d$. Because we can write

$$\frac{x_0}{x_2} = \frac{x_0}{x_1}\frac{x_1}{x_2},$$

we can focus on polynomials in $k[\frac{x_0}{x_1}, \frac{x_2}{x_1}, \frac{x_1}{x_2}]$.

We know that such polynomials cannot have purely powers of x_1 or x_2 since the image of ϕ contains polynomials with only x_1 powers in the denominator and polynomials with only x_2 powers in the denominator. So we know some possible monomials are

$$\left(\frac{x_0}{x_1}\right)^{r_1} \left(\frac{x_1}{x_2}\right)^{r_2}$$

where $1 \leq r_1 \leq d-1$ and $r_2 \in \mathbb{Z}$. We cannot have $r_2 = 0$ because we cannot have purely powers of x_1 in the denominator. We can also assume $r_2 > 0$ because

$$\frac{x_2}{x_1} \oplus \frac{x_1}{x_2} \in \Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \ \mapsto \frac{x_2}{x_1} - \frac{x_1}{x_2} \in \Gamma(U \cap V, \mathcal{O}_X)$$

gets modded out. We now have

$$\left(\frac{x_0}{x_1}\right)^{r_1} \left(\frac{x_1}{x_2}\right)^{r_2}$$

for $1 \leq r_1 \leq d-1$ and $r_2 \in \mathbb{Z}_{>0}$ as the possibilities. We can also get $1 \leq r_2 \leq d-1$ since we modded out by the ideal generated by f for $r_2 \geq d$, the monomial becomes

$$\left(\frac{x_0}{x_1}\right)^{r_1}\left(\frac{x_1}{x_2}\right)^{r_2} = \frac{x_0 x_1^{r_2-r_1}}{x_2^{r_2}}$$

which we know is not allowed. We also know $r_1 \neq r_2$ since

$$\left(\frac{x_0}{x_1}\right)^{r_1}\left(\frac{x_1}{x_2}\right)^{r_2} = \frac{x_0^{r_1}}{x_2^{r_2}}$$

which is not allowed. So the requirement is that $1 \leq r_2 < r_1 \leq d-1$. Of course, there are $\binom{d-1}{2}$ possibilities. Hence the k-dimension is

$$\dim_k H^1(X, \mathcal{O}_X) = \binom{d-1}{2}.$$

□

□

Hartshorne III.4.8 *Cohomological Dimension* (Hartshorne [3]). Let X be a Noetherian separated scheme. We define the *cohomology dimension* if X, denoted $\mathrm{cd}(X)$, to be the least integer n such that $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent $\mathscr{F}$ and all $i > n$. Thus for example, Serre's Theorem (3.7) says that $\mathrm{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\mathrm{cd}(X) \leq \dim X$.

(a) In the definition of $\mathrm{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X. Use (II, Ex. 5.15) and (2.9).

(b) If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X. Use (II, 5.18).

(c) Suppose X has a covering by $r+1$ open affine subsets. Use Čech cohomology to show that $\mathrm{cd}(X) \leq r$.

*(d) If X is a quasi-projective scheme of dimension r over a field k, then X can be covered by $r+1$ open affine subsets. Conclude (independently of (2.7)) that $\mathrm{cd}(X) \leq \dim X$.

(e) Let Y be a set-theoretic complete intersection (I, Ex. 2.17) of codimension r in $X = \mathbb{P}^n_k$. Show that $\mathrm{cd}(X - Y) \leq r - 1$.

Proof. (a) From Hartshorne II.5.15(e), we know every quasi-coherent sheaf is a union of its coherent subsheaves. Taking unions is a form of colimit so we are done by Hartshorne Proposition III.2.9.

(b) Apply Hartshorne Corollary II.5.18 and Proposition III.2.9. Recall that a quasi-projective scheme X simply factors $X \to X' \to \mathrm{Spec}(k)$ where $X \to X'$ is an open immersion and $X' \to \mathrm{Spec}(k)$ is a projective morphism. This does not affect the claim of the result since cohomology is not affected by open immersions. So we reduce to considering locally free coherent sheaves on X.

(c) If X has a covering by $r+1$ open affine subsets, we know that the Čech complex looks like

$$0 \to C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F}) \to \ldots \to C^r(\mathfrak{U}, \mathscr{F}) \to 0$$

and by Hartshorne Theorem III.4.5, $H^i(X, \mathscr{F}) = 0$ for all $i > r$. In other words, $\operatorname{cd}(X) \leq r$.

(d) See Section 22.4.15 of [9] for a proof of the first statement (also related is Theorem 18.2.6 of [9]). From (c), $\operatorname{cd}(X) \leq \dim X = r$.

(e) From Hartshorne I.2.7, $\dim Y \geq n - (r-1) = n - r + 1$ since Y can be cut out by $r-1$ hypersurfaces. Therefore, $\dim X - Y \leq n - (n - r + 1) = r - 1$. From (d), $\operatorname{cd}(X - Y) \leq \dim X - Y \leq r - 1$.

(Another method). First off, $\dim X - Y$ is covered by r open affine subsets. Then $\operatorname{cd}(X - Y) \leq r - 1$ by the same computation via Čech cohomology. □

Hartshorne III.4.9 Let $X = \operatorname{Spec} k[x_1, x_2, x_3, x_4]$ be affine four-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\overline{Y}$ in $\mathbb{P}^4_k$ is also not a set-theoretic complete intersection.
[*Hints*: Use an affine analogue of (Ex. 4.8e). Then show that $H^2(X - Y, \mathcal{O}_X) \neq 0$, by using (Ex. 2.3) and (Ex. 2.4). If $P = Y_1 \cap Y_2$, imitate (Ex. 4.3) to show $H^3(X - P, \mathcal{O}_X) \neq 0$.].

Proof. □

Hartshorne III.4.10 Let X be a nonsingular variety over an algebraically closed field k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathscr{F}$ (II, Ex. 8.7) up to isomorphism, and the group $H^1(X, \mathscr{F} \otimes \mathscr{T})$, where $\mathscr{T}$ is the tangent sheaf of X (II, §8).
[*Hint*: Use (II, Ex. 8.6) and (4.5).]

Proof. □

Hartshorne III.4.11 This exercise shows that Čech cohomology will agree with the usual cohomology whenever the sheaf has no cohomology on any of the open sets. More precisely, let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups, and $\mathfrak{U} = (U_i)$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$, that $H^k(V, \mathscr{F}|_V) = 0$. Then prove that for all $p \geq 0$, the natural maps

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$$

of (4.4) are isomorphisms. Show also that one can recover (4.5) as a corollary of this more general result.

Proof. Assume $H^k(V, \mathscr{F}|_V) = 0$ for any $k > 0$ and V as above. To show $\check{H}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$ for all $p \geq 0$, we mimic the proof of Hartshorne Theorem III.4.5.

First embed $\mathscr{F}$ into injective sheaf $\mathscr{G} \in \mathfrak{Ab}(X)$. Then consider the SES $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0$. Restricting to $U_{i_0,\ldots,i_p}$ and taking the LES of cosmology, the hypothesis says

$$0 \to \mathscr{F}(U_{i_0,\ldots,i_p}) \to \mathscr{G}(U_{i_0,\ldots,i_p}) \to \mathscr{R}(U_{i_0,\ldots,i_p}) \to 0$$

is always exact. The proof of Hartshorne Theorem III.4.5 repeats verbatim.

Indeed, we get a sequence of Čech complexes

$$0 \to C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{U}, \mathscr{G}) \to C^\bullet(\mathfrak{U}, \mathscr{R}) \to 0.$$

Then $\mathscr{G}$ being injective means it is flasque so we get an exact sequence

$$0 \to \check{H}^0(\mathfrak{U}, \mathscr{F}) \to \check{H}^0(\mathfrak{U}, \mathscr{G}) \to \check{H}^0(\mathfrak{U}, \mathscr{R}) \to \check{H}^1(\mathfrak{U}, \mathscr{F}) \to 0$$

and isomorphisms

$$\check{H}^p(\mathfrak{U}, \mathscr{R}) \to \check{H}^{p+1}(\mathfrak{U}, \mathscr{F}) \text{ for each } p \geq 1.$$

The fact that $\check{H}^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^0(X, \mathscr{F})$ is clear. Comparing the first part of the exact sequence in cohomology with the above sequence of Čech cohomology, the 5-Lemma gives $\check{H}^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^1(X, \mathscr{F})$.

Now the rest of the isomorphisms follow by induction. Given $\mathscr{F}$, we know

$$\check{H}^{p+1}(\mathfrak{U}, \mathscr{F}) \cong \check{H}^p(\mathfrak{U}, \mathscr{R}) \cong H^p(X, \mathscr{R}) \cong H^{p+1}(X, \mathscr{F})$$

where the second isomorphism is the inductive hypothesis and the third is by the LES in sheaf cohomology.

Lemma 4.8. We can recover Hartshorne Theorem III.4.5 from the preceding result.

Proof. Let $\mathscr{F}$ be a quasi-coherent sheaf on X and $\mathfrak{U}$ an open affine cover. Since X is separated, any of the intersections $V = U_{i_0} \cap \cdots \cap U_{i_p}$ are affine schemes. Since X is Noetherian, V is always a Noetherian affine scheme. As $\mathscr{F}|_V$ is a quasi-coherent sheaf, Hartshorne Theorem III.3.5 implies $H^k(V, \mathscr{F}|_V) = 0$ for all $k > 0$. The preceding result says

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$$

is an isomorphism. This is Hartshorne Theorem III.4.5. □

□

5. The Cohomology of Projective Space

Hartshorne III.5.1 Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the Euler characteristic of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum (-1)^i \dim_k H^i(X, \mathscr{F})$$

If

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

Proof. By Hartshorne Theorem III.5.2, we can find an n sufficiently large s.t. there is a LES of k-vector spaces

$$\begin{aligned} 0 \to H^0(X, \mathscr{F}') \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}') \to \\ \ldots \to H^n(X, \mathscr{F}') \to H^n(X, \mathscr{F}) \to H^n(X, \mathscr{F}'') \to 0. \end{aligned}$$

Taking the dimension and applying the rank-nullity theorem,

$$\sum_{i=0}^{n} (-1)^i \dim_k H^i(X, \mathscr{F}') + (-1)^{i+1} \dim_k H^i(X, \mathscr{F}) + (-1)^{i+2} \dim_k H^i(X, \mathscr{F}'') = 0$$

which means $\chi(\mathscr{F}) - \chi(\mathscr{F}') + \chi(\mathscr{F}'') = 0$. □

Hartshorne III.5.2 (a) Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbb{Q}[z]$, such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$. We call P **the Hilbert polynomial of** $\mathscr{F}$ with respect to the sheaf $\mathcal{O}_X(1)$.

[*Hints*: Use induction on $\dim \operatorname{Supp} \mathscr{F}$, general properties of numerical polynomials (I, 7.3), and suitable exact sequences

$$0 \to \mathscr{R} \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{Q} \to 0.]$$

(b) Now let $X = \mathbb{P}^r_k$, and let $M = \Gamma_*(\mathscr{F})$, considered as a graded $S = k[x_0, \ldots, x_r]$ module. Use (5.2) to show that the Hilbert polynomial of $\mathscr{F}$ just defined is the same as the Hilbert polynomial of M defined in (I, §7).

Proof. **(a)**[4] As $\mathscr{F}$ is coherent, the multiplication map gives a map $\mathscr{F}(-1) \to \mathscr{F}$ which fits into an exact sequence

$$0 \to \mathscr{R} \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{Q} \to 0.$$

Now we recall Hartshorne Proposition I.7.3. It states that (a) if $P(z)$ is a numerical polynomial, then $P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \cdots + c_n$ for choices of $c_i \in \mathbb{Z}$ and (b) if $f : \mathbb{Z} \to \mathbb{Z}$ s.t. there exists a numerical polynomial $Q(z)$ so that $Q(n) = f(n+1) - f(n)$ for $n \gg 0$, then there is a numerical polynomial, $P(t)$ s.t. $P(n) = f(n)$ for $n \gg 0$.

Lemma 5.1. Following the hint, suppose $\dim \operatorname{Supp} \mathscr{F} = 0$. Then the claimed Hilbert polynomial exists.

Proof. In this case, $\mathscr{F}$ is a direct sum of skyscrapper sheaves $\mathscr{F} = \bigoplus_{P_i \in X, i=1,\ldots,k} i_P A$ where $i_P A$ is the skyscrapper sheaf at P with value A. Because projective schemes are noetherian, this is a finite direct sum. The twisting of a skyscrapper sheaf is isomorphic to the original skyscrapper sheaf the stalks there are the same. One obtains $\chi(\mathscr{F}(n)) = \dim_k \Gamma(X, \mathscr{F}(n)) = \dim_k \bigoplus_{P_i \in X, i=1,\ldots,k} A$ for all n. From the exact sequence

$$0 \to \mathscr{R}(n) \to \mathscr{F}(n-1) \to \mathscr{F}(n) \to \mathscr{Q}(n) \to 0$$

one obtains $\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = \chi(\mathscr{Q}(n)) - \chi(\mathscr{R}(n))$. The RHS is always zero so the RHS equals the zero polynomial at all integer points. Hartshorne Proposition I.7.3 implies there is a numerical polynomial $P(t)$ s.t. $P(n) = \chi(\mathscr{F}(n))$ for $n \gg 0$.

In fact, the RHS shows that $\chi(\mathscr{F}(n))$ is constant for all n so we may upgrade this to say the claimed Hilbert polynomial exists. □

Lemma 5.2. Assume the result for $\dim \operatorname{Supp} \mathscr{F} = \ell$. Then the Hilbert polynomial exists for $\dim \operatorname{Supp} \mathscr{F} = \ell + 1$.

Proof. Assume $\dim \operatorname{Supp} \mathscr{F} = \ell + 1$. Since the map $\mathscr{F}(-1) \to \mathscr{F}$ is always a generic isomorphism (i.e. an isomorphism at the generic point), $\dim \operatorname{Supp} \mathscr{R} = \dim \operatorname{Supp} \mathscr{Q} = \ell$. From

$$\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = \chi(\mathscr{Q}(n)) - \chi(\mathscr{R}(n))$$

we know that the RHS is given by the value of the difference of two Hilbert polynomials. Repeat the proof of Hartshorne Proposition I.7.3 to see that there is a such a Hilbert polynomial. □

[4] See also this post.

Remark 49. Hartshorne asserts there exists a Hilbert polynomial which agrees with $\chi(\mathscr{F}(n))$ for *all* $n \in \mathbb{Z}$. One may come to believe that it is not and find a *false counterexample* (which is false because $\chi(\mathcal{O}_X(n))$ needs not be equal to $\dim_k \Gamma(X, \mathcal{O}_X(n))$ for all $n \in \mathbb{Z}$ sufficiently small).

False "Counterexample". Let $X := \mathbb{P}^1_k$ and $\mathcal{O}_X$ be our $\mathscr{F}$. Then $\chi(\mathcal{O}_X(n)) = \dim_k \Gamma(X, \mathcal{O}_X(n)) = 0$ for all negative n. However, $\chi(\mathcal{O}_X) = 1$. There does not exist any such polynomial satisfying the property $P(n) = 0$ for $n < 0$ and $P(1) = 1$.

(b) Let P_M be the Hilbert polynomial from Hartshorne Theorem I.7.5. Let $P_{\mathscr{F}}$ be the polynomial constructed above. The n'th graded piece of M is $\Gamma(X, \mathscr{F}(n))$. Then $P_M(n) = \dim \Gamma(X, \mathscr{F}(n))$ for $n \gg 0$. OTOH, $P_{\mathscr{F}}(n) = \chi(\mathscr{F}(n)) = \dim \Gamma(X, \mathscr{F}(n))$ if we choose n large enough so that Hartshorne Theorem III.5.2 applies. By uniqueness of P_M, we get $P_M = P_{\mathscr{F}}$. □

Hartshorne III.5.3 *Arithmetic Genus.* Let X be a projective scheme of dimension r over a field k. We define the *arithmetic genus* p_a of X by

$$p_a(X) = (-1)^r \left(\chi\left(\mathcal{O}_X\right) - 1\right).$$

Note that it depends only on X, not on any projective embedding.

(a) If X is integral, and k algebraically closed, show that $H^0\left(X, \mathcal{O}_X\right) \cong k$, so that

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}\left(X, \mathcal{O}_X\right)$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1\left(X, \mathcal{O}_X\right).$$

[*Hint*: Use (I, 3.4).]

(b) If X is a closed subvariety of $\mathbb{P}^r_k$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

(c) If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a *birational* invariant. Conclude that a nonsingular plane curve of degree $d \geqslant 3$ is not rational. (This gives another proof of (II, 8.20.3) where we used the geometric genus.)

Proof. **(a)** X being a projective integral scheme implies it is a projective variety. So $\mathcal{O}(X) = k$ by Hartshorne Theorem I.3.4 and so $H^0(X, \mathcal{O}_X) = k$. Then,

$$\begin{aligned} p_a(X) &= (-1)^r(\chi(\mathcal{O}_X) - 1) = (-1)^r \left(-1 + \sum_{i=0}^{r} (-1)^i \dim_k H^i(X, \mathcal{O}_X)\right) \\ &= (-1)^r \sum_{i=1}^{r} (-1)^i \dim_k H^i(X, \mathcal{O}_X) = \sum_{i=1}^{r} (-1)^{r+i} \dim_k H^i(X, \mathcal{O}_X) \\ &= \sum_{i=0}^{r} (-1)^{r+i+1} = \sum_{i=0}^{r-1} (-1)^{r+1+i} \dim_k H^{i+1}(X, \mathcal{O}_X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X). \end{aligned}$$

By Grothendieck's Vanishing Theorem, we know $H^k(X, \mathcal{O}_X) = 0$ for $k \geq 2$ and so we take $r = 1$ get

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X).$$

(b) Let X be a closed subvariety of $\mathbb{P}^r_k$ of dimension n. From Hartshorne II.5.14, we know $P_X(0) = P_{\mathcal{O}_X}(0)$ since the two polynomials will agree for sufficiently large integer values. It follows

$$(-1)^n(P_X(0) - 1) = (-1)^n(P_{\mathcal{O}_X}(0) - 1) = (-1)^n(\chi(\mathcal{O}_X) - 1)$$

showing this definition of p_a agrees with the old.

(c) A birational map between nonsingular projective curves is always an isomorphism since we can extend the map from the open dense subsets to the whole curve. Since the composite of the two inverse maps restrict to the identity on a dense open subset, the extensions give inverse maps. Therefore, it induces an isomorphism on cohomology implying that $p_a(X)$ is a birational invariant.

From (b) and Hartshorne III.4.7, we know that $p_a(X) = \frac{1}{2}(d-1)(d-2) \geq 1$. However, $\dim_k(H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = 0$ since $\mathbb{P}^1_k \subseteq \mathbb{P}^2_k$ is a subscheme cut out by $x_1 = 0$ and we use Hartshorne III.4.7.

Remark 50. $p_a(X)$ is not a birational invariant if the nonosingularity hypothesis is dropped. Take for instance the nodal cubic C and $\mathbb{P}^1_k$. We know $p_a(\mathbb{P}^1_k) = 0$ and any singular plane cubic curve is rational. However, the nodal cubic $C : y^2 - x^3 - x^2 = 0$ is an elliptic curve so it has arithmetic genus 1.

□

Hartshorne III.5.4 Recall from (II, Ex. 6.10) the definition of the Grothendieck group $K(X)$ of a noetherian scheme X.

(a) Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Show that there is a (unique) additive homomorphism

$$P : K(X) \to \mathbb{Q}[z]$$

such that for each coherent sheaf $\mathscr{F}$ on $X, P(\gamma(\mathscr{F}))$ is the Hilbert polynomial of $\mathscr{F}$ (Ex. 5.2).

(b) Now let $X = \mathbb{P}^r_k$. For each $i = 0, 1, \ldots, r$, let L_i be a linear space of dimension i in X. Then show that

(1) $K(X)$ is the free abelian group generated by $\{\gamma(\mathcal{O}_{L_i}) \mid i = 0, \ldots, r\}$, and
(2) the map $P : K(X) \to \mathbb{Q}[z]$ is injective.

[*Hint*: Show that (1) ⇒ (2). Then prove (1) and (2) simultaneously, by induction on r, using (II, Ex. 6.10c).]

Proof. **(a)** There is a map Ψ from coherent sheaves to $\mathbb{Q}[z]$ sending $\mathscr{F}$ to $P_{\mathscr{F}}$ which is the Hilbert polynomial from Hartshorne III.5.2. From additivity of χ, we know that

$P_{\mathscr{F}''} - P_{\mathscr{F}} + P_{\mathscr{F}'} = 0$ for any give SES $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves. So the universal property of $K(X)$ says there is a $P : K(X) \to \mathbb{Q}[z]$ making the diagram commute

$$\begin{array}{ccc} \mathbb{Z}\{\text{coherent sheaves}\} & & \\ {\scriptstyle\gamma}\big\downarrow & \searrow^{\Phi} & \\ K(X) & \xrightarrow{\exists! P} & \mathbb{Q}[z] \end{array}$$

where $\mathbb{Z}\{\text{coherent sheaves}\}$ is the free abelian group on coherent sheaves.

(b) We being by showing (1)⇒(2) and work towards proving (1) and (2) simultaneously.

Lemma 5.3. If $K(X) \cong \mathbb{Z}\langle\gamma(\mathcal{O}_{L_i}) : i = 0, \dots, r\}$, then $P : K(X) \hookrightarrow \mathbb{Q}[z]$.

Proof. Assume (1). We understand how P behaves so we will need to compute the Hilbert polynomial. One can show[5] that

$$P(\gamma(\mathcal{O}_{\mathbb{P}^r}))(z) = \binom{z+r}{z} \in \mathbb{Q}[z]$$

because $\chi(\mathcal{O}_{\mathbb{P}^r}(m)) = \binom{m+r}{m}$. Then it is clear that the map is injective because there cannot possibly be any linear relations between such polynomials (for if there were, it would have to hold for all z and we take z sufficiently large and compare orders). Hence, (1) always implies (2). □

Lemma 5.4. By an inductive argument, (1) and (2) holds for all r.

Proof. For $r = 0$, we know that $K(X) = K(\mathbb{A}^1_k) = \mathbb{Z}$. A point $L_0 \in \mathbb{P}^1_k$ is just a line $\mathbb{A}^1_k$. So $K(X)$ is generated by $\gamma(\mathcal{O}_{L_0})$ where L_0 is just a point. This is because every f.g. module over $k[x]$ has a two-step free resolution. Therefore the Grothendieck group is $\Gamma(\mathcal{O}_{L_0}) \cong \mathbb{Z}\langle\gamma(\widetilde{k[x]})\rangle$. So we have the base case. Afterwards, (1)⇒(2) gives statement (2) for $r = 0$.

Now we use induction and assume the result for some r and prove it for $r+1$. We know $P : K(\mathbb{P}^r) \to \mathbb{Q}[z]$ is injective. By Hartshorne II.6.10(c), there is an exact sequence

$$K(L_r) \to K(\mathbb{P}^{r+1}_k) \to K(\mathbb{A}^r_k) \to 0.$$

As $L_r \cong \mathbb{P}^r_k$, the sequence is

$$\mathbb{Z}^r \to K(\mathbb{P}^{r+1}_k) \to K(\mathbb{A}^r_k) \to 0.$$

It is a known fact that $K(\mathbb{A}^r_k) \cong \mathbb{Z}$. So the sequence is

$$\mathbb{Z}^r \to K(\mathbb{P}^{r+1}_k) \to K(\mathbb{A}^r_k) \to 0.$$

The LHS map is injective because the elements of $K(\mathbb{P}^r_k)$ have nonzero Hilbert polynomials. So the sequence is a SES

$$0 \to \mathbb{Z}^r \to K(\mathbb{P}^{r+1}_k) \to \mathbb{Z} \to 0$$

and it splits due to $\mathbb{Z}$ being projective. So $K(\mathbb{P}^{r+1}_k) \cong \mathbb{Z}^r \oplus \mathbb{Z}$. Since the basis of $\mathbb{Z}^r$ is $\{\gamma(\mathcal{O}_{L_i}) : i = 0, \dots, r\}$, we show the generator of $K(\mathbb{A}^{r+1}_k)$ is $\gamma(\mathcal{O}_{L_{r+1}})$. Indeed, the generator must be $\gamma(\mathcal{O}_{L_{r+1}})$ because the restriction of $\mathcal{O}_{L_{r+1}} = \mathcal{O}_{\mathbb{P}^{r+1}_k}$ to $\mathbb{A}^r_k$ is precisely the sheaf $\widetilde{k[x_0, \dots, x_r]}$. This proves (1) for the case of $r+1$. The previous lemma proves (2) for $r+1$. □

[5] See these notes by Ravi Vakil.

□

Hartshorne III.5.5 Let k be a field, let $X = \mathbb{P}^r_k$, and let Y be a closed subscheme of dimension $q \geqslant 1$, which is a complete intersection (II, Ex. 8.4). Then: (a) for all $n \in \mathbb{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

is surjective. (This gives a generalization and another proof of (II, Ex. 8.4c), where we assumed Y was normal.)

(b) Y is connected;

(c) $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbb{Z}$;

(d) $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$.[*Hint*: Use exact sequences and induction on the codimension, starting from the case $Y = X$ which is (5.1).]

Proof. **(a)** Consider the case when Y is of codimension 1. By Hartshorne II.8.4, $\mathscr{I}_Y$ is the ideal sheaf cut out by a single hypersurface of degree d. There is a SES $0 \to \mathscr{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$. Twist by n and take cohomology to obtain

$$0 \to H^0(X, \mathscr{I}_Y(n)) \to H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n)) \to H^1(X, \mathscr{I}_Y(n)) \to \dots$$

and we want $H^1(X, \mathscr{I}_Y(n)) = 0$. We know $\mathscr{I}_Y = \mathcal{O}_X(-d)$ and so, $\mathscr{I}_Y(n) = \mathcal{O}_X(n-d)$. This has vanishing cohomology $H^i(X, \mathcal{O}_X(n-d)) = 0$ for $0 < i < r$. Assuming we are not in the vacuous case where $r = 1$, we have $r \geq 2$ and this is clear.

Use this post. **[Incomplete]**

(b) Take $n = 0$ and (a) says we have a surjective map of rings $k \to H^0(Y, \mathcal{O}_Y)$. This map cannot be surjective if Y is disconnected.

(c) We can do the proof inductively on the dimension of Y i.e. $Y = H_1 \cap \dots \cap H_r$ and $\dim Y = n - r$. Let $Z := H_1 \cap \dots \cap H_{r-1}$ and there is an exact sequence

$$0 \to \mathcal{O}_Z(-d) \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

where $\deg H_r = d$. If we knew the result for $\dim Y = n - (r-1) = n - r + 1$, then we can take the LES and use Hartshorne Theorem III.5.1. Indeed, we would get

$$0 \to H^0(X, \mathcal{O}_Z(-d)) \to \dots \to H^i(X, \mathcal{O}_Z(-d)) \to H^i(X, \mathcal{O}_X) \to H^i(Y, \mathcal{O}_Y) \to H^{i+1}(X, \mathcal{O}_Z(-d)) \to \dots$$

and taking $0 < i < n-r$, we know $H^{i+1}(X, \mathcal{O}_Z(-d))$ always vanishes by $0 < i+1 < n-r+1$ and $H^i(X, \mathcal{O}_X)$ does as well. Hence, $H^i(Y, \mathcal{O}_Y)$ vanishes for all $0 < i < n-r$. Now if we had twisted the SES by n, the result would still apply via the induction step. This descending induction fails if we are out of the range of $0 < i < q := n - r$.

(d) The base case for the induction is $\operatorname{codim}(Y, X) = 0$. We get $Y = X$ and need $p_a(Y) = \dim_k H^r(Y, \mathcal{O}_Y)$. This is clear from Hartshorne III.5.3(a) and Hartshorne Theorem III.5.1(b).

Assume the result for some codimension $n < r$ and we prove the claim for Y of codimension $n+1 \leq r$ or just $\dim Y = r - n - 1$. From Hartshorne II.8.4, we can write $Y = H_1 \cap H_2 \cap \dots \cap H_{n+1}$ and assume $\deg H_{n+1} = d$ inside $\mathbb{P}^r_k$. Let $Z := H_1 \cap \dots \cap H_n$. Then there is a SES

$$0 \to \mathcal{O}_Z(-d) \to \mathcal{O}_Z \to i_*\mathcal{O}_Y \to 0.$$

in which $i : Y \hookrightarrow Z$ is the inclusion. **[Incomplete]**

From this SES, we know that $\chi(\mathcal{O}_Z) = \chi(i_*\mathcal{O}_Y)) + \chi(\mathcal{O}_Z(-d))$. **[Incomplete]**

□

Remark 51. A faulty proof of (a) is discussed here.

Hartshorne III.5.6 *Curves on a Nonsingular Quadric Surface.* Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbb{P}^3_k$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that $\operatorname{Pic} Q \cong \mathbb{Z} \oplus \mathbb{Z}$, so we can talk about the type (a, b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathscr{L}(Y)$ by $\mathcal{O}_Q(a, b)$. Thus for any $n \in \mathbb{Z}, \mathcal{O}_Q(n) = \mathcal{O}_Q(n, n)$.
(a) Use the special cases $(q, 0)$ and $(0, q)$, with $q > 0$, when Y is a disjoint union of q lines

$\mathbb{P}^1$ in Q, to show:

(1) if $|a - b| \leqslant 1$, then $H^1(Q, \mathcal{O}_Q(a, b)) = 0$;

(2) if $a, b < 0$, then $H^1(Q_{\mathcal{O}}(a, b)) = 0$;

(3) If $a \leqslant -2$, then $H^1(Q, \mathcal{O}_Q(a, 0)) \neq 0$.

(b) Now use these results to show:

(1) if Y is a locally principal closed subscheme of type (a, b), with $a, b > 0$, then Y is connected;

(2) now assume k is algebraically closed. Then for any $a, b > 0$, there exists an irreducible nonsingular curve Y of type (a, b). Use (II, 7.6.2) and (II, 8.18).

(3) an irreducible nonsingular curve Y of type $(a, b), a, b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a-b| \leqslant 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbb{P}^3$. The simplest is the one of type $(1, 3)$, which is just the rational quartic curve (I, Ex. 3.18).

(c) If Y is a locally principal subscheme of type (a, b) in Q, show that $p_a(Y) = ab - a - b + 1$.

[*Hint*: Calculate Hilbert polynomials of suitable sheaves, and again use the special case $(q, 0)$ which is a disjoint union of q copies of $\mathbb{P}^1$. See (V, 1.5.2) for another method.]

Proof. **[Incomplete]** □

Hartshorne III.5.7 Let X (respectively, Y) be proper schemes over a noetherian ring A. We denote by $\mathscr{L}$ an invertible sheaf.

(a) If $\mathscr{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathscr{L}$ is ample on Y, where $i : Y \to X$ is the inclusion.

(b) $\mathscr{L}$ is ample on X if and only if $\mathscr{L}_{\text{red}} = \mathscr{L} \otimes \mathcal{O}_{X_{\text{red}}}$ is ample on X_{red}.

(c) Suppose X is reduced. Then $\mathscr{L}$ is ample on X if and only if $\mathscr{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

(d) Let $f : X \to Y$ be a finite surjective morphism, and let $\mathscr{L}$ be an invertible sheaf on Y. Then $\mathscr{L}$ is ample on Y if and only if $f^*\mathscr{L}$ is ample on X.

[*Hints*: Use (5.3) and compare (Ex. 3.1, Ex. 3.2, Ex. 4.1, Ex. 4.2). See also Hartshorne [5, Ch. I §4] for more details.]

PROOF. **(a)** By Hartshorne Proposition III.5.3, $\mathscr{L}$ is ample on X iff $H^k(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $k > 0$ and all $n \geq n_0$ with n_0 depending on $\mathscr{F}$. Given $\mathscr{G}$ a coherent sheaf on Y. Apply Hartshorne Lemma III.2.10 and Hartshorne II.5.1(d)

$$H^k(Y, \mathscr{F} \otimes (i^*\mathscr{L})^n) = H^k(X, i_*(\mathscr{F} \otimes (i^*\mathscr{L}^n)) = H^k(X, i_*\mathscr{F} \otimes \mathscr{L}^n)$$

which vanishes because $i_*\mathscr{F}$ is coherent (see Hartshorne Caution II.5.8.1) and we use the criterion above. Indeed, we take $n \gg 0$ large so that the RHS vanishes for all $k > 0$.

(b) ($\Rightarrow$). Observe that we have the following sequences of isomorphisms where $j : X_{red} \to X$ is the map from Hartshorne II.2.3

$$\begin{aligned} H^k(X_{red}, \mathscr{F} \otimes_{\mathcal{O}_{X_{red}}} \mathscr{L}_{red}^n) &\cong H^k(X, j_*(\mathscr{F} \otimes_{\mathcal{O}_{X_{red}}} \mathscr{L}_{red}^n) \\ &\cong H^k(X, j_*(\mathscr{F} \otimes_{\mathcal{O}_{X_{red}}} (\mathscr{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{red}}^n))) \\ &\cong H^k(X, j_*(\mathscr{F} \otimes_{\mathcal{O}_{X_{red}}} j^*\mathscr{L}^n)) \\ &\cong H^k(X, j_*\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n) \qquad \text{(Projection formula)} \\ &\cong H^k(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n). \end{aligned}$$

This uses the fact that X_{red} is closed in X. Now the result follows from Hartshorne Proposition III.5.3 because the LHS vanishes for all $k > 0$ whenever $n \geq n_0$ iff the RHS does the same. It follows that $\mathscr{L}$ being ample on X is implies $\mathscr{L}_{red}$ being ample on X_{red}.
($\Leftarrow$). Let $\mathscr{F}$ be a coherent sheaf on X and $\mathscr{N}$ the nilpotent sheaf of $\mathcal{O}_X$. Then $\mathscr{N}^r = 0$ for large $r > 0$ since X is Noetherian. Take the filtration (as in a previous exercise),

$$\mathscr{F} \supseteq \mathscr{N}\mathscr{F} \supseteq \cdots \supseteq \mathscr{N}^{r-1}\mathscr{F} \supseteq \mathscr{N}^r\mathscr{F} = 0$$

and for each of $i = 1, 2, \ldots, r-1$, there is a SES

$$0 \to \mathscr{N}^i\mathscr{F} \to \mathscr{N}^{i-1}\mathscr{F} \to \mathscr{N}^{i-1}\mathscr{F}/\mathscr{N}^i\mathscr{F} \to 0.$$

Then we get a LES in cohomology

$$\begin{aligned} \ldots \to H^k(X, \mathscr{N}^i\mathscr{F} \otimes \mathscr{L}^n) \to H^p(X, \mathscr{N}^{i-1}\mathscr{F} \otimes \mathscr{L}^n) \\ \to H^k(X, \mathscr{N}^{i-1}\mathscr{F}/\mathscr{N}^i\mathscr{F} \otimes \mathscr{L}^n) \to H^{k+1}(X, \mathscr{N}^i\mathscr{F} \otimes \mathscr{L}^n) \to \ldots \end{aligned}$$

Our goal is to show that $H^k(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $p > 0$ and n sufficiently large. Use descending induction on i since we know $\mathscr{N}^i\mathscr{F} = 0$ for $i = r$. Each $\mathscr{N}^{i-1}\mathscr{F}/\mathscr{N}^i\mathscr{F}$ is a coherent $\mathcal{O}_{X_{red}}$-module for each $i = 1, \ldots, r-1$ so

$$H^k(X, \mathscr{N}^{i-1}\mathscr{F}/\mathscr{N}^i\mathscr{F} \otimes \mathscr{L}^n) = H^k(X_{red}, \mathscr{N}^{i-1}\mathscr{F}/\mathscr{N}^i\mathscr{F} \otimes \mathscr{L}^n) = 0$$

for all $k > 0$ when n is sufficiently large. It follows $H^k(X, \mathscr{N}^r\mathscr{F} \otimes \mathscr{L}^n) = H^k(X, \mathscr{N}^{r-1}\mathscr{F}/\mathscr{N}^r\mathscr{F} \otimes \mathscr{L}^n) = 0$ vanishing for n large implies $H^k(X, \mathscr{N}^{r-1}\mathscr{F} \otimes \mathscr{L}^n)$ vanishes. This is the base case of $i = r$. Then applying descending induction gives the result for all $0 \leq i \leq r$. In particular, the case of $i = 0$ is

$$\exists n_0, \text{ s.t. } H^k(X, \mathscr{F} \otimes \mathscr{L}^n) = 0 \ \forall k > 0 \qquad \& \qquad n \geq n_0.$$

(c) ($\Rightarrow$). Let $j : X_i \to X$ be the inclusion of an irreducible component. Let $\mathscr{F}$ be a coherent sheaf on X_i. If $k > 0$, then

$$\begin{aligned} H^k(X_i, \mathscr{F} \otimes (\mathscr{L} \otimes \mathcal{O}_{X_i})^n) &\cong H^k(Y, j_*(\mathscr{F} \otimes (\mathscr{L} \otimes \mathcal{O}_{X_i})^n) \\ &\cong H^k(X, j_*(\mathscr{F} \otimes (j^*\mathscr{L})^n)) \\ &\cong H^k(X, j_*\mathscr{F} \otimes \mathscr{L}^n) \end{aligned}$$

is zero for $n_0 \leq n$ (note that we use Hartshorne Caution II.5.8.1 to justify that $j_*\mathscr{F}$ is coherent). Here, we made the observation that

$$j^*\mathscr{L}^n \cong j^{-1}\mathscr{L}^n \otimes_{j^{-1}\mathcal{O}_X} \mathcal{O}_X \cong \mathscr{L}^n \otimes_{j^{-1}\mathcal{O}_X} \mathcal{O}_{X_i} \cong \mathscr{L}^n \otimes \mathcal{O}_{X_i} \cong (\mathscr{L} \otimes \mathcal{O}_{X_i})^n.$$

($\Leftarrow$). From (b), assume WLOG X is reduced. Let $X = \bigcup_{i=1}^r X_i$ be its union over irreducible components. Let $\mathscr{I}_i$ be the sheaf of ideals of X_i. If $r = 1$, the result is trivial since $X = X_1$. Assume the inductive hypothesis and consider the SES

$$0 \to \mathscr{I}_1\mathscr{F} \to \mathscr{F} \to \mathscr{F}/\mathscr{I}_1\mathscr{F} \to 0$$

where $\mathscr{F}$ is a coherent sheaf on X. Consider the LES of cohomology

$$\ldots \to H^k(X, \mathscr{I}_1\mathscr{F} \otimes \mathscr{L}^n) \to H^k(X, \mathscr{F} \otimes \mathscr{L}^n) \to H^k(X, \mathscr{F}/\mathscr{I}_1\mathscr{F} \otimes \mathscr{L}^n) \to H^{k+1}(X, \mathscr{I}_1 \otimes \mathscr{L}^n) \to \ldots$$

By hypothesis, we have $H^k(X, \mathscr{F}/\mathscr{I}_1\mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $k > 0$ when n is sufficiently large (since $\mathscr{F}/\mathscr{I}_1\mathscr{F}$ is a sheaf on X_i). The induction hypothesis says that $H^k(X, \mathscr{I}_1\mathscr{F} \otimes \mathscr{L}^n +)$ for all $k > 0$ and n large (since $\mathscr{I}_1\mathscr{F}$ can be computed via $\operatorname{supp}(\mathscr{I}_1\mathscr{F}) \subseteq X_2 \cup \cdots \cup X_r$). It follows $H^k(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $p > 0$ and n large.

(d) This proof is taken from Chapter 1 Section 4 of "Ample Subvarieties of Algebraic Varieties" by Robin Hartshorne. Suppose $f^*\mathscr{L}$ is ample on X. From part (b) and (c), we assume WLOG that both X and Y are integral.

Apply Hartshorne III.4.2(c) to a coherent sheaf $\mathscr{F}$ on Y to find a coherent sheaf $\mathscr{G}$ on X with a map $\beta : f_*\mathscr{G} \to \mathscr{F}^{\oplus m}$ s.t. β induces an isomorphism on the generic point.

We apply Noetherian induction on $\operatorname{supp}(\mathscr{F})$. Let $\mathscr{K}$ and $\mathscr{M}$ be the kernel and cokernel of β i.e. there are SESs

$$0 \to \mathscr{K} \to f_*(\mathscr{G}) \to \operatorname{im}\beta \to 0$$

and

$$0 \to \operatorname{im}\beta \to \mathscr{F}^{\oplus m} \to \mathscr{M} \to 0.$$

In this case, $\operatorname{supp}(\mathscr{K}), \operatorname{supp}(\mathscr{M}) \subsetneq Y$ because β is a generic isomorphism. Since $\mathscr{K}$ and $\mathscr{C}$ are coherent on Y, we know that

$$H^k(Y, \mathscr{K} \otimes \mathscr{L}^n) = H^k(Y, \mathscr{M} \otimes \mathscr{L}^n) = 0 \text{ for all } i > 0 \text{ and } n \text{ sufficiently large.}$$

Consider the LES

$$\ldots \to H^k(Y, \operatorname{im}\beta \otimes \mathscr{L}^n) \to H^k(Y, \mathscr{F}^{\oplus m} \otimes \mathscr{L}^n) \to H^k(Y, \mathscr{M} \otimes \mathscr{L}^n) \to H^{k+1}(Y, \operatorname{im}\beta \otimes \mathscr{L}^n) \to \ldots$$

and the vanishing of $H^k(Y, \mathscr{M} \otimes \mathscr{L}^n)$ for all $k > 0$ and n sufficiently large give the isomorphism

$$H^k(Y, \mathscr{F} \otimes \mathscr{L}^n)^{\oplus m} \cong H^k(Y, \operatorname{im}(\beta) \otimes \mathscr{L}^n).$$

Using the other sequence, we get

$$H^k(Y, \mathscr{F} \otimes \mathscr{L}^n)^{\oplus m} \cong H^k(Y, \operatorname{im}(\beta) \otimes \mathscr{L}^n) \cong H^k(Y, f_*(\mathscr{G}) \otimes \mathscr{L}^n).$$

Apply the projection formula to the RHS to get

$$H^k(Y, \mathscr{F} \otimes \mathscr{L}^n)^{\oplus m} \cong H^k(Y, f_*(\mathscr{G} \otimes f^*\mathscr{L}^n)) \cong H^k(X, \mathscr{G} \otimes (f^*\mathscr{L})^n)$$

and the RHS vanishes for all $i > 0$ whenever n is sufficiently large. □

Hartshorne III.5.8 Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.

(a) If X is irreducible and nonsingular, then X is projective by (II, 6.7).

(b) If X is integral, let $\widetilde{X}$ be its normalization (II, Ex. 3.8). Show that $\widetilde{X}$ is complete and nonsingular, hence projective by (a). Let $f : \widetilde{X} \to X$ be the projection. Let $\mathscr{L}$ be a very ample invertible sheaf on $\widetilde{X}$. Show there is an effective divisor $D = \sum P_i$ on $\widetilde{X}$ with $\mathscr{L}(D) \cong \mathscr{L}$, and such that $f(P_i)$ is a nonsingular point of X, for each i. Conclude that there is an invertible sheaf $\mathscr{L}_0$ on X with $f^*\mathscr{L}_0 \cong \mathscr{L}$. Then use (Ex. 5.7d), (II, 7.6) and (II, 5.16.1) to show that X is projective.

(c) If X is reduced, but not necessarily irreducible, let $X_1, \ldots, X_r$ be the irreducible components of X. Use (Ex. 4.5) to show $\operatorname{Pic} X \to \bigoplus \operatorname{Pic} X_i$ is surjective. Then use (Ex. 5.7c) to show X is projective.

(d) Finally, if X is any one-dimensional proper scheme over k, use (2.7) and (Ex. 4.6) to show that $\operatorname{Pic} X \to \operatorname{Pic} X_{\text{red}}$ is surjective. Then use (Ex. 5.7b) to show X is projective.

Proof. **(a)** Since X is nonsingular, we know X is reduced by checking the stalks. Hence, X is integral. Then Hartshorne Proposition II.6.7 applies to give X projective.

(b) For curves, the normalization $\widetilde{X}$ is always nonsingular. From Hartshorne II.3.8, the normalization $f : \widetilde{X} \to X$ is a finite morphism. By Hartshorne II.4.1, $\widetilde{X} \to X$ is proper. Then $\widetilde{X} \to X \to \operatorname{Spec}(k)$ is a composition of proper morphisms so $\widetilde{X} \to \operatorname{Spec}(k)$ is proper. So (a) says $\widetilde{X}$ is projective.

Now we prove the existence of the effective divisor D. This is done in the following lemma.

Lemma 5.5. There is an effective divisor $D = \sum P_i$ on $\widetilde{X}$ with $\mathscr{L}(D) \cong \mathscr{L}$ and s.t. $f(P_i)$ is a nonsingular point of X for each i.

Proof. [Incomplete] □

The existence of $\mathscr{L}_0$ on X with $f^*\mathscr{L}_0 \cong \mathscr{L}$ follows from the SES in Hartshorne II.6.9(a). By Hartshorne III.5.7(d), $\mathscr{L}_0$ is ample on X. By Hartshorne II.7.6, $\mathscr{L}_0^m$ is very ample relative to $\operatorname{Spec} k$ for some $m > 0$. By Hartshorne Remark II.5.16.1, X is projective.

(c) The map $\operatorname{Pic} X \to \bigoplus \operatorname{Pic} X_i$ is precisely a map $H^1(X, \mathcal{O}_X^*) \to \bigoplus H^1(X_i, \mathcal{O}_{X_i}^*)$. Consider the SES

$$0 \to \mathscr{M} \to \mathcal{O}_X^* \to \bigoplus_{i=1}^{r} (\psi_i)_*(\mathcal{O}_{X_i})^* \to 0$$

where $\psi_i : X_i \hookrightarrow X$. Take the LES of cohomology, use Grothendieck's Vanishing Thoerem, and cohomology's preservation of direct limits to get

$$0 \to \ldots \to H^1(X, \mathcal{O}_X^*) \to \bigoplus_{i=1}^{r} H^1(X, ((\psi_i)_* \mathcal{O}_{X_i})^*) \to 0.$$

Apply Hartshorne III.4.1(a) to get $H^1(X, (\psi_i)_* \mathcal{O}_{X_i})^*) = H^1(X_i, \mathcal{O}_{X_i}^*) = \operatorname{Pic} X_i$. So we have surjecitivty. By Hartshorne III.5.7(c) and the previous part, we can find an ample sheaf on X. The same argument in (c) applies to say X is projective.

(d) See [10, Lemma 09NY] and [10, Proposition 09NZ] for the proof of the claim. In particular, the arguments given there shows that $\operatorname{Pic} X \to \operatorname{Pic} X_{\text{red}}$ is surjective and then the proof is clear. The following is the proof Hartshorne likely wanted.

To do this, we use Hartshorne II.4.6 in the following manner. Consider the spaces $(X, \mathcal{O}_X/\mathscr{I})$, $(X, \mathcal{O}_X/\mathscr{I}^2)$,...,$(X, \mathcal{O}_X/\mathscr{I}^{n+1}), (X, \mathcal{O}_X)$ which we call $X^1, X^2, \ldots, X^{n+1}, X$ respectively. Now we use Hartshorne III.4.6 (since $\mathscr{I}^{2i} = 0$ on $(X, \mathcal{O}_X/\mathscr{I}^{i+1})$) to get

$$\ldots \to H^1(X, \mathscr{I}^i) \to \operatorname{Pic} X^{i+1} \to \operatorname{Pic} X^i \to H^2(X, \mathscr{I}^i) \to \ldots$$

and since $H^2(X, \mathscr{I}^i) = 0$, we get $\operatorname{Pic} X^{i+1} \to \operatorname{Pic} X^i$ surjective. So the composition

$$\operatorname{Pic} X \to \operatorname{Pic} X^{n+1} \to \operatorname{Pic} X^n \to \ldots \to \operatorname{Pic} X^2 \to \operatorname{Pic} X_{red}$$

is a surjection. By functoriality, this is precisely the map we needed to be surjective. □

Hartshorne III.5.9 *A Nonprojective Scheme.* We show the result of (Ex. 5.8) is false in dimension 2. Let k be an algebraically closed field of characteristic 0 , and let $X = \mathbb{P}_k^2$. Let ω be the sheaf of differential 2 -forms (II, §8). Define an infinitesimal extension X' of X by ω by giving the element $\xi \in H^1(X, \omega \otimes \mathscr{T}\ )$ defined as follows (Ex. 4.10). Let x_0, x_1, x_2 be the homogeneous coordinates of X, let U_0, U_1, U_2 be the standard open covering, and let $\xi_{ij} = (x_j/x_i)\, d\,(x_i/x_j)$. This gives a Čech 1-cocycle with values in Ω_X^1, and since $\dim X = 2$, we have $\omega \otimes \mathscr{T} \cong \Omega^1$ (II, Ex. 5.16b). Now use the exact sequence $\ldots \to H^1(X, \omega) \to \operatorname{Pic} X' \to \operatorname{Pic} X \xrightarrow{\delta} H^2(X, \omega) \to \ldots$ of (Ex. 4.6) and show δ is injective. We have $\omega \cong \mathcal{O}_X(-3)$ by (II, 8.20.1), so $H^2(X, \omega) \cong k$. Since char $k = 0$, you need only show that $\delta(\mathcal{O}(1)) \neq 0$, which can be done by calculating in Čech cohomology. Since $H^1(X, \omega) = 0$, we see that Pic $X' = 0$. In particular, X' has no ample invertible sheaves, so it is not projective.

Indeed, let D be an ample divisor on X. Then D determines an element $c_1(D) \in H^1\,(X, \Omega^1)$ which we use to define X', as above. Then for any divisor E on X one can show that $\delta(\mathscr{L}(E)) = (D.E)$, where $(D.E)$ is the intersection number (Chapter V), considered as an element of k. Hence if E is ample, $\delta(\mathscr{L}(E)) \neq 0$. Therefore X' has no ample divisors.

On the other hand, over a field of characteristic $p > 0$, a proper scheme X is projective if and only if X_{red} is!

Note. In fact, this result can be generalized to show that for any nonsingular projective surface X over an algebraically closed field k of characteristic 0 , there is an infinitesimal extension X' of X by ω, such that X' is not projective over k.

Proof. □

Hartshorne III.5.10 Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \ldots \to \mathscr{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geqslant n_0$, the sequence of global sections is exact.

$$\Gamma\left(X, \mathscr{F}^1(n)\right) \to \Gamma\left(X, \mathscr{F}^2(n)\right) \to \ldots \to \Gamma\left(X, \mathscr{F}^r(n)\right).$$

Proof. Twist the sequence by n and we get the induced sequence of global sections

$$\Gamma\left(X, \mathscr{F}^1(n)\right) \stackrel{\phi_1}{\to} \Gamma\left(X, \mathscr{F}^2(n)\right) \stackrel{\phi_2}{\to} \ldots \stackrel{\phi_{r-1}}{\to} \Gamma\left(X, \mathscr{F}^r(n)\right).$$

To prove exactness, we need $\operatorname{im}\phi_i = \ker\phi_{i+1}$. Let us above notation and write $\phi_i : \mathscr{F}^i \to \mathscr{F}^{i+1}$ for the morphism of sheaves as well. For every i, there is a SES of coherent sheaves

$$0 \to \operatorname{im}\phi_i \to \mathscr{F}^{i+1} \to \operatorname{im}\phi_{i+1} \to 0.$$

If we twist this by a sufficiently large $n \geq n_i$ and take cohomology, Hartshorne Theorem III.5.2(b) says

$$0 \to \Gamma(X, \operatorname{im}\phi_i(n)) \to \Gamma(X, \mathscr{F}^{i+1}) \to \Gamma(X, \operatorname{im}\phi_{i+1}(n)) \to 0.$$

Now we note that there is also a SES (by exactness of the sequence of coherent sheaves)

$$0 \to \Gamma(X, \ker\phi_i(n)) \to \Gamma(X, \mathscr{F}^i(n)) \to \Gamma(X, \operatorname{im}\phi_i(n)) \to 0$$

and we can splice these two to obtain the exact sequence

$$0 \to \Gamma(X, \ker\phi_i(n)) \to \Gamma(X, \mathscr{F}^i(n)) \to \Gamma(X, \mathscr{F}^{i+1}(n)) \to \Gamma(X, \operatorname{im}\phi_{i+1}(n)) \to 0.$$

If we splice together everything, we then get the exact sequence

$$0 \to \Gamma(X, \ker\phi_1) \to \Gamma(X, \mathscr{F}^1(n)) \stackrel{\phi_1}{\to} \ldots \stackrel{\phi_{r-1}}{\to} \Gamma(X, \mathscr{F}^r(n)) \to \Gamma(X, \operatorname{coker}\phi_r(n)) \to 0$$

provided we take $n \geq \max_{i=1}^r n_i$. Drop the two terms at the end on the left and right to get the desired sequence. □

6. Ext Groups and Sheaves

Hartshorne III.6.1 Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{F}', \mathscr{F}'' \in \mathfrak{MN}(X)$. An extension of $\mathscr{F}''$ by $\mathscr{F}'$ is a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

in $\mathfrak{Mod}(X)$. Two extensions are **isomorphic** if there is an isomorphism of the short exact sequences, inducing the identity maps on $\mathscr{F}'$ and $\mathscr{F}''$. Given an extension as above consider the long exact sequence arising from $\operatorname{Hom}(\mathscr{F}'', \cdot)$, in particular the map

$$\delta : \operatorname{Hom}(\mathscr{F}'', \mathscr{F}'') \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}'),$$

and let $\xi \in \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ be $\delta(1_{\mathscr{F}''})$. Show that this process gives a one-to-one correspondence between isomorphism classes of extensions of $\mathscr{F}''$ by $\mathscr{F}'$, and elements of the group $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$. For more details, see, e.g., Hilton and Stammbach [1, Ch. III].

Proof. Given an isomorphism class of extensions, naturality of the connecting homomorphism implies $\delta(1_{\mathscr{F}''})$ is well-defined up on such classes.

Given a an element ξ of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$. We need to construct an $\mathscr{F}$. There are enough injectives so choose an $\mathscr{I}$ injective and look at the SES

$$0 \to \mathscr{F}' \to \mathscr{I} \to \mathscr{G} \to 0.$$

Apply $\mathrm{Hom}(\mathscr{F}'', -)$ and take derived functors to get

$$0 \to \mathrm{Hom}(\mathscr{F}'', \mathscr{F}') \to \mathrm{Hom}(\mathscr{F}'', \mathscr{I}) \to \mathrm{Hom}(\mathscr{F}'', \mathscr{G}) \to \mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}') \to 0.$$

So the RHS map is surjective which means we can pick $\beta : \mathscr{F}'' \to \mathscr{G}$ such that $\delta(\beta) = \xi$. Now define a sheaf $\mathscr{H}$ as the pull-back of the RHS square fitting in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{H} & \longrightarrow & \mathscr{F}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{I} & \longrightarrow & \mathscr{G} & \longrightarrow & 0 \end{array}$$

We show that $\mathscr{H}$ maps to $\xi \in \mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}')$. Indeed, one chases the element $1 \in \mathrm{Hom}(\mathscr{F}'', \mathscr{F}')$ and checks that it lands on ξ using the commutative diagram

$$\begin{array}{ccc} 1_{\mathscr{F}''} \in \mathrm{Hom}(\mathscr{F}'', \mathscr{F}') & \longrightarrow & \mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}') \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathscr{F}'', \mathscr{G}) & \longrightarrow & \xi \in \mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}) \end{array}$$

Clearly the two operations are inverse to each other. □

Hartshorne III.6.2 Let $X = \mathbb{P}^1_k$, with k an infinite field.

(a) Show that there does not exist a projective object $\mathscr{P} \in \mathfrak{Mod}(X)$, together with a surjective map $\mathscr{P} \to \mathcal{O}_X \to 0$.

[*Hint*: Consider surjections of the form $\mathcal{O}_V \to k(x) \to 0$, where $x \in X$ is a closed point, V is an open neighborhood of x, and $\mathcal{O}_V = j_!(\mathcal{O}_X|_V)$, where $j : V \to X$ is the inclusion.]

(b) Show that there does not exist a projective object $\mathscr{P}$ in either $Q_{co}(X)$ or $\mathbb{C}\rtimes\simeq(X)$ together with a surjection $\mathscr{P} \to \mathcal{O}_X \to 0$. [*Hint*: Consider surjections of the form $\mathscr{L} \to \mathscr{L} \otimes k(x) \to 0$, where $x \in X$ is a closed point, and $\mathscr{L}$ is an invertible sheaf on X.]

Proof. **[Incomplete]** □

Hartshorne III.6.3 Let X be a noetherian scheme, and let $\mathscr{F}, \mathscr{S} \in \mathfrak{Mod}(X)$.

(a) If $\mathscr{F}, \mathscr{G}$ are both coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent, for all $i \geqslant 0$.

(b) If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent, for all $i \geqslant 0$.

Proof. (a) Combine Hartshorne Proposition II.5.4 and Proposition III.6.2 to find that for any open affine subset $U \subseteq X$, we have

$$\mathscr{E}xt^i_X(\mathscr{F}, \mathscr{G})|_U \cong \mathscr{E}xt^i_U(\mathscr{F}|_U, \mathscr{G}|_U) \cong \mathscr{E}xt^i_U(\widetilde{M}, \widetilde{N})$$

where $\widetilde{M} \cong \mathscr{F}|_U$ and $\widetilde{N} \cong \mathscr{G}|_U$. Our goal is to show that the RHS is $\cong \widetilde{L}$ for some A-module L and $U = \mathrm{Spec}\, A$ for all i. Choose a free resolution $F_\bullet \to M \to 0$ of M. Apply $\sim$ to get $\widetilde{F_\bullet} \to \widetilde{M} \to 0$ exact. Then apply $\mathscr{H}om(-, \widetilde{N})$ and use Hartshorne Proposition III.6.5 to write

$$\mathscr{E}xt^i_U(\mathscr{F}|_U, \mathscr{G}|_U) \cong h^i(\mathscr{H}om(\widetilde{F_\bullet}, \mathscr{G}|_U)).$$

Now the ith cohomology is defined by taking the quotient of the kernel by the image. Combined with the equivalence categories Hartshorne Corollary II.5.5 to deduce that it is a quotient by two quasi-coherent sheaves. By Hartshorne Proposition II.5.7, the quotient is also quasi-coherent. But then $\mathscr{E}xt^i_U(\mathscr{F}|_U, \mathscr{G}|_U)$ is the $\sim$ of some A-module L. The fact that

L is finitely generated follows from the fact that each $F_\bullet$ can be taken to be f.g. free modules and N a f.g. A-module.
(b) Repeat the proof above except note the fact that N need not be a f.g. A-module. Hence, $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})$ is only quasi-coherent for all $i \geq 0$. □

Remark 52. One might ask if $\mathscr{F}$ being coherent is necessary in part (b) and if it fails had we assumed $\mathscr{F}$ were quasi-coherent instead. My suspicion is that it is necessary, but a counterexample eludes me.

Hartshorne III.6.4 Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. In this case we say $\mathfrak{Coh}(X)$ has enough locally frees. Then for any $\mathscr{G} \in \mathfrak{Mod}(X)$, show that the δ-functor $(\mathscr{E}xt^i(\cdot,\mathscr{G}))$, from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(X)$, is a contravariant universal δ-functor. [*Hint*: Show $\mathscr{E}xt^i(\cdot,\mathscr{G})$ is coeffaceable (§1) for $i > 0$.]

Proof. [Incomplete] □

Hartshorne III.6.5 Let X be a noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathscr{F}$ we define the homological dimension of $\mathscr{F}$, denoted $\mathrm{hd}(\mathscr{F})$, to be the least length of a locally free resolution of $\mathscr{F}$ (or $+\infty$ if there is no finite one). Show:

(a) $\mathscr{F}$ is locally free $\Leftrightarrow \mathscr{E}xt^1(\mathscr{F},\mathscr{G}) = 0$ for all $\mathscr{G} \in \mathfrak{Mod}(X)$;

(b) $\mathrm{hd}(\mathscr{F}) \leq n \Leftrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$;

(c) $\mathrm{hd}(\mathscr{F}) = \sup_x \mathrm{pd}_{\mathcal{O}_x} \mathscr{F}_x$.

Proof. **(a)** ($\Rightarrow$). Assume $\mathscr{F}$ is locally free. It suffices to show that $\mathscr{H}om(\mathscr{F},-)$ to see $\mathscr{E}xt^1(\mathscr{F},\mathscr{G}) = 0$ for all $\mathscr{G} \in \mathfrak{Mod}(X)$. Given a SES $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ of coherent sheaves, we always have an exact sequence

$$0 \to \mathscr{H}om(\mathscr{F},\mathscr{G}') \to \mathscr{H}om(\mathscr{F},\mathscr{G}) \to \mathscr{H}om(\mathscr{F},\mathscr{G}'').$$

We want exactness on the right. We use the criterion of Hartshorne II.1.3(a) to prove surjectivity. Let $U \subseteq X$ be an open subset. Let $\phi \in \mathscr{H}om(\mathscr{F},\mathscr{G}'')(U) = \mathrm{Hom}_{\mathcal{O}_{X|U}}(\mathscr{F}|_U, \mathscr{G}''|_U)$. Cover U by the open sets $\{U_i\}$ on which $\mathscr{F}|_{U_i} = \mathcal{O}_X|_{U_i}^n$. Then there is always a SES

$$0 \to \mathrm{Hom}_{\mathcal{O}_{U_i}}(\mathscr{F}|_{U_i}, \mathscr{G}'|_{U_i}) \to \mathrm{Hom}_{\mathcal{O}_{U_i}}(\mathscr{F}|_{U_i}, \mathscr{G}|_{U_i}) \to \mathrm{Hom}_{\mathcal{O}_{U_i}}(\mathscr{F}|_{U_i}, \mathscr{G}''|_{U_i}) \to 0.$$

So we can always lift the sections $\phi_i|_{U_i}$ of the sheaf Hom to $\psi_i \in \mathscr{H}om(\mathscr{F},\mathscr{G}'')(U_i)$ which map to $\phi|_{U_i}$. Hence, the RHS map is surjective.

($\Leftarrow$). Suppose $\mathscr{E}xt^1(\mathscr{F},\mathscr{G}) = 0$ for any $\mathscr{G} \in \mathfrak{Mod}(X)$. To show $\mathscr{F}$ is locally free, it suffices to show $\mathscr{F}_x$ is free. By Hartshorne Proposition III.6.8, we know

$$\mathscr{E}xt^1_{\mathcal{O}_x}(\mathscr{F}_x, \mathscr{G}_x) = 0 \qquad \text{for any } \mathscr{G} \text{ and } x \in X.$$

Since we can always extend coherent sheaves, we deduce that $\mathscr{F}_x$ is a projective $\mathcal{O}_{X,x}$-module. Since $\mathscr{F}_x$ is a f.g. projective module over a local ring, it is free.

(b) ($\Rightarrow$) is clear so we show ($\Leftarrow$). This implication is clear when $n = 0$ by part (a). Now we continue the proof by induction. Take a resolution $\mathscr{L}_\bullet \to \mathscr{F} \to 0$. We want to obtain finiteness and that we can take $\mathscr{L}_k = 0$ for $k \geq n+1$.

Using the method of dimension shifting, we know that $0 = \mathscr{E}xt^{n+1}(\mathscr{F}, \mathscr{G}) = \mathscr{E}xt^1(\mathscr{M}, \mathscr{G})$ where $\mathscr{M}$ is the kernel fitting into the exact sequence

$$0 \to \mathscr{M} \to \mathscr{L}_{n-1} \to \mathscr{L}_{n-2} \to \ldots \to \mathscr{L}_0 \to \mathscr{F} \to 0.$$

Since this holds for all $\mathscr{G}$, we deduce that $\mathscr{M}$ is a locally free sheaf. So we know that $\mathrm{hd}(\mathscr{F}) \leq n$ as desired.

(c) The fact $\mathrm{hd}(\mathscr{F}) \geq \sup_x \mathrm{pd}_x \mathscr{F}_x$ is clear since any locally free resolution of $\mathscr{F}$ gives rise to a projective resolution of $\mathscr{F}_x$.

That this is an equality is clear when $\sup_x \mathrm{pd}_{\mathcal{O}_x} \mathscr{F}_x$ is finite. When $\sup_x \mathrm{pd}_x \mathscr{F}_x = \infty$ then there can be no finite locally free resolution as it would not be terminate at all points. □

Hartshorne III.6.6 Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10A) as follows.

(a) M is projective if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > 0$. [*Hint*: Use (6.11A) and descending induction on i to show that $\mathrm{Ext}^i(M, N) = 0$ for all $i > 0$ and all finitely generated A-modules N. Then show M is a direct summand of a free A-module (Matsumura [2, p.129]).]

(b) Use (a) to show that for any n, pd $M \leqslant n$ if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > n$.

Proof. **(a)** ($\Rightarrow$) is obvious so we show ($\Leftarrow$). By Hartshorne Proposition III.6.11A, we know $\mathrm{pd}(M) \leq \dim A$ and so $n = \mathrm{pd}(M) < \infty$. It follows that $\mathrm{Ext}^i(M, N) = 0$ for all $i > n$ and all N.

The base step for descending induction is covered by $\mathrm{pd}(M) = n$. Now suppose N were finitely generated and we knew the result for $k \leq n$. Then there is a surjection $\bigoplus_{i \in I} A \to N$ and a SES

$$0 \to L \to \bigoplus_{i \in I} A \to N \to 0.$$

Take derived functors of $\mathrm{Hom}(M, -)$ to get a sequence

$$\ldots \to \mathrm{Ext}^{k-1}(M, L) \to \mathrm{Ext}^{k-1}\left(M, \bigoplus_{i \in I} A\right) \xrightarrow{\phi} \mathrm{Ext}^{k-1}(M, N) \to \mathrm{Ext}^k(M, L) \to \ldots$$

Since $\mathrm{Ext}^k(M, L) = 0$ and $\mathrm{Ext}^{k-1}(M, \oplus_{i \in I} A) = 0$ by hypothesis, $\mathrm{Ext}^{k-1}(M, N) = 0$. From the descending induction, $\mathrm{Ext}^i(M, N) = 0$ for all $i > 0$ and f.g. A-modules N.

Let A^n be a free A-module and $A^n \to M$ be a surjection. We get a SES $0 \to L \to A^n \xrightarrow{\gamma} M \to 0$. We would like for this sequence to split. It splits iff there is a lift of the map $A^n \to M$. Apply $\mathrm{Hom}(M, -)$ to get

$$0 \to \mathrm{Hom}(M, L) \to \mathrm{Hom}(M, A^n) \to \mathrm{Hom}(M, M) \to 0.$$

So there is an $\alpha : M \to A^n$ such that $\gamma \circ \alpha : M \to A^n \to M$ is the identity. So we get the splitting and $M \oplus L = A^n$. This condition is equivalent to M being projective.

(b) By Hartshorne Proposition III.6.10(b), we know $\mathrm{pd}(M) \leq n$ iff $\mathrm{Ext}^i(M, N) = 0$ for all $i > n$ and all A-modules and this is iff $\mathrm{Ext}^i(M, A) = 0$ for all $i > n$ by essentially the same argument above. □

Hartshorne III.6.7 Let $X = \operatorname{Spec} A$ be an affine noetherian scheme. Let M, N be A-modules, with M finitely generated. Then and

$$\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)$$
$$\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)^\sim.$$

Proof. From the hypotheses, we know that M is a finitely presented A-module. For the first isomorphism, let $0 \to N \to I_\bullet$ be an injective resolution. Then $0 \to \widetilde{N} \to \widetilde{I_\bullet}$ is a flasque resolution by Hartshorne Proposition III.3.4. There is a natural morphism $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_X(\widetilde{M}, \widetilde{N})$ which is an isomorphism. Since both $\operatorname{Ext}^i_X(\widetilde{M}, -)$ and $\operatorname{Ext}^i_A(M, -)$ are universal δ-functors we get maps going in both directions. Since the $I_\bullet$ gives rise to a flasque resolution, taking cohomology of $0 \to \operatorname{Hom}(M, I_\bullet)$ and $0 \to \operatorname{Hom}(\widetilde{M}, \widetilde{I_\bullet})$ give rise to the same cohomology.

To prove the second isomorphism, we show there is a map $\widetilde{\operatorname{Hom}_A(M, N)} \to \mathscr{H}om_X(\widetilde{M}, \widetilde{N})$. Indeed, a homomorphism $\phi \in \operatorname{Hom}_A(M, N)_{D(f)}$ corresponds to a homomorphism

$$\operatorname{Hom}_{\mathcal{O}_X|_{D(f)}}(\widetilde{M}|_{D(f)}, \widetilde{N}|_{D(F)})$$

via restrictions. Since the RHS of the isomorphism are δ-functors, we get a map $\Phi : \operatorname{Ext}^i_A(M, N)^\sim \to \mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N})$.

Our hypotheses open up the use of Hartshorne Proposition III.6.8. From exactness of $\sim$,

$$(\widetilde{\operatorname{Ext}^i_A(M, N)})_x = (\operatorname{Ext}^i_A(M, N))_x \overset{\Phi_x}{\leftarrow} \operatorname{Ext}^i_{\mathcal{O}_x}(\widetilde{M}_x, \widetilde{N}_x) = \mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N})_x$$

and so the isomorphism follows. □

Hartshorne III.6.8 Prove the following theorem of Kleiman (see Borelli [1]): if X is a noetherian, integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).

(a) First show that open sets of the form X_s, for various $s \in \Gamma(X, \mathscr{L})$, and various invertible sheaves $\mathscr{L}$ on X, form a base for the topology of X. [*Hint*: Given a closed point $x \in X$ and an open neighborhood U of x, to show there is an $\mathscr{L}, s$ such that $x \in X_s \subseteq U$, first reduce to the case that $Z = X - U$ is irreducible. Then let ζ be the generic point of Z. Let $f \in K(X)$ be a rational function with $f \in \mathcal{O}_x, f \notin \mathcal{O}_\zeta$. Let $D = (f)_\infty$, and let $\mathscr{L} = \mathscr{L}(D), s \in \Gamma(X, \mathscr{L}(D))$ correspond to D (II, §6).]

(b) Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus \mathscr{L}_i^{n_i}$ for various invertible sheaves $\mathscr{L}_i$ and various integers n_i.

Proof. **(a)** **[Incomplete]**

(b) **[Incomplete]** □

Hartshorne III.6.9 Let X be a noetherian, integral, separated, regular scheme. (We say a scheme is regular if all of its local rings are regular local rings.) Recall the definition of the Grothendieck group $K(X)$ from (II, Ex. 6.10). We define similarly another group $K_1(X)$ using locally free sheaves: it is the quotient of the free abelian group generated by all locally free (coherent) sheaves, by the subgroup generated by all expressions of the form $\mathscr{E} - \mathscr{E}' - \mathscr{E}''$, whenever $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is a short exact sequence of locally free sheaves. Clearly

there is a natural group homomorphism $\varepsilon : K_1(X) \to K(X)$. Show that ε is an isomorphism (Borel and Serre [1, §4]) as follows.

(a) Given a coherent sheaf $\mathscr{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathscr{E}. \to \mathscr{F} \to 0$. Then use (6.11A) and (Ex. 6.5) to show that it has a finite locally free resolution

$$0 \to \mathscr{E}_n \to \ldots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

(b) For each $\mathscr{F}$, choose a finite locally free resolution $\mathscr{E}. \to \mathscr{F} \to 0$, and let $\delta(\mathscr{F}) = \sum(-1)^i \gamma(\mathscr{E}_i)$ in $K_1(X)$. Show that $\delta(\mathscr{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K(X)$ to $K_1(X)$, and finally, that it is an inverse to ε.

Proof. **(a)** By Hartshorne Exercise III.6.8, we a locally free resolution always exists. Hartshorne III.6.5 and Proposition III.6.11A says

$$\operatorname{hd}(\mathscr{F}) \leq \sup_x \operatorname{pd}_x \mathscr{F}_x \leq \sup_x \dim \mathcal{O}_{X,x} \leq \dim X$$

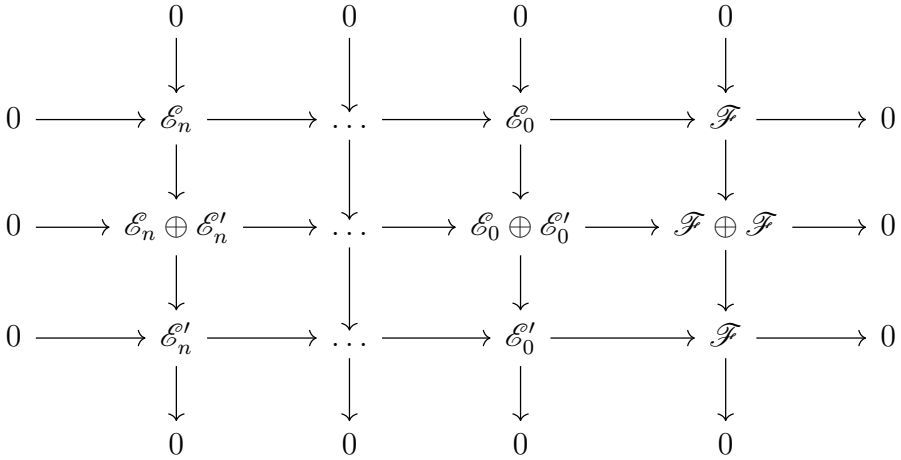
which means we can always take this resolution to be finite.

(b) Let $\mathscr{E}_\bullet \to \mathscr{F} \to 0$ and $\mathscr{E}'_\bullet \to \mathscr{F} \to 0$ be two finite locally free resolutions. We show $\delta(\mathscr{F})$ does not depend on the choice of finite locally free resolution. The resolutions say that $\sum(-1)^i\gamma(\mathscr{E}'_i) = \sum(-1)^i\gamma(\mathscr{E}'_i)$ inside $K(X)$ so we would only need ϵ to be injective.

Assume $\epsilon\left(\sum n_i\gamma_1(\mathscr{G}_i)\right) = \sum n_i\gamma(\mathscr{G}_i) = \gamma(\mathcal{O}_X)$. There must be cancellation of terms via the relations of $K(X)$. The constraints are the quotients involve only the class

[Incomplete]

Take the direct sum and look at the diagram



[Incomplete]

This is clearly a homomorphism: if $\mathscr{E}_\bullet \to \mathscr{F} \to 0$ and $\mathscr{E}'_\bullet \to \mathscr{F}' \to 0$, then we look at $\mathscr{E}_\bullet \oplus \mathscr{E}'_\bullet \to \mathscr{F} \oplus \mathscr{F}' \to 0$ which means $\delta(\mathscr{F} \oplus \mathscr{F}) = \delta(\mathscr{F}) + \delta(\mathscr{F}')$. Checking $\delta \circ \epsilon = \mathrm{id}_{K_1(X)}$ and $\epsilon \circ \delta = \mathrm{id}_{K(X)}$ is obvious. □

Hartshorne III.6.10 *Duality for a Finite Flat Morphism.*
(a) Let $f : X \to Y$ be a finite morphism of noetherian schemes. For any quasicoherent $\mathcal{O}_Y$-module $\mathscr{G}$, $\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})$ is a quasi-coherent $f_*\mathcal{O}_X$-module, hence corresponds to a quasi-coherent $\mathcal{O}_X$-module, which we call $f^!\mathscr{G}$ (II, Ex. 5.17e).

(b) Show that for any coherent $\mathscr{F}$ on X and any quasi-coherent $\mathscr{G}$ on Y, there is a natural isomorphism

$$f_*\mathscr{H}om_X\left(\mathscr{F}, f^!\mathscr{G}\right) \xrightarrow{\sim} \mathscr{H}om_Y\left(f_*\mathscr{F}, \mathscr{G}\right).$$

(c) For each $i \geqslant 0$, there is a natural map

$$\varphi_i : \mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G}).$$

[*Hint*: First construct a map

$$\mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^i_Y(f_*\mathscr{F}, f_*f^!\mathscr{G})$$

Then compose with a suitable map from $f_*f^!\mathscr{G}$ to $\mathscr{G}$.]

(d) Now assume that X and Y are separated, $\mathfrak{Coh}(X)$ has enough locally frees, and assume that $f_*\mathcal{O}_X$ is locally free on Y (this is equivalent to saying f flat-see § 9). Show that φ_i is an isomorphism for all i, all $\mathscr{F}$ coherent on X, and all $\mathscr{G}$ quasi-coherent on Y.

[*Hints*: First do $i = 0$. Then do $\mathscr{F} = \mathcal{O}_X$, using (Ex. 4.1). Then do $\mathscr{F}$ locally free. Do the general case by induction on i, writing $\mathscr{F}$ as a quotient of a locally free sheaf.]

Proof. **(a)** Let $f^!\mathscr{G}$ be the corresponding quasi-coherent $\mathcal{O}_X$-module of $\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})$. From Hartshorne II.5.17(e), the $\mathcal{O}_X$-module structure is given by

$$\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})(U) = \mathrm{Hom}_{\mathcal{O}_V}(f_*\mathcal{O}_X|_V, \mathscr{G}|_V) \qquad \text{for all } U \subseteq X \text{ and } f(U) = V \subseteq Y \text{ is open.}$$

(b) Let $U = \operatorname{Spec} A \subseteq Y$ and $\operatorname{Spec} B = V = f^{-1}(U) \subseteq X$. We define the natural isomorphism over each affine open subset. Define the morphism over each U. That means we need to specify a morphism

$$\mathrm{Hom}_V(\mathscr{F}|_V, f^!\mathscr{G}|_V) \to \mathrm{Hom}_U(f_*\mathscr{F}|_U, \mathscr{G}|_U).$$

By using Hartshone II.2.4, we have natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_V(\mathscr{F}|_V, f^!\mathscr{G}|_V) &\cong \mathrm{Hom}_B(\Gamma(V, \mathscr{F}), \Gamma(V, f^!\mathscr{G})) \\
&= \mathrm{Hom}_B(\Gamma(V, \mathscr{F}), \Gamma(U, \mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})) \\
&= \mathrm{Hom}_B(\Gamma(V, \mathscr{F}), \mathrm{Hom}_U(f_*\mathcal{O}_Y|_U, \mathscr{G}|_U)) \\
&= \mathrm{Hom}_B(\Gamma(V, \mathscr{F}), \mathrm{Hom}_A(B, \Gamma(U, \mathscr{G})) \\
&= \mathrm{Hom}_A(B \otimes_A \Gamma(U, \mathscr{F}), \Gamma(U, \mathscr{G})).
\end{aligned}$$

Meanwhile, the RHS is isomorphic to

$$\begin{aligned}
\mathrm{Hom}_U(f_*\mathscr{F}|_U, \mathscr{G}|_U) &= \mathrm{Hom}_A(\Gamma(U, f_*\mathscr{F}), \Gamma(U, \mathscr{G}) \\
&= \mathrm{Hom}_A(\Gamma(U, f_*\mathscr{F}), \Gamma(U, \mathscr{G})) \\
&= \mathrm{Hom}_A(\Gamma(V, \mathscr{F}), \Gamma(U, \mathscr{G})).
\end{aligned}$$

Of course, $B \otimes_A \Gamma(U, \mathscr{F}) \cong \Gamma(V, \mathscr{F})$ and so the isomorphism is natural. Now glue together these maps to get the global morphism on sheaf-Homs.

Another way to the problem is as follows. We want a natural isomorphism of sheaves of $f_*\mathcal{O}_X$-modules

$$f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G}).$$

We need to define specify the morphism over each open subset $V \subseteq Y$. Let $U := f^{-1}(V)$. Taking sections, the LHS becomes

$$f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G})(V) = \mathrm{Hom}_{\mathcal{O}_U}(\mathscr{F}|_U, f^!\mathscr{G}|_U)$$

while that RHS becomes

$$\mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G})(V) = \mathrm{Hom}_{\mathcal{O}_V}(f_*\mathscr{F}|_V, \mathscr{G}|_V) = f^!\mathscr{G}(V).$$

So we can define the morphism

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathscr{F}|_U, f^!\mathscr{G}|_U) \to f^!\mathscr{G}(U)$$

which takes a morphism $\Phi : \mathscr{F}|_U \to f^!\mathscr{G}|_U$ and maps it to $\Phi(U)(1) \in f^!\mathscr{G}(U)$. Tracing through the definitions and comparing with the first map, we see that we have defined the same maps. So this is an isomorphism.

A third way to do (b), is to observe that there are global morphisms

$$f_*\mathscr{H}om(\mathscr{F}, f^!\mathscr{G}) \to \mathscr{H}om_{f_*\mathcal{O}_X}(f_*\mathscr{F}, f_*f^!\mathscr{G}) \to \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G})$$

where we composed with the map $f_*f^!\mathscr{G} \to \mathscr{G}$.

(c) To construct the first map, we begin at $i = 0$. In this case, we want a map

$$\mathrm{Hom}_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Hom}_Y(f_*\mathscr{F}, f_*f^!\mathscr{G}).$$

There is an obvious map which is defined by taking a $\phi : \mathscr{F} \to f^!\mathscr{G}$ which is given by a compatible system of morphisms $\phi(U) : \mathscr{F}(U) \to f^!\mathscr{G}(U)$ and mapping it to the morphism $\psi : f_*\mathscr{F} \to f_*f^!\mathscr{G}$ whose compatible system of morphisms is $\psi(V) : \mathscr{F}(f^{-1}(U)) \overset{\phi(f^{-1}(U))}{\to} \mathscr{G}^!(f^{-1}(U))$. Now note the fact that f_* is an exact functor. Since the derived functors are universal δ-functor, and by the definition in Hartshorne p. 205, we extend to obtain morphisms $\mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^i_Y(f_*\mathscr{F}, f_*f^!\mathscr{G})$.

Now we specify the map $f_*f^!\mathscr{G} \to \mathscr{G}$. Taking global sections in (b) with $\mathscr{F} = f^!\mathscr{G}$, specifying a morphism $f_*f^!\mathscr{G} \to \mathscr{G}$ is the space as specifying a global section of $f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G})$. The global sections are $\mathrm{Hom}_X(f^!\mathscr{G}, f^!\mathscr{G})$ so we take the morphism $f_*f^!\mathscr{G} \to \mathscr{G}$ corresponding to $f^!\mathscr{G} \to f^!\mathscr{G}$.

(d) Let $i = 0$. We need $\varphi_0 : \mathrm{Hom}_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Hom}_Y(f_*\mathscr{F}, \mathscr{G})$ to be an isomorphism. We simply take the global sections of the natural isomorphism in (b).

Consider the case $\mathscr{F} = \mathcal{O}_X$. By Hartshorne III.4.1 and Hartshorne Proposition III.6.3(c), one has the isomorphisms

$$\mathrm{Ext}^i_X(\mathcal{O}_X, f^!\mathscr{G}) = H^i(X, f^!\mathscr{G}) = H^i(Y, f_*f^!\mathscr{G}) = \mathrm{Ext}^i_Y(\mathcal{O}_X, f_*f^!\mathscr{G}).$$

Meanwhile, one can use Hartshorne II.5.1(b) and the fact that $f_*\mathcal{O}_X$ is locally free to get

$$\begin{aligned}\mathrm{Ext}^i_Y(f_*\mathcal{O}_X, \mathscr{G}) &= \mathrm{Ext}^i_Y(\mathcal{O}_Y, (f_*\mathcal{O}_X)^\smile \otimes \mathscr{G}) = \mathrm{Ext}^i_Y(\mathcal{O}_Y, \mathscr{H}om(f_*\mathcal{O}_X, \mathscr{G})) \\ &= H^i(Y, \mathscr{H}om_Y, (f_*\mathcal{O}_X, \mathscr{G})) \\ &= H^i(Y, f_*f^!\mathscr{G}).\end{aligned}$$

Indeed, we know from (a) that $f_*f^!\mathscr{G} = \mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})$. Hence, the result holds with $\mathscr{F} = \mathcal{O}_X$.

For a general locally free sheaf $\mathscr{F}$, we can do the same thing. Indeed, $\mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) = H^i(X, \mathscr{F}^\smile \otimes f^!\mathscr{G})$ and then proceed as above. So the result clearly holds in the case where $\mathscr{F}$ is locally free. Of course, this natural isomorphism is induced from the maps in (b) and f_*. Therefore, it is the same map as ϕ_i which means ϕ_i is an isomorphism.

Now let $\mathscr{F}$ be an arbitrary coherent sheaf. Let $0 \to \mathscr{K} \to \mathscr{L} \to \mathscr{F} \to 0$ be a SES with $\mathscr{L}$ a locally free sheaf. Assume the result for all $\mathscr{F}$ coherent and up to some i. We need to prove the isomorphism for $i+1$. By naturality, we have a diagram

$$\begin{array}{ccccccccc}
\operatorname{Ext}^i_X(\mathscr{L}, f^!\mathscr{G}) & \longrightarrow & \operatorname{Ext}^i_X(\mathscr{L}, f^!\mathscr{G})) & \longrightarrow & \operatorname{Ext}^{i+1}_X(\mathscr{F}, f^!\mathscr{G})) & \longrightarrow & \operatorname{Ext}^{i+1}_X(\mathscr{L}, f^!\mathscr{G})) & \longrightarrow & \operatorname{Ext}^{i+1}_X(\mathscr{K}, f^!\mathscr{G})) \\
\downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i+1}} & & \downarrow{\scriptstyle \phi_{i+1}} & & \downarrow{\scriptstyle \phi_{i+1}} \\
\operatorname{Ext}^i_Y(f_*\mathscr{L}, \mathscr{G}) & \longrightarrow & \operatorname{Ext}^i_Y(f_*\mathscr{K}, \mathscr{G}) & \longrightarrow & \operatorname{Ext}^{i+1}_Y(f_*\mathscr{F}, \mathscr{G}) & \longrightarrow & \operatorname{Ext}^{i+1}_Y(f_*\mathscr{L}, \mathscr{G}) & \longrightarrow & \operatorname{Ext}^{i+1}_Y(f_*\mathscr{K}, \mathscr{G})
\end{array}$$

The first two vertical maps are isomorphisms by the inductive hypothesis. The fourth is an isomorphism since $\mathscr{L}$ is locally free. By the 5-lemma, the middle vertical map is injective. Since $\mathscr{F}$ arbitrary, we deduce that the last map is also injective. But then the full 5-lemma says that the middle itself is injective. □

Remark 53. For a detailed write-up, see also these solutions.

7. The Serre Duality Theorem

Hartshorne III.7.1 Let X be an integral projective scheme of dimension $\geqslant 1$ over a field k, and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $H^0(X, \mathscr{L}^{-1}) = 0$. (This is an easy special case of Kodaira's vanishing theorem.)

Proof. Since $\mathscr{L}$ is ample, $\mathscr{L}^n$ is very ample so assume WLOG that $\mathscr{L}$ is very ample itself. Then choose a closed embedding $X \to \mathbb{P}^N$ s.t. the pullback of $\mathcal{O}(1)$ is $\mathscr{L}$. Then $\mathscr{L}$ has sections. Then $\Gamma(X, \mathscr{L}^m) \geq \dim X + 1$ for $m \gg 0$. Now if $H^0(X, \mathscr{L}^{-1}) \neq 0$, we know $H^0(X, \mathscr{L}^{-m})$ has a lot of sections too. Choosing sections of $\mathscr{L}^{-m}$ and tensoring these sections with those of $\mathscr{L}^m$, we get linearly independent sections of $\mathcal{O}_X$ contradicting $\dim_k \Gamma(X, \mathcal{O}_X) = 1$. □

Hartshorne III.7.2 Let $f : X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k, and let ω_Y° be a dualizing sheaf for Y.

(a) Show that $f^!(\omega_Y^\circ)$ is a dualizing sheaf for X, where $f^!$ is defined as in (Ex. 6.10).

(b) If X and Y are both nonsingular, and k algebraically closed, conclude that there is a natural trace map $t : f_*\omega_X \to \omega_Y$.

Proof. **(a)** Recall that $f^!(\omega_Y^\circ)$ is the $\mathcal{O}_X$-module associated to the quasi-coherent $f_*\mathcal{O}_X$-module $\mathscr{H}om(f_*\mathcal{O}_X, \omega_Y^\circ)$. Since ω_Y° is coherent and f is finite, $f^!(\omega_Y^\circ)$ is coherent.

Now we define the trace map $t : H^n(X, f^!\omega_Y^\circ) \to k$. From Hartshorne Proposition III.6.3 and Hartshorne III.6.10(c),

$$H^n(X, f^!\omega_Y^\circ) = \operatorname{Ext}^n_X(\mathcal{O}_X, f^!\omega_Y^\circ) = \operatorname{Ext}^n_Y(f_*\mathcal{O}_X, \omega_Y^\circ).$$

Apply Serre Duality to get

$$\operatorname{Ext}^n(f_*\mathcal{O}_X, \omega_Y^\circ) = H^0(Y, f_*\mathcal{O}_X)' = k$$

since X is a projective scheme and the dimension of globals sections of projective schemes is 1. So the trace morphism is defined via these equalities / isomorphisms.
(b) Let X, Y be both nonsingular and k an algebraically closed field. By Hartshorne Corollary III.7.12, the dualizing sheaves are precisely the canonical sheaves.

We know that $f^!\omega_Y$ and ω_X are both dualizing sheaves for X. So there is a natural isomorphism $\eta : \omega_X \to f^!\omega_Y$ obtained from the universal property. By Hartshorne III.6.10(c), there is a natural map

$$\varphi_0 : \mathrm{Hom}_X(\omega_X, f^!\omega_Y) \to \mathrm{Hom}_Y(f_*\omega_X, \omega_Y).$$

Then we take the desired "natural trace map" $t : f_*\omega_X \to \omega_Y$ to be the image of $\varphi_0(\eta)$. □

Remark 54. Hartshorne does not define what it means for $t : f_*\omega_X \to \omega_Y$ to be a trace map.

Hartshorne III.7.3 Let $X = \mathbb{P}^n_k$. Show that $H^q\left(X, \Omega^p_X\right) = 0$ for $p \neq q, k$ for $p = q, 0 \leqslant p, q \leqslant n$.

Proof. For this solution, the second part of the proof follows Exercise 21.5.M of Vakil's Notes.

Lemma 7.1. $H^p(X, \Omega^p_X) = k$ for all $0 \leq p \leq n$.

Proof. First off, if $p = q = n$ then we get $H^n(X, \Omega^n_X) = H^n(X, \mathcal{O}_X(-n-1)) = k$ by Hartshorne Theorem III.5.1. Then apply Hartshorne Corollary III.7.13 to deduce that $H^p(X, \Omega^p_X) = k$ for all $0 \leq p \leq n$. □

Lemma 7.2. $H^p(X, \Omega^q_X) = 0$ for all $0 \leq p, q \leq n$ and $p \neq q$.

Proof. Recall that there is an Euler sequence (see Theorem 21.4.6 of Vakil's notes)

$$0 \to \Omega_{X/k} \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0.$$

By Hartshorne II.5.16(c)(d), we can find a filtration

$$\bigwedge^{n+1} \mathcal{O}_X(-1)^{\oplus n+1} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^{n+1} = 0$$

s.t. the subquotients are $F^i/F^{i+1} \cong \Omega^i_X \otimes \bigwedge^{n-i} \mathcal{O}_X \cong \Omega^i_X$. We may proceed by climbing this filtration since we are interested in the cohomology of F^i/F^{i+1}.

First off, the SES $0 \to F^{n+1} \to F^n \to F^n/F^{n+1} \to 0$ tells us that $F^n \cong \Omega^n_X = \mathcal{O}_X(-n-1)$. Next, look at $0 \to F^n \to F^{n-1} \to F^{n-1}/F^n \to 0$.Take the cohomology LES,

$$0 \to H^0(X, F^n) \to H^0(X, F^{n-1}) \to H^0(X, F^{n-1}/F^n) \to H^1(X, F^n) \to \dots$$

and by Hartshorne Theorem III.5.1, we know $H^i(X, F^n) = 0$ for all $0 < i < n$. It follows that $H^i(X, F^{n-1}) = H^i(X, F^{n-1}/F^n)$ for those $1 < i < n-1$.

[**Incomplete**]I don't know just yet. □

□

Hartshorne III.7.4 *The Cohomology Class of a Subvariety.* Let X be a nonsingular projective variety of dimension n over an algebraically closed field k. Let Y be a nonsingular subvariety of codimension p (hence dimension $n-p$). From the natural map $\Omega_X \otimes \mathcal{O}_Y \to \Omega_Y$ of (II, 8.12) we deduce a map $\Omega^{n-p}_X \to \Omega^{n-p}_Y$. This induces a map on cohomology $H^{n-p}\left(X, \Omega^{n-p}_X\right) \to H^{n-p}\left(Y, \Omega^{n-p}_Y\right)$. Now $\Omega^{n-p}_Y = \omega_Y$ is a dualizing sheaf for Y, so we have the trace map $t_Y : H^{n-p}\left(Y, \Omega^{n-p}_Y\right) \to k$. Composing, we obtain a linear map $H^{n-p}\left(X, \Omega^{n-p}_X\right) \to k$. By (7.13) this corresponds to an element $\eta(Y) \in H^p\left(X, \Omega^p_X\right)$, which we call the **cohomology class** of Y.

(a) If $P \in X$ is a closed point, show that $t_X(\eta(P)) = 1$, where $\eta(P) \in H^n(X, \Omega^n)$ and t_X is the trace map.

(b) If $X = \mathbb{P}^n$, identify $H^p(X, \Omega^p)$ with k by (Ex. 7.3), and show that $\eta(Y) = (\deg Y) \cdot 1$, where $\deg Y$ is its **degree** as a projective variety (I, §7).
[*Hint*: Cut with a hyperplane $H \subseteq X$, and use Bertini's theorem (II, 8.18) to reduce to the case Y is a finite set of points.]

(c) For any scheme X of finite type over k, we define a homomorphism of sheaves of abelian groups $d\log : \mathcal{O}_X^* \to \Omega_X$ by $d\log(f) = f^{-1}df$. Here $\mathcal{O}^*$ is a group under multiplication, and Ω_X is a group under addition. This induces a map on cohomology $\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega_X)$ which we denote by c – see (Ex. 4.5).

(d) Returning to the hypotheses above, suppose $p = 1$. Show that $\eta(Y) = c(\mathscr{L}(Y))$, where $\mathscr{L}(Y)$ is the invertible sheaf corresponding to the divisor Y.

See Matsumura [1] for further discussion.

Proof. (a). The element $\eta(P) \in H^n(X^n, \omega_X)$ is the element corresponding to

$$H^0(X, \mathcal{O}_X) \xrightarrow{\phi} H^0(Y, \mathcal{O}_Y) \xrightarrow{t} k$$

under the Serre duality correspondence. But the morphism above is just the identity so clearly $\eta(P) = 1$.
(b). Cut down by hyperplanes and use induction to make the reduction. In the case Y is a finite set of points, the result is clear from (a).
(c). This is purely a definition.
(d). To be completely honest, this is not a fully rigorous proof. The element $\eta(Y) \in H^1(X, \Omega_X^1)$ is Serre dual to the map $\Phi : H^{n-1}(X, \Omega_X^{n-1}) \to H^{n-1}(Y, \omega_Y) \to k$ in which the first map is induced by the inclusion of Y into X. OTOH, c is the map induced from $d\log$ which provides a logarithmic differential (and such a logarithmic differential is going to come from a function on Y) and so corresponds to Φ i.e. $c(\mathscr{L}(Y)) = \eta(Y)$.

For a fully rigourous proof, see Hartshorne's reference to Matsumura. □

8. Higher Direct Images of Sheaves

Hartshorne III.8.1 Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X, and assume that $R^i f_*(\mathscr{F}) = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geqslant 0$,

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F}).$$

(This is a degenerate case of the Leray spectral sequence—see Godement [1, II, 4.17.1].)

Proof. First, resolve $\mathscr{F}$ by a resolution of injective sheaves $0 \to \mathscr{F} \to \mathscr{I}^\bullet$. Then take global sections $0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{I}^\bullet)$ and cohomology to obtain $H^i(X, \mathscr{F})$. On the other hand, apply f_* to get $0 \to f_*\mathscr{F} \to f_*\mathscr{I}^\bullet$ which is a flasque resolution of $f_*\mathscr{F}$ because $\mathscr{I}^\bullet$ is a flasque resolution and pushforwards preserve flasque sheaves while $R^i f_*\mathscr{F} = 0$ and $R^i f_*\mathscr{I}^j = 0$ for all j and $i > 0$ indicates that exactness is preserved. Taking global sections gives a complex which is the same as $0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{I}^\bullet)$ and so the cohomology is the same. □

Hartshorne III.8.2 Let $f : X \to Y$ be an affine morphism of schemes (II, Ex. 5.17) with X noetherian, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Show that the hypotheses of (Ex. 8.1) are satisfied, and hence that $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$ for each $i \geqslant 0$. (This gives another proof of (Ex. 4.1).)

Proof. The map $f : X \to Y$ is clearly continuous. We only need to check $R^i f * (\mathscr{F}) = 0$ for all $i > 0$. Let $i > 0$ and we know that $R^i f_*(\mathscr{F})$ is the sheaf associated to the presheaf

$$V \mapsto H^i(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)}).$$

Since f is affine and $\mathscr{F}$ is quasi-coherent, we can focus on those open affine subsets of Y s.t. $\mathscr{F}|_{f^{-1}(V)}$ trivialize. So we may assume WLOG that Y is an affine scheme and by Hartshorne Proposition III.8.5,

$$R^i f_*(\mathscr{F}) \cong H^i(X, \mathscr{F})^{\sim}.$$

We show $H^i(X, \mathscr{F}) = 0$ for $i > 0$. Of course, $f : X \to Y$ is an affine morphism onto an affine scheme and so WLOG assume X is affine. Since $\mathscr{F}$ is quasi-coherent, we can apply Serre's Affine Criterion to get $H^i(X, \mathscr{F}) = 0$ in the case X, Y are affine. In the general case, $H^i(X, \mathscr{F})^{\sim}$ is zero since its stalks would be zero. So $R^i f_*\mathscr{F} = 0$.

[Incomplete] □

Hartshorne III.8.3 Let $f : X \to Y$ be a morphism of ringed spaces, let $\mathscr{F}$ be an $\mathcal{O}_X$-module, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Prove the projection formula (cf. (II, Ex. 5.1))

$$R^i f_* (\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}.$$

Proof. For this exercise, we shall use the fact that taking cohomology commutes with applying the functor when the functor is exact. See this blog or Exercise 1.6.H in Vakil's notes.

First, choose an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\bullet$. Apply f_* to get a cochain complex $0 \to f_*\mathscr{F} \to f_*\mathscr{I}^\bullet$. Take cohomology and apply the exact functor $- \otimes \mathscr{E}$ to get the cohomology $R^i f_*((\mathscr{F}) \otimes \mathscr{E}$. On the other hand, we can apply the exact functor $- \otimes \mathscr{E}$ and use Hartshorne II.5.1(e) to get

$$0 \to f_*(\mathscr{F} \otimes f^*\mathscr{G}) \to f_*(\mathscr{I}^0 \otimes f^*\mathscr{G}) \to f_*(\mathscr{I}^1 \otimes f^*\mathscr{G}) \to \dots$$

and taking cohomology gives the cohomology $R^i f_*(\mathscr{F} \otimes f^*\mathscr{G})$. Hence, there are natural isomorphisms

$$R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}.$$

□

Hartshorne III.8.4 Let Y be a noetherian scheme, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of rank $n+1$, $n \geqslant 1$. Let $X = \mathbb{P}(\mathscr{E})$ (II,§7), with the invertible sheaf $\mathcal{O}_X(1)$ and the projection morphism $\pi : X \to Y$.

(a) Then $\pi_*(\mathcal{O}(l)) \cong S^l(\mathscr{E})$ for $l \geqslant 0, \pi_*(\mathcal{O}(l)) = 0$ for $l < 0$ (II, 7.11); $R^i\pi_*(\mathcal{O}(l)) = 0$ for $0 < i < n$ and all $l \in \mathbb{Z}$; and $R^n\pi_*(\mathcal{O}(l)) = 0$ for $l > -n - 1$.

(b) Show there is a natural exact sequence

$$0 \to \Omega_{X/Y} \to (\pi^*\mathscr{E})(-1) \to \mathcal{O} \to 0$$

cf. (II, 8.13), and conclude that the relative canonical sheaf $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ is isomorphic to $(\pi^* \wedge^{n+1} \mathscr{E})(-n-1)$. Show furthermore that there is a natural isomorphism $R^n\pi_* (\omega_{X/Y}) \cong \mathcal{O}_Y$(cf. (7.1.1)).

(c) Now show, for any $l \in \mathbb{Z}$, that

$$R^n\pi_*(\mathcal{O}(l)) \cong \pi_*(\mathcal{O}(-l-n-1))^\vee \otimes \left(\wedge^{n+1}\mathscr{E}\right)^\vee.$$

(d) Show that $p_a(X) = (-1)^n p_a(Y)$ (use (Ex. 8.1)) and $p_g(X) = 0$ (use (II, 8.11)).

(e) In particular, if Y is a nonsingular projective curve of genus g, and $\mathscr{E}$ a locally free sheaf of rank 2 , then X is a projective surface with $p_a = -g, p_g = 0$, and irregularity g(7.12.3). This kind of surface is called a geometrically ruled surface (V, §2).

Proof. (a). I omit the details, but will explain the heuristics. Recall that $\mathbb{P}(\mathscr{E})$ is the relative Proj and so locally the map looks like

$$\mathbb{P}^n_U \subseteq X \to U \subseteq Y.$$

Now $\pi_*(\mathcal{O}(l))$ should locally be pushing forward the degree l graded piece of the homogeneous coordinate ring which is $S^l(\mathscr{E})$ in this case. Since all of the claimed isomorphisms can be done locally, we deduce these results from our knowledge of cohomology of projective spaces and the graded pieces of the homogeneous coordinate ring of projective space.
(b). This is a globalized version of Hartshorne Theorem II.8.13. The fact that $\pi^*\mathscr{E}(-1)$ should correspond to $\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}$ coes frmo the fact that locally, $\pi^*\mathscr{E} = \pi^*\mathcal{O}_Y^{n+1}$ which then locally looks like $\mathcal{O}_{\mathbb{P}^n}^{n+1}$. Then the twist by -1 gives the correct identification. As a warninig, one should note that $\mathcal{O} := \mathcal{O}_X$.

To do this, we need to take determinants and this gives an isomorphism

$$\det(\pi^*\mathscr{E}(-1)) \cong \det(\Omega_{X/Y}) = \wedge^n \Omega_{X/Y} = \omega_{X/Y}.$$

[Incomplete]

For the natural isomorphism, one should think that this is locally the duality result for projective space. **[Incomplete]**
(c).
(d).
(e). By assumption, $n = 1$ and $p_a(Y) = p_g(Y)$ and so $p_a(X) = -g$. Also $p_g(X) = 0$ and the irregularity of X is $H^1(X, \mathcal{O}_X) = g$ via the formula $p_a(X) = (-1)^2(\chi(\mathcal{O}_X) - 1)$. □

9. Flat Morphisms

Hartshorne III.9.1 A flat morphism $f : X \to Y$ of finite type of noetherian schemes is open, i.e, for every open subset $U \subseteq X, f(U)$ is open in Y. [*Hint:* Show that $f(U)$ is constructible and stable under generization (II, Ex. 3.18) and (II, Ex. 3.19).]

Proof. See [10, Tag 02JY]. See also the following note. □

Hartshorne III.9.2 Do the calculation of (9.8.4) for the curve of (I, Ex. 3.14). Show that you get an embedded point at the cusp of the plane cubic curve.

Proof. **[Incomplete]** □

Hartshorne III.9.3 Some examples of flatness and nonflatness.

(a) If $f : X \to Y$ is a finite surjective morphism of nonsingular varieties over an algebraically closed field k, then f is flat.

(b) Let X be a union of two planes meeting at a point, each of which maps isomorphically to a plane Y. Show that f is not flat. For example, let $Y = \operatorname{Spec} k[x, y]$ and $X = \operatorname{Spec} k[x, y, z, w]/(z, w) \cap (x + z, y + w)$.

(c) Again let $Y = \operatorname{Spec} k[x, y]$, but take $X = \operatorname{Spec} k[x, y, z, w]/\left(z^2, zw, w^2, xz - yw\right)$. Show that $X_{\text{red}} \cong Y, X$ has no embedded points, but that f is not flat.

PROOF. **(a)** The question is local so we only need to show the result if $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a finite morphism, with A, B finitely generated k-algebras, with $\operatorname{Spec} A, \operatorname{Spec} B$ nonsingular. That is, all local rings are regular. Localizing, the question is that we need to show $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is a flat extension when $f^{-1}(\mathfrak{p}) = \mathfrak{q}$. Now $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are regular local rings and $B_{\mathfrak{q}}$ is finite over $A_{\mathfrak{p}}$.

But $B_{\mathfrak{q}}$ is finitely generated over $A_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ is a local noetherian ring so by Hartshorne Proposition III.9.1A, we need only check that $B_{\mathfrak{q}}$ is free as an $A_{\mathfrak{p}}$-module. Of course, this follows from a result in commutative algebra (the fact that $B_{\mathfrak{q}}$ is maximal Cohen-Macaulay follows from the fact that $f : X \to Y$ is finite and thereby quasi-finite and so the dimensions of the local rings are the same and then one uses the fact that regular local rings are indeed Cohen-Macaulay over themselves)[6]. I think this uses the fact that $B_{\mathfrak{q}}$ is an integral domain by nonsingularity.

(b) Let $f : X \to Y$ as above. I claim that the ring extension

$$R := k[x, y] \to k[x, y, z, w]/((z, w) \cap (x + z, y + w)) =: S$$

is not flat. Localize at the prime ideal $\mathfrak{p} = (x, y)$ and we just show that $S_{\mathfrak{p}}$ is not a free $R_{\mathfrak{p}}$ (it is easy to deduce that S is finite over R). Indeed, there is a relation $xw - yz = 0$ (which is zero since it sits in (z, w) and $(x + z, y + w)$, in $S_{\mathfrak{p}}$) with coefficients $x, y \in R_{\mathfrak{p}}$ (which are not zero).

(c) The fact that $X_{red} \cong Y$ is easy to check as the nilradical of the ring defining X is (z, w).

The fact that X has no embedded points comes from taking a primary decomposition of $(z^2, zw, w^2, xz - yw)$. This computation is annoying so I shall omit it for now.

The fact that f is not flat is clear. Consider the ring map

$$k[x, y] \xrightarrow{\cdot x} k[x, y]$$

The fact that f is not a flat map is a bit tricky. Localize at the ideal $\mathfrak{p} = (x, y)$ and we need to check that $S := k[x, y, w, z]/(z^2, w^2, zw, xz - yw)$ localized at $\mathfrak{p}$ does not make it a free $k[x, y]_{\mathfrak{p}}$-module. But $S_{\mathfrak{p}}$ would take S and invert z and w at least. Furthermore, that means one can rewrite the relation $xz - yw = 0$ as $x = \frac{w}{z}y$ which means there is no hope for it to be free over $k[x, y]_{\mathfrak{p}}$. □

Hartshorne III.9.4 *Open Nature of Flatness.* Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then $\{x \in X \mid f \text{ is flat at } x\}$ is an open subset of X (possibly empty)-see Grothendieck [EGA IV$_3$, 11.1.1].

[6]Some care must be made though because the extension of a regular local ring need not generally be free e.g. $R \to R/(x)$ for $x \in \mathfrak{m} - \mathfrak{m}^2$.

PROOF. See also Tag 0399. We will have to assume a number of difficult results (see this Stack Exchange post). Essentially, the slogan is that flatness is an open condition follows from generic freeness. □

Hartshorne III.9.5 *Very Flat Families.* For any closed subscheme $X \subseteq \mathbb{P}^n$, we denote by $C(X) \subseteq \mathbb{P}^{n+1}$ the projective cone over X (I, Ex. 2.10). If $I \subseteq k[x_0, \ldots, x_n]$ is the (largest) homogeneous ideal of X, then $C(X)$ is defined by the ideal generated by I in $k[x_0, \ldots, x_{n+1}]$.

(a) Give an example to show that if $\{X_t\}$ is a flat family of closed subschemes of $\mathbb{P}^n$, then $\{C(X_t)\}$ need not be a flat family in $\mathbb{P}^{n+1}$.

(b) To remedy this situation, we make the following definition. Let $X \subseteq \mathbb{P}^n_T$ be a closed subscheme, where T is a noetherian integral scheme. For each $t \in T$, let $I_t \subseteq S_t = k(t)[x_0, \ldots, x_n]$ be the homogeneous ideal of X_t in $\mathbb{P}^n_{k(t)}$. We say that the family $\{X_t\}$ is very flat if for all $d \geqslant 0$,

$$\dim_{k(t)} (S_t/I_t)_d$$

is independent of t. Here $()_d$ means the homogeneous part of degree d.

(c) If $\{X_t\}$ is a very flat family in $\mathbb{P}^n$, show that it is flat. Show also that $\{C(X_t)\}$ is a very flat family in $\mathbb{P}^{n+1}$, and hence flat.

(d) If $\{X_{(t)}\}$ is an algebraic family of projectively normal varieties in $\mathbb{P}^n_k$, parametrized by a nonsingular curve T over an algebraically closed field k, then $\{X_{(t)}\}$ is a very flat family of schemes.

PROOF. **(a)**
(b)
(c)
(d) □

Hartshorne III.9.6 Let $Y \subseteq \mathbb{P}^n$ be a nonsingular variety of dimension $\geqslant 2$ over an algebraically closed field k. Suppose $\mathbb{P}^{n-1}$ is a hyperplane in $\mathbb{P}^n$ which does not contain Y, and such that the scheme $Y' = Y \cap \mathbb{P}^{n-1}$ is also nonsingular. Prove that Y is a complete intersection in $\mathbb{P}^n$ if and only if Y' is a complete intersection in $\mathbb{P}^{n-1}$. [Hint: See (II, Ex. 8.4) and use (9.12) applied to the affine cones over Y and Y'.]

PROOF. **[Incomplete]** □

Hartshorne III.9.7 Let $Y \subseteq X$ be a closed subscheme, where X is a scheme of finite type over a field k. Let $D = k[t]/t^2$ be the ring of dual numbers, and define an infinitesimal deformation of Y as a closed subscheme of X, to be a closed subscheme $Y' \subseteq X \times \times_k D$, which is flat over D, and whose closed fibre is Y. Show that these Y' are classified by $H^0(Y, \mathcal{N}_{Y/X})$, where

$$\mathcal{N}_{Y/X} = \mathscr{H}om(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathcal{O}_Y).$$

PROOF. **[Incomplete]** □

Hartshorne III.9.8 Let A be a finitely generated k-algebra. Write A as a quotient of a polynomial ring P over k, and let J be the kernel:

$$0 \to J \to P \to A \to 0.$$

Consider the exact sequence of (II, 8.4A)

$$J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0.$$

Apply the functor $\mathrm{Hom}_A(\cdot, A)$, and let $T^1(A)$ be the cokernel:

$$\mathrm{Hom}_A\left(\Omega_{P/k} \otimes A, A\right) \to \mathrm{Hom}_A\left(J/J^2, A\right) \to T^1(A) \to 0.$$

Now use the construction of (II, Ex. 8.6) to show that $T^1(A)$ classifies infinitesimal deformations of A, i.e., algebras A' flat over $D = k[t]/t^2$, with $A' \otimes_D k \cong A$. It follows that $T^1(A)$ is independent of the given representation of A as a quotient of a polynomial ring P. (For more details, see Lichtenbaum and Schlessinger [1].)

Proof. [Incomplete] □

Hartshorne III.9.9 A k-algebra A is said to be rigid if it has no infinitesimal deformations, or equivalently, by (Ex. 9.8) if $T^1(A) = 0$. Let $A = k[x, y, z, w]/(x, y) \cap (z, w)$, and show that A is rigid. This corresponds to two planes in $\mathbb{A}^4$ which meet at a point.

Proof. [Incomplete] □

Hartshorne III.9.10 A scheme X_0 over a field k is rigid if it has no infinitesimal deformations.

(a) Show that $\mathbb{P}^1_k$ is rigid, using (9.13.2).

(b) One might think that if X_0 is rigid over k, then every global deformation of X_0 is locally trivial. Show that this is not so, by constructing a proper, flat morphism $f : X \to \mathbb{A}^2$ over k algebraically closed, such that $X_0 \cong \mathbb{P}^1_k$, but there is no open neighborhood U of 0 in $\mathbb{A}^2$ for which $f^{-1}(U) \cong U \times \mathbb{P}^1$.

*(c) Show, however, that one can trivialize a global deformation of $\mathbb{P}^1$ after a flat base extension, in the following sense: let $f : X \to T$ be a flat projective morphism, where T is a nonsingular curve over k algebraically closed. Assume there is a closed point $t \in T$ such that $X_t \cong \mathbb{P}^1_k$. Then there exists a nonsingular curve T', and a flat morphism $g : T' \to T$, whose image contains t, such that if $X' = X \times_T T'$ is the base extension, then the new family $f' : X' \to T'$ is isomorphic to $\mathbb{P}^1_{T'} \to T'$.

Proof. **(a)**

(b)

(c)

(d) □

Hartshorne III.9.11 Let Y be a nonsingular curve of degree d in $\mathbb{P}^n_k$, over an algebraically closed field k. Show that

$$0 \leqslant p_a(Y) \leqslant \frac{1}{2}(d-1)(d-2).$$

[Hint: Compare Y to a suitable projection of Y into $\mathbb{P}^2$, as in (9.8.3) and (9.8.4).]

Proof. From Hartshorne Example III.9.8.3 the projection from a point gives a flat family so degree is preserved when we take projections. From Hartshorne Example III.9.8.4, we know the projection $\varphi(Y)$ could possibly lead to singularities. But we know that if we take the normalization of such a curve, the arithmetic genus may increase. And a smooth nonsingular curve of degree d in $\mathbb{P}^2_k$ has arithmetic genus $\frac{1}{2}(d-1)(d-2)$. So we get the claimed inequality. □

10. Smooth Morphisms

Hartshorne III.10.1 Over a nonperfect field, smooth and regular are not equivalent. For example, let k_0 be a field of characteristic $p > 0$, let $k = k_0(t)$, and let $X \subseteq \mathbb{A}^2_k$ be the curve defined by $y^2 = x^p - t$. Show that every local ring of X is a regular local ring, but X is not smooth over k.

PROOF. [Incomplete] □

Hartshorne III.10.2 Let $f : X \to Y$ be a proper, flat morphism of varieties over k. Suppose for some point $y \in Y$ that the fibre X_y is smooth over $k(y)$. Then show that there is an open neighborhood U of y in Y such that $f : f^{-1}(U) \to U$ is smooth.

PROOF. [Incomplete] □

Hartshorne III.10.3 A morphism $f : X \to Y$ of schemes of finite type over k is étale if it is smooth of relative dimension 0 . It is unramified if for every $x \in X$, letting $y = f(x)$, we have $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$, and $k(x)$ is a separable algebraic extension of $k(y)$. Show that the following conditions are equivalent:

(i) f is étale;

(ii) f is flat, and $\Omega_{X/Y} = 0$;

(iii) f is flat and unramified.

PROOF. [Incomplete] □

Hartshorne III.10.4 Show that a morphism $f : X \to Y$ of schemes of finite type over k is étale if and only if the following condition is satisfied: for each $x \in X$, let $y = f(x)$. Let $\widehat{\mathcal{O}}_x$ and $\widehat{\mathcal{O}}_y$ be the completions of the local rings at x and y. Choose fields of representatives (II, 8.25A) $k(x) \subseteq \widehat{\mathcal{O}}_x$ and $k(y) \subseteq \widehat{\mathcal{O}}_y$ so that $k(y) \subseteq k(x)$ via the natural map $\widehat{\mathcal{O}}_y \to \widehat{\mathcal{O}}_x$. Then our condition is that for every $x \in X, k(x)$ is a separable algebraic extension of $k(y)$, and the natural map is an isomorphism.

$$\widehat{\mathcal{O}}_y \otimes_{k(y)} k(x) \to \widehat{\mathcal{O}}_x.$$

PROOF. [Incomplete] □

Hartshorne III.10.5 If x is a point of a scheme X, we define an étale neighborhood of x to be an étale morphism $f : U \to X$, together with a point $x' \in U$ such that $f(x') = x$. As an example of the use of étale neighborhoods, prove the following: if $\mathscr{F}$ is a coherent sheaf on X, and if every point of X has an étale neighborhood $f : U \to X$ for which $f^*\mathscr{F}$ is a free $\mathcal{O}_U$-module, then $\mathscr{F}$ is locally free on X.

PROOF. [Incomplete] □

Hartshorne III.10.6 Let Y be the plane nodal cubic curve $y^2 = x^2(x + 1)$. Show that Y has a finite étale covering X of degree 2 , where X is a union of two irreducible components, each one isomorphic to the normalization of Y (Fig. 12).

PROOF. The proof itself is tedious if one tries to do it through equations. Let me "cheat" and explain why this is a degree 2 étale covering.

First off, locally at every point in the étale cover, there is an isomorphism. In particular, $f : X \to Y$ is flat, and the geometric fibres have dimension 0. The fact that this map is of

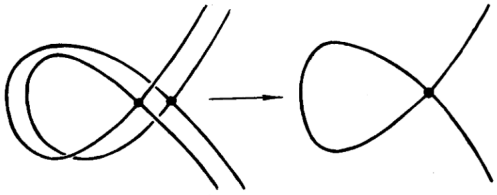


Figure 12. A finite étale covering.

degree 2 follows from the fact that every fibre has two points. Meanwhile, $f : X \to Y$ is finite since it is finite locally. □

Hartshorne III.10.7 *(Serre). A linear system with moving singularities.* Let k be an algebraically closed field of characteristic 2 . Let $P_1, \ldots, P_7 \in \mathbb{P}^2_k =: X$ be the seven points of the projective plane over the prime field $\mathbb{F}_2 \subseteq k$. Let $\mathfrak{d}$ be the linear system of all cubic curves in X passing through $P_1, \ldots, P_7$.
(a) $\mathfrak{d}$ is a linear system of dimension 2 with base points $P_1, \ldots, P_7$, which determines an inseparable morphism of degree 2 from $X - \{P_i\}$ to $\mathbb{P}^2$.
(b) Every curve $C \in \mathfrak{d}$ is singular. More precisely, either C consists of 3 lines all passing through one of the P_i, or C is an irreducible cuspidal cubic with cusp $P \neq$ any P_i. Furthermore, the correspondence $C \mapsto$ the singular point of C is a $1-1$ correspondence between $\mathfrak{d}$ and $\mathbb{P}^2$. Thus the singular points of elements of $\mathfrak{d}$ move all over.

Proof. (a) To determine the dimension of $\mathfrak{d}$, we first compute the dimension of space of all cubic curves in $\mathbb{P}^2_k$. This is given by the complete linear system associated to $\mathcal{O}_X(3)$ which has dimension $\binom{5}{3} - 1 = 9$. Requiring all cubics to pass through the seven (distinct) points drops the dimension by seven and so $\dim_k \mathfrak{d} = 2$. The base points are clearly $P_1, \ldots, P_7$ since they are in the support of all cubics in this linear system. So we get a morphism of degree 2

$$f : X - \{P_i\}_{i=1}^7 \to \mathbb{P}^2_k.$$

We check inseparability.[Incomplete]
(b) The fact that the curve $C \in \mathfrak{d}$ is always singular can be checked via writing equations and using the fact that we are working in characteristic zero.

It should then be readily verifiable (I am too lazy to write this out) that there only two types of irreducible cubic curves up to isomorphism. One can do this as follows. A singular cubic can have at most one singularity and so we can have either three lines through one point, a cusp, or a node. We can rule out a nodal cubic by looking at the equation explicitly.

We show that $C \mapsto$the singular point of C is a correspondence between $\mathfrak{d}$ and $\mathbb{P}^2$. This can be (easily?) seen by noting that the irreducible cubics with cusp $P \neq$ any P_i correspond to $X - \{P_i\}_{i=1}^7$ and the reducible cubics correspond to the $\{P_i\}_{i=1}^7$. So the singular points of elements of $\mathfrak{d}$ cover all of $\mathbb{P}^2_k$. □

Hartshorne III.10.8 A linear system with moving singularities contained in the base locus (any characteristic). In affine 3 -space with coordinates x, y, z, let C be the conic $(x-1)^2 + y^2 = 1$ in the xy-plane, and let P be the point $(0, 0, t)$ on the z-axis. Let Y_t be the closure in $\mathbb{P}^3$ of the cone over C with vertex P. Show that as t varies, the surfaces $\{Y_t\}$ form a linear system of dimension 1, with a moving singularity at P. The base locus of this linear system is the conic C plus the z-axis.

Proof. [Incomplete] □

Hartshorne III.10.9 Let $f : X \to Y$ be a morphism of varieties over k. Assume that Y is regular, X is Cohen-Macaulay, and that every fibre of f has dimension equal to $\dim X - \dim Y$. Then f is flat.

[*Hint:* Imitate the proof of (10.4), using (II, 8.21A).]

Proof. [Incomplete] □

11. The Theorem on Formal Functions

Hartshorne III.11.1 Show that the result of (11.2) is false without the projective hypothesis. For example, let $X = \mathbb{A}^n_k$, let $P = (0, \dots, 0)$, let $U = X - P$, and let $f : U \to X$ be the inclusion. Then the fibres of f all have dimension 0 , but $R^{n-1}f_*\mathcal{O}_U \neq 0$.

Proof. Clearly all fibres have dimension 0. Now we check that $R^{n-1}f_*\mathcal{O}_U \neq 0$. This corresponds to computing cohomology since $R^{n-1}f_*\mathcal{O}_U$ is the sheaf on X associated to the presheaf

$$V \mapsto H^{n-1}(f^{-1}(V), \mathcal{O}_U|_{f^{-1}(V)}).$$

For ease, let's take the case of $n = 2$. Then if $V := X$, then

$$R^1 f_*\mathcal{O}_U(V) = H^1(\mathbb{A}^2_k \setminus P, \mathcal{O}_{\mathbb{A}^2_k \setminus P}) \neq 0$$

by Hartshorne III.4.3. □

Hartshorne III.11.2 Show that a projective morphism with finite fibres (=quasi-finite (II, Ex. 3.5)) is a finite morphism.

Proof. Everything we did in this section is w.r.t. noetherian schemes so assume $f : X \to Y$ is a projective quasi-finite morphism of noetherian schemes. Take a Stein factorization $f = f' \circ g'$. Since g' is already finite, we just need to check f' is finite. Now, f' is a projective morphism with connected fibres, so having finite fibres implies that the fibres of f' are singleton.

Let $f' : X \to Y'$ factor as $X \to \mathbb{P}^n_{Y'} \to Y'$ where $X \to \mathbb{P}^n_{Y'}$ is a closed immersion. It suffices to show f' is a finite morphism. [Incomplete]

The question is local on target so replace Y' by an affine scheme $\operatorname{Spec} B$. Then we WTS $f' : X \to \operatorname{Spec} B$ that is projective and has singleton fibres is necessarily finite. The factorization looks like $X \to \mathbb{P}^n_B \to \operatorname{Spec}(B)$. Since $f' : X \to \operatorname{Spec} B$ is necessarily of finite type by Hartshorne Theorem II.4.9, we may replace X by $f'^{-1}(\operatorname{Spec}(B))$ to reduce the problem to the following form.

Lemma 11.1. If $f' : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ is a projective morphism of finite type which is bijective, then the morphism is finite.

Proof. We get $\operatorname{Spec}(A) \to \mathbb{P}^n_B \to \operatorname{Spec}(B)$. We know B is a f.g. A-algebra and so $B = A[y_1, \dots, y_r]/(f_1, \dots, f_s)$. The first map $\operatorname{Spec}(A) \to \mathbb{P}^n_B$ being a closed immersion implies that it is finite. Let Z be the image in $\mathbb{P}^n_B$ and our job is to check $Z \to \operatorname{Spec}(B)$ is finite. Since the map must be bijective and $\mathbb{P}^n_B \to \operatorname{Spec}(B)$ is the structure morphism, Z must be precisely a point in $\mathbb{P}^n_B$ i.e. $Z \to \operatorname{Spec}(B)$ is an isomorphism. So this is map is certainly finite. □

[Incomplete]

Lemma 11.2. Let $f' : X \to Y'$ be a projective morphism that is bijective. Then this is a finite morphism.

PROOF. Some general thoughts on the difficulty of this result can be found here. □

[Incomplete] □

Hartshorne III.11.3 Let X be a normal, projective variety over an algebraically closed field k. Let $\mathfrak{d}$ be a linear system (of effective Cartier divisors) without base points, and assume that $\mathfrak{d}$ is not composite with a pencil, which means that if $f : X \to \mathbb{P}^n_k$ is the morphism determined by $\mathfrak{d}$, then $\dim f(X) \geqslant 2$. Then show that every divisor in $\mathfrak{d}$ is connected. This improves Bertini's theorem (10.9.1).

[Hints: Use (11.5), (Ex. 5.7) and (7.9).]

PROOF. Hartshorne Corollary III.7.9 asserts that the support of an ample divisor is connected. Meanwhile Hartshorne III.5.7 gives various criterions for ampleness. Lastly, Hartshorne III.11.5 is Stein factorization.

After summarizing the main results, let's prove the claim. A divisor in $\mathfrak{d}$ is a hyperplane section of $f(X)$ in $\mathbb{P}^n_k$. Now $n \geq 2$ at least and $\dim X \geq 2$. Let $D \in \mathfrak{d}$ be a divisor correspond to the pull-back $f^{-1}(H)$ of some hyperplane in $\mathbb{P}^n_k$.

I want to show that $f^{-1}(H)$ corresponds to an effective ample divisor. Effectiveness is clear so we check ampleness. But of course, $f^{-1}(H)$ is really just the divisor $f^*(H)$ and Hartshorne III.5.7(d) asserts that this is a also ample if $f : X \to \mathbb{P}^n_k$ is a finite surjective morphism. Of course, this has no hope of being surjective so what we really do is look at $f : X \to f(X)$ and instead of H, we really look at the divisor $H \cap f(X)$. This is still an ample divisor and the map is certainly surjective. Furthermore, f is a projective morphism with finite fibres by construction. So Hartshorne III.11.2 says it is finite. □

Hartshorne III.11.4 *Principle of Connectedness.* Let $\{X_t\}$ be a flat family of closed subschemes of $\mathbb{P}^n_k$ parametrized by an irreducible curve T of finite type over k. Suppose there is a nonempty open set $U \subseteq T$, such that for all closed points $t \in U, X_t$ is connected. Then prove that X_t is connected for all $t \in T$.

PROOF. [Incomplete] □

Hartshorne III.11.5 Let Y be a hypersurface in $X = \mathbb{P}^N_k$ with $N \geqslant 4$. Let $\hat{X}$ be the formal completion of X along Y (II, §9). Prove that the natural map $\operatorname{Pic} \hat{X} \to \operatorname{Pic} Y$ is an isomorphism.

[Hint: Use (II, Ex. 9.6), and then study the maps $\operatorname{Pic} X_{n+1} \to \operatorname{Pic} X_n$ for each n using (Ex. 4.6) and (Ex. 5.5).]

PROOF. First off, we can use Hartshorne II.9.6 to write $\operatorname{Pic} \widehat{X} = \varprojlim_n \operatorname{Pic} X_{n+1}$ where $X_{n+1} := (Y, \mathcal{O}_X/\mathscr{I}^n)$ for $\mathscr{I}$ the ideal sheaf associated to Y. Then, we can look at the SES's

$$0 \to \mathscr{I}^n/\mathscr{I}^{n+1} \to (\mathcal{O}_X/\mathscr{I}^{n+1})^* \to (\mathcal{O}_X/\mathscr{I}^n)^* \to 0$$

and use Hartshorne III.4.6 to get an exact sequence

$$H^1(X, \mathscr{I}^n/\mathscr{I}^{n+1}) \to \operatorname{Pic}(X_{n+1}) \to \operatorname{Pic}(X_n) \to H^2(X, \mathscr{I}^n/\mathscr{I}^{n+1}).$$

Now observe that $\mathscr{I} \cong \mathcal{O}_X(-d)$ and therefore $\mathscr{I}^n = \mathcal{O}_X(-dn)$ which implies $\mathscr{I}^n/\mathscr{I}^{n+1} = \mathscr{I}^n \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathscr{I} \cong \mathcal{O}_Y(-dn)$ as sheaves on X. Therefore, Hartshorne III.5.5 says

$$H^i(X, \mathscr{I}^n/\mathscr{I}^{n+1}) = 0 \text{ for all } 0 < i < N - 1.$$

Since $N \geq 4$ we get the result at least for $0 < i < 3$ i.e. the $H^1(X, \mathscr{I}^n/\mathscr{I}^{n+1}) = H^2(X, \mathscr{I}^n/\mathscr{I}^{n+1}) = 0$ vanish. Hence, $\operatorname{Pic}(X_{n+1}) \to \operatorname{Pic}(X_n)$ is an isomorphism and the inverse limit is isomorphic to $\operatorname{Pic} X_1$ which is just $\operatorname{Pic} Y$ and the natural map on Picard groups is an isomorphism. □

Hartshorne III.11.6 Again let Y be a hypersurface in $X = \mathbb{P}_k^N$, this time with $N \geqslant 2$.

(a) If $\mathscr{F}$ is a locally free sheaf on X, show that the natural map

$$H^0(X, \mathscr{F}) \to H^0(\hat{X}, \hat{\mathscr{F}})$$

is an isomorphism.

(b) Show that the following conditions are equivalent:

(i) for each locally free sheaf $\mathscr{F}$ on $\hat{X}$, there exists a coherent sheaf $\mathscr{F}$ on X such that $\mathfrak{F} \cong \hat{\mathscr{F}}$ (i.e., $\mathcal{F}$ is algebraizable);

(ii) for each locally free sheaf $\mathfrak{F}$ on $\hat{X}$, there is an integer n_0 such that $\mathfrak{F}(n)$ is generated by global sections for all $n \geqslant n_0$.

[Hint: For (ii) $\Rightarrow$ (i), show that one can find sheaves $\mathscr{E}_0, \mathscr{E}_1$ on X, which are direct sums of sheaves of the form $\mathcal{O}(-q_i)$, and an exact sequence $\hat{\mathscr{E}}_1 \to \hat{\mathscr{E}}_0 \to \mathfrak{F} \to 0$ on $\hat{X}$. Then apply (a) to the sheaf $\mathscr{H}\mathrm{om}\,(\mathscr{E}_1, \mathscr{E}_0)\cdot$]

(c) Show that the conditions (i) and (ii) of (b) imply that the natural map $\operatorname{Pic} X \to \operatorname{Pic} \hat{X}$ is an isomorphism.

Note. In fact, (i) and (ii) always hold if $N \geqslant 3$. This fact, coupled with (Ex. 11.5) leads to Grothendieck's proof [SGA 2] of the Lefschetz theorem which says that if Y is a hypersurface in $\mathbb{P}_k^N$ with $N \geqslant 4$, then $\operatorname{Pic} Y \cong \mathbb{Z}$, and it.is generated by $\mathcal{O}_{\mathbb{Y}}(1)$. See Hartshorne [5, Ch. IV] for more details.

Proof. (a)
(b) The implication (i)⇒(ii)
(c) □

Hartshorne III.11.7 Now let Y be a curve in $X = \mathbb{P}_k^2$.
(a) Use the method of (Ex. 11.5) to show that $\operatorname{Pic} \hat{X} \to \operatorname{Pic} Y$ is surjective, and its kernel is an infinite-dimensional vector space over k.
(b) Conclude that there is an invertible sheaf $\mathfrak{L}$ on $\hat{X}$ which is not algebraizable.
(c) Conclude also that there is a locally free sheaf $\mathfrak{F}$ on $\hat{X}$ so that no twist $\mathfrak{F}(n)$ is generated by global sections. Cf. (II, 9.9.1)

Proof. [Incomplete] □

Hartshorbne III.11.8 Let $f : X \to Y$ be a projective morphism, let $\mathscr{F}$ be a coherent sheaf on X which is flat over Y, and assume that $H^i(X_y, \mathscr{F}_y) = 0$ for some i and some $y \in Y$. Then show that $R^i f_*(\mathscr{F})$ is 0 in a neighborhood of y.

Proof. [Incomplete] □

12. The Semicontinuity Theorem

This section itself has some solutions online which can be found here.

Hartshorne III.12.1 Let Y be a scheme of finite type over an algebraically closed field k. Show that the function

$$\varphi(y) = \dim_k \left(\mathfrak{m}_y/\mathfrak{m}_y^2\right)$$

is upper semicontinuous on the set of closed points of Y.

Proof. By definition, $\mathfrak{m}_y/\mathfrak{m}_y^2$ is an $A_y := \mathcal{O}_{Y,y}/\mathfrak{m}_y$-module since it is $A_y \otimes_{\mathcal{O}_{Y,y}} \mathfrak{m}_y$ (and $A_y = k(y) = k$). Since Y is of finite type, $\mathfrak{m}_y/\mathfrak{m}^2$ is always finite dimensional over k.

Pick generators $\bar{m}_1, \ldots, \bar{m}_\ell$ of $\mathfrak{m}_y/\mathfrak{m}_y^2$ as a k-vector space. Lift these to a set of generators of $\mathfrak{m}_y$ i.e. $m_1, \ldots, m_\ell$ generate $\mathfrak{m}_y$.

Now to show upper semicontinuity, it suffices to show that

$$S := \{y' \in Y \text{ closed} : \varphi(y') < \varphi(y)\} \text{ is open.}$$

To do this, it suffices to show that the further localizations of $\mathcal{O}_{Y,y}$ have strictly less number of generators for its maximal ideals. Of course, a further localization $\mathcal{O}_{Y,y'}$ gives a map $\operatorname{Spec}(\mathcal{O}_{Y,y'}) \hookrightarrow \operatorname{Spec}(\mathcal{O}_{Y,y})$ and the maximal ideal $\mathfrak{m}_{Y,y'}$ must be generated by strictly less than ℓ-elements. But $\operatorname{Spec}(\mathcal{O}_{Y,y'})$ is an open subscheme of S containing a particular point.

For another reference, see McKernan's Lecture. One can actually use the results in McKernan's lectures in our case since we can WLOG assume Y is affine and so Y is a scheme of finite type over an algebraically closed field k. Since we are concerned with closed points we may as well assume Y is an affine algebraic subset of some affine space. Then a computation using the Jacobian matrix is doable. □

Hartshorne III.12.2 Let $\{X_t\}$ be a family of hypersurfaces of the same degree in $\mathbb{P}_k^n$. Show that for each i, the function $h^i(X_t, \mathcal{O}_{X_t})$ is a constant function of t.

Proof. From the proof of Hartshorne Proposition I.7.6, the Hilbert polynomial of a hypersurface in $\mathbb{P}_k^n$ is determined by the degree of the hypersurface. Since all of our hypersurfaces have the same degree, we can use Hartshorne Theorem III.9.9 to conclude that the family is indeed flat. Since the family is flat, we obtain some numerical invariants. Since hypersurfaces are all complete intersections, Hartshorne III.5.5 says that $h^i(X_t, \mathcal{O}_{X_t}) = 0$ for all t and $0 < i < q$.

For $h^0(X_t, \mathcal{O}_{X_t})$ to all be the same, we need each X_t to have the same number of connected components. This i can be guaranteed since we have a flat family.

For $h^{n-1}(X_t, \mathcal{O}_{X_t})$, we use the fact that the arthmetic genus is constant for flat families and due to our vanishing results, $p_a(X_t) = (-1)^{n-1}h^{n-1}(X_t, \mathcal{O}_{X_t})$. So we get the claim for h^{n-1}.

For all $i > n-1$, the cohomologies vanish by Grothendieck vanishing. □

Hartshorne III.12.3 Let $X_1 \subseteq \mathbb{P}_k^4$ be the *rational normal quartic curve* (which is the 4-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^4$). Let $X_0 \subseteq \mathbb{P}_k^3$ be a nonsingular rational quartic curve, such as the one in (I, Ex. 3.18b). Use (9.8.3) to construct a flat family $\{X_t\}$ of curves in $\mathbb{P}^4$, parametrized by $T = \mathbb{A}^1$, with the given fibres X_1 and X_0 for $t = 1$ and $t = 0$. Let $\mathscr{I} \subseteq {C_{\mathbb{P}}}^4 \times T$ be the ideal sheaf of the total family $X \subseteq \mathbb{P}^4 \times T$. Show that $\mathscr{I}$ is flat over T. Then show that

$$h^0(t, \mathscr{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$

and also

$$h^1(t, \mathscr{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$

This gives another example of cohomology groups jumping at a special point.

PROOF. [Incomplete] □

Hartshorne III.12.4 Let Y be an integral scheme of finite type over an algebraically closed field k. Let $f : X \to Y$ be a flat projective morphism whose fibres are all integral schemes. Let $\mathscr{L}, \mathscr{M}$ be invertible sheaves on X, and assume for each $y \in Y$ that $\mathscr{L}_y \cong \mathscr{M}_y$ on the fibre X_y. Then show that there is an invertible sheaf $\mathscr{N}$ on Y such that $\mathscr{L} \cong \mathscr{M} \otimes f^*\mathscr{N}$. [Hint: Use the results of this section to show that $f_*(\mathscr{L} \otimes \mathscr{M}^{-1})$ is locally free of rank 1 on Y.]

PROOF. See Mumford's Abelian Varieties textbook for some details for this idea. In particular, this is the idea used to determine the Picard scheme/variety.

Let's delve into the proof. Let $\mathscr{F} := \mathscr{L} \otimes \mathscr{M}^{-1}$. We want ot show that $f_*(\mathscr{L} \otimes \mathscr{M}^{-1})$ is an invartible sheaf on Y. We shall use Grauert's theorem to prove that $f_*(\mathscr{F})$ is locally free on Y of rank 1. We already know $f : X \to Y$ satisfies the hypoetheses and $\mathscr{F}$ is certainly coherent on X and flat over Y (flatness is local). We need to show that $h^0(y, \mathscr{F})$ is constant on Y.

Our fibres are projective and integral and thereby proper. So By Hartshorne II.4.5(d) and our hypothes is that $\mathscr{L}_y \cong \mathscr{M}_y$, we get

$$H^0(X_y, \mathscr{F}_y) = H^0(X_y, \mathcal{O}_{X_y}) = k$$

for all $y \in Y$. Now since we have locally free, we know $f_*(\mathscr{F})$ is locally free of rank 1 by a stalk computation.

To check that $f^*f_*(\mathscr{F})$ is what we want, we need to use an adjunction. There is a map

$$\mathscr{F} \to f^*f_*(\mathscr{F})$$

and because of our hypotheses (it is an isomorphism on stalks), $\mathscr{F} \cong f^*f_*(\mathscr{F})$. So, $\mathscr{N} \cong f_*\mathscr{F}$ works for our choice of $\mathscr{N}$. □

Hartshorne III.12.5 Let Y be an integral scheme of finite type over an algebraically closed field k. Let $\mathscr{E}$ be a locally free sheaf on Y, and let $X = \mathbb{P}(\mathscr{E})$-see (II, § 7). Then show that Pic $X \cong$ (Pic Y) $\times Z$. This strengthens (II, Ex. 7.9).

PROOF. [Incomplete] □

Hartshorne III.12.6 Let X be an integral projective scheme over an algebraically closed field k, and assume that $H^1(X, \mathcal{O}_X) = 0$. Let T be a connected scheme of finite type over k.
(a) If $\mathscr{L}$ is an invertible sheaf on $X \times T$, show that the invertible sheaves $\mathscr{L}_t$ on $X = X \times \{t\}$ are isomorphic, for all closed points $t \in T$.
(b) Show that $\operatorname{Pic}(X \times T) = \operatorname{Pic} X \times \operatorname{Pic} T$. (Do not assume that T is reduced!) Cf. (IV, Ex. 4.10) and (V, Ex. 1.6) for examples where $\operatorname{Pic}(X \times T) \neq \operatorname{Pic} X \times \operatorname{Pic} T$.

[Hint: Apply (12.11) with $i = 0, 1$ for suitable invertible sheaves on $X \times T$.]

PROOF. [Incomplete] □

CHAPTER 4

Chapter IV Exercises

1. Riemann-Roch Theorem

Hartshorne IV.1.1 Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

PROOF. Choose an $n \gg 0$ s.t. $\dim |nP| = n \deg P - g$ where. Since $\dim |nP| = l(nP) - 1$, we deduce that $l(nP) = n \deg P - g - 1$. An element of $H^0(X, \mathscr{L}(nP))$ is just a rational function $f \in K(X)$ s.t. $\operatorname{div}(f) \geq -nP$. So, f is regular everywhere except possibly by at P where it has at worse a pole of order n. Since $l(nP) < l((n+1)P)$, there is at least one function $f \in K(X)$ with $\operatorname{ord}_P(f) = -(n+1)$ and f is regular everywhere else. □

Hartshorne IV.1.2 Again let X be a curve, and let $P_1, \ldots, P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_i, and regular elsewhere.

PROOF. Use the same trick as above. Consider $D_1 = nP_1 + P_2 + \cdots + P_r$ and let n be sufficiently large so that $l(D) = \deg D - g - 1$. Then by increasing n, one deduces there exists an $f_1 \in K(X)$ such that $\operatorname{div}(f) \geq -nP_1 - P_2 - \cdots - P_r$ and $\operatorname{ord}_{P_1}(f) = -n$ and f is regular everywhere. Look at the divisor $D_2 = P_1 + nP_2 + P_3 + \cdots + P_r$ and do the same to get an f_2. After obtaining $f_1, \ldots, f_r$, the product $f_1 f_2 \ldots f_r$ is the desired rational function. □

Hartshorne IV.1.3 Let X be an integral, separated, regular, one-dimensional scheme of finite type over k, which is not proper over k. Then X is affine.

[Hint: Embed X in a (proper) curve $\bar{X}$ over k, and use (Ex. 1.2) to construct a morphism $f : \bar{X} \to \mathbb{P}^1$ such that $f^{-1}(\mathbb{A}^1) = X$]

PROOF. Embed X into a proper curve $\overline{X}$ by using Hartshorne Proposition I.6.8.

Using Hartshorne IV.1.2, we can construct a morphism $f : \overline{X} \to \mathbb{P}^1$ as follows. Let $P_1, \ldots, P_l \in \overline{X} \setminus X$. Choose a function g with poles precisely at these P_i. Define the morphism

$$f(Q) = \begin{cases} [f(P) : 1] & \text{if } f \text{ is regular at } P \\ [1 : 0] & \text{if } f \text{ not regular at } P \end{cases}.$$

Then if we pull back the affine chart $\mathbb{A}^1 \ni [x : y]$ where $y \neq 0$, we get X. This shows that X is birationally equivalent to $\mathbb{P}^1$, but it cannot be isomorphic since it is not proper over k. By Hartshorne I.6.1 (a), X is isomorphic to an open subset of $\mathbb{A}^1$ and Hartshorne I.6.1(b) says that X is thereby affine. □

Hartshorne IV.1.4 Show that a separated, one-dimensional scheme of finite type over k, none of whose irreducible components is proper over k, is affine.

[Hint: Combine (Ex. 1.3) with (III, Ex. 3.1, Ex. 3.2, Ex. 4.2).]

PROOF. By Hartshorne III.3.1, assume X is reduced. By Hartshorne III.3.2, assume X is irreducible. So, X is an integral scheme.

Let Y be the blow-up of X. The blow-up map $f : Y \to X$ is a finite surjective morphism of noetherian separated schemes since X is 1-dimensional. Since Y is also regular and not proper over k, Hartshorne IV.1.3 says Y is affine. By Hartshorne III.4.2, X is affine.

Remark 55. Instead of a blow-up, we could have also taken the normaliation.

□

Hartshorne IV.1.5 For an effective divisor D on a curve X of genus g, show that $\dim |D| \leqslant \deg D$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

PROOF. Let D be an effective divisor. Let $D' = D + -(1 + \deg D)P$ which has $\deg D' < 0$. Hence, $l(D') = 0$. It follows

$$\dim_k \left(H^0(X, \mathscr{L}(D))/H^0(X, \mathscr{L}(D')\right) \leq \deg D+1 \Rightarrow \ell(D) \leq \deg D+1 \Rightarrow \dim_k |D| \leq \deg D.$$

For the second part, we analyze when equality may occur.
($\Rightarrow$). Suppose $\dim |D| = \deg D$. Equality is clear if $D = 0$ so assume $D \neq 0$ and we show $g = 0$. Then $l(D) - 1 = \deg D$ is nonzero. So $l(D) = 1 + \deg D \geq 2$ implies there are nonconstant functions inside $H^0(X, \mathscr{L}(D))$. Choose such a function f. Then define a map $X \to \mathbb{P}^1$ by

$$P \mapsto \begin{cases} [f(P) : 1] \text{ if } f(P) \text{ regular at } P \\ [1 : 0] \text{ if } f \text{ is not regular at } P \end{cases}$$

This defines a birational map to $\mathbb{P}^1$ since we know $\operatorname{div}(f) \geq -D$ has at most finitely many poles. But X itself was a nonsingular complete curve which is now birational to $\mathbb{P}^1_k$. Since g is a birational invariant, $p_g(X) = g = 0$ as desired,
($\Leftarrow$). Suppose $D = 0$ or $g = 0$. If $D = 0$, then $\dim |D| = \deg D$ by Hartshorne Remark IV.1.3.2. Now let $g = 0$. Then

$$l(D) - l(K - D) = \deg D + 1 \Rightarrow \dim |D| - l(K - D) = \deg D \Rightarrow \dim |D| \geq \deg D.$$

Since $\dim |D| \leq \deg D$ is always true, we get equality.

□

Hartshorne IV.1.6 Let X be a curve of genus g. Show that there is a finite morphism $f : X \to \mathbb{P}^1$ of degree $\leqslant g + 1$. (Recall that the degree of a finite morphism of curves $f : X \to Y$ is defined as the degree of the field extension $[K(X) : K(Y)]$ (II, §6).)

PROOF. Apply the Riemann-Roch Theorem to get

$$l((g+1)P) = l(K - (g+1)P) + \deg((g+1)P) + 1 - g = l(K - (g+1)P) + g + 1 + 1 - g \geq 2.$$

Then $\dim_k |(g + 1)P| \geq 1$. Choose a nonconstant rational function f in the linear system. It defines a finite morphism $f : X \to \mathbb{P}^1$ as in our proof of Hartshorne IV.1.5 and $\deg(f) = \deg((g + 1)P) = g + 1$. □

Hartshorne IV.1.7 A curve X is called *hyperelliptic* if $g \geqslant 2$ and there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree 2.

(a) If X is a curve of genus $g = 2$, show that the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1, without base points. Use (II, 7.8.1) to conclude that X is hyperelliptic.

(b) Show that the curves constructed in (1.1.1) all admit a morphism of degree 2 to $\mathbb{P}^1$. Thus there exist hyperelliptic curves of any genus $g \geqslant 2$.

Note. We will see later (Ex. 3.2) that there exist nonhyperelliptic curves. See also (V, Ex.2.10).

PROOF. **(a)** We know that $\deg K = 2g - 2 = 2$ since $g = 2$. Meanwhile, we know $l(K) = g = 2$ by definition so that $\dim |K| = g - 1 = 1$.

To check that K is basepoint free, we can just specify a morphism into projective space. Since $f : X \to \mathbb{P}^1$ is a finite morphism of degree 2, it suffices to show that this must be determined by K. Pull-back $f^*\mathcal{O}(1)$ which is a sheaf on X of degree 2 whose associated linear system must is of dimension 1. It suffices to check only K does this.

Assume $\deg D = 2$ and $l(D) - 1 = 1$. Then $l(D) = 2$ and the Riemann-Roch Theorem gives

$$2 - l(D - K) = 2 + 1 - g = 1.$$

This forces $l(D - K) = 1$. We know $\deg D - K = \deg D - \deg K = 0$. Hartshorne Lemma IV.1.2 says that $l(D - K) \neq 0$ and $\deg(D - K) = 0$ implies $D - K \sim 0$. Hence, $D \sim K$. So $f^*\mathcal{O}(1) \cong \mathcal{O}_X(D) \cong \mathcal{O}_X(K)$.

By Hartshorne Remark II.7.8.1, there is a degree 2 finite morphism $f : X \to \mathbb{P}^1$. So X is hyperelliptic.

(b) A divisor of type $(g + 1, 2)$ on a nonsingular quadric surface is associated to a curve X. Now the morphism $p_2^* : \operatorname{Cl} \mathbb{P}^1 \to \operatorname{Cl} Q$ in the second coordinate determines a divisor of degree 2 on the curve X which we denote by D. Additionally, it induces an isomorphism $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \cong H^0(X, p_2^*(\mathcal{O}_Q(D))|_X)$. But the former has dimension 2 and so $\mathcal{O}_Q(D)|_X$ determines a linear system of dimension 1 on X with a degree 2 divisor. Hence, we get the claimed morphism to $\mathbb{P}^1$. □

Hartshorne IV.1.8 p_a of a Singular Curve. Let X be an integral projective scheme of dimension 1 over k, and let $\tilde{X}$ be its normalization (II, Ex. 3.8). Then there is an exact sequence of sheaves on X,

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{\tilde{X}} \to \sum_{P \in X} \tilde{\mathcal{O}}_P/\mathcal{O}_P \to 0$$

where $\tilde{\mathcal{O}}_P$ is the integral closure of $\mathcal{O}_P$. For each $P \in X$, let $\delta_P = \operatorname{length}\left(\tilde{\mathcal{O}}_P/\mathcal{O}_P\right)$.

(a) Show that $p_a(X) = p_a(\tilde{X}) + \sum_{P \in X} \delta_P$. [Hint: Use (III, Ex. 4.1) and (III, Ex. 5.3).]

(b) If $p_a(X) = 0$, show that X is already nonsingular and in fact isomorphic to $\mathbb{P}^1$. This strengthens (1.3.5).

*(c) If P is a node or an ordinary cusp (I, Ex. 5.6, Ex. 5.14), show that $\delta_P = 1$.

[Hint: Show first that δ_P depends only on the analytic isomorphism class of the singularity at P. Then compute δ_P for the node and cusp of suitable plane cubic curves. See (V, 3.9.3) for another method.]

PROOF. First, we verify the existence of such a short exact sequence. We only need to prove the cokernel of $\mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}}$ is $\sum_{P \in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P$. The normalization $\widehat{X} \to X$ is an isomorphism on a dense open subset. So the sheaf of rings $\mathcal{O}_X$ is modified only at points and the cokernel

must be a skyscrapper sheaf. Taking stalks, we deduce that the cokernel has stalks $\widetilde{\mathcal{O}_P}/\mathcal{O}_P$ at the point P.
(a) Hartshorne III.5.3 says $p_a(X) = 1 - \chi(\mathcal{O}_X) = \dim_k H^1(X, \mathcal{O}_X)$. So additivity of $\chi(-)$ combined with $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^i(X, f_*\mathcal{O}_{\widetilde{X}})$ yields

$$p_a(X) = 1 - \chi(\mathcal{O}_X) = 1 - \chi(f_*\mathcal{O}_X) + \chi\left(\sum_{P\in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P\right) = 1 - \chi(\mathcal{O}_{\widetilde{X}}) + \chi\left(\sum_{P\in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P\right)$$

$$= p_a(\mathcal{O}_{\widetilde{X}}) + \chi\left(\sum_{P\in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P\right) = p_a(\widetilde{X}) + \sum_{P\in X} \delta_P.$$

Here, we used the fact that $\chi\left(\widetilde{\mathcal{O}_P}/\mathcal{O}_P\right) = \text{length}(\widetilde{\mathcal{O}_P}/\mathcal{O}_P)$.
(b) Assume $p_a(X) = 0$. Then $p_a(\widetilde{X}) = 0$ and $\sum_{P\in X} \delta_P = 0$. Therefore $\widetilde{\mathcal{O}_P} = \mathcal{O}_P$ for all $P \in X$ which implies X is already nonsingular. Because $p_a(X) = p_g(X) = 0$ in this case, it must be $\mathbb{P}^1$.
(c) Taking completions should preserve the length and so δ_P is preserved for analytic isomorphism classes. Then we take two curves, the nodal cubic and the cuspidal cubic and we get the claim. For instance, the equation of the cuspudal cubic is $x^2 = y^3$. The $\widetilde{\mathcal{O}_P}/\mathcal{O}_P$ is the same as $k[t]_{(t)}/k[t^2, t^3]_{(t)}$ which has length 1 over k ($0 \subseteq (t) = \widetilde{\mathcal{O}_P}/\mathcal{O}_P$). The same computation works for the nodal-cubic since we know that it is analytically isomorphic to $xy = 0$ the intersection of two lines. □

Hartshorne IV.1.9 *Riemann-Roch for Singular Curves.* Let X be an integral projective scheme of dimension 1 over k. Let X_{reg} be the set of regular points of X.
(a) Let $D = \sum n_i P_i$ be a divisor with support in X_{reg}, i.e., all $P_i \in X_{\text{reg}}$. Then define $\deg D = \sum n_i$. Let $\mathscr{L}(D)$ be the associated invertible sheaf on X, and show that

$$\chi(\mathscr{L}(D)) = \deg D + 1 - p_a.$$

(b) Show that any Cartier divisor on X is the difference of two very ample Cartier divisors. (Use (II, Ex. 7.5).)
(c) Conclude that every invertible sheaf $\mathscr{L}$ on X is isomorphic to $\mathscr{L}(D)$ for some divisor D with support in X_{reg}.
(d) Assume furthermore that X is a locally complete intersection in some projective space. Then by (III, 7.11) the dualizing sheaf ω_X° is an invertible sheaf on X, so we can define the canonical divisor K to be a divisor with support in X_{reg} corresponding to ω_X°. Then the formula of (a) becomes

$$l(D) - l(K - D) = \deg D + 1 - p_a.$$

Proof. **(a)** This follows immediately from the Riemann-Roch Theorem i.e. X_{reg} is a nonsingular curve and so we can use the theorem. One needs to check that $p_a(X) = p_a(X_{reg})$. Of course, the set of singular points of X form a set of finite points and so p_a stays the same.
(b) Let D be a Cartier divisor, $\mathscr{M} = \mathscr{L}(D)$ its associated invertible sheaf, and let $\mathscr{L}$ a very ample invertible sheaf.

Choose $n > 0$ such that $\mathscr{M} \otimes \mathscr{L}^n$ is generated by global sections. By Hartshorne II.7.5(d), $\mathscr{M} \otimes \mathscr{L}^{n+1}$ and $\mathscr{L}^{n+1}$ are very ample. By Hartshorne Proposition II.6.15, $\mathscr{M} \otimes \mathscr{L}^{n+1} \cong \mathscr{L}(D')$ for some Cartier divisor D'. Similarly, $\mathscr{L}^{n+1} \cong \mathscr{L}(D'')$. Then,

$$\mathscr{M} = \mathscr{M} \otimes \mathscr{L}^{n+1} \otimes \mathscr{L}^{-n-1} = \mathscr{L}(D') \otimes \mathscr{L}(-D'')$$

which means $D' - D'' \sim D$. So we can make is so that D is the difference between two vary ample Cartier divisors.

(c) Using (b), we may only need to prove this when $\mathscr{L} \cong \mathscr{L}(D)$ where D is an effective very ample Cartier divisor. BY definition of very ample, D is the pullback of a hyperplane H, which can be made to miss the singular locus of X (pick another hyperplane linearly equivalent to H via Bertini's Theorem). In that case, $\operatorname{Supp} D \subseteq X_{\text{reg}}$.

[**Incomplete**]

(d) There is nothing to show here – Harthorne is just making a statement regarding the Riemann-Roch Theorem. I suppose we could check Cohen-Macaulayness, but that is not too interesting to do. □

Hartshorne IV.1.10 Let X be an integral projective scheme of dimension 1 over k, which is locally complete intersection, and has $p_a = 1$. Fix a point $P_0 \in X_{\text{reg.}}$. Imitate (1.3.7) to show that the map $P \to \mathscr{L}(P - P_0)$ gives a one-to-one correspondence between the points of X_{reg} and the elements of the group $\operatorname{Pic}^\circ X$. This generalizes (II, 6.11.4) and (II, Ex. 6.7).

Proof. This generalizes the group law on elliptic curves to curves with potentially singular points.

In mimicking the proof of Hartshorne Example IV.1.3.7, we only need to use the previous exercise to replace the use of the Riemann-Roch Theorem. □

2. Hurwitz's Theorem

Hartshorne IV.2.1 Use (2.5.3) to show that $\mathbb{P}^n$ is simply connected.

Proof. See this Stack-Exchange post. The idea of the proof is to do induction. Suppose $f : X \to \mathbb{P}^n$ is a finite étale cover. Assume WLOG that X is connected. Then if $H \subseteq \mathbb{P}^n$ is a hyperplane, then $f^*H \to H$ is a finite étale cover of $H \cong \mathbb{P}^{n-1}$. So, $f^*H \to H$ is an isomorphism (by Hartshorne Theorem III.7.9 we know f^*H is connected and then we use our inductive hypothesis since $H \cong \mathbb{P}^{n-1}$). Since H was an arbitrary hyperplane, we deduce that f must be a degree 1 map. Therefore, $f : X \to \mathbb{P}^n$ is an isomorphism. □

Hartshorne IV.2.2 *Classification of Curves of Genus 2.* Fix an algebraically closed field k of characteristic $\neq 2$.

(a) If X is a curve of genus 2 over k, the canonical linear system $|K|$ determines a finite morphism $f : X \to \mathbb{P}^1$ of degree 2 (Ex. 1.7). Show that it is ramified at exactly 6 points, with ramification index 2 at each one. Note that f is uniquely determined, up to an automorphism of $\mathbb{P}^1$, so X determines an (unordered) set of 6 points of $\mathbb{P}^1$, up to an automorphism of $\mathbb{P}^1$.

(b) Conversely, given six distinct elements $\alpha_1, \ldots, \alpha_6 \in k$, let K be the extension of $k(x)$ determined by the equation $z^2 = (x - \alpha_1) \cdots (x - \alpha_6)$. Let $f : X \to \mathbb{P}^1$ be the corresponding morphism of curves. Show that $g(X) = 2$, the map f is the same as the one determined by the canonical linear system, and f is ramified over the six points $x = \alpha_i$ of $\mathbb{P}^1$, and nowhere else. (Cf. (II, Ex. 6.4).)

(c) Using (I, Ex. 6.6), show that if P_1, P_2, P_3 are three distinct points of $\mathbb{P}^1$, then there exists a unique $\varphi \in \operatorname{Aut} \mathbb{P}^1$ such that $\varphi(P_1) = 0, \varphi(P_2) = 1, \varphi(P_3) = \infty$. Thus in (a), if we order the six points of $\mathbb{P}^1$, and then normalize by sending the first three to $0, 1, \infty$, respectively,

we may assume that X is ramified over $0, 1, \infty, \beta_1, \beta_2, \beta_3$, where $\beta_1, \beta_2, \beta_3$ are three distinct elements of $k, \neq 0, 1$.

(d) Let Σ_6 be the symmetric group on 6 letters. Define an action of Σ_6 on sets of three distinct elements $\beta_1, \beta_2, \beta_3$ of $k, \neq 0, 1$, as follows: reorder the set $0, 1, \infty, \beta_1, \beta_2, \beta_3$ according to a given element $\sigma \in \Sigma_6$, then renormalize as in (c) so that the first three become $0, 1, \infty$ again. Then the last three are the new $\beta_1', \beta_2', \beta_3'$.

(e) Summing up, conclude that there is a one-to-one correspondence between the set of isomorphism classes of curves of genus 2 over k, and triples of distinct elements $\beta_1, \beta_2, \beta_3$ of $k, \neq 0, 1$, modulo the action of Σ_6 described in (d). In particular, there are many non-isomorphic curves of genus 2 . We say that curves of genus 2 depend on three parameters, since they correspond to the points of an open subset of $\mathbb{A}^3_k$ modulo a finite group.

PROOF. (a) Since k is algebraically closed and has characteristic $\neq 2$, any ramification which occurs is tame if we know $1 \leq e_P \leq 2$. Indeed $1 \leq e_P \leq 2$ because the degree of f is 2. By Hartshorne Corollary IV.2.4 (from now on referred to as the Hurwitz theorem), we get

$$2 = -4 + \deg R = -4 + \sum_{P \in X} (e_P - 1) \;\Rightarrow\; \sum_{P \in X} (e_P - 1) = 6.$$

There are at least six points in which $e_P > 1$.

Clearly, f is determined up to an automorphism of $\mathbb{P}^1$ which means X determines up to automorphism an unordered set of 6 points in $\mathbb{P}^1$ which are the ramified points.

(b) Let $\alpha_1, \ldots, \alpha_6 \in k$ be distinct points. Let $K/k(x)$ be determined by the equation $z^2 = \prod_{i=1}^6 (x - \alpha_i)$. Then

$$K := k(x)[z]/(z^2 - (x - \alpha_1) \ldots (x - \alpha_6))$$

which is a degree 2 algebraic extension of $k(x)$ and a function field of dimension 1. So $f : X \to \mathbb{P}^1$ defines a finite separable morphism of degree 2. Apply the Hurwitz Formula to get $g(X) = -1 + \frac{1}{2} \deg R$. We want $\deg R = 6$.

We can explicitly describe X as the projective closure of Spec $\frac{k[x,z]}{(z^2 - \prod_{i=1}^6 (x - \alpha_i)}$ inside of $\mathbb{P}^2_k$ and we blow-up all singularities that occur. In particular, we consider X as the curve cut out by $Y^2 = \prod_{i=1}^6 (X - \alpha_i Z)$ inside weighted projective space $\mathbb{P}^2(1, 3, 1)$.

The morphism $f : X \to \mathbb{P}^1$ is then given by projection onto the x-axis i.e. $f([x : y : z]) = [x : 1]$.

The image lands entirely in one affine chart. Let $f((a, b)) = [a : 1]$. The uniformizer at $[a : 1]$ can be computed from

$$\mathcal{O}_{\mathbb{P}^1, [a:1]} \cong \operatorname{Spec} k[x]_{(x-a)}$$

which has uniformizer $x - a$. Now the uniformizer of (a, b) for X is either $x - a$ or $z - b$ depending on if $\frac{\partial g}{\partial z}(a, b) \neq 0$ or $\frac{\partial g}{\partial x}(a, b) \neq 0$ respectively. Here, $g = z^2 - (x - \alpha_1) \ldots (x - \alpha_6)$. So the two conditions are

$$\frac{\partial g}{\partial x}(a, b) = 2b \neq 0 \iff b \neq 0 \qquad \& \qquad \frac{\partial g}{\partial x}(a, b) = \sum_{i=1}^{6} (x - \alpha_1) \ldots \widehat{(x - \alpha_i)} \ldots (x - \alpha_6)|_{(a,b)} \neq 0.$$

In the first case, $b \neq 0$ gives the uniformizer $x - a$. But the valuation of $x - a$ with uniformizer $x - a$ is just 1. Hence, there is no ramification. Assume $b = 0$. Then we compute

the uniformizer of $(a,0)$. Since $(a,0)$ must satisfy $g(a,0)=0$, we deduce that $a=\alpha_i$ for some i. Assume WLOG that $a=\alpha_1$. The local ring is

$$\mathcal{O}_{X,(a,0)} = (k[x,z]/(g))_{(x-a,z)}.$$

The maximal ideal is $\mathfrak{m} = (x-a,z)$ generated in $k[x,z]/(g)$. Also, $\mathfrak{m}/\mathfrak{m}^2$ must be 1-dimensional so we show $x-a$ has a relation in terms of z as a vector space over the residue field. Indeed, $(x-a)(x-\alpha_2)\dots(x-\alpha_6) = z^2$. So $(x-a) = \frac{1}{(x-\alpha_2)\dots(x-\alpha_6)}z^2$ is a scalar multiple of $z^2 \in \mathfrak{m}$. So eliminate $x-a$ from the spanning set of $\mathfrak{m}/\mathfrak{m}^2$. This gives the basis generated by z and so, z is the uniformizer. Of course, the valuation

$$v_z(x-a) = v_z(x-\alpha_1) = 2.$$

So there is ramification at $(a,0)\mapsto[a:1]$ which occurs only at the points $a\in\{\alpha_i\}_{i=1}^6$.

Now that we know there is ramification at the six points $[1:\alpha_i]$ of $\mathbb{P}^1$ and nowhere else, the Hurwtiz formula tells us $g(X)=2$.

Now we show that the corresponding morphism of curves $f: X\to\mathbb{P}^1$ is the same at that in (a). The canonical divisor on an algebraic curve of genus 2 is can be taken as any divisor D of degree 2 with $l(D)=2$. Let K be any degree 2 divisor and then, Riemann-Roch gives

$$l(K) - l(0) = 1 \Rightarrow l(K) = 1 + l(0) = 2.$$

It also follows that $\dim|K|=1$ and so we only need to find a nonzero element to determine the morphism of the canonical system. Let us take

$$K = (\alpha_1,0)+(\alpha_2,0)\in \mathrm{Cl}X \quad \text{where } \alpha_1\neq\alpha_2$$

as the canonical divisor. Then the canonical linear system is $\left(H^0(X,\mathscr{L}(K))-\{0\}\right)/k^*$. An element of this is the rational function x. So the map defined by the canonical linear system is

$$X\to\mathbb{P}^1 \quad (x,y,z)\mapsto[x:1].$$

This is the map in (a) and we observe that this is precisely the map determined by $K/k(x)$.

(c) Let $P_1,P_2,P_3\in\mathbb{P}^1$ be distinct points. To construct the choice of $\varphi\in\operatorname{Aut}\mathbb{P}^1$, we show there exists a unique fractional linear transformation s.t. $\varphi(P_1)=0$, $\varphi(P_2)=1$, and $\varphi(P_3)=\infty$. We identify $\mathbb{P}^1=\mathbb{A}^1\cup\{\infty\}$ to make this computation easier. The conditions we want are that

$$\varphi(x)=\frac{ax+b}{cx+d} \text{ with } ad-bc\neq 0 \text{ and } \frac{aP_1+b}{cP_1+d}=0,\ \frac{aP_2+b}{cP_2+d}=1,\ \frac{aP_3+b}{cP_3+d}=\infty.$$

We use some knowledge about linear fractional transformations from complex analysis. There is an implicit form to define a map $(z_1,z_2,z_3)\mapsto(w_1,w_2,w_3)$ and that tells us

$$w=\frac{(z-P_1)(P_2-P_3)}{(z-P_3)(P_2-P_1)}$$

is the correct linear fractional transformation.

From (a), we apply an automorphism to assume that ramification is over the six points $0,1,\infty,\beta_1,\beta_2,\beta_3$ for $\beta_1,\beta_2,\beta_3\in k\setminus\{0,1\}$ distinct.

(d) There is nothing to prove.

(e) There is a one-to-one correspondence between isomorphism classes of curves of genus 2 over k and triples of distinct elements β_1,β_2,β_3 of $k\setminus\{0,1\}$ modulo the action in (d).

Indeed, each isomorphism class of curve of genus 2 gets associated to such a triple of points by (a),(c), and (d). Conversely, (b) with (c) shows each triple has a corresponding class of curves.

There are many non-isomorphic curves of genus 2 because there are many choices of triples $\beta_1, \beta_2, \beta_3$. So curves of genus 2 depend on three parameters as they correspond to points on an open subset of $\mathbb{A}^2_k$ modulo a Σ_6-action. □

Hartshorne IV.2.3 *Plane Curves.* Let X be a curve of degree d in $\mathbb{P}^2$. For each point $P \in X$, let $T_P(X)$ be the tangent line to X at P (*I*, Ex. 7.3). Considering $T_P(X)$ as a point of the dual projective plane $(\mathbb{P}^2)^*$, the map $P \to T_P(X)$ gives a morphism of X to its dual curve X^* in $(\mathbb{P}^2)^*$ (I, Ex. 7.3). Note that even though X is nonsingular, X^* in general will have singularities. We assume char $k = 0$ below.

(a) Fix a line $L \subseteq \mathbb{P}^2$ which is not tangent to X. Define a morphism $\varphi : X \to L$ by $\varphi(P) = T_P(X) \cap L$, for each point $P \in X$. Show that φ is ramified at P if and only if either (1) $P \in L$, or (2) P is an inflection point of X, which means that the intersection multiplicity (I, Ex. 5.4) of $T_P(X)$ with X at P is $\geqslant 3$. Conclude that X has only finitely many inflection points.

(b) A line of $\mathbb{P}^2$ is a multiple tangent of X if it is tangent to X at more than one point. It is a bitangent if it is tangent to X at exactly two points. If L is a multiple tangent of X, tangent to X at the points $P_1, \ldots, P_r$, and if none of the P_i is an inflection point, show that the corresponding point of the dual curve X^* is an ordinary r-fold point, which means a point of multiplicity r with distinct tangent directions (I, Ex. 5.3). Conclude that X has only finitely many multiple tangents.

(c) Let $O \in \mathbb{P}^2$ be a point which is not on X, nor on any inflectional or multiple tangent of X. Let L be a line not containing O. Let $\psi : X \to L$ be the morphism defined by projection from O. Show that ψ is ramified at a point $\mathrm{P} \in X$ if and only if the line OP is tangent to X at P, and in that case the ramification index is 2. Use Hurwitz's theorem and (I, Ex. 7.2) to conclude that there are exactly $d(d-1)$ tangents of X passing through O. Hence the degree of the dual curve (sometimes called the class of X) is $d(d-1)$.

(d) Show that for all but a finite number of points of X, a point O of X lies on exactly $(d+1)(d-2)$ tangents of X, not counting the tangent at O.

(e) Show that the degree of the morphism φ of (a) is $d(d-1)$. Conclude that if $d \geqslant 2$, then X has $3d(d-2)$ inflection points, properly counted. (If $T_P(X)$ has intersection multiplicity r with X at P, then P should be counted $r-2$ times as an inflection point. If $r = 3$ we call it an ordinary inflection point.) Show that an ordinary inflection point of X corresponds to an ordinary cusp of the dual curve X^*.

(f) Now let X be a plane curve of degree $d \geqslant 2$, and assume that the dual curve X^* has only nodes and ordinary cusps as singularities (which should be true for sufficiently general X). Then show that X has exactly $\frac{1}{2}d(d-2)(d-3)(d+3)$ bitangents.

[Hint: Show that X is the normalization of X^*. Then calculate $p_a(X^*)$ two ways: once as a plane curve of degree $d(d-1)$, and once using (Ex. 1.8).]

(g) For example, a plane cubic curve has exactly 9 inflection points, all ordinary. The line joining any two of them intersects the curve in a third one.

(h) A plane quartic curve has exactly 28 bitangents. (This holds even if the curve has a tangent with four-fold contact, in which case the dual curve X^* has a tacnode.)

PROOF. **(a)** There are at most finitely many ramified points so the conclusion is easy.
[**Incomplete**]

(b)

(c)

(d)

(e)

(f)

(h) □

Hartshorne IV.2.4 *A Funny Curve in Characteristic p.* Let X be the plane quartic curve $x^3y + y^3z + z^3x = 0$ over a field of characteristic 3 . Show that X is nonsingular, every point of X is an inflection point, the dual curve X^* is isomorphic to X, but the natural map $X \to X^*$ is purely inseparable.

PROOF. Nonsingularity is easy by the Jacobian criterion.

Dehomogenize w.r.t. z to get the equation $x^3y + y^3 + x = 0$. Now the tangent line at the origin is determined by $x = 0$ i.e. the vertical line. This gives an inflection point since we intersect a degree 4 curve with a degree 1 curve (or alternatively, the order of vanishing that the point of intersection $x = 0, y = 0$ has highest order of vanishing 3).

This shows we have an inflection at the point $[0 : 0 : 1]$. We want to show there is an inflection at any point $[a : b : c]$ of the curve. But a linear change of coordinates would never change the form of the equation and so it has an inflection point everywhere.

Now we check that the dual X^* is isomorphic to X. The line tangent to X at the point $p = [a : b : c]$ is given by the formula

$$\frac{\partial f}{\partial x}|_p x + \frac{\partial f}{\partial y}|_p y + \frac{\partial f}{\partial z}|_p z = 0.$$

Of course, this just gives the equation

$$(3a^2b + c^3)x + (3b^2c + a^3)y + (b^3 + 3c^2a)z = 0 \;\Rightarrow\; c^3x + a^3y + b^3z = 0$$

and so $[a : b : c] \to [c : a : b]$. This map is clearly an isomorphism then.

Now we show that $X \to X^*$ is purely inseparable. This comes down to showing the extension of fields is purely inseparable i.e. study $K(X^*)/K(X)$.
[**Incomplete**] □

Hartshorne IV.2.5 *Automorphisms of a Curve of Genus* $\geqslant 2$. Prove the theorem of Hurwitz [1] that a curve X of genus $g \geqslant 2$ over a field of characteristic 0 has at most $84(g-1)$ automorphisms. We will see later (Ex. 5.2) or (V, Ex. 1.11) that the group $G = \operatorname{Aut} X$ is finite. So let G have order n. Then G acts on the function field $K(X)$. Let L be the fixed field. Then the field extension $L \subseteq K(X)$ corresponds to a finite morphism of curves $f : X \to Y$ of degree n.

(a) If $P \in X$ is a ramification point, and $e_P = r$, show that $f^{-1}f(P)$ consists of exactly n/r points, each having ramification index r. Let $P_1, \ldots, P_s$ be a maximal set of ramification points of X lying over distinct points of Y, and let $e_{P_i} = r_i$. Then show that Hurwitz's theorem implies that

$$(2g-2)/n = 2g(Y) - 2 + \sum_{i=1}^{s} (1 - 1/r_i)$$

(b) (b) Since $g \geqslant 2$, the left hand side of the equation is > 0. Show that if $g(Y) \geqslant 0$, $s \geqslant 0, r_i \geqslant 2, i = 1, \ldots, s$ are integers such that

$$2g(Y) - 2 + \sum_{i=1}^{s} (1 - 1/r_i) > 0$$

then the minimum value of this expression is $1/42$. Conclude that $n \leqslant 84(g-1)$. See (Ex. 5.7) for an example where this maximum is achieved.

Note: It is known that this maximum is achieved for infinitely many values of g (Macbeath [1]). Over a field of characteristic $p > 0$, the same bound holds, provided $p > g+1$, with one exception, namely the hyperelliptic curve $y^2 = x^p - x$, which has $p = 2g+1$ and $2p\,(p^2-1)$ automorphisms (Roquette [1]). For other bounds on the order of the group of automorphisms in characteristic p, see Singh [1] and Stichtenoth [1].

Proof. We start the proof with a lemma.

Lemma 2.1. The finite morphism of curves $f : X \to Y$ of degree n can described explicitly.

Proof. First off, the extension is of degree n. To see this, note that $|G| \leq [k(X) : L]$ in general. So we check that equality does occur and that the extension is at least algebraic. □

(a) We show $f^{-1}f(P)$ has n/r points. We know that the field extension $L \subseteq K(X)$ gave $f : X \to Y$ with degree n so $K(Y) = L$. Since the map is of degree n, apply Riemann-Hurwitz[1] to find that

$$2 \leq 2g(X) - 2 = n(2g(Y) - 2) + \deg R.$$

since $g(X) \geq 2$.

Now each point of $f^{-1}(f(P))$ maps to $f(P)$ and since $P \to f(P)$ has ramification index of r, so do all other $Q \to f(P)$ in $f^{-1}(f(P))$.

[Incomplete]

[1]Check hypotheses?

Let $P_1, \ldots, P_s$ be a maximal set of ramification points of X lying over distinct points of Y. Apply Hurwitz's formula to get

$$\frac{2g-2}{n} = 2g(Y) - 2 + \frac{1}{n}\frac{n}{r_i}\sum_{i=1}^{s}(r_i - 1) = 2g(Y) - 2 + \sum_{i=1}^{s}(r_i - 1)$$

using the fact that $f^{-1}f(P)$ has size n/r whenever P is a ramified point. Essentially, the point is that over the branch point $f(P)$ there are n/r ramified points with ramification index r to account for.

(b) We can minimize the result based on $g(Y)$. If $g(Y) > 1$, then we clearly cannot attain the minimal value. If $g(Y) = 1$, then we are trying to minimize $\sum_{i=1}^{s}(1 - \frac{1}{r_i}) > 0$. But then the smallest value we can hope for occurs when $s = 1$ and $r_i = 2$ i.e. $\frac{1}{2}$. We can do better if $g(Y) = 0$. In this case, we need $\sum_{i=1}^{s}(1 - \frac{1}{r_i}) > 2$ for the value to be positive. This cannot occur if $s = 1, 2$.

If $s = 4$, we can take $r_i = i + 1$. Then this is minimal amongst than any possible value we can get if $s \geq 4$ i.e. $-2 + 5 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} = \frac{43}{60}$.

Now we can consider the case of $s = 3$ which is going to do better than $s \geq 4$. Of course, we can take $r_1 = 2, r_2 = 3, r_3 = 7$ to get the minimum of $\frac{1}{42}$ (as one cannot have $r_3 \in \{4, 5, 6\}$). □

Hartshorne IV.2.6 *f_* for Divisors.* Let $f : X \to Y$ be a finite morphism of curves of degree n. We define a homomorphism $f_* : \operatorname{Div} X \to \operatorname{Div} Y$ by $f_*\left(\sum n_i P_i\right) = \sum n_i f\left(P_i\right)$ for any divisor $D = \sum n_i P_i$ on X.

(a) For any locally free sheaf $\mathscr{E}$ on Y, of rank r, we define $\det \mathscr{E} = \bigwedge^r \mathscr{E} \in \operatorname{Pic} Y$ (II, Ex. 6.11). In particular, for any invertible sheaf $\mathscr{M}$ on X, $f_*\mathscr{M}$ is locally free of rank n on Y, so we can consider $\det f_*\mathscr{M} \in \operatorname{Pic} Y$. Show that for any divisor D on X,

$$\det\left(f_*\mathscr{L}(D)\right) \cong \left(\det f_*\mathcal{O}_X\right) \otimes \mathscr{L}\left(f_*D\right).$$

Note in particular that $\det\left(f_*\mathscr{L}(D)\right) \neq \mathscr{L}\left(f_*D\right)$ in general!

[Hint: First consider an effective divisor D, apply f_* to the exact sequence $0 \to \mathscr{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$, and use (II, Ex. 6.11).]

(b) Conclude that f_*D depends only on the linear equivalence class of D, so there is an induced homomorphism $f_* : \operatorname{Pic} X \to \operatorname{Pic} Y$. Show that $f_* \circ f^* : \operatorname{Pic} Y \to \operatorname{Pic} Y$ is just multiplication by n.

(c) Use duality for a finite flat morphism (III, Ex. 6.10) and (III, Ex. 7.2) to show that $\det f_*\Omega_X \cong \left(\det f_*\mathcal{O}_X\right)^{-1} \otimes \Omega_Y^{\otimes n}$.

(d) Now assume that f is separable, so we have the ramification divisor R. We define the branch divisor B to be the divisor f_*R on Y. Show that

$$\left(\det f_*\mathcal{O}_X\right)^2 \cong \mathscr{L}(-B).$$

Proof. **(a)** Let D be an effective divisor. The SES $0 \to \mathscr{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ twists to $0 \to \mathcal{O}_X \to \mathscr{L}(D) \to \mathcal{O}_D \to 0$ since $\mathcal{O}_D \otimes \mathscr{L}(D) \cong \mathcal{O}_D$. Indeed, $\mathcal{O}_D$ is a direct sum of skyscrapper sheaves so tensoring by an invertible sheaf preserves it. By Hartshorne

Proposition III.8.1 and f a finite morphism, $R^1 f_*(\mathcal{O}_X) = 0$ so there a SES of locally free sheaves

$$0 \to f_*\mathcal{O}_X \to f_*\mathscr{L}(D) \to f_*\mathcal{O}_D \to 0,$$

and the determinant gives

$$\det \gamma(f_*\mathscr{L}(D)) \cong \det \gamma(f_*\mathcal{O}_X) \otimes \det \gamma(f_*\mathcal{O}_D).$$

One can observe $f_*\mathcal{O}_D = \mathcal{O}_{f_*D} = \psi(f_*D)$. Hartshorne II.6.10 can be applied to obtain

$$\det f_*\mathscr{L}(D) = (\det f_*\mathcal{O}_X) \otimes \mathscr{L}(f_*D).$$

Now let $D = D_1 - D_2$ be an arbitrary divisor with D_1 and D_2 effective divisors. There is a SES $0 \to \mathscr{L}(-D_2) \to \mathcal{O}_X \to \mathcal{O}_{D_2} \to 0$ which we twist by $\mathscr{L}(D_1)$ to get

$$0 \to \mathscr{L}(D) \to \mathscr{L}(D_1) \to \mathcal{O}_{D_2}(D_1) \to 0.$$

But $\mathcal{O}_{D_2}(D_1) \cong \mathcal{O}_{D_2}$ and applying f_* gives

$$0 \to f_*\mathscr{L}(D) \to f_*\mathscr{L}(D_1) \to f_*\mathcal{O}_{D_2} \to 0.$$

Apply det with our result for effective divisors,

$$\det f_*\mathscr{L}(D_1) = \det f_*\mathscr{L}(D) \otimes \det f_*\mathcal{O}_{D_2} = \det f_*\mathscr{L}(D) \otimes \mathscr{L}(f_*D_2).$$

The LHS is isomorphic to $(\det f_*\mathcal{O}_X) \otimes \mathscr{L}(f_*D_1)$ and tensoring by $\mathscr{L}(-f_*D_1)$,

$$(\det f_*\mathcal{O}_X) \otimes \mathscr{L}(f_*D) \cong \det f_*\mathscr{L}(D).$$

(b) Let $D' \sim D$ and so $\mathscr{L}(D) \cong \mathscr{L}(D')$. From (a), we get two isomorphisms

$$\mathscr{L}(f_*D) = (\det f_*\mathcal{O}_X)^{-1} \otimes \det f_*\mathscr{L}(D)$$
$$\mathscr{L}(f_*D') = (\det f_*\mathcal{O}_X)^{-1} \otimes \det f_*\mathscr{L}(D')$$

Of course, the two RHS's are isomorphic to each other. So we get a well-defined map $f_* : \operatorname{Pic} Y \to \operatorname{Pic} X$ by identitfying the Picard group with the divisor class group.

To compute $f_* \circ f^*$, we will do it purely at the level of divisors. Since both of the maps are linear, we deal with the case of a prime divisor Q. Indeed,

$$f_* \circ f^*(Q) = f_* \left(\sum_{f(P)=Q} v_P(t) Q \right) = \sum_{f(P)=Q} v_P(t) f(P) = \sum_{f(P)=Q} v_P(t) Q.$$

Now it is a matter of showing that sums of the valuations $v_P(t)$ add to n where t is a uniformizer $t \in \mathcal{O}_Q$. Here, n is the degree of f. Of course, this simply boils down to the fundamental identity (see Proposition I.8.2 of [7]).

(c) WTS

$$\det f_*\Omega_X \cong (\det f_*\mathcal{O}_X)^{-1} \otimes \Omega_Y^{\otimes n} \cong (\det f_*\mathcal{O}_X)^{-1} \otimes f_*f^*\Omega_Y$$

Since X and Y are complete curves, $\Omega_X = \omega_X$ and $\Omega_Y = \omega_Y$ are the dualizing sheaves. Indeed, note that by Hartshorne Corollary III.7.12, the dualizing sheaves are precisely the canonical sheaves and for complete nonsingular curves X, $\omega_X = \Omega_X$. By Hartshorne III.7.2, $f^!\Omega_Y = \Omega_X$. From Hartshorne III.6.10(a), we know

$$f_*\mathscr{H}om_X(\mathcal{O}_X, \Omega_X) \cong \mathscr{H}om_Y(f_*\mathcal{O}_X, \Omega_Y).$$

The LHS is isomorphic to $f_*\Omega_X$ while the RHS is isomorphic to $(f_*\mathcal{O}_X)^\smile \otimes \Omega_Y$ from Hartshorne II.5.1(b). Therefore $f_*\Omega_X \cong (f_*\mathcal{O}_X)^\smile \otimes \Omega_Y$. By a property of determiants combined with the fact $f_*\mathcal{O}_X$ is locally free of rank n and Ω_Y is invertible,

$$\det f_*\Omega_X \cong (\det f_*\mathcal{O}_X)^{-1} \otimes (\det \Omega_Y)^n \cong (\det f_*\mathcal{O}_X) \otimes \Omega_Y.$$

(d) Assume f is separable and R is the ramification divisor. Let $B := f_*R$ be the branch divisor. By Hartshorne Proposition IV.2.3, we know $K_X \sim f^*K_Y + R$. Apply f_* to get $f_*K_X \sim nK_Y + f_*R = nK_Y + B$. So,

$$\mathscr{L}(-B) \cong \mathscr{L}(-f_*K_X) \otimes \mathscr{L}(-nK_Y). \tag{33}$$

From (a) and (c),

$$\det(f_*\mathscr{L}(K_X)) \otimes (\det(f_*\mathcal{O}_X))^{-1} \cong \mathscr{L}(f_*K_X) \qquad \& \qquad \det f_*\mathscr{L}(K_X) \otimes \det(f_*\mathcal{O}_X) \cong \Omega_Y^{\otimes n}.$$

Using these isomorphisms on (33), we obtain

$$\begin{aligned}\mathscr{L}(-B) &\cong \mathscr{L}(-f_*K_X) \otimes \Omega_Y^{\otimes n} \\ &\cong (\det f_*\mathscr{L}(K_X))^{-1} \otimes \det(f_*\mathcal{O}_X) \otimes \det f_*\mathscr{L}(K_X) \otimes \det(f_*\mathcal{O}_X) \\ &\cong (\det f_*\mathcal{O}_X)^2.\end{aligned}$$

□

Hartshorne IV.2.7 *Étale Covers of Degree 2.* Let Y be a curve over a field k of characteristic $\neq 2$. We show there is a one-to-one correspondence between finite étale morphisms $f : X \to Y$ of degree 2, and 2-torsion elements of Pic Y, i.e., invertible sheaves $\mathscr{L}$ on Y with $\mathscr{L}^2 \cong \mathcal{O}_Y$.

(a) Given an étale morphism $f : X \to Y$ of degree 2 , there is a natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Let $\mathscr{L}$ be the cokernel. Then $\mathscr{L}$ is an invertible sheaf on $Y, \mathscr{L} \cong \det f_*\mathcal{O}_X$, and so $\mathscr{L}^2 \cong \mathcal{O}_Y$ by (Ex. 2.6). Thus an étale cover of degree 2 determines a 2-torsion element in Pic Y.

(b) Conversely, given a 2-torsion element $\mathscr{L}$ in Pic Y, define an $\mathcal{O}_Y$-algebra structure on $\mathcal{O}_Y \oplus \mathscr{L}$ by $\langle a, b\rangle \cdot \langle a', b'\rangle = \langle aa' + \varphi(b \otimes b'), ab' + a'b\rangle$, where φ is an isomorphism of $\mathscr{L} \otimes \mathscr{L} \to \mathcal{O}_Y$. Then take $X = \mathbf{Spec}(\mathcal{O}_Y \oplus \mathscr{L})$ (II, Ex. 5.17). Show that X is an étale cover of Y.

(c) Show that these two processes are inverse to each other.

[Hint: Let $\tau : X \to X$ be the involution which interchanges the points of each fibre of f. Use the trace map $a \mapsto a + \tau(a)$ from $f_*\mathcal{O}_X \to \mathcal{O}_Y$ to show that the sequence of $\mathcal{O}_Y$-modules in (a)

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathscr{L} \to 0$$

is split exact.

Note. This is a special case of the more general fact that for $(n, \operatorname{char} k) = 1$, the étale Galois covers of Y with group $\mathbb{Z}/n\mathbb{Z}$ are classified by the étale cohomology group $H^1_{\text{et}}(Y, \mathbb{Z}/n\mathbb{Z})$, which is equal to the group of n-torsion points of Pic Y. See Serre [6].

Proof. **(a)** We have an exact sequence $0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathscr{L} \to 0$ of $\mathcal{O}_Y$-modules since f is flat. Taking determinants, we get $\det f_*\mathcal{O}_X \cong \det \mathscr{L} \otimes \mathcal{O}_Y \cong \det \mathscr{L}$. To get $\det \mathscr{L} \cong \mathscr{L}$, it suffices to show that $\mathscr{L}$ is invertible.

Lemma 2.2. $\mathscr{L}$ is an invertible $\mathcal{O}_Y$-module.

PROOF. First off, we can take stalks to get a SESs

$$0 \to \mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y \to \mathscr{L}_y \to 0 \qquad \text{for all } y \in Y.$$

[Incomplete] □

With this all said and done, Hartshorne IV.2.6(d) says $\mathscr{L}^2 \cong \mathscr{L}(-B) = \mathcal{O}_Y$ because f is an étale morphism and therefore unramified by Hartshorne III.10.3.

(b) Suppose $\mathscr{L} \in \operatorname{Pic} Y$ satisfies $\mathscr{L} \otimes \mathscr{L} \cong \mathcal{O}_Y$. Define a $\mathcal{O}_Y$-algebra structure on $\mathcal{O}_Y \oplus \mathscr{L}$ as stated where $\varphi : \mathscr{L} \otimes \mathscr{L} \to \mathcal{O}_Y$ is a fixed isomorphism.

The multiplication operation defined in the problem statement gives $\mathcal{O}_Y \oplus \mathscr{L}$ a ring structure. For the $\mathcal{O}_Y$-module structure, we simply use the fact that $\mathcal{O}_Y, \mathscr{L}$ are both $\mathcal{O}_Y$-modules. We check that this indeed gives an $\mathcal{O}_Y$-algebra structure. One has

$$c\langle a, b\rangle \cdot \langle a', b'\rangle = \langle caa' + c\varphi(b \otimes b'), cab' + ca'b\rangle = \langle ca, cb\rangle\langle a', b'\rangle$$

and this equals $\langle a, b\rangle \cdot \langle ca', cb'\rangle$. So the action of $\mathcal{O}_Y$ commutes with this multiplication structure. This multiplication structure also turns $\mathcal{O}_Y \oplus \mathscr{L}$ into a sheaf of rings.

Let $X := \mathbf{Spec}(\mathcal{O}_Y \oplus \mathscr{L})$. First, there is a natural map $f : X \to Y$ from Hartshorne II.5.17(d) such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathscr{L}$. We would like this to be a finite étale morphism.

Since the morphism is affine by Hartshorne II.5.17(d), we check finiteness affine locally. From Hartshorne II.5.17(b), we know that if $V = \operatorname{Spec} A \subseteq Y$ is an open affine, then

$$f^{-1}(V) \cong \operatorname{Spec}((\mathcal{O}_Y \oplus \mathscr{L})(V)) \to V =: \operatorname{Spec} A.$$

It suffices to show that the corresponding morphism $A \to (\mathcal{O}_Y \oplus \mathscr{L})(V) = \mathcal{O}_Y(V) \oplus \mathscr{L}(V)$ is a finite A-module morphism. Of course, this is clear since $A \to A \oplus \mathscr{L}(A)$ and $\mathscr{L}$ being an invertible sheaf on $\mathscr{L}$ means this is finite.

For this morphism to be étale, we need flat and unramified. For flatness, we need

$$f^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \qquad \text{to be a flat extension for all } y = f(x).$$

Suppose $x \in \operatorname{Spec}((\mathcal{O}_Y \oplus \mathscr{L})(V)) = f^{-1}(V) \subseteq X$. Then this stalk computation is

$$f^\# : A_y \to (\mathcal{O}_Y \oplus \mathscr{L})(V)_x = \mathcal{O}_Y(V)_x \oplus \mathscr{L}(V)_x$$

Of course, this if a flat extension because $\mathcal{O}_{X,x}$ is a free $\mathcal{O}_{Y,y} = A_y$ module of rank 2. For unramifiedness, we need to show that

$$\mathfrak{m}_y \cdot \mathcal{O}_{X,x} = \mathfrak{m}_x \qquad \& \qquad k(x)/k(y) \text{ is a separable field extension.}$$

[Incomplete]

To show that $f : X \to Y$ is étale, we use Hartshorne III.10.3 and just show that $f : X \to Y$ is smooth of relative dimension 0.

First off, X is a smooth curve and $\deg(f) = 2$. By Hartshorne III.9.3(a), $f : X \to Y$ is flat. On the other hand, the sequence $f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$ implies $\Omega_{X/Y} = 0$. Indeed, this is because the morphism is unramified.

[Incomplete]

(c) Let $\tau : X \to X$ be the involution that interchanges the two points of each fibre of f (since f is a degree 2 morphism).

We want to show that the map t given by $a \mapsto a + \tau(a)$ defines a section of the map $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$. Indeed,

$$a \mapsto a + \tau(a) \mapsto \langle a, \tau(a) \rangle$$

[Incomplete]

Since the sequence splits, $f_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathscr{L}$. Suppose we are given an étale morphism $f : X \to Y$ of degree 2. We obtain from this an invertible 2-torsion sheaf $\mathscr{L} \cong \det f_*\mathcal{O}_X$. Applying the construction in part (b), we need to show $X = \mathbf{Spec}(\mathcal{O}_Y \oplus \mathscr{L})$. It is equivalent to show that $X = \mathbf{Spec}(\det f_*\mathcal{O}_X)$. By Hartshorne II.5.17(d), this is clear.

Conversely, suppose we are given a 2-torsion sheaf $\mathscr{L}$. From (b), $X := \mathbf{Spec}(\mathcal{O}_X \oplus \mathscr{L}) \to Y$ is an étale cover of Y of degree 2. Then (a) recovers

$$\det f_*\mathcal{O}_X = \det(\mathcal{O}_Y \oplus \mathscr{L}) = \det(\mathcal{O}_Y) \otimes \det \mathscr{L} = \det \mathscr{L} = \mathscr{L}$$

which is a 2-torsion sheaf. This shows the processes are mutually inverse to each other. □

3. Embeddings in Projective Space

Hartshorne IV.3.1 If X is a curve of genus 2, show that a divisor D is very ample $\Leftrightarrow$ $\deg D \geq 5$. This strengthens (3.3.4).

Proof. One direction is clear from Hartshorne IV.3.3.4. Now assume D is very ample. First, Hartshorne Proposition IV.3.3(b) gives $0 \leq \dim |D - P - Q| = \dim |D| - 2$ which means $\dim |D| \geq 2$. Hence, $l(D) \geq 3$.

If $\deg D = 4$, we know $\deg D > 2g - 2 = 2$ and so D is nonspecial. That is, $l(K - D) = 0$ and the Riemann-Roch Theorem gives $l(D) = 3$. So $\dim |D| = 2$ is achieved. So we get a morphism $f : X \to \mathbb{P}^2$ of degree 4. Since X is a curve, we know X has codimension 1 inside $\mathbb{P}^2$. Using Hartshorne III.4.7, we deduce that $p_(X) = \dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(4-1)(4-2) = 1$ which is clearly not the case.

If $\deg D = 3$, then D is nonspecial. So $l(K - D) = 0$ and the Riemann-Roch Theorem gives $l(D) = 2$. So $\dim |D| = 1$. By Hartshorne Proposition IV.3.1(b), D cannot possibly be very ample.

If $\deg D = 2$ and D were very ample, then $\mathscr{L}(D) \otimes \mathscr{L}(D)$ is also very ample. So, $\mathscr{L}(2D)$ is also very ample. But we know from above that there exists no such divisors as $\deg 2D = 4$. Similarly, if $\deg D = 1$, then $\mathscr{L}(3D)$ is also very ample. But we know there are no degree 3 very ample degree 3 divisors. □

Hartshorne IV.3.2 Let X be a plane curve of degree 4 .

(a) Show that the effective canonical divisors on X are exactly the divisors $X.L$, where L is a line in $\mathbb{P}^2$.

(b) If D is any effective divisor of degree 2 on X, show that $\dim |D| = 0$.

(c) Conclude that X is not hyperelliptic (Ex. 1.7).

Proof. (a) First off, X being a plane curve of degree 4 means that $g(X) = \frac{1}{2}(4-1)(4-2) = 3$. Any divisor of degree 4 can be taken as the canonical divisor. Let K be an effective canonical divisor and so $\deg K = 2g(X) - 2 = 4$. Every such effective canonical divisor K determines a line L in which $X.L = K$.

(b) From Hartshorne IV.1.5, we know $\dim|D| \leq \deg D = 2$. So we need to show that $\dim|D| \notin \{1, 2\}$. Add two points $D + P + Q = K_X$ to get a canonical divisor which is very ample. It follows

$$\dim|D| = \dim|K_X - P - Q| = \dim|K_X| - 2.$$

By the Riemann-Roch Theorem,

$$\dim|K_X| = l(K_X) - 1 = \deg K_X + 1 - g = 2g - 2 + 1 - g = g - 1 = 3 - 1 = 2.$$

It follows $\dim|D| = 0$.

(c) A curve X is hyperelliptic if $g \geq 2$ and there is a finite morphism $f : X \to \mathbb{P}^1$ of degree 2. The existence of such a finite morphism corresponded to an effective divisor of degree 2 on X. However, all such divisors D have $\dim|D| = 0$. So by Hartshorne Proposition IV.3.1(a), there cannot exist such a morphism. □

Hartshorne IV.3.3 If X is a curve of genus $\geqslant 2$ which is a complete intersection (II, Ex. 8.4) in some $\mathbb{P}^n$, show that the canonical divisor K is very ample. Conclude that a curve of genus 2 can never be a complete intersection in any $\mathbb{P}^n$. Cf. (Ex. 5.1).

Proof. If so, Hartshorne II.8.4 says that $\omega_X = \mathcal{O}_X(\sum_{i=1}^{n-1} d_i - n - 1)$. For the genus to be ≥ 2, I know that the degree of $\omega_X \geq 2$ and so $\deg \mathcal{O}_X(\sum_{i=1}^{n-1} d_i - n - 1) \geq 2$. But then $\sum_{i=1}^{n-1} d_i - n - 1 \geq 2$. In this case, the canonical divisor is very ample.

A curve X of genus 2 cannot have very ample canonical divisor so it can never be a complete intersection in $\mathbb{P}^n$ (use Hartshorne IV.3.1 and $\deg K_X = 2$.) □

Hartshorne IV.3.4 Let X be the d-uple embedding (I, Ex. 2.12) of $\mathbb{P}^1$ in $\mathbb{P}^d$, for any $d \geqslant 1$. We call X the rational normal curve of degree d in $\mathbb{P}^d$.

(a) Show that X is projectively normal, and that its homogeneous ideal can be generated by forms of degree 2 .

(b) If X is any curve of degree d in $\mathbb{P}^n$, with $d \leqslant n$, which is not contained in any $\mathbb{P}^{n-1}$, show that in fact $d = n, g(X) = 0$, and X differs from the rational normal curve of degree d only by an automorphism of $\mathbb{P}^d$. Cf. (II. 7.8.5).

(c) In particular, any curve of degree 2 in any $\mathbb{P}^n$ is a conic in some $\mathbb{P}^2$.

(d) A curve of degree 3 in any $\mathbb{P}^n$ must be either a plane cubic curve, or the twisted cubic curve in $\mathbb{P}^3$.

Proof. (a) The d-uple embedding is projectively normal by Hartshorne II.5.14(c). Further, the homogeneous ideal can be generated by forms of degree 2 because we can say $[x_0 : x_1] \to [x_0^d : x_0^{d-1}x_1 : \cdots : x_1^d]$ and label the coordinates on the RHS according to the power of x_0 and x_1. Indeed, let $y_{ij} := x_0^i x_1^j$. Then the relations are precisely

$$y_{ij}y_{kl} = y_{d0}y_{0d}$$

where $i + k = d, j + l = d$.

(b) Suppose not and $d < n$. A generic hyperplane will intersect X at d points, but since $d < n$, this does not completely determine a hyperplane. So, there is a hyperplane that does

not intersect X at all. Projection from this hyperplane defines an embedding X into some $\mathbb{P}^{n-1}$. Contradiction.

Since $d = n$, we can compute $g(X)$ using Riemann-Roch as follows. We see that $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n}(d)$. But then $g(X) = \dim H^0(X, \omega_X) = \dim H^0(X, \mathcal{O}_{\mathbb{P}^n}(d-d-1))) = H^0(X, \mathcal{O}_{\mathbb{P}^n}(-1)) = 0$. So the genus is zero.

Now we seek an automorphism of $\mathbb{P}^d$ sending X to the rational normal curve Y of degree d. We modify the argument of Hartshorne Example II.7.8.5.

An embedding of X is determined by a linear system $\mathfrak{d}$ of hyperplane sections of X. This is d-dimensional since X is not in any hyerpplane. Hence, the map $\Gamma(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1)) \to \Gamma(X, i^*\mathcal{O}_{\mathbb{P}^d}(1))$ is injective for $i : X \hookrightarrow \mathbb{P}^d$.

Meanwhile, $\mathfrak{d}$ is a linear system of degree d since X is a curve of degree d. So thinking of X as $\mathbb{P}^1$ (since $g(X) = 0$), the linear system must correspond to a $d+1$-dimensional subspace $V \subseteq \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d+1))$ and since this itself has dimension $d+1$, we get that $V = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. So the embedding is determined by $\mathfrak{d}$ and the choice of basis of V, so X is the same as Y except by an automorphism of $\mathbb{P}^d$.

(c) Let X be a curve of degree 2 in $\mathbb{P}^n$. Then embed X as a curve of degree 2 in $\mathbb{P}^3$ by projecting down. If X sits in some $\mathbb{P}^2 \subseteq \mathbb{P}^3$, then we are done. If not, (b) says $2 = n$ and $g(X) = 0$ and X and we are also once again done.

(d) Embed a curve X of degree 3 into $\mathbb{P}^3$. If it is in a $\mathbb{P}^2 \subseteq \mathbb{P}^3$, then it is a plane cubic curve. If not, X differs from the rational normal curve of degree 3 only by an automorphism of $\mathbb{P}^3$. But that just says X is isomorphic to a twisted cubic. □

Hartshorne IV.3.5 Let X be a curve in $\mathbb{P}^3$, which is not contained in any plane.

(a) If $O \notin X$ is a point, such that the projection from O induces a birational morphism φ from X to its image in $\mathbb{P}^2$, show that $\varphi(X)$ must be singular.

[Hint: Calculate $\dim H^0(X, \mathcal{O}_X(1))$ two ways.]

(b) If X has degree d and genus g, conclude that $g < \frac{1}{2}(d-1)(d-2)$. (Use (Ex. 1.8).)

(c) Now let $\{X_t\}$ be the flat family of curves induced by the projection (III, 9.8.3) whose fibre over $t = 1$ is X, and whose fibre X_0 over $t = 0$ is a scheme with support $\varphi(X)$. Show that X_0 always has nilpotent elements. Thus the example (III, 9.8.4) is typical.

PROOF. **(a)** Suppose not and $\varphi(X)$ were nonsingular. Since X is nonsingular, $X \cong \varphi(X)$. Since X is not in any hyperplane,

$$\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to \Gamma(X, \mathcal{O}_X(1))$$

induced from the pull-back is injective. Hence, $\dim \Gamma(X, \mathcal{O}_X(1)) \geq 4$.

Now $\varphi(X)$ is a complete intersection (cut out by itself in $\mathbb{P}^2$), and so by Hartshorne II.5.5(a), $\dim H^0(\varphi(X), \mathcal{O}_{\varphi(X)}(1)) \leq \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 3$ But this gives a contradiction. **(b)** Let X have degree d and genus g. Project from a point not on X and so $\varphi(X)$ has degree d as well. Now X is the normalization of $\varphi(X)$ and so Hartshorne IV.1.8 says that

$$\frac{(d-1)(d-2)}{2} = p_a(X) = p_a(\widetilde{X}) + \sum \delta_P > p_a(\widetilde{X}) = g(X).$$

(c) Suppose not and X_0 does not have nilpotents. Then X_0 would be the curve given by its support $\varphi(X)$. But then genus of $\varphi(X)$ is larger than the genus of X (as it is singular) which contradicts the fact that the fibres of a flat family have the same Hilbert polynomial. □

Hartshorne IV.3.6 Curves of Degree 4.

(a) If X is a curve of degree 4 in some $\mathbb{P}^n$, show that either

(1) $g = 0$, in which case X is either the rational normal quartic in $\mathbb{P}^4$ (Ex. 3.4) or the rational quartic curve in $\mathbb{P}^3$ (II, 7.8.6), or

(2) $X \subseteq \mathbb{P}^2$, in which case $g = 3$, or

(3) $X \subseteq \mathbb{P}^3$ and $g = 1$.

(b) In the case $g = 1$, show that X is a complete intersection of two irreducible quadric surfaces in $\mathbb{P}^3$ (I, Ex. 5.11).

[Hint: Use the exact sequence $0 \to \mathscr{I}_X \to \mathcal{O}_{\mathbb{P}3} \to \mathcal{O}_X \to 0$ to compute $\dim H^0(\mathbb{P}^3, \mathscr{I}_X(2))$, and thus conclude that X is contained in at least two irreducible quadric surfaces.]

Proof. **(a)** Fix X a curve of degree 4 in $\mathbb{P}^n$. If X is not contained in any plane $\mathbb{P}^{n-1}$, then we get the situation in Hartshorne IV.3.4 i.e. we get $n = 4$ and the first situation in (1).

Now supposed X is contained in a plane $\mathbb{P}^{n-1}$. We can project down to embed X in $\mathbb{P}^3$. Now we split into cases.

Suppose $X \subseteq \mathbb{P}^3$ is contained in some hyperplane $\mathbb{P}^2$. Then we get situation (2) using $g(X) = \frac{(4-1)(4-2)}{2} = 3$.

Suppose X is not in any hyperplane $\mathbb{P}^2 \subseteq \mathbb{P}^3$. Then Hartshorne IV.3.5 says that $g < \frac{1}{2}(4-1)(4-2) = 3$. So the only possibilities are $g = 0, 1, 2$. We first rule out $g = 2$. Indeed, if $g = 2$, then Hartshorne Remark IV.3.11.1 says that is has 1 node. But this is absurd since X is nonsingular and was embedded in $\mathbb{P}^3$.

We have either $g = 0$ or $g = 1$. If $g = 0$, we get a rational quartic curve in $\mathbb{P}^3$. If not, then we get (3).

(b) Consider

$$0 \to \mathscr{I}_X \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

Twist by $\mathcal{O}_{\mathbb{P}^3}(2)$ and take cohomology

$$0 \to H^0(X, \mathscr{I}_X(2)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(X, \mathcal{O}_X(2)) \to \dots$$

and the middle space has dimension 10. The space $H^0(X, \mathcal{O}_X(2))$ can be computed using the Riemann-Roch Theorem since the embedding of X into $\mathbb{P}^3$ would be determined by a divisor D of degree 4 with $|D|$ of dimension 3 Since $\deg 2D > 2g(X) - 2$, we know $2D$ is nonspecial and so

$$l(2D) = \deg 2D + 1 - g(X) = 2 \deg D = 8.$$

And so the $\dim_k H^0(X, \mathcal{O}_X(2)) = 8$ which means $\dim_k H^0(X, \mathscr{I}_X(2)) \geq 2$. So there are at least two quadric surfaces containing X. □

Hartshorne IV.3.7 In view of (3.10), one might ask conversely, is every plane curve with nodes a projection of a nonsingular curve in $\mathbb{P}^3$? Show that the curve $xy + x^4 + y^4 = 0$ (assume char $k \neq 2$) gives a counterexample.

PROOF. This curve has a single node. Homogenize so that it is a curve in $\mathbb{P}^2$ given by equations $XYZ^2 + X^4 + Y^4 = 0$. The partial derivatives are

$$YZ^2 + 4X^3, \qquad XZ^2 + 4Y^3, \qquad 2XYZ.$$

For all of these to vanish, we need one of the variables to be zero. If $X = 0$, then $YZ^2 = 4Y^3 = 0$. That means $Y = 0$ and then Z is a free variable. So there is a singularity at $[0 : 0 : 1]$. If $Y = 0$, then $4X^3 = 0$ implies $X = 0$ and therefore Z is a free variable which is the singularity we had before. If $Z = 0$, then $4Y^3 = 0$ and $4X^3 = 0$ which cannot be a point in projective space. So there is a single singularity and this singularity is actually that of a node.

Now Hartshorne Remark IV.3.11.1 says that the genus of the curve's normalization is

$$\frac{(4-1)(4-2)}{2} - 1 = 2$$

since there is a single node and it is a degree 4 curve. So the curve X projecting onto this has genus $g(X) = 2$ and degree 4 (projection is degree-preserving). But no such curve exists by Hartshorne IV.3.6(a) statement 3. □

Hartshorne IV.3.8 We say a (singular) integral curve in $\mathbb{P}^n$ is *strange* if there is a point which lies on all the tangent lines at nonsingular points of the curve.

(a) There are many singular strange curves, e.g., the curve given parametrically by $x = t, y = t^p, z = t^{2p}$ over a field of characteristic $p > 0$.

(b) Show, however, that if char $k = 0$, there aren't even any singular strange curves besides $\mathbb{P}^1$.

PROOF. **(a)** We need to pick a point that lies on all tangent lines at nonsingular points of the curve.

We need to determine the nonsingular points of this curve. It sits inside $\mathbb{P}^3$. The equations are $x^p - y = 0, y^2 - z = 0$ and homogenized is $X^p - YW^{p-1} = 0$, $Y^2 - ZW = 0$. Now the Jacobian matrix is

$$\begin{pmatrix} 0 & -1 & 0 & -Y \\ 0 & 2Y & -W & -Z \end{pmatrix}$$

and this has rank 2 everywhere except at the point $(0 : 0 : 0 : 0)$ which is a singular point.

Consider the chart where $W = 1$. Now a vector in the direction of the tangent line at the point given by some t is given by $(1, 0, 0)$ (derivative w.r.t. t in each coordinate). So all the tangent lines point in the same direction. So pick the point $(0, 0, 0)$ and observe that it lies on all tangent lines. Indeed, following the direction of $(1, 0, 0)$ from a general point (t, t^p, t^{2p}) towards the origin implies that we only ever intersect the origin.

(b) I think, geometrically, the proof is to take projections from the point $p \in C$ laying on all of the tangent lines. If one does such a projection, then one can "complete" the image by compactifying and then project from the new point. If one does this repeatedly, one obtains $\mathbb{P}^1$ and since each of these projections are isomorphisms, we get the claim. □

Hartshorne IV.3.9 Prove the following lemma of Bertini: if X is a curve of degree d in $\mathbb{P}^3$, not contained in any plane, then for almost all planes $H \subseteq \mathbb{P}^3$ (meaning a Zariski open subset of the dual projective space $(\mathbb{P}^3)^*$), the intersection $X \cap H$ consists of exactly d distinct points, no three of which are collinear.

PROOF. The condition that X is not in any plane ensures that there isn't an infinite family of planes that just cut through X at 1 point i.e. take planes perpendicular to the plane containing X.

For a plane H to intersect X at points of multiplicity 1, it suffices to require that H does not contain the tangent line of those points.

[Incomplete]I don't know.

Three points are collinear iff there is a multisecant line passing through all three points. A hyperplane in $\mathbb{P}^3$ intersects X at d points iff the hyperplane does not pass through any tangent lines. Avoiding tangent lines is an open condition. The space of multisecant lines is ≤ 1 dimensional the union of these two spaces is a proper closed subset of $(\mathbb{P}^3)^*$.

The proof that the space of multisecant lines is ≤ 1 dimensional is not obvious. To determine it, we must give at least three points on X which the line cuts through. First, we specify two points p_1, p_2 of X. This gives 2 dimensions. as X is a curve. The two points determine a line which we need to cut through X at one new point. Use p_1 and project along p_1 to get a new curve $\varphi(X)$ in $\mathbb{P}^2$. Now the line through p_1 determines a point $q \in \varphi(X)$ and this point must be a node for the original line to have been a multisecant. The condition that this is a node drops dimension by 1. So the space of multisecant lines is 1-dimensional. □

Hartshorne IV.3.10 Generalize the statement that "not every secant is a multisecant" as follows. If X is a curve in $\mathbb{P}^n$, not contained in any $\mathbb{P}^{n-1}$, and if char $k = 0$, show that for almost all choices of $n-1$ points $P_1, \ldots, P_{n-1}$ on X, the linear space L^{n-2} spanned by the P_i does not contain any further points of X.

PROOF. □

Hartshorne IV.3.11 (a) If X is a nonsingular variety of dimension r in $\mathbb{P}^n$, and if $n > 2r+1$, show that there is a point $O \notin X$, such that the projection from O induces a closed immersion of X into $\mathbb{P}^{n-1}$.

(b) If X is the Veronese surface in $\mathbb{P}^5$, which is the 2-uple embedding of $\mathbb{P}^2$ (I, Ex. 2.13), show that each point of every secant line of X lies on infinitely many secant lines. Therefore, the secant variety of X has dimension 4 , and so in this case there is a projection which gives a closed immersion of X into $\mathbb{P}^4$ (II, Ex. 7.7). (A theorem of Severi [1] states that the Veronese surface is the only surface in $\mathbb{P}^5$ for which there is a projection giving a closed immersion into $\mathbb{P}^4$. Usually one obtains a finite number of double points with transversal tangent planes.)

PROOF. (a) We generalize Hartshorne Proposition IV.3.4 and IV.3.5.

First off, we determine the condition in which projection from O induces a closed immersion. Fix $O \notin X$. Then the morphism φ is determined by the linear system cut out on X of hyperplanes of $\mathbb{P}^n$ passing through O. This is a closed immersion iff it separates points and tangent vectors.

Separating points requires that O is not on the line PQ just as in Hartshorne Proposition IV.3.4. Separating tangent vectors at P means the same thing as well i.e. H contains O and P and meets X at P with multiplcity 1.

Consider the secant variety. Then it is a locally closed subset of $\mathbb{P}^n$ with dimension $\leq 2r+1$ while the tangent variety has dimension $\leq r+1$. So the union is not all of $\mathbb{P}^n$ provided $n > 2r+1$. This proves (a).

(b) **[Incomplete]** □

Hartshorne IV.3.12 For each value of $d = 2, 3, 4, 5$ and r satisfying $0 \leqslant r \leqslant \frac{1}{2}(d-1)(d-2)$, show that there exists an irreducible plane curve of degree d with r nodes and no other singularities.

PROOF. If $d = 2$, then $r = 0$. So just pick any nonsingular quadric.

If $d = 3$, then $r \in \{0, 1\}$ and we know there are smooth cubics as well as a cubic with a node e.g. the nodal cubic.

If $d = 4$, then $r \in \{0, 1, 2, 3\}$. There are certainly smooth quartic curves so $r = 0$ is easy. The case of $r = 3$ is done in Hartshorne Remark IV.3.11.1. The curve in $\mathbb{A}^2$ cut out by $xy = x^4 + y^4$ gives the case of $r = 1$. The hard part is $r = 2$. **[Incomplete]** □

4. Elliptic Curves

Hartshorne IV.4.1 Let X be an elliptic curve over k, with char $k \neq 2$, let $P \in X$ be a point, and let R be the graded ring $R = \bigoplus_{n \geqslant 0} H^0(X, \mathcal{O}_X(nP))$. Show that for suitable choice of t, x, y

$$R \cong k[t, x, y] / \left(y^2 - x\left(x - t^2\right)\left(x - \lambda t^2\right)\right)$$

as a graded ring, where $k[t, x, y]$ is graded by setting $\deg t = 1, \deg x = 2$, $\deg y = 3$.

PROOF. □

Hartshorne IV.4.2 If D is any divisor of degree $\geqslant 3$ on the elliptic curve X, and if we embed X in $\mathbb{P}^n$ by the complete linear system $|D|$, show that the image of X in $\mathbb{P}^n$ is projectively normal.

Note. It is true more generally that if D is a divisor of degree $\geqslant 2g + 1$ on a curve of genus g, then the embedding of X by $|D|$ is projectively normal (Mumford [4, p. 55])

PROOF. □

Hartshorne IV.4.3 Let the elliptic curve X be embedded in $\mathbb{P}^2$ so as to have the equation $y^2 = x(x-1)(x-\lambda)$. Show that any automorphism of X leaving $P_0 = (0, 1, 0)$ fixed is induced by an automorphism of $\mathbb{P}^2$ coming from the automorphism of the affine (x, y)-plane given by

$$\begin{cases} x' = ax + b \\ y' = cy \end{cases}$$

In each of the four cases of (4.7), describe these automorphisms of $\mathbb{P}^2$ explicitly, and hence determine the structure of the group $G = \mathrm{Aut}(X, P_0)$.

PROOF. □

Hartshorne IV.4.4 Let X be an elliptic curve in $\mathbb{P}^2$ given by an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

Show that the j-invariant is a rational function of the a_i, with coefficients in $\mathbb{Q}$. In particular, if the a_i are all in some field $k_0 \subseteq k$, then $j \in k_0$ also. Furthermore, for every $\alpha \in k_0$, there exists an elliptic curve defined over k_0 with j-invariant equal to α.

PROOF. This is done in Silverman's "The Arithmetic of Elliptic Curves" p. 45. The computation is quite tedious so let me describe the idea.

To rewrite the equation into Weierstrass form, we need to do a series of change of coordinates as in Hartshorne Proposition IV.4.6. Then one uses the formula for the j-invariant

as given before Hartshorne Theorem IV.4.1. Using this formula, it is also easily deduced that for every $\alpha \in k_0$, there exists an elliptic curve defined over k_0 with j-invariant equal to α. □

Hartshorne IV.4.5 Let X, P_0 be an elliptic curve having an endomorphism $f : X \to X$ of degree 2 .

(a) If we represent X as a 2-1 covering of $\mathbb{P}^1$ by a morphism $\pi : X \to \mathbb{P}^1$ ramified at P_0, then as in (4.4), show that there is another morphism $\pi' : X \to \mathbb{P}^1$ and a morphism $g : \mathbb{P}^1 \to \mathbb{P}^1$, also of degree 2 , such that $\pi \circ f = g \circ \pi'$.

(b) For suitable choices of coordinates in the two copies of $\mathbb{P}^1$, show that g can be taken to be the morphism $x \to x^2$.

(c) Now show that g is branched over two of the branch points of π, and that g^{-1} of the other two branch points of π consists of the four branch points of π'. Deduce a relation involving the invariant λ of X.

(d) Solving the above, show that there are just three values of j corresponding to elliptic curves with an endomorphism of degree 2 , and find the corresponding values of λ and j. [Answers: $j = 2^6 \cdot 3^3; j = 2^6 \cdot 5^3; j = -3^3 \cdot 5^3$.]

Proof. (a) Let $\pi : X \to \mathbb{P}^1$ be a $2-1$ covering of $\mathbb{P}^1$. Applying the Hurwitz formula, we deduce that

$$0 = -2\deg(\pi) + \deg R \;\Rightarrow\; 2\deg(\pi) = \deg(R).$$

The degree of the morphism π is 2 and so the degree of the ramification divisor is 4. **[Incomplete]**

We use Hartshorne Lemma IV.4.4 to deduce the existence of a morphism $f : X \to X$ and $g : \mathbb{P}^1 \to \mathbb{P}^1$ s.t.

[Incomplete]

(b) **[Incomplete]**

(c) **[Incomplete]**

(d) □

Hartshorne IV.4.6 (a) Let X be a curve of genus g embedded birationally in $\mathbb{P}^2$ as a curve of degree d with r nodes. Generalize the method of(Ex. 2.3) to show that X has $6(g-1)+3d$ inflection points. A node does not count as an inflection point. Assume char $k = 0$.

(b) Now let X be a curve of genus g embedded as a curve of degree d in $\mathbb{P}^n, n \geqslant 3$, not contained in any $\mathbb{P}^{n-1}$. For each point $P \in X$, there is a hyperplane H containing P, such that P counts at least n times in the intersection $H \cap X$. This is called an osculating hyperplane at P. It generalizes the notion of tangent line for curves in $\mathbb{P}^2$. If P counts at least $n+1$ times in $H \cap X$, we say H is a hyperosculating hyperplane, and that P is a hyperosculation point. Use Hurwitz's theorem as above, and induction on n, to show that X has $n(n+1)(g-1) + (n+1)d$ hyperosculation points.

(c) If X is an elliptic curve, for any $d \geqslant 3$, embed X as a curve of degree d in $\mathbb{P}^{d-1}$, and conclude that X has exactly d^2 points of order d in its group law.

Proof. □

Hartshorne IV.4.7 *The Dual of a Morphism.* Let X and X' be elliptic curves over k, with base points P_0, P_0'.

(a) If $f : X \to X'$ is any morphism, use (4.11) to show that $f^* : \operatorname{Pic} X' \to \operatorname{Pic} X$ induces a homomorphism $\hat{f} : (X', P_0') \to (X, P_0)$. We call this the dual of f.

(b) If $f : X \to X'$ and $g : X' \to X''$ are two morphisms, then $\widehat{(g \circ f)} = \widehat{f} \circ \widehat{g}$.

(c) Assume $f(P_0) = P_0'$, and let $n = \deg f$. Show that if $Q \in X$ is any point, and $f(Q) = Q'$, then $\hat{f}(Q') = n_X(Q)$. (Do the separable and purely inseparable cases separately, then combine.) Conclude that $f \circ \hat{f} = n_X$, and $\hat{f} \circ f = n_X$.

*(d) If $f, g : X \to X'$ are two morphisms preserving the base points P_0, P_0', then $(f+g)^\wedge = \hat{f} + \hat{g}$.

[Hints: It is enough to show for any $\mathscr{L} \in \operatorname{Pic} X'$, that $(f+g)^*\mathscr{L} \cong f^*\mathscr{L} \otimes g^*\mathscr{L}$. For any f, let $\Gamma_f : X \to X \times X'$ be the graph morphism. Then it is enough to show (for $\mathscr{L}' = p_2^*\mathscr{L}$) that

$$\Gamma_{f+g}^*(\mathscr{L}') = \Gamma_f^*\mathscr{L}' \otimes \Gamma_g^*\mathscr{L}'$$

Let $\sigma : X \to X \times X'$ be the section $x \to (x, P_0')$. Define a subgroup of $\operatorname{Pic}(X \times X')$ as follows: $\operatorname{Pic}_\sigma^\circ = \{\mathscr{L} \in \operatorname{Pic}(X \times X') \mid \mathscr{L}$ has degree 0 along each fibre of p_1, and $\sigma^*\mathscr{L} = 0$ in $\operatorname{Pic} X\}$. Note that this subgroup is isomorphic to the group $\operatorname{Pic}^\circ(X'/X)$ used in the definition of the Jacobian variety. Hence there is a 1-1 correspondence between morphisms $f : X \to X'$ and elements $\mathscr{L}_f \in \operatorname{Pic}_\sigma$ (this defines $\mathscr{L}_f$). Now compute explicitly to show that $\Gamma_g^*(\mathscr{L}_f) = \Gamma_f^*(\mathscr{L}_g)$ for any f, g. Use the fact that $\mathscr{L}_{f+g} = \mathscr{L}_f \otimes \mathscr{L}_g$, and the fact that for any $\mathscr{L}$ on X', $p_2^*\mathscr{L} \in \operatorname{Pic}_\sigma^\circ$ to prove the result.]

(e) Using (d), show that for any $n \in \mathbb{Z}, \hat{n}_X = n_X$. Conclude that $\deg n_X = n^2$.

(f) Show for any f that $\deg \hat{f} = \deg f$.

Proof. All of these results can be found in Silverman's *The Arthmetic of Elliptic Curves.* We omit the proofs of these and review the reader to Silverman's phenomenal textbook. □

Hartshorne IV.4.8 For any curve X, the *algebraic fundamental group* $\pi_1(X)$ is defined as $\lim \operatorname{Gal}(K'/K)$, where K is the function field of X, and K' runs over all Galois extensions of K such that the corresponding curve X' is étale over X (III, Ex. 10.3). Thus, for example, $\pi_1(\mathbb{P}^1) = 1$ (2.5.3). Show that for an elliptic curve X,

$$\pi_1(X) = \prod_{l \text{ prime}} \mathbb{Z}_l \times \mathbb{Z}_l \quad \text{if char } k = 0$$

$$\pi_1(X) = \prod_{l \neq p} \mathbb{Z}_l \times \mathbb{Z}_l \quad \text{if char } k = p \text{and Hasse } X = 0;$$

$$\pi_1(X) = \mathbb{Z}_p \times \prod_{l \neq p} \mathbb{Z}_l \times \mathbb{Z}_l \text{if char } k = p \text{and Hasse } X \neq 0$$

where $\mathbb{Z}_l = \varprojlim \mathbb{Z}/l^M\mathbb{Z}$. is the l-adic integers.

[Hints: Any Galois étale cover X' of an elliptic curve is again an elliptic curve. If the degree of X' over X is relatively prime to p, then X' can be dominated by the cover $n_X : X \to X$

for some integer n with $(n,p)=1$. The Galois group of the covering n_X is $\mathbb{Z}/n \times \mathbb{Z}/n$. Étale covers of degree divisible by p can occur only if the Hasse invariant of X is not zero.]

Note: More generally, Grothendieck has shown [SGA 1, X, 2.6, p. 272] that the algebraic fundamental group of any curve of genus g is isomorphic to a quotient of the completion, with respect to subgroups of finite index, of the ordinary topological fundamental group of a compact Riemann surface of genus g, i.e., a group with $2g$ generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ and the relation $\left(a_1 b_1 a_1^{-1} b_1^{-1}\right) \cdots \left(a_g b_g a_g^{-1} b_g^{-1}\right) = 1$.

PROOF. That a Galois étale cover X' of an elliptic curve is again an elliptic curve is due to the general result that a Galois étale cover of an abelian variety is again an abelian variety. Now we follow the hint.

Let $X' \to X$ be a Galois cover of degree relatively prime to p. Then X' is dominated by the cover $n_X : X \to X$ s.t. $(n,p) = 1$ if we consider our proof of IV.4.9(a) below. Everything also follows through it the degree is not relatively prime to p except that we can only dominate by a cover $n_X : X \to X$ s.t. $(n,p) = 1$. But this will not later affect the calculation of the inverse limit.

The Galois group of the covering n_X is $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ because the Galois group can be identified with the fibres. Now in the case where $\operatorname{char} k = 0$, we get the first claim. In the case where $\operatorname{char} k = p$ and the Hasse $X = 0$, there is no p-torsion and so the claim follows. In the last case, there will be p-torsion and the claim follows. □

Hartshorne IV.4.9 We say two elliptic curves X, X' are isogenous if there is a finite morphism $f : X \to X'$.

(a) Show that isogeny is an equivalence relation.

(b) For any elliptic curve X, show that the set of elliptic curves X' isogenous to X, up to isomorphism, is countable.

[Hint: X' is uniquely determined by X and $\ker f$.]

PROOF. (a) Let $f : X \to X'$ be an isogeny and $g : X' \to X''$ another isogeny. Then $g \circ f$ is certainly an isogeny since composition of finite morphism are finite. Certainly, the identity morphism is an isogeny. So all that is left is to show that is the reflexivity axiom.

Let $f : X \to X'$ be an isogeny. Then the kernel $\ker(f)$ is finite. Assume for now that $|\ker(f)|$ does not divide the field's characteristic. Then, $\ker(f) \subseteq \ker(n_X) = X_n$ for some sufficiently large n. Then consider the multiplication by n map $n_X : X \to X$ and we get a factorization

$$X \to X/\ker(f) \to X/\ker(n_X) \to X$$

and since $X/\ker(f) \cong Y$ by Zariski's Main Theorem, we have our map $Y \to X$ which is an isogeny.
(b) Let X be an elliptic curve and $\mathfrak{S}$ the set of elliptic curves isogenous to X. Our claim is that X and $\ker f$ determine X'.

If the claim is true, then we use the fact that $\ker(f)$ is finite and abelian to deduce from the classification of finite abelian groups that the set $\mathfrak{S}$ is countable.

Let $f : X' \to X$ and $g : X'' \to X$ be two isogenies s.t. $\ker(f) \cong \ker(g)$. Let $\widehat{f} : X \to X'$ be the dual isogeny. □

Hartshorne IV.4.10 If X is an elliptic curve, show that there is an exact sequence

$$0 \to p_1^* \operatorname{Pic} X \oplus p_2^* \operatorname{Pic} X \to \operatorname{Pic}(X \times X) \to R \to 0,$$

where $R = \operatorname{End}(X, P_0)$. In particular, we see that $\operatorname{Pic}(X \times X)$ is bigger than the sum of the Picard groups of the factors. Cf. (III, Ex. 12.6), (V, Ex. 1.6).

PROOF. The fact that the first map is injective is clear and it is defined by sending $(p_1^*\mathscr{L}, p_2^*\mathscr{M}) \to p_1^*\mathscr{L} \otimes p_2^*\mathscr{M}$. Indeed, we know that if

$$p_1^*\mathscr{L} \otimes p_2^*\mathscr{M} \cong \mathcal{O}_{X\times X},$$

then we can consider the Künneth Theorem which therefore implies that $p_1^*\mathscr{L}$ and $p_2^*\mathscr{L}$ have no higher cohomology except $h^0 = 1$ and the same is true of their inverse. Since an elliptic curve is necessarily a proper integral scheme, this implies that these line bundles are trivial themselves.

For the second part of the question, I refered to the following post on the Stack Exchange[2].

Let me define $\pi : X \times X \to X \times X$ to be (P_0, id_X) i.e. we have the constant trivial map in the first coordinate. Now, the endomorphism ring $\operatorname{End}(X, P_0)$ consists of morphisms $f : X \to X$ that fix P_0. From the definition and existence of the Jacobian variety, we deduce that giving an endomorphism $f : X \to X$ fixing P_0 is equivalent to specifying a line bundle $\mathscr{M} \in \operatorname{Pic}^\circ(X/X)$ s.t. $f^*\mathscr{L} \cong \mathscr{M}$. So we want to define a map $\operatorname{Pic}(X \times X)$ to $\operatorname{Pic}^\circ(X/X)$. The candidate we pick is the morphism $(1 - \pi)^*$ and we check that this actually works.

Let $\mathscr{M}$ be an arbitrary line bundle on $X \times X$. Apply $(1 - \pi)^*(\mathscr{M})$. Then if we restrict to $\{P\} \times X$, we get a degree zero line bundle necause $(1 - \pi)^*X|_{\{P\}\times X} = \mathscr{M}|_{\mathrm{pt}\times X}$ which is the pull-back of a line bundle on X. So this is trivial in $\operatorname{Pic}^\circ(X/X)$. We conclude that there is a unique morphism $f : X \to X$ s.t. $f(P_0) = P_0$. Furthermore, it follows from definition of $\operatorname{Pic}^\circ(X/X)$ that the kernel consists is $p_1^* \operatorname{Pic}(X) \oplus p_2^* \operatorname{Pic}(X)$. □

Hartshorne IV.4.11 Let X be an elliptic curve over $\mathbb{C}$, defined by the elliptic functions with periods $1, \tau$. Let R be the ring of endomorphisms of X.

(a) If $f \in R$ is a nonzero endomorphism corresponding to complex multiplication by α, as in (4.18), show that $\deg f = |\alpha|^2$.

(b) If $f \in R$ corresponds to $\alpha \in \mathbb{C}$ again, show that the dual $\hat{f}$ of (Ex. 4.7) corresponds to the complex conjugate $\bar{\alpha}$ of α.

(c) If $\tau \in \mathbb{Q}(\sqrt{-d})$ happens to be integral over $\mathbb{Z}$, show that $R = \mathbb{Z}[\tau]$.

PROOF. A complete solution (a), (b), and (c) can be found in Section 1.3 of Diamond and Shurman *A First Course in Modular Forms*.

(a) This exercise is essentially an exercise in linear algebra and the geometric of lattice points. First off, we shall apply a linear transformation to $\mathbb{C}$ to assume that $1, \tau$ is $1, i$. Then multiplication by α is defined by the matrix $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ and the determinant is $c^2 + d^2 = |\alpha|^2$. I claim that this implies there are $|\alpha|^2$ points that map to the lattice points. As far as I can tell, a proof of this appears in Neukirch's Algebraic Number Theory Chapter 1.

[2]The issue was that all of the methods I had tried before were not entirely helpful e.g. considering restriction of a line bundle to the diagonal and/or restriction to a fibre of a projection.

(b) We shall use Hartshorne IV.4.7 freely in our computation. First off, we know that $\deg(f) = |\alpha|^2$ and we want $\widehat{f}$ to have the property that $\widehat{f} \circ f = n_X$ and $f \circ \widehat{f} = n_X$. If $\widehat{f}$ corresponds to complete multiplication by β, we deduce that $\alpha\beta = |\alpha|^2$ and Hartshorne IV.4.7(d) implies $\deg \beta = |\beta|^2 = \deg \alpha$. These conditions imply $\beta = \overline{\alpha}$.
(c) Assume that τ is integral over $\mathbb{Z}$. There are two cases to consider e.g. $-d \equiv 1 \bmod 4$ and $-d \equiv 2, 3 \bmod 4$. Then the ring of integers are respectively $\mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$ and $\mathbb{Z}[\sqrt{-d}]$. Then from Hartshorne Theorem IV.4.19 that X has complex multiplication and R is determined in said theorem. Indeed, it implies $R = \mathbb{Z}[\tau]$.

□

Hartshorne IV.4.12 Again let X be an elliptic curve over C determined by the elliptic functions with periods $1, \tau$, and assume that τ lies in the region G of (4.15 B).

(a) If X has any automorphisms leaving P_0 fixed other than ± 1, show that either $\tau = i$ or $\tau = \omega$, as in (4.20.1) and (4.20.2). This gives another proof of the fact (4.7) that there are only two curves, up to isomorphism, having automorphisms other than ± 1.

(b) Now show that there are exactly three values of τ for which X admits an endomorphism of degree 2. Can you match these with the three values of j determined in (Ex. 4.5)? [Answers: $\tau = i; \tau = \sqrt{-2}; \tau = \frac{1}{2}(-1 + \sqrt{-7})$.]

Proof. □

Hartshorne IV.4.13 If $p = 13$, there is just one value of j for which the Hasse invariant of the corresponding curve is 0 . Find it. [Answer: $j = 5 (\bmod 13)$.]

Proof. The j-value talks values in the field $\mathbb{F}_{13}$. Then use the form of Hartshorne Proposition IV.4.6, the formula for the j-ivariant, and Hartshorne Proposition IV.4.21 to compute that there is just a single value s.t. the Hasse invariant is 0. That is, $j \equiv 5 \bmod 13$. □

Hartshorne IV.4.14 The Fermat curve $X : x^3 + y^3 = z^3$ gives a nonsingular curve in characteristic p for every $p \neq 3$. Determine the set $\mathfrak{P} = \{p \neq 3 \mid X_{(p)} \text{ has Hasse invariant } 0\}$, and observe (modulo Dirichlet's theorem) that it is a set of primes of density $\frac{1}{2}$.

Proof. The first claim is clear via the Jacobian criterion on $\mathbb{P}^2$. Secondly, we observe from Hartshorne Proposition IV.4.21 that $X_{(p)}$ has Hasse invariant 0 iff the coefficient of $(xyz)^{p-1}$ in $(x^3 + y^3 - z^3)^{p-1}$ is 0. The coefficient is equal to 1 if $p - 1 \equiv 0 \bmod 3$ and 0 if not. But there are infinitely many primes $p \equiv 1 \bmod 3$ and applying Dirichlet's Theorem, we deduce that that the set of such primes has density $\frac{1}{2}$ in the collection of primes (indeed, there are only two integers $0 \leq n \leq 3$ coprime to 3). □

Hartshorne IV.4.15 Let X be an elliptic curve over a field k of characteristic p. Let $F' : X_p \to X$ be the k-linear Frobenius morphism (2.4.1). Use (4.10.7) to show that the dual morphism $\hat{F}' : X \to X_p$ is separable if and only if the Hasse invariant of X is 1 . Now use (Ex. 4.7) to show that if the Hasse invariant is 1, then the subgroup of points of order p on X is isomorphic to $\mathbb{Z}/p$; if the Hasse invariant is 0, it is 0 .

Proof. Let $F' : X_p \to X$ be the k-linear Frobenius morphism i.e. it is the identity on topological spaces while it maps functions $f \mapsto f^p$. ($\Rightarrow$). Assume $\widehat{F}' : X \to X_p$ is separable. Then the map on tangent spaces cannot be zero (as the map on the tangent spaces would need to be sujective). But then we know that the tangent space to X here is $H^1(X, \mathcal{O}_X)$

while the tangent space of X_p here is $H^1(X_p, \mathcal{O}_{X_p}) = H^1(X, \mathcal{O}_X)$. In this case, the Frobenius endomorphism is nonzero on $H^1(X, \mathcal{O}_X)$. ($\Leftarrow$). Is immediate by writing the previous argument backwards.

Now assume Hartshorne IV.4.7 and that the Hasse invariant on X is 1. The preceding part implies that $\widehat{F'}$ is separable, so that degrees make sense, and Hartshorne IV.4.7(f) implies that $\deg(\widehat{F'}) = \deg(F')$. In this case, the degree of the Frobenius is p and so $\deg(\widehat{F'}) = \deg(F') = p$. Since the Frobenius morphism is an isogeny of degree p, we get that

$$\#X[p] := \deg_s(p_X) = \deg_s(\widehat{F'} \circ \widehat{F}) = \deg_s(\widehat{F'}) = p.$$

In this case, we $X[p] \cong \mathbb{Z}/p\mathbb{Z}$.

Now assume that the $\widehat{F'}_X$ is inseparable i.e. the Hasse invariant of X is 0. Then the preceding equation gives $\#X[p] = 1$ since $\widehat{F'}$ is thereby inseparable. □

Hartshorne IV.4.16 Again let X be an elliptic curve over k of characteristic p, and suppose X is defined over the field $\mathbb{F}_q$ of $q = p^r$ elements, i.e., $X \subseteq \mathbb{P}^2$ can be defined by an equation with coefficients in $\mathbb{F}_q$. Assume also that X has a rational point over $\mathbb{F}_q$. Let $F' : X_q \to X$ be the k-linear Frobenius with respect to q.

(a) Show that $X_q \cong X$ as schemes over k, and that under this identification, $F' : X \to X$ is the map obtained by the qth-power map on the coordinates of points of X, embedded in $\mathbb{P}^2$.

(b) Show that $1_X - F'$ is a separable morphism and its kernel is just the set $X(\mathbb{F}_q)$ of points of X with coordinates in $\mathbb{F}_q$.

(c) Using (Ex. 4.7), show that $F' + \hat{F}' = a_X$ for some integer a, and that $N = q - a + 1$, where $N = \#X(\mathbb{F}_q)$.

(d) Use the fact that $\deg(m + nF') > 0$ for all $m, n \in \mathbb{Z}$ to show that $|a| \leqslant 2\sqrt{q}$. This is Hasse's proof of the analogue of the Riemann hypothesis for elliptic curves (App. C, Ex. 5.6).

(e) Now assume $q = p$, and show that the Hasse invariant of X is 0 if and only if $a \equiv 0 (\mathrm{mod} p)$. Conclude for $p \geq 5$ that X has Hasse invariant 0 if and only if $N = p + 1$.

Proof. The proof of this result is given inside Mumford's Abelian Varieties. Let me write out the arguments here as well.
(a) We prove that $X_q \cong X$. This will follow from the fact that X is defined over the field $\mathbb{F}_q$. First off, X is defined by the Weierstrass form. It has equation

$$y^2 = x(x-1)(x-\lambda)$$

on the chart where $z \neq 0$. One can write the equations for the other charts. It follows that X is locally given by $\operatorname{Spec} \mathbb{F}_q[x, y]/(y^2 - x(x-1)(x-\lambda))$ and the Frobenius morphism on this affine chart is defined by raising everything to the qth power. But any element of $\mathbb{F}_q$ raised to the qth power is itself. So on the level of sheaves, our morphism is an isomorphism[3].
(b) It suffices to prove that $1_X - F'$ has nonzero action on the tangent space of X at the origin. But 1_X induces the identity map on the tangent space while $(F')^* : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is 0 since we know that $F' : X \to X$ defines the isomorphism.

[3]More details are warranted but this is enough for now.

The fact that the kernel is just $X(\mathbb{F}_q)$ is clear. The fixed points fo the action of the Frobenius are preceisely those coordinates in the field $\mathbb{F}_q$.
(c) We shall use Hartshorne IV.4.7 for this part. The first claim is that $F' + \widehat{F'} = a_X$ for some integer a. Consider the map $1 + F'$ and we deduce

$$\deg(1+F')_X = (1+F') \circ (\widehat{1} + \widehat{F'}) = \deg(1_X)_X + F' \circ \widehat{1_X} + 1_X \circ \widehat{F'} + \deg(F')_X$$

and so $F' + \widehat{F'} = \deg(1+F')_X - 1_X - \deg(F')_X = a_X$ where $a = \deg(1+F') - 1 - \deg(F')$.

Now we want to compute $\#X(\mathbb{F}_q) = N$. We know that

(34)

$$N_X := \deg(1_X - F') = (1_X - F') \circ (\widehat{1_X} - \widehat{F'}) = 1_X \circ \widehat{1_X} - 1_X \circ \widehat{F'} - F' \circ \widehat{1_X} + F' \circ \widehat{F'}$$

(35)
$$= 1 - 1_X \circ \widehat{F'} - F' \circ \widehat{1_X} + q = 1 - a_X + q_X$$

which implies that $N = q - a + 1$. Now that we used the fact that $\deg(F') = q$.
(d) The result here is the bound of Hasse and can be found in Silverman's The Arithemtic of Elliptic Curves. In particular, it is Theorem 1.11 and so the proof is omitted.
(e) Fix $q = p$. If the Hasse invariant is 0, we know there is a single p-torsion point. Furthermore, $p_X : X \to X$ maps $\mathbb{F}_q$-points to $\mathbb{F}_q$-points. Therefore, p divides $N - 1$ (as there is a single fixed point which is the single p-torsion point). So, p divides $p - q$ and hence, p divides a. This prove ($\Rightarrow$). To prove the other direction, simply reverse the argument.

Let $p \geq 5$. We know that $|a| \leq 2\sqrt{p}$. The preceding part says that X has Hasse invariant zero iff $a \equiv 0 \bmod p$. But for $p \geq 5$, we know $2\sqrt{p} < p$ and so $a \equiv 0 \bmod p$ with the bound implies $a = 0$. So X has Hasse invariant 0 iff $N = p + 1$. □

Hartshorne IV.4.17 Let X be the curve $y^2 + y = x^3 - x$ of (4.23.8).

(a) If $Q = (a, b)$ is a point on the curve, compute the coordinates of the point $P + Q$, where $P = (0, 0)$, as a function of a, b. Use this formula to find the coordinates of $nP, n = 1, 2, \ldots, 10$. [Check: $6P = (6, 14)$.]

(b) This equation defines a nonsingular curve over $\mathbb{F}_p$ for all $p \neq 37$.

Proof. □

Hartshorne IV.4.18 Let X be the curve $y^2 = x^3 - 7x + 10$. This curve has at least 26 points with integer coordinates. Find them (use a calculator), and verify that they are all contained in the subgroup (maybe equal to all of $X(\mathbb{Q})$?) generated by $P = (1, 2)$ and $Q = (2, 2)$.

Proof. □

Hartshorne IV.4.19 Let X, P_0 be an elliptic curve defined over $\mathbb{Q}$, represented as a curve in $\mathbb{P}^2$ defined by an equation with integer coefficients. Then X can be considered as the fibre over the generic point of a scheme $\bar{X}$ over Spec $\mathbb{Z}$. Let $T \subseteq \operatorname{Spec} \mathbb{Z}$ be the open subset consisting of all primes $p \neq 2$ such that the fibre $X_{(p)}$ of $\bar{X}$ over p is nonsingular. For any n, show that $n_X : X \to X$ is defined over T, and is a flat morphism. Show that the kernel of n_X is also flat over T. Conclude that for any $p \in T$, the natural map $X(\mathbb{Q}) \to X_{(p)}(\mathbb{F}_p)$ induced on the groups of rational points, maps the n-torsion points of $X(\mathbb{Q})$ injectively into the torsion subgroup of $X_{(p)}(\mathbb{F}_p)$, for any $(n, p) = 1$.

By this method one can show easily that the groups $X(\mathbb{Q})$ in (Ex. 4.17) and (Ex. 4.18) are torsion-free.

PROOF. □

Hartshorne IV.4.20 Let X be an elliptic curve over a field k of characteristic $p > 0$, and let $R = \mathrm{End}\,(X, P_0)$ be its ring of endomorphisms.

(a) Let X_p be the curve over k defined by changing the k-structure of X(2.4.1). Show that $j\,(X_p) = j(X)^{1/p}$. Thus $X \cong X_p$ over k if and only if $j \in \mathbb{F}_p$.

(b) Show that p_X in R factors into a product $\pi\hat{\pi}$ of two elements of degree p if and only if $X \cong X_p$. In this case, the Hasse invariant of X is 0 if and only if π and $\hat{\pi}$ are associates in R (i.e., differ by a unit). (Use (2.5).)

(c) If Hasse $(X) = 0$ show in any case $j \in \mathbb{F}_{p^2}$.

(d) For any $f \in R$, there is an induced map $f^* : H^1\,(\mathcal{O}_X) \to H^1\,(\mathcal{O}_X)$. This must be multiplication by an element $\lambda_f \in k$. So we obtain a ring homomorphism $\varphi : R \to k$ by sending f to λ_f. Show that any $f \in R$ commutes with the (nonlinear) Frobenius morphism $F : X \to X$, and conclude that if Hasse $(X) \neq 0$, then the image of φ is $\mathbb{F}_p$. Therefore, R contains a prime ideal $\mathfrak{p}$ with $R/\mathfrak{p} \cong \mathrm{F}_p$.

PROOF. □

Hartshorne IV.4.21 Let O be the ring of integers in a quadratic number field $\mathbb{Q}(\sqrt{-d})$. Show that any subring $R \subseteq O, R \neq \mathbb{Z}$, is of the form $R = \mathbb{Z} + f \cdot O$, for a uniquely determined integer $f \geqslant 1$. This integer f is called the *conductor* of the ring R.

PROOF. □

Hartshorne IV.4.22 If $X \to \mathbb{A}^1_{\mathbb{C}}$ is a family of elliptic curves having a section, show that the family is trivial.
[Hints: Use the section to fix the group structure on the fibres. Show that the points of order 2 on the fibres form an étale cover of $\mathbb{A}^1_{\mathbb{C}}$, which must be trivial, since $\mathbb{A}^1_{\mathbb{C}}$ is simply connected. This implies that λ can be defined on the family, so it gives a map $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}} - \{0, 1\}$. Any such map is constant, so λ is constant, so the family is trivial.]

PROOF. Let $\sigma : \mathbb{A}^1_{\mathbb{C}} \to X$ be a section i.e. $\pi \circ \sigma = \mathrm{id}_{\mathbb{A}^1_{\mathbb{C}}}$ for $\pi : X \to \mathbb{A}^1_{\mathbb{C}}$ the projection. Using the section, we can fix the group structure on each fibre e.g. if $\sigma(\lambda) = P_0$, then we can give X_λ the group structure where P_0 is the identity element. Now consider the points of order two on the fibres and let Y denote their set.

Lemma 4.1. The projection $Y \to \mathbb{A}^1_{\mathbb{C}}$ is an étale cover of $\mathbb{A}^1_{\mathbb{C}}$.

PROOF. That the projection map is flat is clear from Hartshorne Proposition III.9.7. and/or Hartshorne Theorem III.9.11.

Next, we need to check that the morphism is unramified. That there is a separable algebraic extension of residue fields at every point is clear from the characteristic 0 assumption. That $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$ for $\pi(x) = y$ is a bit trickier, but *should* follow from the existence of a section.

Having shown that $Y \to \mathbb{A}^1_{\mathbb{C}}$ forms an étale cover, we deduce that it must be trivial due to simply connectedness of $\mathbb{A}^1_{\mathbb{C}}$. We conclude from this that the family of elliptic curves does not degenerate.

Then, the invariant λ can be defined on the whole of the family i.e. there is a morphism $\lambda : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}} - \{0, 1\}$ which sends a point $t \in \mathbb{A}^1_{\mathbb{C}}$ to the corresponding λ value of the elliptic curve X_t. It follows we have a morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}} - \{0, 1\}$. But this morphism must be constant and so the family itself must be constant. □

□

5. Curves

Hartshorne IV.5.1 Show that a hyperelliptic curve can never be a complete intersection in any projective space. Cf. (Ex. 3.3).

Proof. Suppose it were a complete intersection and so Hartshorne IV.3.3 says that the canonical divisor is K_X very ample (hyperelliptic curves must have genus ≥ 2). But Hartshorne Proposition IV.5.1 says that K_X very ample implies X is not hyerelliptic. Contradiction. □

Hartshorne IV.5.2 If X is a curve of genus $\geqslant 2$ over a field of characteristic 0 , show that the group Aut X of automorphisms of X is finite.
[Hint: If X is hyperelliptic, use the unique g^1_2 and show that Aut X permutes the ramification points of the 2 -fold covering $X \to \mathbb{P}^1$. If X is not hyperelliptic, show that Aut X permutes the hyperosculation points (Ex. 4.6) of the canonical embedding. Cf. (Ex. 2.5).]

Proof. If X is hyperelliptic, fix $f : X \to \mathbb{P}^1$ as the unique g^1_2. Any automorphism would permute the ramification points of the 2-fold covering f. But we know from Hurwitz's formula that

$$2g(X) - 2 = -4 + \#\text{of ramification points} \geq 2.$$

Therefore, the degree of the ramification divisor is

$$2g(X) + 2 = \deg(R) \geq 6.$$

Since the map is a degree 2 map, the ramificatin indices are at most 2 and so there are at least six ramification points. By uniqueness of the g^1_2, any automorphism of X must permute the ramification points on $\mathbb{P}^1$ i.e. permute the ramification points of the 2-fold cover $X \to \mathbb{P}^1$. But there can be most finitely many permutations since an automorphism of $\mathbb{P}^1$ is determined by where one maps $0, 1, \infty$.

Now suppose X is not hyperelliptic. Then $|K_X|$ defines an embedding $X \to \mathbb{P}^{g-1}$. Then by Hartshorne IV.4.6(b), there are

$$(g-1)g(g-1) + (g-1+1)(2g-2) = g(g-1)^2 + (2g-2)g \geq 2 + (2g-2)g \geq 6$$

hyperosculating points and there are at least 6 hyperosculating points. One can check via Hartshorne IV.4.6(b) that any automorphism must permute the hyperosculation points of the embedding and is uniquely determined by such a permutation (this is a key detail I am omitting). So Aut(X) is finite. □

Hartshorne IV.5.3 *Moduli of Curves of Genus 4.* The hyperelliptic curves of genus 4 form an irreducible family of dimension 7. The nonhyperelliptic ones form an irreducible family

of dimension 9. The subset of those having only one g^1_3 is an irreducible family of dimension 8.

[Hint: Use (5.2.2) to count how many complete intersections $Q \cap F_3$ there are.]

PROOF. If we have a hyperelliptic curve of genus 4, say X, then it has a unique g^1_2, so let $f : X \to \mathbb{P}^1$ be the degree 2 map. Then by the Hurwitz formula,

$$2g(X) - 2 = 2(-2) + \deg R \;\Rightarrow\; \deg R = 10.$$

So $f : X \to \mathbb{P}^1$ is ramified over 10 different points (recall that the map is degree 2 is the ramification indices are at worst 1). But then by picking an automorphism so that f is ramified over $0, 1, \infty$, we see that X is uniquely determined by the remaining seven ramification points. So we have a family of dimension 7. The fact that the family is irreducible follows from the fact that we parameterize using $\mathbb{A}^1 \setminus \{0, 1, \infty\}/S_7$.

[Incomplete]

Now let X be a nonhyperelliptic curve. We use Hartshorne Example IV.5.2.2 to determine the dimension of the family. First, X is contained in a unique quadric surface and the space of all such quadric surfaces has dimension

$$\dim_k \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 10.$$

We need to take a quadric surface Q and intersect it with a cubic surface to get the curve X. From Hartshorne Example IV.5.2.2, we know that there is a single 1-dimensional family of cubics that we can choose from. So the space of nonhyperelliptic crves form an irreducible family of dimension 9.

[Incomplete] □

Hartshorne IV.5.4 Another way of distinguishing curves of genus g is to ask, what is the least degree of a birational plane model with only nodes as singularities (3.11)? Let X be nonhyperelliptic of genus 4 . Then:

(a) if X has two g^1_3's, it can be represented as a plane quintic with two nodes, and conversely;

(b) if X has one g^1_3, then it can be represented as a plane quintic with a tacnode (I, Ex. 5.14 d), but the least degree of a plane representation with only nodes is 6.

PROOF. (a) Assume X has two g^1_3's. Since X is nonhyperelliptic of genus 4, Hartshorne Example IV.5.5.2, says that X sits on a unique irreducible nonsingular quadric surface Q and the two family of lines cut out the g^1_3's on X. The curve X has type $(3, 3)$ on Q. Now we want to project onto $\mathbb{P}^2$.

If we project from a point, we get a plane quintic (degree of X in $\mathbb{P}^3$ plus degree of the surface). Now we want to show that the only singularities we get are nodes.

Suppose $Proj_{P_0} : Q \to \mathbb{P}^2$ is the projection from the point P_0. Then we get a singularity if there are points mapping to the same point. Suppose $P_1, P_2, \ldots, P_r \in X$ map to the same point. Then they all sit on the same line i.e. $\overline{P_0 P_1 P_2 \ldots P_r} = L$ gives a line. If $r \geq 2$, then the line sits inside the quadric. Then the line L intersects the curve at 3 points counting multiplicity due to Bezout's Theorem. But those 3 points would determine a g^1_3 which we know there are only two of.

Conversely, suppose $C \subseteq \mathbb{P}^2$ is a plane quintic with a tacnode. A quintic has genus 6 and so the normalization has genus 4. Now a line through a node $P \in C$ will intersect three other points. So the three other points will define a g^1_3 (up to linear equivalence).

(b) Assume X has a single g^1_3 and so it sits on a unique singular irreducible surface. Then we have two cases. If we project from a point of the curve that is not passing through a singularity, then we get a plane quintic with tacnode. [**Incomplete**][4]

□

Hartshorne IV.5.5 *Curves of Genus 5.* Assume X is not hyperelliptic.

(a) The curves of genus 5 whose canonical model in $\mathbb{P}^4$ is a complete intersection $F_2 \cdot F_2 \cdot F_2$ form a family of dimension 12 .

(b) X has a g^1_3 if and only if it can be represented as a plane quintic with one node. These form an irreducible family of dimension 11.

[Hint: If $D \in g^1_3$, use $K - D$ to map $X \to \mathbb{P}^2$.]

(c) In that case, the conics through the node cut out the canonical system (not counting the fixed points at the node). Mapping $\mathbb{P}^2 \to \mathbb{P}^4$ by this linear system of conics, show that the canonical curve X is contained in a cubic surface $V \subseteq \mathbb{P}^4$, with V isomorphic to $\mathbb{P}^2$ with one point blown up (II, Ex. 7.7). Furthermore, V is the union of all the trisecants of X corresponding to the g^1_3 (5.5.3), so V is contained in the intersection of all the quadric hypersurfaces containing X. Thus V and the g^1_3 are unique.

Note. Conversely, if X does not have a g^1_3, then its canonical embedding is a complete intersection, as in (a). More generally, a classical theorem of Enriques and Petri shows that for any nonhyperelliptic curve of genus $g \geqslant 3$, the canonical model is projectively normal, and it is an intersection of quadric hypersurfaces unless X has a g^1_3 or $g = 6$ and X has a g^2_5. See Saint-Donat [1].

Proof. □

Hartshorne IV.5.6 Show that a nonsingular plane curve of degree 5 has no g^1_3. Show that there are nonhyperelliptic curves of genus 6 which cannot be represented as a nonsingular plane quintic curve.

Proof. Fix a X nonsingular plane curve of degree 5. Then $g(X) = \frac{4\cdot 3}{2} = 6$. Then the Hurwitz formula

$$10 = -6 + \deg R \;\Rightarrow\; \deg R = 16.$$

So there are 16 ramification points on X.

[**Incomplete**] □

Hartshorne IV.5.7 (a) Any automorphism of a curve of genus 3 is induced by an automorphism of $\mathbb{P}^2$ via the canonical embedding.

*(b) Assume char $k \neq 3$. If X is the curve given by

$$x^3y + y^3z + z^3x = 0$$

the group Aut X is the simple group of order 168 , whose order is the maximum $84(g-1)$ allowed by (Ex. 2.5). See Burnside [1, §232] or Klein [1].

*(c) Most curves of genus 3 have no automorphisms except the identity.

[4] I do not know how to get only nodes.

[Hint: For each n, count the dimension of the family of curves with an automorphism T of order n. For example, if $n = 2$, then for suitable choice of coordinates, T can be written as $x \to -x, y \to y, z \to z$. Then there is an 8-dimensional family of curves fixed by T; changing coordinates there is a 4-dimensional family of such T, so the curves having an automorphism of degree 2 form a family of dimensional 12 inside the 14-dimensional family of all plane curves of degree 4.]

Note: More generally it is true (at least over $\mathbf{C}$) that for any $g \geqslant 3$, a "sufficiently general" curve of genus g has no automorphisms except the identity-see Baily [1].

PROOF. □

6. Classification of Curves in $\mathbb{P}^3$

Hartshorne IV.6.1 A rational curve of degree 4 in $\mathbb{P}^3$ is contained in a unique quadric surface Q, and Q is necessarily nonsingular.

PROOF. Fix X to be the said rational curve of degree 4 in $\mathbb{P}^3$. Let $\mathscr{I}$ the ideal sheaf of X inside $\mathbb{P}^3$. Look at the SES

$$0 \to \mathscr{I} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0.$$

Twist by 2 and take cohomology to get the sequence

$$0 \to H^0(\mathbb{P}^3, \mathscr{I}(2)) \to H^0(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(X, \mathcal{O}_X(2)) \to H^1(\mathbb{P}^3, \mathscr{I}(2)) \to 0$$

Now we attempt to count dimensions as before. The embedding of the curve X is given by a divisor of degree 5 on $\mathbb{P}^1$. So $\mathcal{O}_X(2)$ is really $\mathcal{O}_X(2D)$ where $\deg 2D = 10$. This is nonspecial as a divisor so by Riemann-Roch

$$\dim H^0(X, \mathcal{O}_X(2)) = 10 - 1 + 0 = 9.$$

It follows that $\dim H^0(\mathbb{P}^3, \mathscr{I}(2)) \geq 1$. A form in the space $H^0(\mathbb{P}^3, \mathscr{I}(2))$ will give us the quadric surface Q.

Now if X is in two quadrics $V(f), V(g)$, we get a SES

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathscr{I}_X \to 0.$$

Taking cohomology of the first exact sequence and then the exact sequence above, we deduce that $H^1(X, \mathcal{O}_X)$ has dimension 1 which contradicts X being rational.

Now take the quadric Q. WLOG and by change of coordinates, it has equations $X_0^2, X_0^2 + X_1^2, X_0^2 + X_1^2 + X_3^2, X_0^2 + X_1^2 + X_2^2 + X_3^2$ as its only possibilities. The first cannot occur since X is of degree 4. The same can be said about the second equation since it becomes a union of two lines over an algebraically closed field.

Now we are left with the last two equations. For the equation $X_0^2 + X_1^2 + X_2^2$ we know that the quadric is then a cone Q. By Hartshorne Remark IV.6.4.1, d must be a complete intersection of Q with a surface of degree a and $g(X) = a^2 - 2a + 1$. But ths is absurd since $a^2 - 2a + 1 = 0$ has no integer solutions.

So we are left with the last one. By Hartshorne I.5.12,(c), the singular locus must be empty. □

Hartshorne IV.6.2 A rational curve of degree 5 in $\mathbb{P}^3$ is always contained in a cubic surface, but there are such curves which are not contained in any quadric surface.

PROOF. A rational curve of degree 5 in $\mathbb{P}^3$ being contained in a cubic curve is clear by our usual arguments thus far. Take the exact sequence

$$0 \to \mathscr{I}_X \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

where $\mathscr{I}_X$ is the ideal sheaf. Twist by $\mathcal{O}_{\mathbb{P}^3}(3)$ to get a SES

$$0 \to \Gamma(\mathbb{P}^3, \mathscr{I}_X(3)) \to \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \to \Gamma(\mathbb{P}^3, \mathcal{O}_X(3)) \to H^1(\mathbb{P}^3, \mathscr{I}_X(3)).$$

Now $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20$ and $h^0(\mathcal{O}_X(3))$ can be computed via the Riemann-Roch Theorem. Indeed,

$$h^0(\mathcal{O}_X(3)) = h^0(\mathcal{O}_{\mathbb{P}^1}(15)) = 16 + 1 - g(\mathbb{P}^1) = 17.$$

It follows that $\dim_k \Gamma(\mathbb{P}^3, \mathscr{I}_X(3)) > 0$ and so X is contained in a cubic surface.

We want to find a rational curve of degree 5 that does not sit in any quadric surface. The method of proof above no longer works, so we come up with a particular example.

Take the rational curve $\mathbb{P}^1 \to \mathbb{P}^3$ defined by $[s : t] \to [s^5 : s^4t + s^3t^2 : s^2t^3 + st^4 : t^5]$ and one can check that it has an ideal generated by elements of degrees $0, 3, 3, 3, 3, 4$ and so therefore cannot be inside a quadric surface (this can be done via Macaulay2). □

Hartshorne IV.6.3 A curve of degree 5 and genus 2 in $\mathbb{P}^3$ is contained in a unique quadric surface Q. Show that for any abstract curve X of genus 2 , there exist embeddings of degree 5 in $\mathbb{P}^3$ for which Q is nonsingular, and there exist other embeddings of degree 5 for which Q is singular.

PROOF. □

Hartshorne IV.6.4 There is no curve of degree 9 and genus 11 in $\mathbb{P}^3$.

[Hint: Show that it would have to lie on a quadric surface, then use (6.4.1).]

PROOF. That it lies on a quadric surface is clear. Applying Hartshonre Remark IV.6.4.1, we deduce that $d = 9 = 2 \cdot 4 + 1$ implies $a = 4$ and the remark says that $g = 4^2 - 4 = 12$ whcih is not equal to 11. Contradiction. □

Hartshorne IV.6.5 If X is a complete intersection of surfaces of degrees a, b in $\mathbb{P}^3$, then X does not lie on any surface of degree $< \min(a, b)$.

PROOF. Let $\mathscr{I}_X$ be the ideal sheaf of X. Then there is resolution via

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a-b) \xrightarrow{\begin{pmatrix} f \\ -g \end{pmatrix}} \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} \mathscr{I}_X \to 0$$

Then the result follows from sheaf cohomology i.e. if we twist by k for $k < \min(a, b)$, then $H^0(\mathbb{P}^3, \mathscr{I}_X(k)) = 0$. □

Hartshorne IV.6.6 Let X be a projectively normal curve in $\mathbb{P}^3$, not contained in any plane. If $d = 6$, then $g = 3$ or 4 . If $d = 7$, then $g = 5$ or 6 . Cf. (II, Ex. 8.4) and (III, Ex. 5.6).

PROOF. If $d = 6$, we can then look at the SES $0 \to \mathscr{I}_X(1) \to \mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_X(1) \to 0$ and then take cohomology. We get the surjection

$$H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_X(1)) \to 0$$

and by the Riemann-Roch Theorem,

$$h^0(\mathcal{O}_X(1)) = 7 - g + h^1(\mathcal{O}_X(1)) \geq 7 - g.$$

Since $h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$, we deduce that

$$4 \geq h^0(\mathcal{O}_X(1)) \geq 7 - g \;\Rightarrow\; g \geq 3.$$

Finally, we apply Hartshorne Theorem IV.6.4 to deduce that $g \leq \frac{1}{4}(6)^2 - 6 + 1 = 4$ finishing off the proof.

From the same method but for $d = 7$, we deduce that

$$4 \geq h^0(\mathcal{O}_X(1)) \geq 8 - g \;\Rightarrow\; g \geq 4.$$

Once more by Catelnuovo's Theorem, we deduce that $g \leq \frac{1}{4}(7^2 - 1) - 7 + 1 = \frac{48}{4} - 7 + 1 = 6$. All that is left is to rule out the case of $g = 4$. This can be ruled out by using Hartshorne Corollary IV.6.2 and Proposition IV.6.3 as follows.

Suppose $g = 4$. Then we can look at

$$0 \to \mathscr{I}_X(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_X(2) \to 0.$$

Taking global sections, and using projectively normal,

$$0 \to H^0(\mathscr{I}_X(2) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2)) \to 0.$$

The latter two spaces have dimensions $\binom{5}{3} = 10$ and $h^0(\mathcal{O}_X(2)) = 11 + \dim H^1(X, \mathcal{O}_X(2))$ respectively. It follows that X is contained a in quadric surface. Apply a linear change of coordinates so that X sits in the nonsingular quadric surface Q of Hartshorne III.5.6.

Now we use Hartshorne Remark IV.6.4.1. Since $d = 7$ is odd, our curve X has genus $3^2 - 3 = 6$ which is a contradiction. □

Hartshorne IV.6.7 The line, the conic, the twisted cubic curve and the elliptic quartic curve in $\mathbb{P}^3$ have no multisecants. Every other curve in $\mathbb{P}^3$ has infinitely many multisecants.

[Hint: Consider a projection from a point of the curve to $\mathbb{P}^2$.]

Proof. The statement for a line is clear.

Case of a conic. By Hartshorne Example IV.6.2, the only curve of degree 2 is the conic in $\mathbb{P}^2$ and this certainly has no multisecants.

Case of a twisted cubic curve. Embed X inside $\mathbb{P}^3$ via the 3-uple embedding $[s : t] \to [s^3 : s^2t : st^2 : t^3]$. Now $g(X) = 0$ and we know from Hartsone IV.3.5 that $g(X) < \frac{1}{2}(3-1)(3-2) = 1$. That is, there can be at best a node in the image of any projection away from a point. Then use Hartshorne Proposition IV.3.7.

Case of an elliptic quartic curve. Let X be an elliptic quartic curve in $\mathbb{P}^3$. Now X is the intersection of two quadric surfaces in $\mathbb{P}^3$. A general line will intersect a quadric surface at 2 distinct points. Now pick a point $O \in \mathbb{P}^3$ away from the two quadric surfaces. Then apply a projection from this point. Then O will intersect either one or two points of each quadric surface. The worst case scenario is that it intersects two points each and both point are on the curve while all the other cases give us the claim. In this worst case scenario, we get a node and so we are done.

Case of all other curves. We assume that our curve is none of the above. We shall use Hartshorne Example IV.6.4.2 fo rthe moment. We have already discussed the situation of $d = 1$ and $d = 2$ above. In the $d = 3$ case, we ruled out the twisted cubic and clearly the plane cubic has multisecants. In the $d = 4$ case, we had rational quartics, plane quartics, and the elliptic quartic. We can easily rule out the plane quartic since it has many obvious multisecants. For the rational quartic, we know its genus is 0 but it has degree 4. Now,

$\mathbb{P}^1 \to \mathbb{P}^3$ must be given by a linear system $V \subseteq |\mathcal{O}(3)|$ and so it suffices to check that we get a multisecant for any particula choice of 3-dimensional subspace of $|\mathcal{O}(3)|$.

[Incomplete]

All that remains now is the case where $d \geq 5$ and of some genus g. Take a point not on the curve and consider the projection away from the point. We know as before $g(X) < \frac{1}{2}(d-2)(d-1)$. and $\frac{1}{2}(d-2)(d-1) \geq 6$.

[Incomplete] □

Hartshorne IV.6.8 A curve X of genus g has a nonspecial divisor D of degree d such that $|D|$ has no base points if and only if $d \geqslant g+1$.

Proof. The result for when $g = 0, 1$ is done by Hartshorne Examples IV.3.3.1 and IV.3.3.3 and some easy work. Fix $g(X) \geq 2$.

($\Leftarrow$). Fixing $d \geq g+1$, we show that there exists a basepoint free special divisor D of said degree.

Halphen's theorem says that X has a nonspecial very ample divisor E iff $d \geq g+3$. So if $d \geq g+3$, pick that nonspecial very ample divisor and we are done. Very ample implies basepoint free for example. Let H be said divisor.

Now look at $H - P$ where $P \in X$ is a general point. Then $H - P$ is still basepoint free and nonspecial. Nonspecialty follows because

$$\dim |H - P - Q| = \dim |H - P| - 1$$

due to the fact that H was very ample. So we get the claim for when $d = g+2$.

Now suppose $d = g+1$. Pick our H once more and look at $H - P - Q$ to get the claim.

($\Rightarrow$). Suppose that D is a nonspecial divisor with no base points. Suppose not and that $\deg D < g+1$. Then $\deg D + P + Q < g+3$. The divisor $D + P + Q$ is still nonspecial and is very ample since D has no basepoints. But then we get a contradiction of Halphen's theorem. □

Hartshorne IV.6.9 Let X be an irreducible nonsingular curve in $\mathbb{P}^3$. Then for each $m >> 0$, there is a nonsingular surface F of degree m containing X. [Hint: Let $\pi : \tilde{\mathbb{P}} \to \mathbb{P}^3$ be the blowing-up of X and let $Y = \pi^{-1}(X)$. Apply Bertini's theorem to the projective embedding of $\tilde{\mathbb{P}}$ corresponding to $\mathscr{I}_Y \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(m)$.]

Proof. Let $\pi : \widetilde{\mathbb{P}} \to \mathbb{P}^3$ be the blow-up along $X \subseteq \mathbb{P}^3$. By Hartshorne II.7.5(b), $\mathscr{I}_Y \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(m)$ is very ample for $m \gg 0$ and $\mathscr{I}_Y$ is an invertible sheaf because $Y \cong \mathbb{P}^2$ is a divisor on $\mathbb{P}^3$. Now look at the projective embedding $\phi : \widetilde{\mathbb{P}} \to \mathbb{P}^N$ corresponding to this very ample sheaf. By Bertini's Theorem, there is a hyperplane section that is nonsingular and irreducible. Such a hyperplane section pulls back along the embedding to $\widetilde{\mathbb{P}}$ and projects down along π to a degree m surface in $\mathbb{P}^3$.

Now we check nonsingularity. The pullback to $\widetilde{\mathbb{P}}$ is still nonsingular for obvious reasons. It can be made to not intersect Y by moving the hyperplane section we choose. So the pullback to $\widetilde{\mathbb{P}}$ is isomorphic to its image along π.

The fact that the nonsingular surface F has degree m follows from tracing through the linear systems.

The fact that F contains X follows from the the sheaf we chose. The functions on the pullback $\widetilde{F}$ to $\widetilde{\mathbb{P}}$ which vanish must also vanish on $\pi^{-1}(X)$ and so pushing forward along π, we see that the functions that vanish on F must also vanish on X. So $X \subseteq F$ as desired. □

CHAPTER 5

Chapter V Exercises

1. Geometry on a Surface

Hartshorne V.1.1 Let C, D be any two divisors on a surface X, and let the corresponding invertible sheaves be $\mathscr{L}, \mathscr{M}$. Show that

$$C.D = \chi(\mathcal{O}_X) - \chi\left(\mathscr{L}^{-1}\right) - \chi\left(\mathscr{M}^{-1}\right) + \chi\left(\mathscr{L}^{-1} \otimes \mathscr{M}^{-1}\right).$$

Proof. Recall that the Euler characteristic of an invertible sheaf $\mathscr{F}$ on X is given by

$$\chi(\mathscr{F}) = \dim_k H^0(X, \mathscr{F}) - \dim_k H^1(X, \mathscr{F}) + \dim_k H^2(X, \mathscr{F}).$$

By definition, $\mathscr{L} = \mathscr{L}(C)$ and $\mathscr{M} = \mathscr{L}(D)$. Computing the Euler characteristic as in the proof of Hartshorne Theorem V.1.6 (from now on called the Riemann-Roch Theorem),

$$\begin{aligned}
\chi(\mathscr{L}^{-1}) &= \frac{1}{2}(-C).(-C-K) + 1 + p_a = \frac{1}{2}C^2 2 + \frac{1}{2}C.K + 1 + p_a \\
\chi(\mathscr{M}^{-1}) &= \frac{1}{2}(-D).(-D-K) + 1 + p_a = \frac{1}{2}D^2 + \frac{1}{2}D.K + 1 + p_a \\
\chi(\mathscr{L}^{-1} \otimes \mathscr{M}^{-1}) &= \frac{1}{2}(-C-D).(-C-D-K) + 1 + p_a \\
&= \frac{1}{2}C^2 + \frac{1}{2}C.D + \frac{1}{2}C.K + \frac{1}{2}D.C + \frac{1}{2}D^2 + \frac{1}{2}D.K + 1 + p_a \\
\chi(\mathcal{O}_X) &= p_a + 1.
\end{aligned}$$

Now if one cancels each of the terms, it follows that the RHS equals $\frac{1}{2}C.D + \frac{1}{2}D.C$ and since the intersection number is symmetric, we get $C.D$. □

Hartshorne V.1.2 Let H be a very ample divisor on the surface X, corresponding to a projective embedding $X \subseteq \mathbb{P}^N$. If we write the Hilbert polynomial of X (III, Ex. 5.2) as

$$P(z) = \frac{1}{2}az^2 + bz + c$$

show that $a = H^2, b = \frac{1}{2}H^2 + 1 - \pi$, where π is the genus of a nonsingular curve representing H, and $c = 1 + p_a$. Thus the degree of X in $\mathbb{P}^N$, as defined in $(I, §7)$, is just H^2. Show also that if C is any curve in X, then the degree of C in $\mathbb{P}^N$ is just $C.H$.

Proof. First off, if $z = 0$, we know that $P(0) = \chi(X, \mathcal{O}_X)$ which is just $1 + p_a(X)$. Second, a should be the degree of X. The means we need to compute the degree of the very ample divisor H on X. But that is just the intersection of H with a hyperplane section aka H^2.

The difficult part is in determining b. For this, we notice that

$$P(-1) = \chi(X, \mathcal{O}_X(-1)) = \frac{1}{2}H^2 - b + 1 + p_a = \frac{1}{2}H^2 - b + \chi(X, \mathcal{O}_X).$$

It follow that

$$\chi(X, \mathcal{O}_X(-1)) - \chi(X, \mathcal{O}_X) = \frac{1}{2}H^2 - b.$$

The LHS is just $-\chi(X, \mathcal{O}_H)$ because of the SES

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_H \to 0.$$

To determine the Euler characteristic of the curve, we only need to compute H^0 and H^1. First off, $h^0(X, \mathcal{O}_H)$ is just 1. OTOH,

$$h^1(X, \mathcal{O}_H) = h^1(H, \mathcal{O}_H) = h^0(H, K_H) = g(H) = \frac{1}{2}H^2 + \frac{1}{2}H.K_X + 1.$$

It follows that

$$\frac{1}{2}H^2 + \frac{1}{2}H.K_X = -1 + \frac{1}{2}H^2 + \frac{1}{2}H.K_X + 1 = -\chi(X, \mathcal{O}_H) = \frac{1}{2}H^2 - b.$$

It therefore follows that

$$b = -\frac{1}{2}H.K_X = \frac{1}{2}H^2 - 1 + g(H) = \frac{1}{2}C^2 - 1 + g(C)$$

as claimed. □

Hartshorne V.1.3 Recall that the *arithmetic genus* of a projective scheme D of dimension 1 is defined as $p_a = 1 - \chi(\mathcal{O}_D)$ (III, Ex. 5.3).

(a) If D is an effective divisor on the surface X, use (1.6) to show that $2p_a - 2 = D.(D + K)$.

(b) $p_a(D)$ depends only on the linear equivalence class of D on X.

(c) More generally, for any divisor D on X, we define the virtual arithmetic genus (which is equal to the ordinary arithmetic genus if D is effective) by the same formula: $2p_a - 2 = D.(D + K)$. Show that for any two divisors C, D we have and

$$p_a(-D) = D^2 - p_a(D) + 2p_a(C + D) = p_a(C) + p_a(D) + C.D - 1.$$

Proof. **(a)** We know $1 - \chi(\mathcal{O}_D) = p_a$ and applying Hartshorne Theorem V.1.6,

$$\chi(\mathscr{L}(D)) = \frac{1}{2}(D.D + K) + 1 + p_a = \frac{1}{2}(D.D + K) + \chi(\mathcal{O}_X).$$

Since D is effective, $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D))$ and $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X)$. Therefore,

$$\chi(\mathcal{O}_D) = \frac{1}{2}(D.D + K) \;\Rightarrow\; 2p_a - 2 = \frac{1}{2}(D.D + K).$$

(b) Clear from (a) and if we write D as the difference of two effective divisors.

(c) For the first formula, note that $p_a(D) = \frac{(D.D+K)}{2} + 1$ and so the first formula is a trivial computation. The same can be said for the second formula. □

Hartshorne V.1.4 (a) If a surface X of degree d in $\mathbb{P}^3$ contains a straight line $C = \mathbb{P}^1$, show that $C^2 = 2 - d$.

(b) Assume char $k = 0$, and show for every $d \geqslant 1$, there exists a nonsingular surface X of degree d in $\mathbb{P}^3$ containing the line $x = y = 0$.

PROOF. **(a)** By the Hartshorne Proposition V.1.5 and C a nonsingular curve of genus 0, we get $-2 = C.C(+K) = C^2 + C.K$. So $C^2 = -2 - C.K$ and we want $C.K = -4 + d$.

Here, $K := K_X$. By Hartshorne Example II.8.20.3 gives $\omega_X \cong \mathcal{O}_X(d-3-1) \cong \mathcal{O}_X(d-4)$. So $\mathscr{L}(K_X) = \mathcal{O}_X(d-4)$ and applying the adjunction formula

$$C.K_X = \deg_C(\mathscr{L}(K_X) \otimes \mathcal{O}_C) = \deg_C(\mathcal{O}_X(d-4) \otimes \mathcal{O}_C) = \deg_C(\mathcal{O}_C(d-4)) = d-4.$$

(b) This is a partial converse to (a). Fix $d \geq 0$ arbitrary. Inside $\mathbb{P}^3$, take surface X cut out by the equation $x^d + y^d = 0$. Since $\operatorname{char} k = 0$, we know $x^d + y^d = 0$ is irreducible. By Hartshorne Proposition I.7.6(d), X is of degree d. Nonsingularity is clear by checking that the stalks are regular local rings in the canonical open affines U_0, U_1, U_2 of $\mathbb{P}^3$. Clearly $x = y = 0$ is in X. □

Hartshorne V.1.5 (a) If X is a surface of degree d in $\mathbb{P}^3$, then $K^2 = d(d-4)^2$.

(b) If X is a product of two nonsingular curves C, C', of genus g, g' respectively, then $K^2 = 8(g-1)(g'-1)$. Cf. (II, Ex. 8.3).

PROOF. **(a)** By the adjunction formula, $K_X = \mathcal{O}_X(-4+d)$. Then the self-intersection is then $d(4-d)^2$ because X has degree d.

(b) One can use Hartshorne II.8.3 and then compute the intersection from Hartshorne V.1.1. □

Hartshorne V.1.6 (a) If C is a curve of genus g, show that the diagonal $\Delta \subseteq C \times C$ has self-intersection $\Delta^2 = 2 - 2g$. (Use the definition of $\Omega_{C/k}$ in (II, §8).)

(b) Let $l = C \times \text{ pt}$ and $m = \text{pt} \times C$. If $g \geqslant 1$, show that l, m, and Δ are linearly independent in $\operatorname{Num}(C \times C)$. Thus $\operatorname{Num}(C \times C)$ has rank $\geqslant 3$, and in particular, $\operatorname{Pic}(C \times C) \neq p_1^* \operatorname{Pic} C \oplus p_2^* \operatorname{Pic} C$. Cf. (III, Ex. 12.6), (IV, Ex. 4.10).

PROOF. **(a)** Let $\Phi : C \to C \times C$ be the diagonal map. By definition, $\Phi^*(\mathscr{I}/\mathscr{I}^2) = \Omega_{C/k}$ where $\mathscr{I}$ is the ideal sheaf of $\Delta = \Phi(C)$. View $\mathscr{I}/\mathscr{I}^2 = \mathscr{I} \otimes \mathcal{O}_{C\times C}/\mathscr{I}$. Now $\mathscr{I}/\mathscr{I}^2$ has dual

$$(\mathscr{I}/\mathscr{I}^2)^\vee = \mathcal{N}_{\Delta/C\times C}.$$

So,

$$\Delta^2 = \deg_C \mathcal{N}_{\Delta/C\times C} = -\deg_C \mathscr{I}/\mathscr{I}^2 = -(2g-2) = 2g-2.$$

Here is another way to do (a). By the adjunction formula, and because Δ is smooth,

$$2g - 2 = \Delta.(\Delta + K_S)$$

where $S := C \times C$. Now $K_S = p_1^* K_C + p_2^* K_C$ by Hartshorne II.8.3(a). But it can be checked geometrically that $p_i^* K_C.\Delta = 2g-2$ for $i = 1, 2$. So, the formula above becomes

$$2g - 2 = \Delta^2 + 2g - 2 + 2g - 2 \;\Rightarrow\; \Delta^2 = 2 - 2g.$$

(b) We compute $\operatorname{Num}(C \times C)$ and I think it is best to see this geometrically. Clearly, $l.m = 1$ and $l.\Delta = 1$ and $m.\Delta = 1$. So assume some linear independence and then take intersections e.g. suppose

$$a_1 l + a_2 m + a_3 \Delta = 0.$$

Intersect with l to get $a_1+a_3=0$, with m to get $a_2+a_3=0$, and Δ to get $a_1+a_2+(2-2g)a_3=0$. These three conditions imply $a_1=a_2=-a_3$ and so $-2ga_3=0$. So either $g=0$ or $a_3=0$. By hypothesis, $g(C)\neq 0$ and so $a_3=0$. This means we did not have a linear dependency. □

Hartshorne V.1.7 *Algebraic Equivalence of Divisors.* Let X be a surface. Recall that we have defined an algebraic family of effective divisors on X, parametrized by a nonsingular curve T, to be an effective Cartier divisor D on $X\times T$, flat over T (III, 9.8.5). In this case, for any two closed points $0,1\in T$, we say the corresponding divisors D_0, D_1 on X are prealgebraically equivalent. Two arbitrary divisors are prealgebraically equivalent if they are differences of prealgebraically equivalent effective divisors. Two divisors D, D' are algebraically equivalent if there is a finite sequence $D=D_0, D_1, \ldots, D_n=D'$ with D_i and D_{i+1} prealgebraically equivalent for each i.

(a) Show that the divisors algebraically equivalent to 0 form a subgroup of $\operatorname{Div} X$.

(b) Show that linearly equivalent divisors are algebraically equivalent.
[Hint: If (f) is a principal divisor on X, consider the principal divisor $(tf-u)$ on $X\times\mathbb{P}^1$, where t,u are the homogeneous coordinates on $\mathbb{P}^1$.]

(c) Show that algebraically equivalent divisors are numerically equivalent.
[Hint: Use (III, 9.9) to show that for any very ample H, if D and D' are algebraically equivalent, then $D.H=D'.H$.]

Note. The theorem of Néron and Severi states that the group of divisors modulo algebraic equivalence, called the Néron-Severi group, is a finitely generated abelian group. Over $\mathbb{C}$ this can be proved easily by transcendental methods (App. B, §5) or as in (Ex. 1.8) below. Over a field of arbitrary characteristic, see Lang and Néron [1] for a proof, and Hartshorne [6] for further discussion. Since $\operatorname{Num} X$ is a quotient of the Néron-Severi group, it is also finitely generated, and hence free, since it is torsion-free by construction.

Proof. (a) There are a lot of checks but everything should be straightforward. See also Lazarsfeld's textbook.
(b) Let $(tf-u)$ be a principal divisor on $X\times\mathbb{P}^1$ as in the hint. For the point $[1:0]\in\mathbb{P}^1$, we get (f). But for $[0:1]\in X\times\mathbb{P}^1$, the divisor is (1) which is trivial.
(c) Using Hartshorne Theorem III.9.9, I know that the Hilbert polynomials of the various fibres D_t are independent. Then for H very ample, then $D.H=D'.H$ via the intersection inside the embedding determined by H. □

Hartshorne V.1.8 *Cohomology Class of a Divisor.* For any divisor D on the surface X, we define its cohomology class $c(D)\in H^1(X,\Omega_X)$ by using the isomorphism $\operatorname{Pic} X\cong H^1(X,\mathcal{O}_X^*)$ of (III, Ex. 4.5) and the sheaf homomorphism $d\log:\mathcal{O}^*\to\Omega_X$ (III, Ex. 7.4c). Thus we obtain a group homomorphism $c:\operatorname{Pic} X\to H^1(X,\Omega_X)$. On the other hand, $H^1(X,\Omega)$ is dual to itself by Serre duality (III, 7.13), so we have a nondegenerate bilinear map

$$\langle\quad,\quad\rangle : H^1(X,\Omega)\times H^1(X,\Omega)\to k.$$

(a) Prove that this is compatible with the intersection pairing, in the following sense: for any two divisors D, E on X, we have

$$\langle c(D), c(E)\rangle = (D.E)\cdot 1$$

in k.

[Hint: Reduce to the case where D and E are nonsingular curves meeting transversally. Then consider the analogous map $c : \operatorname{Pic} D \to H^1(D, \Omega_D)$, and the fact (III, Ex. 7.4) that c (point) goes to 1 under the natural isomorphism of $H^1(D, \Omega_D)$ with k.]

(b) If char $k = 0$, use the fact that $H^1(X, \Omega_X)$ is a finite-dimensional vector space to show that Num X is a finitely generated free abelian group.

Proof. □

Hartshorne V.1.9 (a) If H is an ample divisor on the surface X, and if D is any divisor, show that

$$\left(D^2\right)\left(H^2\right) \leqslant (D.H)^2.$$

(b) Now let X be a product of two curves $X = C \times C'$. Let $l = C \times \text{pt}$, and $m = \text{pt} \times C'$. For any divisor D on X, let $a = D.l, b = D.m$. Then we say D has type (a, b). If D has type (a, b), with $a, b \in \mathbb{Z}$, show that

$$D^2 \leqslant 2ab$$

and equality holds if and only if $D \equiv bl + am$.

[Hint: Show that $H = l + m$ is ample, let $E = l - m$, let $D' = \left(H^2\right)\left(E^2\right)D - \left(E^2\right)(D.H)H - \left(H^2\right)(D.E)E$, and apply (1.9). This inequality is due to Castelnuovo and Severi. See Grothendieck [2].]

Proof. (a) See Mumford's Lectures on Curves on an Algebraic Surface. It is important to note that H being ample is absolutely necessary. Take D, H to be of type $(-1, 1)$ and $(1, -1)$ respectively on $\mathbb{P}^1 \times \mathbb{P}^1$.
(b) To show that $H = l + m$ is ample, we use the Nakai-Moishezon Criterion. This means we check $H^2 > 0$ and $H.C > 0$ for all irreducible curves on X. The fact that $H.C > 0$ is clear for $C \neq H$. Now we check $H^2 > 0$ and so

$$(l + m)^2 = l^2 + 2l.m + m^2 > l^2 + m^2 \geq 0$$

because $l.m = 1$ and $l^2 + m^2 \geq 0$ since $l^2, m^2 = 0$. One still needs to check $l^2, m^2 = 0$ but this step is omitted.

Now let D' be as above. We can compute $(D')^2$ and it turns out that

$$(D') = 16D^2 - 8(D.h)^2 + 8(D.E)^2.$$

Due to the Hodge Index Theorem (if we can check also that $D'.H = 0$ for any H ample), we see that

$$16D^2 \leq 8(D.H)^2 - 8(D.E)^2.$$

Since $D.H = a + b$ and $D.E = a - b$, the above inequality implies

$$16D^2 \leq 8(a + b)^2 + 8(a - b)^2 \;\Rightarrow\; D^2 \leq 2ab.$$

The statement regarding the inequality is clear from a direct computation. □

Hartshorne V.1.10 *Weil's Proof* [2] *of the Analogue of the Riemann Hypothesis for Curves.* Let C be a curve of genus g defined over the finite field $\mathbb{F}_q$, and let N be the number of points of C rational over $\mathbb{F}_q$. Then $N = 1 - a + q$, with $|a| \leqslant 2g\sqrt{q}$.

To prove this, we consider C as a curve over the algebraic closure k of $\mathbb{F}_q$. Let $f : C \to C$ be the k-linear Frobenius morphism obtained by taking qth powers, which makes sense since

C is defined over $\mathbb{F}_q$, so $X_q \cong X$(IV, 2.4.1). Let $\Gamma \subseteq C \times C$ be the graph of f, and let $\Delta \subseteq C \times C$ be the diagonal.

Show that $\Gamma^2 = q(2-2g)$, and $\Gamma.\Delta = N$. Then apply (Ex. 1.9) to $D = r\Gamma + s\Delta$ for all r and s to obtain the result.

See (App. C, Ex. 5.7) for another interpretation of this result.

Proof. □

Hartshorne V.1.11 In this problem, we assume that X is a surface for which Num X is finitely generated (i.e., any surface, if you accept the Néron-Severi theorem (Ex. 1.7)).

(a) If H is an ample divisor on X, and $d \in \mathbb{Z}$, show that the set of effective divisors D with $D.H = d$, modulo numerical equivalence, is a finite set.
[Hint: Use the adjunction formula, the fact that p_a of an irreducible curve is $\geqslant 0$, and the fact that the intersection pairing is negative definite on $H^\perp$ in Num X.]

(b) Now let C be a curve of genus $g \geqslant 2$, and use (a) to show that the group of automorphisms of C is finite, as follows. Given an automorphism σ of C, let $\Gamma \subseteq X = C \times C$ be its graph. First show that if $\Gamma \equiv \Delta$, then $\Gamma = \Delta$, using the fact that $\Delta^2 < 0$, since $g \geqslant 2$ (Ex. 1.6). Then use (a). Cf. (IV, Ex. 2.5).

Proof. (a) [**Incomplete**]
(b) We prove the first claim, assume $\Gamma \equiv \Delta$. It follows that $\Gamma.\Delta < 0$. Now Γ and Δ are both effective divisors on $C \times C$, and the intersection of two effective divisors is ≥ 0 whenever the two divisors are distinct (it can be shown that curves with negative self-intersection are not linearly equivalent to any other curve). So $\Gamma = \Delta$.

To show that the group of automorphisms is finite, it suffices to show that it is f.g. and then that every element is of finite order. Let $\sigma \in \mathrm{Aut}(C)$. Look at the set $S := \{\Gamma_{\sigma^k} : k \in \mathbb{N}\}$. It can be checked that $\Gamma_{\sigma^k}.\Delta$ does not depend on k. Then by (a) and the fact that the diagonal is an ample divisor, S is a finite set and thereby $\sigma^k = \mathrm{id}_C$ for some k.

We have now shown that every element of $\mathrm{Aut}(C)$ has finite order. Now it is necessary to show that there are at most finitely many d's in which $\Gamma_\sigma.\Delta = d$ can occur. This can be done via the fact that Num X is finitely generated. [**Incomplete**] □

Hartshorne V.1.12 If D is an ample divisor on the surface X, and $D' \equiv D$, then D' is also ample. Give an example to show, however, that if D is very ample, D' need not be very ample.

Proof. See the following post describing the example here. So let me just explain why there exists curves with divisors which are of the same degree but one is very ample while the other is only ample.

For the first statement, we simply use the fact that ampleness can be described via the intersection pairing due to the Nakai-Moishezon criterion. □

2. Ruled Surfaces

Hartshorne V.2.1 If X is a birationally ruled surface, show that the curve C, such that X is birationally equivalent to $C \times \mathbb{P}^1$, is unique (up to isomorphism).

PROOF. Suppose X is birational to $C \times \mathbb{P}^1$ and $C' \times \mathbb{P}^1$. I want $C \cong C'$. We can restrict to a dense open subset of X, $C \times \mathbb{P}^1$, and $C' \times \mathbb{P}^1$ which are all isomorphic. Fixing a point of $\mathbb{P}^1$, we can restrict down to a birational morphism between C' and C. But then C' and C are isomorphic (this step needs a decent amount of work).

First off, it can be shown that C and C' have the genus (look at the irregularity of the two surfaces and use the Künneth formula). Now if $g(C) = g(C') = 0$, there is nothing to show.

Suppose $g(C) > 0$. Suppose $g(C) \geq 2$. Pick a birational map $f : C \times \mathbb{P}^1 \to C' \times \mathbb{P}^1$ and f^{-1} be its birational inverse. Then there is a regular locus on which they both make sense $U \subseteq C \times \mathbb{P}^1$ and $B \subseteq C' \times \mathbb{P}^1$ and in which can compose the two. It follows that f cannot collapse almost all the fibres i.e. there is some fibre of $C \times \mathbb{P}^1 \to C$ which is birationally to a fibre of $C' \times \mathbb{P}^1 \to C'$. So the restriction of f to some C must not be constant.

In the case that the genus $g(C) > 1$, we get first a morphism (we are on a curve so we can extend birational maps to regular morphisms) and then an isomorphism between C and C' by the Hurwitz formula i.e. the degree must be 1 and there must be no ramification for the formula to hold.

Now assume $g(C) = g(C') = 1$. The idea here is to just show that the preimage will contain a single point in general. Let $f : C \to C'$ be such the map. If $Q, Q' \in C$ map to $f(Q) = f(Q')$, then they are inside the same fibre of the section $s : C' \to \mathbb{P}^1 \times C'$. But that means they have to be the same point in the section i.e. in the $\mathbb{P}^1$ fibre of s. But that means $Q = Q'$ are the same points to begin with.

See also this post here and/or here. □

Hartshorne V.2.2 Let X be the ruled surface $\mathbb{P}(\mathscr{E})$ over a curve C. Show that $\mathscr{E}$ is decomposable if and only if there exist two sections C', C'' of X such that $C' \cap C'' = \varnothing$.

PROOF. ($\Rightarrow$). Is clear. Assume $\mathscr{E}$ is decomposable and $\mathscr{E} = \mathscr{F}_1 \oplus \mathscr{F}_2$. Then $\mathbb{P}(\mathscr{E}) = \mathbb{P}(\mathscr{F}_1) \oplus \mathbb{P}(\mathscr{F}_2)$ so there are two obvious distinguished disjoint sections.

($\Leftarrow$). Suppose there are two sections C', C'' which are disjoint. Look at the SES(s)

$$0 \to \mathscr{N}_i \to \mathscr{E} \to \mathscr{L}_i \to 0$$

corresponding to the two sections $i = 1, 2$. Recall that $\mathscr{N}_1 = \pi_*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-C'))$ and $\mathscr{N}_2 = \pi_*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-C''))$. Since the two sections are disjoint it follosw that

$$\mathscr{N}_1 \oplus \mathscr{N}_2 \hookrightarrow \mathscr{E}$$

is injective and since they are both of rank 2, this is an isomorphism. □

Hartshorne V.2.3 (a) If $\mathscr{E}$ is a locally free sheaf of rank r on a (nonsingular) curve C, then there is a sequence

$$0 = \mathscr{E}_0 \subseteq \mathscr{E}_1 \subseteq \ldots \subseteq \mathscr{E}_r = \mathscr{E}$$

of subsheaves such that $\mathscr{E}_i / \mathscr{E}_{i-1}$ is an invertible sheaf for each $i = 1, \ldots,$ r. We say that $\mathscr{E}$ is a successive extension of invertible sheaves.

[Hint: Use (II, Ex. 8.2).]

(b) Show that this is false for varieties of dimension $\geqslant 2$. In particular, the sheaf of differentials Ω on $\mathbb{P}^2$ is not an extension of invertible sheaves.

PROOF. (a) The idea is to do this inductively and the claim is easy to see from Hartshorne II.8.2.
(b) Take the sheaf of differentials Ω on $\mathbb{P}^2$ which is a rank 2 sheaf. Suppose it is an extension

$$0 \to \mathcal{O}_{\mathbb{P}^2}(d) \to \Omega \to \mathcal{O}_{\mathbb{P}^2}(e) \to 0$$

of invertible sheaves. Take cohomology and note that $\dim \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = \binom{d+2}{2}$. In particular, we get an exact sequence

$$0 \to \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \to \Gamma(\mathbb{P}^2, \Omega) \to \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e)) \to 0$$

and so $\dim_k \Gamma(\mathbb{P}^2, \Omega) = \binom{e+2}{2} - \binom{d+2}{2}$. But the sheaf of Kähler differentials on $\mathbb{P}^2$ has no global sections by the Euler sequence. So $e = d$ and $e, d < 0$. The clas of this extension is determined by

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(e), \mathcal{O}_{\mathbb{P}^2}(e)) \cong \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}) \cong H^1(X, \mathcal{O}_{\mathbb{P}^2}) = 0.$$

Then we obtain that $\Omega = \mathcal{O}_{\mathbb{P}^2}(d)^2$. Taking determinants,

$$\omega = \mathcal{O}_{\mathbb{P}^2}(d)^2$$

and so we must have $-3 = 4d$ by taking degrees. But this is impossible for any integer $d < 0$. □

Hartshorne V.2.4 Let C be a curve of genus g, and let X be the ruled surface $C \times \mathbb{P}^1$. We consider the question, for what integers $s \in \mathbb{Z}$ does there exist a section D of X with $D^2 = s$? First show that s is always an even integer, say $s = 2r$.

(a) Show that $r = 0$ and any $r \geqslant g + 1$ are always possible. Cf. (IV, Ex. 6.8).

(b) If $g = 3$, show that $r = 1$ is not possible, and just one of the two values $r = 2, 3$ is possible, depending on whether C is hyperelliptic or not.

PROOF. First, we show that $D^2 = s$ is always an even integer. For ease let's write $C := D$ instead. We know

$$2g - 2 = C.(C + K)$$

so it suffices to show that $C.K$ is even. From Hartshorne Corllary V.2.11, I know that

$$C.K = C.(-2C^2 + (2g - 2 - e)f) = -2C + 2g - 2 - C^2$$

[Incomplete]
(a) To get $r = 0$, we can take the section $\sigma : C \to X$ which sends $\sigma(p) = (p, 1)$. Then we can compute the self-intersection and get alinearly equivalent $D \sim C$ s.t $D \cap C = 0$.

For $r \geq g + 1$, we use Hartshorne IV.6.8.. **[Incomplete]**
(b) Let $g = 3$ and we check that $r = 1$ is not possible. **[Incomplete]**

□

Hartshorne V.2.5 Values of e. Let C be a curve of genus $g \geqslant 1$.

(a) Show that for each $0 \leqslant e \leqslant 2g - 2$ there is a ruled surface X over C with invariant e, corresponding to an indecomposable $\mathscr{E}$. Cf. (2.12).

(b) Let $e < 0$, let D be any divisor of degree $d = -e$, and let $\xi \in H^1(\mathscr{L}(-D))$ be a nonzero element defining an extension

$$0 \to \mathcal{O}_C \to \mathscr{E} \to \mathscr{L}(D) \to 0$$

Let $H \subseteq |D+K|$ be the sublinear system of codimension 1 defined by $\ker \xi$, where ξ is considered as a linear functional on $H^0(\mathscr{L}(D+K))$. For any effective divisor E of degree $d-1$, let $L_E \subseteq |D+K|$ be the sublinear system $|D+K-E|+E$. Show that $\mathscr{E}$ is normalized if and only if for each E as above, $L_E \nsubseteq H$. Cf. proof of (2.15).

(c) Now show that if $-g \leqslant e < 0$, there exists a ruled surface X over C with invariant e.

[Hint: For any given D in (b), show that a suitable ξ exists, using an argument similar to the proof of (II, 8.18).]

(d) For $g = 2$, show that $e \geqslant -2$ is also necessary for the existence of X. Note. It has been shown that $e \geqslant -g$ for any ruled surface (Nagata [8]).

PROOF. □

Hartshorne V.2.6 Show that every locally free sheaf of finite rank on $\mathbb{P}^1$ is isomorphic to a direct sum of invertible sheaves.

[Hint: Choose a subinvertible sheaf of maximal degree, and use induction on the rank.]

PROOF. This is just the Grothendieck-Birkhoff Theorem and many proofs can be found online. My favorite proof is the one □

Hartshorne V.2.7 On the elliptic ruled surface X of (2.11.6), show that the sections C_0 with $C_0^2 = 1$ form a one-dimensional algebraic family, parametrized by the points of the base curve C, and that no two are linearly equivalent.

PROOF. □

Hartshorne V.2.8 A locally free sheaf $\mathscr{E}$ on a curve C is said to be stable if for every quotient locally free sheaf $\mathscr{E} \to \mathscr{F} \to 0, \mathscr{F} \neq \mathscr{E}, \mathscr{F} \neq 0$, we have

$$(\deg \mathscr{F})/\operatorname{rank} \mathscr{F} > (\deg \mathscr{E})/\operatorname{rank} \mathscr{E}.$$

Replacing $>$ by $\geqslant$ defines semistable.

(a) A decomposable $\mathscr{E}$ is never stable.

(b) If $\mathscr{E}$ has rank 2 and is normalized, then $\mathscr{E}$ is stable (respectively, semistable) if and only if $\deg \mathscr{E} > 0$ (respectively, $\geqslant 0$).

(c) Show that the indecomposable locally free sheaves $\mathscr{E}$ of rank 2 that are not semistable are classified, up to isomorphism, by giving (1) an integer $0 < e \leqslant 2g-2$, (2) an element $\mathscr{L} \in \operatorname{Pic} C$ of degree $-e$, and (3) a nonzero $\xi \in H^1(\mathscr{L}^\vee)$, determined up to a nonzero scalar multiple.

PROOF. (a) Suppose $\mathscr{E} = \mathscr{F} \oplus \mathscr{G}$ were a decomposition. One can easily show that if $\mu_{\mathscr{E}} := \frac{\deg \mathscr{E}}{\operatorname{rank} \mathscr{E}}$, then

$$\min\{\mu_{\mathscr{F}}, \mu_{\mathscr{G}}\} \leq \mu_{\mathscr{E}} \leq \max\{\mu_{\mathscr{F}}, \mu_{\mathscr{G}}\}.$$

Therefore a decomposable vector bundle has no hope to be stable (flip Hartshorne's definition for the usual definition of stable).
(b) Assume $\mathscr{E}$ has rank 2 and is normalized i.e. $H^0(\mathscr{E}) \neq 0$ and $H^0(\mathscr{E} \otimes \mathscr{L}) = 0$ for all invertible sheaves $\mathscr{L}$ with $\deg \mathscr{L} < 0$.

($\Rightarrow$). If $\mathscr{E}$ is stable, then it is clearly of positive degree. This follows more easily from knowing the inclusion definition of stability. Assume $\mathscr{E}$ is stable. Then the slope of any line subbundle is $<$ the slope of $\mathscr{E}$. If $\deg \mathscr{E} \leq 0$ instead, then we can find a subbundle that contradicts this.

($\Leftarrow$). Suppose that $\deg \mathscr{E} > 0$. Look at a SES

$$0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{F} \to 0.$$

We want to show $\deg \mathscr{E} < 2 \deg \mathscr{F}$. Since $0 < \deg \mathscr{E} = \deg \mathscr{G} + \deg \mathscr{F}$, it suffices to show that $\deg \mathscr{F} \geq \deg \mathscr{G}$. Also, we know at least one of $\deg \mathscr{G}$ or $\deg \mathscr{F}$ is > 0. Suppose not and that $\deg \mathscr{F} < \deg \mathscr{G}$ and so $\deg \mathscr{G} > 0$. Then $\deg \mathscr{G}^\vee < 0$. Tensor by $\mathscr{G}^\vee$ to get

$$0 \to \mathcal{O}_C \to \mathscr{E} \otimes \mathscr{G}^\vee \to \mathscr{F} \otimes \mathscr{G}^\vee \to 0.$$

Taking cohomology and using the fact that $\mathscr{E}$ is normalized, we get a contradiction. The analogous statement for semistability follows immediately by swapping $>$ with $\geq$ appropriately.

(c) Fix an indecomposable locally free sheaf $\mathscr{E}$ of rank 2 which is not semistable. After normalizing $\mathscr{E}$ by twisting by a line bundle, we know semistability is equivalent to $\deg \mathscr{E} \geq 0$. It suffice to show that the conditions in (c) determine this completely.

Since $\xi \in H^1(\mathscr{L}^\vee) = \operatorname{Ext}^1(\mathcal{O}_X, \mathscr{L}^\vee)$ is nonzero, there is a unique extension

$$0 \to \mathcal{O}_X \to \mathscr{E} \to \mathscr{L}^\vee \to 0.$$

Since $\mathscr{L}$ has degree $-e$, we deduce that $\deg \mathscr{E} = e$. Since $0 < e \leq 2g - 2$,we take this $\mathscr{E}$ to be the desired normalized $\mathscr{E}$ (since we know there is the upper bound by $2g - 2$ from Hartshorne Theorem V.2.12.). □

Hartshorne V.2.9 Let Y be a nonsingular curve on a quadric cone X_0 in $\mathbb{P}^3$. Show that either Y is a complete intersection of X_0 with a surface of degree $a \geqslant 1$, in which case $\deg Y = 2a, g(Y) = (a-1)^2$, or, $\deg Y$ is odd, say $2a+1$, and $g(Y) = a^2 - a$. Cf. (IV, 6.4.1).

[Hint: Use (2.11.4).]

Proof. □

Hartshorne V.2.10 For any $n > e \geqslant 0$, let X be the rational scroll of degree $d = 2n - e$ in $\mathbb{P}^{d+1}$ given by (2.19). If $n \geqslant 2e - 2$, show that X contains a nonsingular curve Y of genus $g = d + 2$ which is a canonical curve in this embedding. Conclude that for every $g \geqslant 4$, there exists a nonhyperelliptic curve of genus g which has a g^1_3. Cf. (IV, §5).

Proof. □

Hartshorne V.2.11 Let X be a ruled surface over the curve C, defined by a normalized bundle $\mathscr{E}$, and let e be the divisor on C for which $\mathscr{L}(e) \cong \bigwedge^2 \mathscr{E}$(2.8.1). Let b be any divisor on C.

(a) If $|b|$ and $\mid b$ section $D \sim C_0 + \mathrm{b}f$, and $|D|$ has no base points.

(b) If $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are very ample on C, and for every point $P \in C$, we have $\mathfrak{b} - P$ and $\mathfrak{b} + \mathfrak{e} - P$ nonspecial, then $C_0 + \mathfrak{b}f$ is very ample.

Proof. □

Hartshorne V.2.12 Let X be a ruled surface with invariant e over an elliptic curve C, and let b be a divisor on C.

(a) If $\deg \mathfrak{b} \geqslant e+2$, then there is a section $D \sim C_0 + \mathfrak{b}f$ such that $|D|$ has no base points.

(b) The linear system $|C_0 + \mathfrak{b}f|$ is very ample if and only if $\deg \mathfrak{b} \geqslant e+3$.

Note. The case $e=-1$ will require special attention.

PROOF. □

Hartshorne V.2.13 For every $e \geqslant -1$ and $n \geqslant e+3$, there is an elliptic scroll of degree $d = 2n-e$ in $\mathbb{P}^{d-1}$. In particular, there is an elliptic scroll of degree 5 in $\mathbb{P}^4$.

PROOF. □

Hartshorne V.2.14 Let X be a ruled surface over a curve C of genus g, with invariant $e<0$, and assume that char $k = p > 0$ and $g \geqslant 2$.

(a) If $Y \equiv aC_0 + bf$ is an irreducible curve $\neq C_0, f$, then either $a=1, b \geqslant 0$, or $2 \leqslant a \leqslant p-1, b \geqslant \frac{1}{2}ae$, or $a \geqslant p, b \geqslant \frac{1}{2}ae + 1 - g$.

(b) If $a>0$ and $b > a\left(\frac{1}{2}e + (1/p)(g-1)\right)$, then any divisor $D \equiv aC_0 + bf$ is ample. On the other hand, if D is ample, then $a>0$ and $b > \frac{1}{2}ae$.

PROOF. □

Hartshorne V.2.15 Funny behavior in characteristic p. Let C be the plane curve $x^3y + y^3z + z^3x = 0$ over a field k of characteristic 3 (IV, Ex. 2.4).

(a) Show that the action of the k-linear Frobenius morphism f on $H^1(C, \mathcal{O}_C)$ is identically 0 (Cf. (IV, 4.21)).

(b) Fix a point $P \in C$, and show that there is a nonzero $\xi \in H^1(\mathscr{L}(-P))$ such that $f^*\xi = 0$ in $H^1(\mathscr{L}(-3P))$.

(c) Now let $\mathscr{E}$ be defined by ξ as an extension

$$0 \to \mathcal{O}_C \to \mathscr{E} \to \mathscr{L}(P) \to 0$$

and let X be the corresponding ruled surface over C. Show that X contains a nonsingular curve $Y \equiv 3C_0 - 3f$, such that $\pi : Y \to C$ is purely inseparable. Show that the divisor $D = 2C_0$ satisfies the hypotheses of (2.21 b), but is not ample.

PROOF. □

Hartshorne V.2.16 Let C be a nonsingular affine curve. Show that two locally free sheaves $\mathscr{E}, \mathscr{E}'$ of the same rank are isomorphic if and only if their classes in the Grothendieck group $K(X)$ (II, Ex. 6.10) and (II, Ex. 6.11) are the same. This is false for a projective curve.

PROOF. The direction ($\Rightarrow$) is obvious so suppose $[\mathscr{E}]$ and $[\mathscr{E}']$ are the same class in the Grothendieck group and that $\mathscr{E}, \mathscr{E}'$ are of the same rank. Since C is nonsingular and affine, we know that $\mathscr{E} = \widetilde{M}$ and $\mathscr{E}' = \widetilde{M'}$ for M, M' f.g. modules over A where $C := \operatorname{Spec} A$. In particular, M, M' are projective modules of rank 1 satisfying certain conditions.

I think it should be readily clear that since the two can be shown to be isomorphic in this case. Indeed, they represent the same element in the Grothendieck group so they fit into the same SESs. Then we can localize said SESs to show that they must be isomorphic on the same open sets (which we must choose). The rough idea of this argument is due to Jacob Wood (but I did not flush out all the details). □

Hartshorne V.2.17 (a) Let $\varphi : \mathbb{P}^1_k \to \mathbb{P}^3_k$ be the 3-uple embedding (I, Ex. 2.12). Let $\mathscr{I}$ be the sheaf of ideals of the twisted cubic curve C which is the image of φ. Then $\mathscr{I}/\mathscr{I}^2$ is a locally free sheaf of rank 2 on C, so $\varphi^*(\mathscr{I}/\mathscr{I}^2)$ is a locally free sheaf of rank 2 on $\mathbb{P}^1$. By (2.14), therefore, $\varphi^*(\mathscr{I}/\mathscr{I}^2) \cong \mathcal{O}(l) \oplus \mathcal{O}(m)$ for some $l, m \in \mathbb{Z}$. Determine l and m.

(b) Repeat part (a) for the embedding $\varphi : \mathbb{P}^1 \to \mathbb{P}^3$ given by $x_0 = t^4, x_1 = t^3u$, $x_2 = tu^3, x_3 = u^4$, whose image is a nonsingular rational quartic curve. [Answer: If char $k \neq 2$, then $l = m = -7$; if char $k = 2$, then $l, m = -6, -8$.]

PROOF. (a) From the SES

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \mathcal{O}_C \otimes \Omega_{\mathbb{P}^3} \to \Omega_C \to 0.$$

Since Ω_C and $\mathcal{O}_C \otimes \Omega_{\mathbb{P}^3}$ are locally free of rank 1 and 3 respectively, we deduce that $\mathscr{I}/\mathscr{I}^2$ is locally free of rank 2. Pulling back and using the fact that φ is a closed embedding, $\varphi^*(\mathscr{I}/\mathscr{I}^2)$ is locally free of rank 2.

Apply (2.14) so that $\varphi^*(\mathscr{I}/\mathscr{I}^2) = \mathcal{O}(l) \oplus \mathcal{O}(m)$. Now pulling back this sequence, we get

$$0 \to \varphi^*\mathscr{I}_C/\mathscr{I}_C^2 \to \varphi^*O_C \otimes \varphi^*\Omega_{\mathbb{P}^3} \to \varphi^*\Omega_C \to 0.$$

Now $\varphi^*\Omega_C \cong \omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and since $\mathscr{I}_C$ is cut out by quadrics, we get

$$0 \to \varphi^*\mathscr{I}_C/\mathscr{I}_C^2 \to \varphi^*\Omega_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^1}(-2) \to 0.$$

Now, taking the Euler sequence and pulling back along φ^* we get

$$0 \to \varphi^*\Omega_{\mathbb{P}^3} \to O_{\mathbb{P}^1}(-3)^4 \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

Taking a LES in cohomology, we get

$$0 \to \Gamma(\mathscr{F}) \to 0 \to k \to H^1(\mathscr{F}) \to 0 \to 0 \to H^1(\varphi^*\Omega_{\mathbb{P}^3}) \to 0 \to 0 \to \dots.$$

Now if $\varphi^*\Omega_{\mathbb{P}^3} = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$. Since $H^1(\mathscr{F})$ is 1-dimensional, we know that one of a, b, c is $= -2$ and $\Gamma(\mathscr{F}) = 0$ implies $a, b, c < 0$. WLOG, we can assume $c = -2$ and $a, b \neq -2$. Twist by $\mathcal{O}(2)$ and take cohomology

$$0 \to \Gamma(\mathcal{O}_C(a+2) \oplus \mathcal{O}_C(b+2)) \oplus k \to \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)^4) \to \Gamma(\mathcal{O}_{\mathbb{P}^1}(2)) \to H^1(\mathcal{O}(a+2) \oplus \mathcal{O}(b+2)) \to 0 \to 0.$$

Now if one countes dimensions of this sequence, we know $0 \to ? \to 8 \to 3 \to ?? \to 0$ and deduce that

$$0 = ? - 8 + 3 - ?? \;\Rightarrow ? - ?? = 5.$$

Now we deal with the different values of a and b. Now we know $a, b < 0$ and $a, b \neq -4$ (the fact that $a, b \neq -4$ is easy to check). Then $? - ?? = 5$ implies $? = 5$ and so

$H^0(\mathcal{O}(a+2)\oplus\mathcal{O}(b+2)) = k^4$. Hence, $a, b = -1$. Returning to the sequence at the start, we know $\varphi^*\Omega_{\mathbb{P}^3} = \mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}(-2)$ now so we get

$$0 \to \varphi^*\mathscr{I}/\mathscr{I}^2 \to \mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}(-2) \to \mathcal{O}(-2) \to 0.$$

Now computing cohomology, we know $\mathrm{Ext}^1(\mathcal{O}(-2), \varphi^*\mathscr{I}/\mathscr{I}^2) = 0$ so we know that there is a splitting of the middle term. Furthermore, computing degrees, we know the degree of $\varphi^*\mathscr{I}/\mathscr{I}^2 = -2$. Hence, the only options are $a, b = -1$ and so

$$\varphi^*\mathscr{I}/\mathscr{O}^2 = \mathcal{O}(-1)\oplus\mathcal{O}(-1).$$

(b) □

3. Monoidal Transformations

Hartshorne V.3.1 Let X be a nonsingular projective variety of any dimension, let Y be a nonsingular subvariety, and let $\pi : \tilde{X} \to X$ be obtained by blowing up Y. Show that $p_a(\tilde{X}) = p_a(X)$.

Proof. The idea tof the proof is to generalize Hartshorne's Proposition V.3.1, V.3.2, V.3.3, and V.3.4. Of course, the arguments came down to understanding the exceptional divisors. If one blows up a nonsingular subvariety of codimension r, then the exceptional divisor is a projective space of dimension $r-1$. The main thing to bare in mind here is the fact that the exceptional divisor is the projectivization of the normal bundle. The normal bundle is locally free iff we know that Y is nonsingular which we assumed here. Then the cohomology of projective bundles plus the proof of Hartshorne Proposition V.3.4 give the result. □

Hartshorne V.3.2 Let C and D be curves on a surface X, meeting at a point P. Let $\pi : \tilde{X} \to X$ be the monoidal transformation with center P. Show that $\widetilde{C}.\widetilde{D} = C.D - \mu_P(C)\cdot\mu_P(D)$. Conclude that $C.D = \sum \mu_P(C)\cdot\mu_P(D)$, where the sum is taken over all intersection points of C and D, including infinitely near intersection points.

Proof. The strict transforms of C and D we given by

$$\widetilde{C} = \pi^*C - \mu_P(C)E.$$

It follows that

$$\begin{aligned}\widetilde{C}.\widetilde{D} &= \pi^*C.\pi^*D - \mu_P(C)E.\pi^*D - \mu_P(D)E.\pi^*C + \mu_P(C)\mu_P(D)E^2\\ &= \mu^*C.\mu^*D - \mu_P(C)\mu_P(D)\\ &= C.D - \mu_P(C)\mu_P(D).\end{aligned}$$

This formula implies that

$$C.D = \mu_P(C).\mu_P(D) + \widetilde{C}.\widetilde{D}.$$

We need to compute $\widetilde{C}.\widetilde{D}$. This entails blowing up all the intersection points that remain, and since these are the strict transforms, we also need to blow-up the infinitely near intersection points. So,

$$C.D = \sum_{P\in C\cap D,\text{ or } P \text{ an infinitely near points}} \mu_P(C).\mu_P(D).$$

□

Hartshorne V.3.3 Let $\pi : \tilde{X} \to X$ be a monoidal transformation, and let D be a very ample divisor on X. Show that $2\pi^*D - E$ is ample on $\tilde{X}$.

[Hint: Use a suitable generalization of (I, Ex. 7.5) to curves in $\mathbb{P}^n$.]

Proof. □

Hartshorne V.3.4 Multiplicity of a Local Ring. (See Nagata [7, Ch III, §23] or Zariski-Samuel [1, vol 2, Ch VIII, §10].) Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. For any $l > 0$, let $\psi(l) = \text{length}\left(A/\mathfrak{m}^l\right)$. We call ψ the Hilbert-Samuel function of A.

(a) Show that there is a polynomial $P_A(z) \in \mathbb{Q}[z]$ such that $P_A(l) = \psi(l)$ for all $l \gg 0$. This is the Hilbert-Samuel polynomial of A.

[Hint: Consider the graded ring $\text{gr}_{\mathfrak{m}}A = \bigoplus_{d \geqslant 0} \mathfrak{m}^d/\mathfrak{m}^{d+1}$, and apply (I, 7.5).]

(b) Show that $\deg P_A = \dim A$.

(c) Let $n = \dim A$. Then we define the multiplicity of A, denoted $\mu(A)$, to be $(n!)$. (leading coefficient of P_A). If P is a point on a noetherian scheme X, we define the multiplicity of P on X, $\mu_P(X)$, to be $\mu\left(\mathcal{O}_{P,X}\right)$.

(d) Show that for a point P on a curve C on a surface X, this definition of $\mu_P(C)$ coincides with the one in the text just before (3.5.2).

(e) If Y is a variety of degree d in $\mathbb{P}^n$, show that the vertex of the cone over Y is a point of multiplicity d.

Proof. □

Hartshorne V.3.5 Let $a_1, \ldots, a_r, r \geqslant 5$, be distinct elements of k, and let C be the curve in $\mathbb{P}^2$ given by the (affine) equation $y^2 = \prod_{i=1}^{r}(x - a_i)$. Show that the point P at infinity on the y-axis is a singular point. Compute δ_P and $g(\tilde{Y})$, where $\tilde{Y}$ is the normalization of Y. Show in this way that one obtains hyperelliptic curves of every genus $g \geqslant 2$.

Proof. □

Hartshorne V.3.6 Show that analytically isomorphic curve singularities (I, 5.6.1) are equivalent in the sense of (3.9.4), but not conversely.

Proof. □

Hartshorne V.3.7 For each of the following singularities at $(0, 0)$ in the plane, give an embedded resolution, compute δ_P, and decide which ones are equivalent.
(a) $x^3 + y^5 = 0$.
(b) $x^3 + x^4 + y^5 = 0$.
(c) $x^3 + y^4 + y^5 = 0$.
(d) $x^3 + y^5 + y^6 = 0$.
(e) $x^3 + xy^3 + y^5 = 0$.

Proof. □

Hartshorne V.3.8 Show that the following two singularities have the same multiplicity, and the same configuration of infinitely near singular points with the same multiplicities, hence the same δ_P, but are not equivalent.
(a) $x^4 - xy^4 = 0$.
(b) $x^4 - x^2y^3 - x^2y^5 + y^8 = 0$.

PROOF. □

4. The Cubic Surface in $\mathbb{P}^3$

Hartshorne V.4.1 The linear system of conics in $\mathbb{P}^2$ with two assigned base points P_1 and P_2 (4.1) determines a morphism ψ of X' (which is $\mathbb{P}^2$ with P_1 and P_2 blown up) to a nonsingular quadric surface Y in $\mathbb{P}^3$, and furthermore X' via ψ is isomorphic to Y with one point blown up.

PROOF. □

Hartshorne V.4.2 Let φ be the quadratic transformation of (4.2.3), centered at P_1, P_2, P_3. If C is an irreducible curve of degree d in $\mathbb{P}^2$, with points of multiplicity r_1, r_2, r_3 at P_1, P_2, P_3, then the strict transform C' of C by φ has degree $d' = 2d - r_1 - r_2 - r_3$, and has points of multiplicity $d - r_2 - r_3$ at $Q_1, d - r_1 - r_3$ at Q_2 and $d - r_1 - r_2$ at Q_3. The curve C may have arbitrary singularities.

[Hint: Use (Ex. 3.2).]

PROOF. □

Hartshorne V.4.3 Let C be an irreducible curve in $\mathbb{P}^2$. Then there exists a finite sequence of quadratic transformations, centered at suitable triples of points, so that the strict transform C' of C has only *ordinary* singularities, i.e., multiple points with all distinct tangent directions (I, Ex. 5.14). Use (3.8).

PROOF. □

Hartshorne V.4.4 (a) Use (4.5) to prove the following lemma on cubics: If C is an irreducible plane cubic curve, if L is a line meeting C in points P, Q, R, and L' is a line meeting C in points P', Q', R', let P'' be the third intersection of the line PP' with C, and define Q'', R'' similarly. Then P'', Q'', R'' are collinear.

(b) Let P_0 be an inflection point of C, and define the group operation on the set of regular points of C by the geometric recipe "let the line PQ meet C at R, and let P_0R meet C at T, then $P + Q = T''$ as in (II, 6.10.2) and (II, 6.11.4). Use (a) to show that this operation is associative.

PROOF. □

Hartshorne V.4.5 Prove Pascal's theorem: if A, B, C, A', B', C' are any six points on a conic, then the points $P = AB' \cdot A'B, Q = AC' \cdot A'C$, and $R = BC' \cdot B'C$ are collinear (Fig. 1).

PROOF. We know if there are 8 points, no 4 colinear and no 7 on a conic, then there is a uniquely determined 9th point in which any cubic through the 8th point crosses through the 9th (Hartshorne Corollary V.4.5).

Using the picture for ease, let the 8 points be $A, B, C, A', B', C', Q, R$.

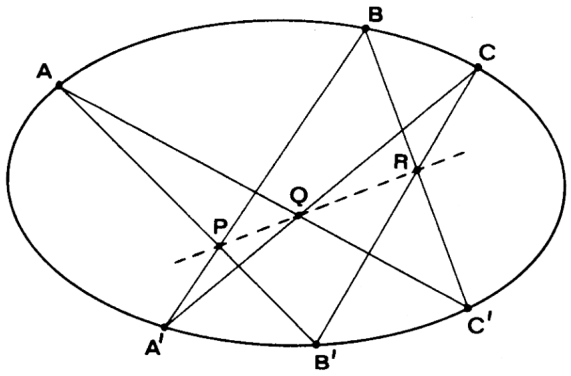


FIGURE 1. Figure 22 in Hartshorne.

The first cubic we consider is the one determined by the conic and the line through Q and R. So there is unique 9th point it crosses through which we call P'. This P' cannot be on the conic so it is on the line.

The second cubic we consider is the one determined by the three lines AC', $A'B$, and $B'C$. These three lines contain the points $A, B, C, A', B', C', Q, R$ and so there is a unique 9th point P''. So $P' = P''$ by the corollary. But P' is the the third point from the intersection of pairs of lines. So we get the claim. □

Hartshorne V.4.6 Generalize (4.5) as follows: given 13 points $P_1, \ldots, P_{13}$ in the plane, there are three additional determined points P_{14}, P_{15}, P_{16}, such that all quartic curves through $P_1, \ldots, P_{13}$ also pass through P_{14}, P_{15}, P_{16}. What hypotheses are necessary on P, P_{13} also pass through P.

PROOF. □

Hartshorne V.4.7 If D is any divisor of degree d on the cubic surface (4.7.3), show that

$$p_a(D) \leqslant \begin{cases} \frac{1}{6}(d-1)(d-2) & \text{if } d \equiv 1, 2 (\text{mod} 3) \\ \frac{1}{6}(d-1)(d-2) + \frac{2}{3} & \text{if } d \equiv 0 (\text{mod} 3). \end{cases}$$

Show furthermore that for every $d > 0$, this maximum is achieved by some irreducible nonsingular curve.

PROOF. □

Hartshorne V.4.8 Show that a divisor class D on the cubic surface contains an irreducible curve $\Leftrightarrow$ it contains an irreducible nonsingular curve $\Leftrightarrow$ it is either (a) one of the 27 lines, or (b) a conic (meaning a curve of degree 2) with $D^2 = 0$, or (c) $D \cdot L \geqslant 0$ for every line L, and $D^2 > 0$.

[Hint: Generalize (4.11) to the surfaces obtained by blowing up $2, 3, 4$, or 5 points of $\mathbb{P}^2$, and combine with our earlier results about curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and the rational ruled surface X_1, (2.18).]

PROOF. □

Hartshorne V.4.9 If C is an irreducible non-singular curve of degree d on the cubic surface, and if the genus $g > 0$, then

$$g \geqslant \begin{cases} \frac{1}{2}(d-6) & \text{if } d \text{ is even, } d \geqslant 8 \\ \frac{1}{2}(d-5) & \text{if } d \text{ is odd, } d \geqslant 13 \end{cases}$$

and this minimum value of $g > 0$ is achieved for each d in the range given.

Proof. □

Hartshorne V.4.10 A curious consequence of the implication (iv) ⇒ (iii) of (4.11) is the following numerical fact: Given integers $a, b_1, \ldots, b_6$ such that $b_i > 0$ for each $i, a - b_i - b_j > 0$ for each i, j and $2a - \sum_{i \neq j} b_i > 0$ for each j, we must necessarily have $a^2 - \sum b_i^2 > 0$. Prove this directly (for $a, b_1, \ldots, b_6 \in \mathbb{R}$) using methods of freshman calculus.

Proof. Certainly an analysis problem is not appearing in this textbook right? Of course, it does because this exercise is not a easy as it seems! □

Hartshorne V.4.11 The Weyl Groups. Given any diagram consisting of points and line segments joining some of them, we define an abstract group, given by generators and relations, as follows: each point represents a generator x_i. The relations are $x_i^2 = 1$ for each i; $(x_i x_j)^2 = 1$ if i and j are not joined by a line segment, and $(x_i x_j)^3 = 1$ if i and j are joined by a line segment.

(a) The Weyl group $\mathbb{A}_n$ is defined using the diagram

[Incomplete]

(b) The Weyl group $\mathbb{E}_6$ is defined using the diagram

[Incomplete]

Call the generators $x_1, \ldots, x_5$ and y. Show that one obtains a surjective homomorphism $\mathbb{E}_6 \to G$, the group of automorphisms of the configuration of 27 lines (4.10.1), by sending $x_1, \ldots, x_5$ to the permutations $(12), (23), \ldots, (56)$ of the E_i, respectively, and y to the element associated with the quadratic transformation based at P_1, P_2, P_3.

*(c) Estimate the number of elements in $\mathbb{E}_6$, and thus conclude that $\mathbb{E}_6 \cong G$. Note: See Manin [3, §25, 26] for more about Weyl groups, root systems, and exceptional curves.

Proof. □

Hartshorne V.4.12 Use (4.11) to show that if D is any ample divisor on the cubic surface X, then $H^1(X, \mathcal{O}_X(-D)) = 0$. This is Kodaira's vanishing theorem for the cubic surface (III, 7.15).

Proof. I'm going to be cheeky here and say that this can be done via Kodaira's Vanishing Theorem and call it a day. □

Hartshorne V.4.13 Let X be the Del Pezzo surface of degree 4 in $\mathbb{P}^4$ obtained by blowing up 5. points of $\mathbb{P}^2$(4.7).

(a) Show that X contains 16 lines.

(b) Show that X is a complete intersection of two quadric hypersurfaces in $\mathbb{P}^4$ (the converse follows from (4.7.1)).

Proof. □

Hartshorne V.4.14 Using the method of (4.13.1), verify that there are nonsingular curves in $\mathbb{P}^3$ with $d = 8, g = 6, 7; d = 9, g = 7, 8, 9; d = 10, g = 8, 9, 10, 11$. Combining with (IV §6), this completes the determination of all posible g for curves of degree $d \leqslant 10$ in $\mathbb{P}^3$.

Proof. □

Hartshorne V.4.15 Let $P_1, \ldots, P_r$ be a finite set of (ordinary) points of $\mathbb{P}^2$, no 3 collinear. We define an admissible transformation to be a quadratic transformation (4.2.3) centered at some three of the P_i (call them P_1, P_2, P_3). This gives a new $\mathbb{P}^2$, and a new set of r points, namely Q_1, Q_2, Q_3, and the images of $P_4, \ldots, P_r$. We say that $P_1, \ldots, P_r$ are in general position if no three are collinear, and furthermore after any finite sequence of admissible transformations, the new set of r points also has no three collinear.

(a) A set of 6 points is in general position if and only if no three are collinear and not all six lie on a conic.

(b) If $P_1, \ldots, P_r$ are in general position, then the r points obtained by any finite sequence of admissible transformations are also in general position.

(c) Assume the ground field k is uncountable. Then given $P_1, \ldots, P_r$ in general position, there is a dense subset $V \subseteq \mathbb{P}^2$ such that for any $P_{r+1} \in V, P_1, \ldots, P_{r+1}$ will be in general position.

[Hint: Prove a lemma that when k is uncountable, a variety cannot be equal to the union of a countable family of proper closed subsets.]

(d) Now take $P_1, \ldots, P_r \in \mathbb{P}^2$ in general position, and let X be the surface obtained by blowing up $P_1, \ldots, P_r$. If $r = 7$, show that X has exactly 56 irreducible nonsingular curves C with $g = 0, C^2 = -1$, and that these are the only irreducible curves with negative self-intersection. Ditto for $r = 8$, the number being 240 .

(e) For $r = 9$, show that the surface X defined in (d) has infinitely many irreducible nonsingular curves C with $g = 0$ and $C^2 = -1$.

[Hint: Let L be the line joining P_1 and P_2. Show that there exist finite sequences of admissible transformations such that the strict transform of L becomes a plane curve of arbitrarily high degree.] This example is apparently due to Kodaira-see Nagata [5, II, p. 283].

Proof. □

Hartshorne V.4.16 For the Fermat cubic surface $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$, find the equations of the 27 lines explicitly, and verify their incidence relations. What is the group of automorphisms of this surface?

Proof. See this Wikipedia page on special cubic surfaces. □

5. Birational Transformations

Hartshorne V.5.1 Let f be a rational function on the surface X. Show that it is possible to "resolve the singularities of f " in the following sense: there is a birational morphism $g : X' \to X$ so that f induces a morphism of X' to $\mathbb{P}^1$.

[Hints: Write the divisor of f as $(f) = \sum n_i C_i$. Then apply embedded resolution (3.9) to the curve $Y = \bigcup C_i$. Then blow up further as necessary whenever a curve of zeros meets a curve of poles until the zeros and poles of f are disjoint.]

Proof. Let $f : X \to \mathbb{P}^1$ be a rational function X. Let $(f) = \sum n_i C_i$ and consider $Y := \bigcup C_i$. Take an embedded resolution so we get a $\widetilde{Y}$ on a surface X'' in which the only singularities are normal crossings. Then blow-up those crossings until the zeros and poles are disjoint – so we get a birational morphism $g : X' \to X$. Then $f : X' \to \mathbb{P}^1$ is obtained from $f \circ g : X' \to \mathbb{P}^1$. □

Hartshorne V.5.2 Let $Y \cong \mathbb{P}^1$ be a curve in a surface X, with $Y^2 < 0$. Show that Y is contractible (5.7.2) to a point on a projective variety X_0 (in general singular).

Proof. The only part of the proof of Castelnuovo's Theorem where we explicitly needed $Y^2 = -1$ was to show that the contracted variety was nonsingular. Outside of that, we only ever needed $Y^2 < 0$. □

Hartshorne V.5.3 If $\pi : \tilde{X} \to X$ is a monoidal transformation with center P, show that $H^1\left(\tilde{X}, \Omega_{\tilde{X}}\right) \cong H^1\left(X, \Omega_X\right) \oplus k$. This gives another proof of (5.8).

[Hints: Use the projection formula (III, Ex. 8.3) and (III, Ex. 8.1) to show that $H^i\left(X, \Omega_X\right) \cong H^i\left(\tilde{X}, \pi^* \Omega_X\right)$ for each i. Next use the exact sequence

$$0 \to \pi^* \Omega_X \to \Omega_{\tilde{X}} \to \Omega_{\tilde{X}/X} \to 0$$

and a local calculation with coordinates to show that there is a natural isomorphism $\Omega_{\tilde{X}/X} \cong \Omega_E$, where E is the exceptional curve. Now use the cohomology sequence of the above sequence (you will need every term) and Serre duality to get the result.]

Proof. This follows from a sequence of lemmas.

Lemma 5.1. We show $H^i(X, \Omega_X) \cong H^i(\widetilde{X}, \pi^* \Omega_X)$ for all i

Lemma 5.2. $\Omega_{\widetilde{X}/X} = \Omega_E$ by a local calculation with coordinates and the SES above.

Lemma 5.3. $H^1(\widetilde{X}, \Omega_{\widetilde{X}}) \cong H^1(X, \Omega_X) \oplus k$.

Proof. Take the LES in cohomology

$$\begin{aligned} 0 \to H^0(\pi^* \Omega_X) \to H^0(\Omega_{\widetilde{X}}) \to H^0(\Omega_{\widetilde{X}/X}) \\ \to H^1(\pi^* \Omega_X)) \to H^1(\Omega_{\widetilde{X}}) \to H^1(\Omega_{\widetilde{X}/X}) \\ \to H^2(\pi^* \Omega_X) \to H^2(\Omega_{\widetilde{X}}) \to H^2(\Omega_{\widetilde{X}/X}) \end{aligned}$$

□

Then using the hints above, the sequence becomes

$$\begin{aligned} 0 \to H^0(X, \Omega_X) \to H^0(\Omega_{\widetilde{X}}) \to H^0(\Omega_E) \\ \to H^1(X, \Omega_X) \to H^1(\Omega_{\widetilde{X}}) \to H^1(E) \\ \to H^2(X, \Omega_X) \to H^2(\Omega_{\widetilde{X}}) \to 0 \end{aligned}$$

Now $E \cong \mathbb{P}^1$ and so $\Omega_E = \mathcal{O}_{\mathbb{P}^1}(-2)$. We know from the cohomology of projective space that $H^1(E, \Omega_E) = k$ in this case. So what we get is $H^0(X, \Omega_X) \cong H^0(\widetilde{X}\Omega_{\widetilde{X}})$ and that

$$0 \to H^1(X, \Omega_X) \to H^1(\widetilde{X}, \Omega_{\widetilde{X}}) \to k \to H^2(X, \Omega_X) \to H^2(\widetilde{X}, \Omega_{\widetilde{X}}) \to 0.$$

It suffices to show that $H^2(X, \Omega_X)$ and use the fact that we work with vector spaces to get the sequence to split. Applying Serre Duality,

$$H^2(\widetilde{X}, \Omega_{\widetilde{X}}) = H^0(\widetilde{X}, \omega_{\widetilde{X}} \otimes \Omega_{\widetilde{X}}^{-1}) = 0$$

because the degree of $\omega_{\widetilde{X}} \otimes \Omega_{\widetilde{X}}^{-1}$ is now zero. The same can said of $H^2(X, \Omega_X)$. so we get the desired SES. □

Hartshorne V.5.4 Let $f : X \to X'$ be a birational morphism of nonsingular surfaces.

(a) If $Y \subseteq X$ is an irreducible curve such that $f(Y)$ is a point, then $Y \cong \mathbb{P}^1$ and $Y^2 < 0$.

(b) (Mumford [6].) Let $P' \in X'$ be a fundamental point of f^{-1}, and let $Y_1, \ldots, Y_r$ be the irreducible components of $f^{-1}(P')$. Show that the matrix $\|Y_i.Y_j\|$ is negative definite.

Proof. (a) Since $f : X \to X'$ is a birational morphism of surfaces, let $g : X \dashrightarrow X'$ be its birational inverse. Then g's set of fundamental points (see p. 410) includes the point $p := f(Y)$. Then by Hartshorne Proposition II.5.3, $f : X \to X'$ factors through the monoidal transformation $Bl_P(X') \to X'$ (the blow-up with center $P \in X'$) i.e.

$$X \to Bl_P(X') \to X'$$

is our morphism f. In fact, we could use Hartshorne Theorem II.5.5 to factor the whole morphism $f : X \to X'$. The fact that $f(Y)$ is a point implies that we never have an inverse to the blow-up $Bl_P(X') \to X'$ or more specifically, a morphism which negates the effect of this blow-up. In particular, Y must be a copy of $\mathbb{P}^1$ inside X. On the other hand, the self-intersection Y^2 is less than 0 because of our results on the intersection theory of surfaces.
(b) See Mumford's paper [6] for the proof. □

Hartshorne V.5.5 Let C be a curve, and let $\pi : X \to C$ and $\pi' : X' \to C$ be two geometrically ruled surfaces over C. Show that there is a finite sequence of elementary transformations (5.7.1) which transform X into X'.

[Hints: First show if $D \subseteq X$ is a section of π containing a point P, and if $\widetilde{D}$ is the strict transform of D by elm_P, then $\widetilde{D}^2 = D^2 - 1$ (Fig. 23). Next show that X can be transformed into a geometrically ruled surface X'' with invariant $e \gg 0$. Then use (2.12), and study how the ruled surface $\mathbb{P}(\mathscr{E})$ with $\mathscr{E}$ decomposable behaves under elm_P.]

Proof. The claim that $\widetilde{D} = D^2 - 1$ follows from Fig. 23. Indeed, we can compute $\widetilde{D}^2$ by looking at pull-back along g to $\widetilde{X}$. Then compute the self-intersection of $g^*\widetilde{D}$ and use the fact that $g^*\widetilde{D} = D + \widetilde{L}$ there.

Recall that the invariant e was given by $-\deg \bigwedge_{i=1}^{2} \mathscr{E}$ i.e. it is the degree of an invertible sheaf. This invertible sheaf defines a divisor on X when we pull-back e.g. we use Proposition V.2.9. Now if we use our result from the previous paragraph, we can decrease the degree of the rank 2 locally free sheaf defining the projective bundle by 1 each time and in that case we increase the value of e. So via a sequence of transformations, we can make $e \gg 0$.

By Hartshorne Theorem V.2.12, we know $\mathscr{E}$ is indecomposable when $-2g \leq e \leq 2g-2$. So for e sufficiently large, $\mathscr{E}$ is decomposable.

Now let $\mathscr{E} = \mathscr{L} \oplus \mathscr{M}$ be a decomposable rank 2 locally free sheaf on C.

[Incomplete]

[Incomplete] □

Hartshorne V.5.6 Let X be a surface with function field K. Show that every valuation ring R of K/k is one of the three kinds described in (II, Ex. 4.12).

[Hint: In case (3), let $f \in R$. Use (Ex. 5.1) to show that for all $i \gg 0, f \in \mathcal{O}_{X_i}$, so in fact $f \in R_0$.]

Proof. □

Hartshorne V.5.7 Let Y be an irreducible curve on a surface X, and suppose there is a morphism $f : X \to X_0$ to a projective variety X_0 of dimension 2 , such that $f(Y)$ is a point P and $f^{-1}(P) = Y$. Then show that $Y^2 < 0$.

[Hint: Let $|H|$ be a very ample (Cartier) divisor class on X_0, let $H_0 \in |H|$ be a divisor containing P, and let $H_1 \in |H|$ be a divisor not containing P. Then consider f^*H_0, f^*H_1 and $\widetilde{H}_0 = f * (H_0 - P)^-$.]

Proof. □

Hartshorne V.5.8 A surface singularity. Let k be an algebraically closed field, and let X be the surface in $\mathbb{A}_k^3$ defined by the equation $x^2 + y^3 + z^5 = 0$. It has an isolated singularity at the origin $P = (0, 0, 0)$.

(a) Show that the affine ring $A = k[x, y, z]/(x^2 + y^3 + z^5)$ of X is a unique factorization domain, as follows. Let $t = z^{-1}; u = t^3x$, and $v = t^2y$. Show that z is irreducible in $A; t \in k[u, v]$, and $A[z^{-1}] = k[u, v, t^{-1}]$. Conclude that A is a UFD.

(b) Show that the singularity at P can be resolved by eight successive blowings-up. If $\tilde{X}$ is the resulting nonsingular surface, then the inverse image of P is a union of eight projective lines, which intersect each other according to the Dynkin diagram $\mathbb{E}_8$:

[Incomplete]

Here each circle denotes a line, and two circles are joined by a line segment whenever the corresponding lines intersect.

Note. This singularity has interesting connections with local algebra, invariant theory, and topology.

In case $k = \mathbb{C}$, Mumford [6] showed that the completion $\hat{A}$ of the ring A at the maximal ideal $m = (x, y, z)$ is also a UFD. This is remarkable, because in general the completion of a local UFD need not be UFD, although the converse is true (theorem of Mori)-see Samuel [3]. Brieskorn [2] showed that the corresponding analytic local ring $\mathbb{C}\{x, y, z\}/(x^2 + y^3 + z^5)$ is the only nonregular normal 2-dimensional analytic local ring

which is a UFD. Lipman [2] generalized this as follows: over any algebraically closed field k of characteristic $\neq 2, 3, 5$, the only nonregular normal complete 2-dimensional local ring which is a UFD is $k[[x, y, z]]/(x^2 + y^3 + z^5)$. See also Lipman [3] for a report on recent work connected with UFD's.

This singularity arose classically out of Klein's work on the icosahedron. The group I of rotations of the icosahedron, which is isomorphic to the simple group of order 60 , acts naturally on the 2-sphere. Identifying the 2-sphere with $\mathbb{P}^1_{\mathbb{C}}$ by stereographic projection, the group I appears as a finite subgroup of Aut $\mathbb{P}^1_{\mathbb{C}}$. This action lifts to give an action of the binary icosahedral group $\bar{I}$ on $\mathbb{C}^2$ by linear transformations of the complex variables t_1 and t_2. Klein $[2, \mathrm{I}, 2, \S 13, \mathrm{p}.62]$ found three invariant polynomials x, y, z in t_1 and t_2, related by the equation $x^2 + y^3 + z^5 = 0$. Thus the surface X appears as the quotient of $\mathbb{A}^2_{\mathbb{C}}$ by the action of the group $\bar{I}$. In particular, the local fundamental group of X at P is just $\bar{I}$.

With regard to the topology of algebraic varieties over $\mathbb{C}$, Mumford [6] showed that a normal algebraic surface over $\mathbb{C}$, whose underlying topological space (in its "usual" topology) is a topological manifold, must be nonsingular. Brieskorn showed that this is not so in higher dimensions. For example, the underlying topological space of the hypersurface in C^4 defined by $x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0$ is a manifold. Later Brieskorn [1] showed that if one intersects such a singularity with a small sphere around the singular point, then one may get a topological sphere whose differentiable structure is not the standard one. Thus for example, by intersecting the singularity

$$x_1^2 + x_2^2 + x_3^2 + x_4^3 + x_5^{6k-1} = 0$$

in C^5 with a small sphere around the origin, for $k = 1, 2, \ldots, 28$, one obtains all 28 possible differentiable structures on the 7 -sphere. See Hirzebruch and Mayer [1] for an account of this work.

PROOF. □

6. Classification of Surfaces

Hartshorne V.6.1 Let X be a surface in $\mathbb{P}^n, n \geqslant 3$, defined as the complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{n-2}$, with each $d_i \geqslant 2$. Show that for all but finitely many choices of $(n, d_1, \ldots, d_{n-2})$, the surface X is of general type. List the exceptional cases, and where they fit into the classification picture.

PROOF. By the adjunction formula (or an exercise from Hartshorne II.8) we know $\omega_X = \mathcal{O}_X(\sum_{i=1}^{n-2} d_i - n - 1)$.

Case $n = 3$. Then $-n - 1 = -4$ and since $d_i \geq 2$, we can pick $d_1 = 2$ or $d_1 = 3$ at best to get that $\omega_X = \mathcal{O}_X(-2)$ or $\mathcal{O}_X(-3)$. In both cases, K_X is a negative multiple of $\mathcal{O}_X(1)$ and so by Hartshorne Theorem V.6.1, $|12K_X| = \emptyset$. But that means X is rational or ruled over some curve of genus $g(C) \geq 1$. If $d_1 = 4$, then $K_X \equiv 0$ which gives a K3 surface. Any other choice of d_1 would give a surface of general type.

Case $n = 4$. In this case, we are forced to pick $d_1 + d_2 \leq 5$ to get something that is not of general type. We can pick $d_1 = 2$ and $d_2 = 3$ to get a K3 surface.

Case $n = 5$. We can pick $d_1 + d_2 + d_3 \leq 6$ to get something that is not of general type. If $d_1 = d_2 = d_3 = 2$, then we get a K3 surface.

Case $n = 6$. We need to pick $d_i \geq 2$ s.t. $d_1 + d_2 + d_3 + d_4 \leq 7$ to get something not of general type. This is impossible. So we always get something of general type.

Case $n \geq 7$. We will always get surfaces of general type. □

Hartshorne V.6.2 Prove the following theorem of Chern and Griffiths. Let X be a non-singular surface of degree d in $\mathbb{P}^{n+1}_{\mathbb{C}}$, which is not contained in any hyperplane. If $d < 2n$, then $p_g(X) = 0$. If $d = 2n$, then either $p_g(X) = 0$, or $p_g(X) = 1$ and X is a K3 surface.

[Hint: Cut X with a hyperplane and use Clifford's theorem (IV, 5.4). For the last statement, use the Riemann-Roch theorem on X and the Kodaira vanishing theorem (III, 7.15).]

Proof. There is a solution to this exercise in Andrew Egbert's set of solutions. I personally do not follow the arguments completely. □

Not everys ection in the appendices has an exercise. For each of the sections without an exercise, I have chosen to write out several references in the literature that are useful for learning said topics.

APPENDIX A

Intersection Theory

1. Intersection Theory

2. Properties of the Chow Ring

3. Chern Classes

4. The Riemann-Roch Theorem

5. Complements and Generalizations

6. Exercises

Hartshorne A.6.1 Show that the definition of rational equivalence in §1 is equivalent to the equivalence relation generated by the following relation: two cycles Y, Z of codimension r on X are equivalent if there exists a cycle W of codimension r on $X \times \mathbf{A}^1$, which intersects $X \times \{0\}$ and $X \times \{1\}$ properly, and such that $Y = W.(X \times \{0\})$, $Z = W.(X \times \{1\})$.

Proof. Rational equivalence means that the pushforward of the normalization are linearly equivalennt as Weil divisors.

[Incomplete] □

Hartshorne A.6.2 Prove the following result about Weil divisors, which generalizes (IV, Ex. 2.6), and which is needed to show that f_* is well-defined modulo rational equivalence (A3): Let $f : X \to X'$ be a proper, generically finite map of normal varieties, and let D_1 and D_2 be linearly equivalent Weil divisors on X. Then f_*D_1 and f_*D_2 are linearly equivalent Weil divisors on X'. [Hint: Remove a subset of codimension $\geqslant 2$ from X' so that f becomes a finite flat morphism, then generalize (IV, Ex. 2.6).]

Proof. □

Hartshorne A.6.3 Show directly that any subvariety of degree d in $\mathbf{P}^n$ is rationally equivalent to d times a linear space of the same dimension, by using a projection argument similar to (III, 9.8.3).

Proof. The result is clear for $d = 0$ where the subvariety is just a linear variety. □

Hartshorne A.6.4 Let $\pi : X \to C$ be a ruled surface (V, §2) over a nonsingular curve C. Show that the group $A^2(X)$ of zero-cycles modulo rational equivalence is isomorphic to $\mathrm{Pic}(C)$.

Proof. The group $A^2(X)$ is the group of codimension 2 cycles modulo rational equivalence. These correspond to points of X. We get a map

$$\pi_* : A^2(X) \to A^1(C)$$

as $\pi : X \to C$ is a proper morphism. Since C is nonsingular, $A^1(C) \cong \mathrm{Pic}(C)$. So our job is to show that pi_* is actually an isomorphism. That this is true will follow from consideration of an exact sequence

$$0 \to K \to A^2(X) \xrightarrow{\pi_*} \mathrm{Pic}(C) \to L \to 0.$$

Now π_* is certainly surjective so L is zero. The fact that $\pi \circ \sigma : C \to C$ is the identity implies $A^2(X) \cong \mathrm{Pic}(C) \oplus K$. The kernel K consists of those cycles $Z \in A^2(X)$ that collapse to the trivial line bundle $\mathcal{O}_C$ i.e. $\pi_*(Z) = \mathcal{O}_C$ where Z is a zero-cycle on X. But that means $\pi_*(Z)$ is linearly equivalent to (f) where $f : C \to \mathbb{A}^1$ is a rational function on C. Pulling back the function f to $\pi^* f : X \to C \to \mathbb{A}^1$, we deduce that Z itself is rationally equivalent to a trivial cycle. □

Hartshorne A.6.5 Let X be a surface, let $P \in X$ be a point, and let $\pi : \tilde{X} \to X$ be the monoidal transformation with center P(V, §3). Show that $A(\tilde{X}) \cong \pi^* A(X) \oplus \mathbf{Z}$, where $\mathbf{Z}$ is generated by the exceptional curve $E \in A^1(X)$, and the intersection theory is determined by $E^2 = -\pi^*(P)$.

Proof. [Incomplete] □

Hartshorne A.6.6 Let X be a nonsingular projective variety of dimension n, and let $\Delta \subseteq X \times X$ be the diagonal. Show that $c_n(\mathscr{T}_X) = \Delta^2$ in $A^n(X)$, under the natural isomorphism of X with Δ.

Hartshorne A.6.7 Let X be a nonsingular projective 3 -fold, with Chern classes c_1, c_2, c_3. Show that

$$1 - p_a = \frac{1}{24} c_1 c_2$$

and for any divisor D,

$$\chi(\mathscr{L}(D)) = \frac{1}{12} D \cdot (D - K) \cdot (2D - K) + \frac{1}{12} D \cdot c_2 + 1 - p_a.$$

Proof. Since X is a nonsingular projective 3-fold, we know that $p_a = 1 - \chi(\mathcal{O}_X)$. Then the LHS is precisely $\chi(\mathcal{O}_X)$. We need to compute c_1 and c_2. Apply Hirzebruch-Riemann-Roch to get

$$\chi(\mathcal{O}_X) = \deg(ch(\mathcal{O}_X).td(\mathscr{T}))_n.$$

The tangent bundle on a three-fold is dual to $\Omega_{X/k}$ we know that

$$c_1(\mathscr{T}_X) = -c_1(\Omega_X) = -c_1(\bigwedge_1^3 \Omega_X) = -c_1(\omega_X) = -K.$$

Then to compute the Todd class, we know

$$\mathrm{td}(\mathscr{T}_X) = 1 + \frac{1}{2} c_1(\mathscr{T}) + \frac{1}{12}(c_1(\mathscr{T})^2 + c_2(\mathscr{T})) + \frac{1}{24}(c_1(\mathscr{T}) c_2(\mathscr{T})). = 1 - \frac{1}{2} K + \frac{1}{12}(K^2 + c_2) + \frac{1}{24}(-K) c_2$$

Now we look at the $\mathrm{ch}(\mathcal{O}_X)$. In this case, we know that

$$\mathrm{ch}(\mathcal{O}_X) = 1 + c_1(\mathcal{O}_X) + \frac{1}{2}(c_1(\mathcal{O}_X)^2 + \frac{1}{6}(c_1(\mathcal{O}_X)^3).$$

Now applying Hirzebruch-Riemann-Roch, we get

$$(\mathrm{ch}(\mathcal{O}_X)\,\mathrm{td}(\mathscr{T}_X))_3 = \frac{1}{12} c_1(\mathcal{O}_X)(c_1(\mathscr{T})^2 + c_2(\mathscr{T})) + \frac{1}{6} c_1(\mathcal{O}_X)^3 + \frac{1}{4} c_1(\mathcal{O}_X)^2 c_1(\mathscr{T}) + \frac{1}{24} c_1(\mathscr{T}) c_2(\mathscr{T})$$

and taking degrees (using that $c_1(\mathcal{O}_X)$ has degree 0 and so do all of its powers),

$$1 - p_a = \chi(\mathcal{O}_X)) = \frac{1}{24} c_1(\mathscr{T}_X) c_2(\mathscr{T}_X).$$

In the general case, we can compute the result as follows. We know that

$$(\mathrm{ch}(\mathcal{O}_X(D)).\,\mathrm{td}(\mathscr{T}))_3 = \frac{1}{12} D.(c_1^2 + c_2) + \frac{1}{6} D^3 + \frac{1}{4} D^2.c_2 + \frac{1}{24} c_1 c_2$$

where c_1, c_2 are the Chern classes of the tangent sheaf. Using the preceding result and rewriting the above, we get the claim (i.e. clear denominators to get $\frac{1}{12}$ appearing). □

Hartshorne A.6.8 Let $\mathscr{E}$ be a locally free sheaf of rank 2 on $\mathbf{P}^3$, with Chern classes c_1, c_2. Since $A(\mathbf{P}^3) = \mathbf{Z}[h]/h^4$, we can think of c_1 and c_2 as integers. Show that $c_1 c_2 \equiv 0 \pmod 2$. [Hint: In the Riemann-Roch theorem for $\mathscr{E}$, the left-hand side is automatically an integer, while the right-hand side is a priori only a rational number.]

Proof. Applying Hirzebruch-Riemann-Roch Theorem. we get

$$\chi(\mathscr{E}) = \frac{1}{24} c_1 c_2 + \frac{1}{12} c_1(\mathscr{E})(c_1^2 + c_2) + \frac{1}{4} c_1(c_1(\mathscr{E})^2 - 2c_2(\mathscr{E})) + \frac{1}{6}(c_1(\mathscr{E})^3 - 3c_1(\mathscr{E}) c_2(\mathscr{E}))$$

Multply everything by 24 and taking the result mod 2, we get the claim. □

Hartshorne A.6.9 *Surfaces in* $\mathbf{P}^4$. (a) Verify the formula of (4.1.3) for the rational cubic scroll in $P^4(\ \mathrm{V}, 2.19.1)$.

(b) If X is a K3 surface in $\mathbf{P}^4$, show that its degree must be 4 or 6 . (Examples of such are the quartic surface in $\mathbf{P}^3$, and the complete intersection of a quadric and a cubic hypersurface in $\mathbf{P}^4$.)

(c) If X is an abelian surface in $\mathbf{P}^4$, show that its degree must be 10 . (Horrocks and Mumford [1] have shown that such abelian surfaces exist.)

*(d) Determine which of the rational ruled surfaces $X_e, e \geqslant 0(\ \mathrm{V}, \S 2)$, admit an embedding in $\mathbf{P}^4$.

Proof. (a)
(b) A $K3$-surface has $p_a = p_g = 0$ and $K \equiv 0$ and $q = 0$. The formula on p. 434 then becomes

$$d^2 - 10d + 24 = 0 \ \Rightarrow (d-4)(d-6) = 0 \ \Rightarrow d = 4, 6.$$

(c) An abelian surface has $p_a = -1$ and $p_g = 1$. Furthermore, $H.K$ Then, the formula becomes

$$d^2 - 10d - 5H.K - 2K^2 = 0.$$

[Incomplete]
(d) [Incomplete] □

Hartshorne A.6.10 Use the fact that the tangent sheaf on an abelian variety is free, to show that it is impossible to embed an abelian 3-fold in $\mathbb{P}^5$.

Proof. Suppose we can embed an abelian 3-fold into $\mathbb{P}^5$. □

APPENDIX B

Transcendental Methods

1. The Associated Complex Analytic Space

2. Comparison of the Algebraic and Analytic Categories

3. When is a Compact Complex Manifold Algebraic?

4. Kählher Manifolds

5. The Exponential Sequence

6. Exercises

Hartshorne B.6.1 Show that the unit disc in $\mathbf{C}$ is not isomorphic to X_h for any scheme X.
Hartshorne B.6.2 Let $z_1, z_2, \ldots$ be an infinite sequence of complex numbers with $|z_n| \to \infty$ as $n \to \infty$. Let $\mathfrak{I} \subseteq \mathcal{O}_{\mathbf{c}}$ be the sheaf of ideals of holomorphic functions vanishing at all of the z_n. Show that there is no coherent algebraic sheaf of ideals $\mathscr{I} \subseteq \mathcal{O}_X$, where $X = \mathbf{A}^1_{\mathbf{C}}$ is the affine line, such that $\mathfrak{I} = \mathscr{F}_h$ as an ideal in $\mathscr{O}_{\mathbf{C}}$. Show on the other hand that there is a coherent sheaf $\mathscr{F}$ on X such that $\mathscr{F}_h \cong \mathfrak{J}$ as coherent sheaves.
Hartshorne B.6.3 (Serre [12].) On $\mathbf{C}^2 - \{0,0\}$, we define an invertible analytic sheaf $\mathfrak{S}$ as follows: $\mathfrak{L} \cong \mathcal{O}$ when $z \neq 0$; $\mathfrak{L} \cong \mathcal{O}$ when $w \neq 0$, and when both $z, w \neq 0$, the two copies of $\mathcal{O}$ are glued by multiplication by $e^{-1/zw}$ in the local ring at the point (z, w).
Hartshorne B.6.4 Show that there is no invertible algebraic sheaf $\mathscr{L}$ on $\mathbf{A}^2 - \{0,0\}$ with $\mathscr{L}_h \cong \mathfrak{L}$. Show directly that if X is a scheme which is reduced and proper over $\mathbf{C}$, then $H^0(X, \mathcal{O}_X) \cong H^0(X_h, \mathcal{O}_{X_h})$. Conversely, show that if X is not proper over $\mathbf{C}$, then there is a coherent sheaf $\mathscr{F}$ on X with $H^0(X, \mathscr{F}) \neq H^0(X_h, \mathscr{F}_h)$.
Hartshorne B.6.5 If X, X' are nonsingular affine algebraic curves, with $X_h \cong X'_h$, show that $X \cong X'$.
Hartshorne B.6.6 Show that if X and Y are projective schemes over C, and $f : X_h \to Y_h$ is a morphism of analytic spaces, then there exists a (unique) morphism $f : X \to Y$ with $f_h = \hat{\mathfrak{f}}$. [Hint: First reduce to the case $Y = \mathbf{P}^n$. Then consider the invertible analytic sheaf $\mathfrak{L} = \mathfrak{f}^* \mathcal{O}(1)$ on X_h, use (2.1), and the techniques of (II, §7).]

APPENDIX C

The Weil Conjectures

1. The Zeta Function and the Weil Conjecture

2. History of Work on the Weil Conjectures

3. The l-adic Cohomology

4. Cohomological Interpretations of the Weil Conjectures

5. Exercises

Hartshorne C.5.1 Let X be a disjoint union of locally closed subschemes X_i. Then show that

$$Z(X,t) = \prod Z\left(X_i, t\right).$$

Hartshorne C.5.2 Let $X = \mathbf{P}^n_k$, where $k = \mathbf{F}_q$, and show from the definition of the zeta function that

$$Z\left(\mathbf{P}^n, t\right) = \frac{1}{(1-t)(1-qt)\cdots(1-q^n t)}$$

Verify the Weil conjectures for $\mathbf{P}^n$.

Hartshorne C.5.3 Let X be a scheme of finite type over $\mathbf{F}_q$, and let $\mathbf{A}^1$ be the affine line. Show that

$$Z\left(X \times \mathbf{A}^1, t\right) = Z(X, qt).$$

Hartshorne C.5.4 The Riemann zeta function is defined as

$$\zeta(s) = \prod \frac{1}{1-p^{-s}},$$

for $s \in \mathbf{C}$, the product being taken over all prime integers p. If we regard this function as being associated with the scheme $\operatorname{Spec} \mathbf{Z}$, it is natural to define, for any scheme X of finite type over $\operatorname{Spec} \mathbf{Z}$,

$$\zeta_X(s) = \prod \left(1 - N(x)^{-s}\right)^{-1}$$

where the product is taken over all closed points $x \in X$, and $N(x)$ denotes the number of elements in the residue field $k(x)$. Show that if X is of finite type over $\mathbf{F}_q$, then this function is connected to $Z(X,t)$ by the formula

$$\zeta_X(s) = Z\left(X, q^{-s}\right).$$

[Hint: Take $d\log$ of both sides, replace q^{-s} by t, and compare.]

Hartshorne C.5.5 Let X be a curve of genus g over k. Assuming the statements (1.1) to (1.4) of the Weil conjectures, show that $N_1, N_2, \ldots, N_g$ determine N_r for all $r \geqslant 1$.

Hartshorne C.5.6 Use (IV, Ex. 4.16) to prove the Weil conjectures for elliptic curves. First note that for any r

$$N_r = q^r - \left(f^r + \hat{f}^r\right) + 1,$$

where $f = F'$. Then calculate $Z(t)$ formally and conclude that

$$Z(t) = \frac{(1 - ft)(1 - \hat{f}t)}{(1 - t)(1 - qt)}$$

and hence

$$Z(t) = \frac{1 - at + qt^2}{(1 - t)(1 - qt)}$$

where $f + \hat{f} = a_X$. This proves rationality immediately. Verify the functional equation. Finally, if we write

$$1 - at + qt^2 = (1 - \alpha t)(1 - \beta t),$$

show that $|a| \leqslant 2\sqrt{q}$ if and only if $|\alpha| = |\beta| = \sqrt{q}$. Thus the analogue of the Riemann hypothesis is just (IV, Ex. 4.16d).

Hartshorne C.5.7 Use (V, Ex. 1.10) to prove the analogue of the Riemann hypothesis (1.3) for any curve C of genus g defined over $\mathbf{F}_q$. Write $N_r = 1 - a_r + q^r$. Then according to (V, Ex. 1.10),

$$|a_r| \leqslant 2g\sqrt{q^r}.$$

On the other hand, by (4.2) the zeta function of C can be written

$$Z(t) = \frac{P_1(t)}{(1 - t)(1 - qt)}$$

where

$$P_1(t) = \prod_{i=1}^{2g} (1 - \alpha_i t)$$

is a polynomial of degree $2g = \dim H^1(C, \mathbf{Q}_l)$.

(a) Using the definition of the zeta function and taking logs, show that

$$a_r = \sum_{i=1}^{2g} (\alpha_i)^r$$

for each r.

(b) Next show that

$$|a_r| \leqslant 2g\sqrt{q^r} \text{ for all } r \quad \Leftrightarrow \quad |\alpha_i| \leqslant \sqrt{q} \text{ for all } i.$$

[Hint: One direction is easy. For the other, use the power series expansion

$$\sum_{i=1}^{2g} \frac{\alpha_i t}{1 - \alpha_i t} = \sum_{r=1}^{\infty} a_r t^r$$

for suitable $t \in \mathbf{C}$.]

(c) Finally, use the functional equation (4.4) to show that $|\alpha_i| \leqslant \sqrt{q}$ for all i implies that $|\alpha_i| = \sqrt{q}$ for all i.

CHAPTER 4

Extra Exercises

Exercise 1. Let k be a field, and let $F = x^n + y^n - z^n \in k[x, y, z]$, where $n \geq 1$ is prime to $\text{char}(k)$. Determine $H^0\left(X, \Omega^1_{X/k}\right)$ for $X = V(F) \subset \mathbb{P}^2_k$.

PROOF. For ease, we assume that $\text{char}(k) = 0$. Deal with each case separately.

If $n = 1$, then F is a curve of degree 1 in $\mathbb{P}^2_k$ which has genus $g = 0$ by $g = \frac{(d-1)(d-2)}{2}$. So F is birational to $\mathbb{P}^1$ and $H^0(X, \Omega^1_{X/k}) \cong H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1/k})$ as vector spaces. So it has dimension 0 here.

If $n = 2$, then the situation is the same since $g = 0$.

If $n = 3$, then the situation is a bit different. In this case, $g = 1$ and F corresponds to a degree 3 curve. This curve is nonsingular, (as is always in the case when n is arbitrary), and so Serre Duality holds. That means $\Omega^1_{X/k}$ is the dualizing sheaf and

$$H^0(X, \Omega^1_{X/k}) \cong H^1(X, \mathcal{O}_X)$$

and the dimension of $H^1(X, \mathcal{O}_X)$ is the geometric genus g. Hence, the dimension is 1. □

Exercise 2.

Bibliography

[1] J. Harris. *Algebraic Geometry: A First Course*. Graduate texts in mathematics. Springer-Verlag, 1992. ISBN: 9783540977162. URL: https://books.google.com/books?id=INW6QgAACAAJ.

[2] M.F. Atiyah and I.G. MacDonald. *Introduction To Commutative Algebra*. Addison-Wesley series in mathematics. Avalon Publishing, 1994. ISBN: 9780813345444. URL: https://books.google.com/books?id=HOASFid4x18C.

[3] Q. Liu and R. Erne. *Algebraic Geometry and Arithmetic Curves*. Oxford Graduate Texts in Mathematics (0-19-961947-6). Oxford University Press, 2006. ISBN: 9780191547805. URL: https://books.google.com/books?id=ePpzBAAAQBAJ.

[4] U. Görtz and T. Wedhorn. *Algebraic Geometry: Part I: Schemes. With Examples and Exercises*. Advanced Lectures in Mathematics. Vieweg+Teubner Verlag, 2010. ISBN: 9783834897220. URL: https://books.google.com/books?id=XEiLudn6sq4C.

[5] D. Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer New York, 2011. ISBN: 9781461253518. URL: https://books.google.com/books?id=JrQGswEACAAJ.

[6] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer New York, 2013. ISBN: 9781475738490. URL: https://books.google.com/books?id=7z4mBQAAQBAJ.

[7] J. Neukirch and N. Schappacher. *Algebraic Number Theory*. basic teaching of mathematical sciences. Springer Berlin Heidelberg, 2013. ISBN: 9783662039830. URL: https://books.google.com/books?id=hS3qCAAAQBAJ.

[8] I.R. Shafarevich and M. Reid. *Basic Algebraic Geometry 1: Varieties in Projective Space*. SpringerLink : Bücher. Springer Berlin Heidelberg, 2013. ISBN: 9783642379567. URL: https://books.google.com/books?id=tyK4BAAAQBAJ.

[9] Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. 2017. URL: http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf.

[10] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2020.

[11] ashpool (https://math.stackexchange.com/users/4341/ashpool). *Direct and inverse limits of sheaves*. Mathematics Stack Exchange. eprint: https://math.stackexchange.com/q/193816.

[12] Harry Gindi (https://mathoverflow.net/users/1353/harry-gindi). *"Pick up a homological algebra book and prove all of the theorems yourself"; (exercise from Lang's Algebra)*. MathOverflow. eprint: https://mathoverflow.net/q/10897.

[13] Coskun. *Nike's Trick*. URL: http://homepages.math.uic.edu/~coskun/Nike.pdf.

[14] user138661. *Map of coherent sheaves inducing isomorphism on the stalks at the generic point*. MathOverflow. URL:https://mathoverflow.net/q/329828 (version: 2019-04-27).

eprint: https://mathoverflow.net/q/329828. URL: https://mathoverflow.net/q/329828.