Lecture Notes on Algebraic Topology

Notes written by Kevin D.T. Dao
Based on lectures by Zhouli Xu and Justin Roberts

These notes are derived from Math 290ABC taken at UCSD for the 2021-2022 academic year. All mistakes are my own fault and not that of the instructors. Furthermore, these notes do not always follow the lecture precisely, but may include more detail than was presented in the lecture. Comments are welcomed (email to ktdao@ucsd.edu).

Some words about the structure of these notes. At the very start of each lecture, I have written an "Abstract" to help the reader know what was covered in that specific lecture. Also, everything is hyperlinked using the hyperref package in LaTeXso this document is best used on a computer. The index has been kept up to date, but not every instance of a word is indexed (otherwise it would be far too large!). There is also a "Chapter" 4 which includes my solutions to some exercises from Hatcher. These were placed there for convenience and are not meant to be a part of the canonical material. The reader is urged to work through the problems in Hatcher and Miller by themselves before referring to that section. Ideally, the reader would never have to open the chapter.

Last updated: June 4, 2022.

Contents

CHAPTER 1

Math 290A - UCSD

ABSTRACT. The course is being taught by Zhouli Xu. The course is being taught asynchronously and most work shall be done autonomously. There are no official textbooks though we will mainly follow Hatcher's text. References for this course are listed as follows. Algebraic Topology, by Allen Hatcher: here. A Concise Course in Algebraic Topology, by J.P. May: here. Lectures on Algebraic Topology, by H. Miller: here.

1. Lecture 1: September 24th, 2021

ABSTRACT. We reviewed the course logistics and went over some aspects of the course material that will be covered.

Again, the entirety of the course is online. For the first two quarters at least and the third quarter will depend on if Justin Roberts chooses to do in-person lectures.

Motivation: Algebraic topology studies the relationship between topology and algebra. Namely, topology is "difficult" and algebra is a good tool to solve problems in the category of topological spaces. Indeed, we will be interested in constructing functors from the category of topological spaces to algebraic objects (and here, we mean functors in the categorical sense). The following are the key functors in this course sequence.

- π_1 the fundamental group
- H_* the homology groups (abelian groups)
- H^* cohomology groups (commutative rings)
- π_n homotopy groups

The first two are studied in 290A, the second and third in 290B, and the last in 290C. Now, we need to balance the complexity of our algebraic objects in comparison to how much information we lose from the functor. Indeed, working with f.g. $\mathbb{Z}$-modules is much easier than working with something like f.g. $\mathbb{C}$-algebras due to the classification of modules over PIDs.

Zhouli discusses a few interesting results of algebraic topology that got him hooked in his respective course as an undergraduate and graduate student:

- the Brouwer fixed point theorem;
- the hairy ball theorem;
- the Borsuk-Ulam theorem.

The first theorem is proved using π_1 while the third uses knowledge of π_n.

Zhouli also posed an interesting problem that can be solved using topological methods.

Question: Does there exist matrices $A_i \in M_n(\mathbb{R})$ s.t. for all nonzero vectors $\gamma \in \mathbb{R}^n$, the set of vectors

$$\{A_1\gamma, \ldots, A_n\gamma\}$$

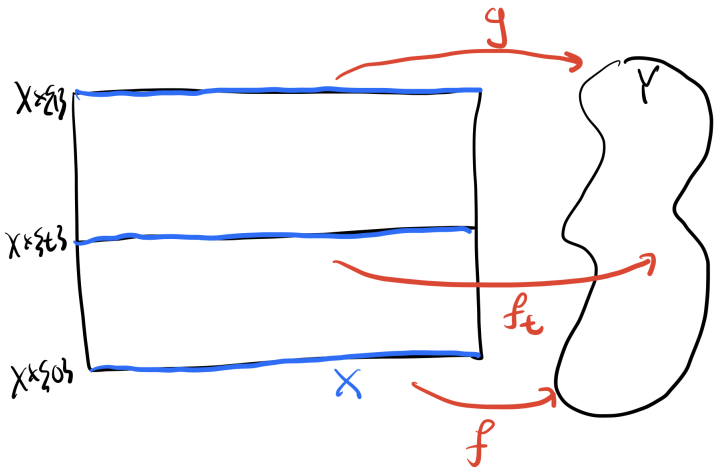


FIGURE 1. The blue coloring indicates X and the $f_t : X \to Y$ are continuous maps. Along the vertical, we are moving along t continuously while along the horizontal, we are varying over X.

is always linearly independent? Hint: the result depends on n. Zhouli asked the class to let him know if we have any ideas on this one, but he of course already knows the answer – this question is not open.

2. Lecture 2: September 27th, 2021

ABSTRACT. We work towards defining the fundamental group (at a basepoint) of a topological space. We define homotopy equivalences, contractible spaces, homotopy relative to a subspace, and the fundamental group at a basepoint.

To classify topological spaces, we need something to classify them w.r.t. and the ideal would be doing it up to homeomorphism. However, this requires a lot of information since we preserve all of the information from the topology. So, we focus on what is called homotopy equivalence which loses some information, but is somewhat easier to work with.

Definition 2.1. Two continuous maps $f, g : X \to Y$ are **homotopic** if there exists a continuous map $H : X \times I \to Y$ s.t.

(1) $H(x, 0) = f(x)$ for all $x \in X$,
(2) $H(x, 1) = g(x)$ for all $x \in X$.

We call H a **homotopy** from f to g. Intuitively, this means we can continuously deform f to g through a family of maps, $H(x, t) = f_t(x)$. We will assume from here on that all maps are continuous.

Example 2.2. Any two maps $f, g : X \to \mathbb{R}^n$ are homotopy via

$$H(x, t) = (1 - t)f(x) + tg(x),$$

a straight line homotopy.

Proposition 2.3. Homotopy is an equivalence relation.

PROOF. The proof is easy to show. If we view homotopy through means of the square in figure 1 above, then transitive property is simply stacking the two squares and reparameterizing. □

Definition 2.4. Two topological spaces X, Y are **homotopy equivalent** if there exists $f : X \to Y$, $g : Y \to X$ maps s.t. fg is homotopic to id_Y and gf is homotopic to id_X.

The pair f, g is called a **homotopy equivalence**.

Proposition 2.5. Homotopy equivalence is an equivalence relation.

Proof. Let f, g be a homotopy equivalence for $X \simeq Y$ with $f : X \to Y$ and $g : Y \to X$. Let f', g' be a homotopy equivalence for $Y \simeq Z$ with $f' : Y \to Z$ and $g' : Z \to Y$.

Clearly, $X \simeq X$ by taking the identity map both ways. Furthermore, $X \simeq Y$ iff $Y \simeq X$ by using the same homotopy equivalence. So we show transitivity: $X \simeq Z$. We claim $f' \circ f, g \circ g'$ is a homotopy equivalence. Indeed,

$$(f' \circ f) \circ (g \circ g') = f' \circ \mathrm{id}_Y \circ g' = f' \circ g' = \mathrm{id}_Z$$
$$(g \circ g') \circ (f' \circ f) = g \circ \mathrm{id}_Y \circ f = \mathrm{id}_X.$$

□

Example 2.6. It is easy to deduce that $\mathbb{R}^n \simeq \{*\}$ (our notation for homotopy equivalent topological spaces) where $\{*\}$ is a one point set. Since homotopy equivalence is an equivalence relation, we deduce $\mathbb{R}^m \simeq \mathbb{R}^n$ for all $(m, n) \in \mathbb{N}^2$.

Remark 1. We will write $f \simeq g$ for homotopic maps and $X \simeq Y$ for homotopy equivalent spaces.

Definition 2.7. A space X is **contractible** if $X \simeq \{*\}$. A map f is a **null-homotopic** map if $f \simeq c$ where c is a constant map.

Proposition 2.8. A space X is contractible iff id_X is null-homotopic.

Proof. ($\implies$): If $X \simeq \{*\}$, there exist a homotopy equivalence $f : X \to \{*\}$, $g : \{*\} \to X$. Observe that $g \circ f$ maps $X \to \{*\} \hookrightarrow x \in X$ so that means $\mathrm{im}(g \circ f) = x$ where $g(*) = x$. Thus, $g \circ f$ is a constant map and $g \circ f \simeq \mathrm{id}_X$ implies id_X is null-homotopic.

($\impliedby$): If X is contractible, $\mathrm{id}_X \simeq c$ for c a constant map $c : X \to y$ for some y. Let $H(x, t)$ be the homotopy. Now, consider $f \circ g : \{*\} \to \{*\}$ with $g : \{y\} \hookrightarrow X$ and $f : X \to \{y\}$. So, $f \circ g \simeq \mathrm{id}_{\{y\}}$. But, $g \circ f$ is a constant map (and constant maps are homotopic) $\implies g \circ f \simeq c \simeq \mathrm{id}_X$. □

Example 2.9. The sphere $S^n \subseteq \mathbb{R}^{n+1}$ is not contractible. The proof of this fact requires a but of work which we will see later.

Other examples are the **torus** and the coffee mug.

Definition 2.10. Maps $f, g : X \to Y$ are **homotopic relative to a subspace** Z of X if there exists a homotopy $H : X \times I \to Y$ s.t. for all $x \in Z$,

$$H(x, 0) = H(x, 1) = f(x) = g(x).$$

That is, $f|_Z = g|_Z$.

Definition 2.11. A **path** in X is a map $f : I \to X$. It is a loop if $f(0) = f(1)$. We will say two paths or loops are homotopic if they are homotopic as maps relative to $\partial I = \{0, 1\}$.

Definition 2.12. Given two paths $f, g : I \to X$ s.t. $f(1) = g(0)$, we may form the composition $f \cdot g$ by

$$f \cdot g := \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1 \end{cases} \tag{1}$$

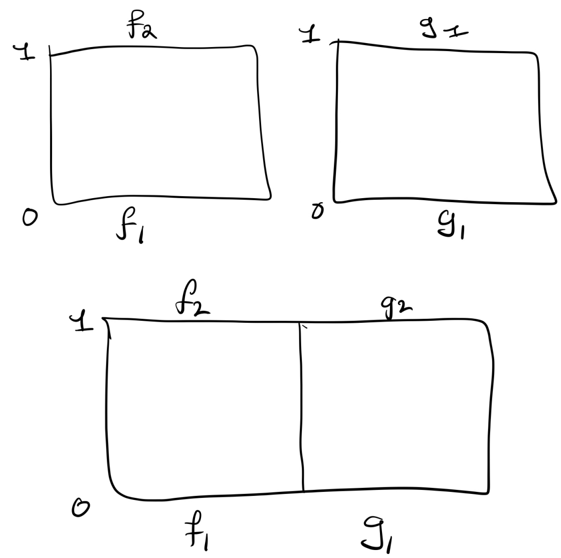


FIGURE 2. The idea is to just combine the homotopies and then reparameterize maps.

which is a map $f : I \to X$.

Proposition 2.13. If $f_1 \simeq f_2$ and $g_1 \simeq g_2$, then $f_1 \cdot g_1 \simeq f_2 \cdot g_2$.

PROOF. A visual proof can be found in figure 2. $\square$

Definition 2.14. We will denote $[f]$ for the homotopy class of paths and the preceding proposition shows $[f] \cdot [g] = [f \cdot g]$ is well-defined (and as Professor Zhouli says, "legit").

Definition 2.15. Let $x_0 \in X$ be a point which we call a **basepoint**. Then define

$$\pi_1(X, x_0) = \{[f] : f : I \to X \text{ s.t. } f(0) = f(1) = x_0\}.$$

Define a binary operation

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0) \tag{2}$$

given by $([f], [g]) \mapsto [f] \cdot [g]$.

Theorem 2.16. The pair $(\pi_1(X, x_0), \cdot)$ forms a group called the **fundamental group**.

3. Lecture 3: September 29th, 2021

ABSTRACT. The goal of this lecture is prove the last theorem of the previous lecture. That is, we show $\pi_1(X, x_0)$ with the multiplication defined by combining paths is a group.

Definition 3.1. A **reparameterization** is a map $\varphi : I \to I$ s.t. $\varphi(0) = 0$ and $\varphi(1) = 1$. If $\alpha : I \to X$ is a path, then $\alpha \circ \varphi$ is a reparameterization of α.

Lemma 3.2. With the notation as above, $\alpha \simeq \alpha \circ \varphi$.

PROOF. Take $H(s, t) = \alpha(ts + (1 - t)\varphi(s))$. $\square$

Now we prove lemmas that will help show that $\pi_1(X, x_0)$ is actually a group.

Lemma 3.3. Let α, β, γ be loops based at $x_0 \in X$. Then $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$.

Lemma 3.4. If $a \in X$, let $C_a : I \to X$ be a constant path at a. If γ is a path from a to b, then $C_a \cdot \gamma \simeq \gamma$ and $\gamma \cdot C_b \simeq \gamma$.

Lemma 3.5. Let $\overline{\alpha} = \alpha(1-s)$ for $\alpha : I \to X$ a path. Then,

$$\alpha \cdot \overline{\alpha} = C_{\alpha(0)} \qquad \& \qquad \overline{\alpha} \cdot \alpha \simeq C_{\alpha(0)}.$$

Proof of Theorems and Lemmas. Each of the lemmas can be proved using a homotopy diagram. Furthermore, the lemmas imply the theorem in obvious ways. □

Note that the basepoint of the fundamental group is important. However, if x_0, x_1 are in the same path component of X, then the respective fundamental groups are isomorphic.

Indeed, let γ be a path from x_0 to x_1. Define

$$[\gamma]_\# : \pi_1(X, x_1) \to \pi_1(X, x_0) \qquad \text{by } [\alpha] \mapsto [\gamma][\alpha][\overline{\gamma}]. \tag{3}$$

Lemma 3.6. The **change of base-point homomorphism** $[\gamma]_\#$ is a group isomorphism.

Definition 3.7. We say X is simply connected if X is path connected and $\pi_1(X, x_0) = 0$. The first condition is usually indicated by $\pi_0(X, x_0) = 0$.

Proposition 3.8. The space X is simply connected iff for all $x_0, x_1 \in X$, there always exists a unique homotopy class of paths from x_0 to x_1.

Proof. Trivial. □

As an exercise, show that X is simply connected iff $\pi_0(X) = 0$ and $f : S^1 \to X$ is null homotopic. Furthermore, $\mathbb{R}^n$ is simply connected for all n, but $\mathbb{R}^2 - \{0\}$ is not simply connected. This second example is actually the case for the following theorem.

Theorem 3.9. $\pi(S^1) \cong \mathbb{Z}$.

4. Lecture 4: October 1st, 2021

Abstract. In this lecture, we work towards showing that $\pi(S^1) \cong \mathbb{Z}$. We first prove some general facts about covering maps and covering spaces. Notably, we prove the UHLP which asserts that covering maps have unique homotopy lifting. This then implies a path lifting property which will be used to show $\pi_1(S^1) \cong \mathbb{Z}$.

It is often convenient to identity S^1 as a subset of $\mathbb{C}$ by $S^1 = \{z : |z| = 1\}$. Pick the base point to be $x_0 = 1 \in S^1$. Then, there exists a map

$$p : \mathbb{R}^1 \to S^1, \qquad \theta \mapsto e^{2\pi i \theta}.$$

We know that $\pi_1(\mathbb{R}^1) = 0$ so our goal is to relate the fundamental groups in some natural way. The preimage of a point $a \in S^1$ is infinite and in the case of x_0, $p^{-1} = \mathbb{Z}$. Furthermore, any small neighborhood of a point of S^1 has preimage which is a disjoint union of sets in $\mathbb{R}^1$. Indeed, if $U \ni a$ is a small neighborhood, then $p^{-1}(U) = \sqcup_{i \in \mathbb{Z}} U_i$ s.t. $p|_{U_i} : U_i \to U$ is a homeomorphism.

There is a more general notion of this which we now define. We also define some related notions as well.

Definition 4.1. A map $p : \widetilde{X} \to X$ is a **covering map** if for every $a \in X$, there exists an open $U \ni x$ s.t. $p^{-1}(U) = \sqcup_{i\in I} U_i$ with $p|_{U_i} : U_i \to U$ a homeomorphism. We say U is **evenly covered** and $\widetilde{X}$ a **covering space**. The covering space is called n**-fold** if $p^{-1}(a)$ contains n points for each a and therefore, $|I| = n$ for all U.

Example 4.2.
(1) Let A be a space with discrete topology. Then $A \times X \to X$ defined by $(a, x) \mapsto x$ is a covering space for X. If $|A| < \infty$, then it is $|A|$-fold. This is a **product covering space**.
(2) If $n \in \mathbb{Z}^+$, define $S^1 \to S^1$ by $z \mapsto z^n$. This is an n-fold cover.
(3) The map $\mathbb{R}^1 \to S^1$ given by $\theta \mapsto e^{2\pi i\theta}$ is a covering map.

Definition 4.3. Let $p : Z \to X$ and a **lift** of $f : Y \to Z$ is a map $\widetilde{f} : Y \to Z$ s.t. $f = p \circ \widetilde{f}$ i.e.

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes. Existence is not obvious and uniqueness is not generally true.

In the case of covering maps, the issues indicated above do not arise.

Theorem 4.4 (**Unique Homotopy Lifting Property (UHLP)**)**.** If $p : \widetilde{X} \to X$ is a covering map, $f : Y \to X$ is a map with lift $\widetilde{f} : Y \to \widetilde{X}$, and $H : Y \times I \to X$ is a homotopy s.t. $H|_{Y\times\{0\}} = f$, then there exists a unique lift $\widetilde{H} : Y \times I \to \widetilde{X}$ s.t. $p \circ \widetilde{H} = H$ and $\widetilde{H}|_{Y\times\{0\}} = \widetilde{f}$.

Essentially, if $i_0(y) = (y, 0)$ so that $f = H \circ i_0$, then the theorem says that if the outer square of the following diagram commutes, then the dotted arrow exists and the smaller triangles commute:

$$\begin{tikzcd} Y \arrow[r, "\widetilde{f}"] \arrow[d, hook, "i_0"'] & \widetilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & X \end{tikzcd}$$

Corollary 4.4.1 (Path lifting property)**.** Let $p : \widetilde{X} \to X$ be a covering, $a \in X$, and $\widetilde{a} \in p^{-1}(a)$. Then

(1) for all paths $\gamma : I \to X$ with $a = \gamma(0)$, there exists a unique lift $\widetilde{\gamma} : I \to \widetilde{X}$ s.t. $\widetilde{\gamma}(0) = \widetilde{a}$;
(2) if $\gamma \simeq \gamma'$, then $\widetilde{\gamma} \simeq \widetilde{\gamma}'$ (and so $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$).

Proof. For (1), let Y be a point in the theorem. For (2), let $Y := I$. $\square$

Warning, the lifted loop may not be a loop in $\widetilde{X}$. This can already be seen with $\mathbb{R}^1 \to S^1$.

Proof of UHLP. The proof of UHLP can be found in Hatcher Theorem 1.17 Property (c).

First, the UHLP is true if the image of H is contained in an evenly covered neighborhood. We may assume Y is path connected and $\widetilde{f}$ is defined on a path component. These assumptions aid in the proof of this fact.

Next, the idea is to break $Y \times I$ into small pieces and apply the lemma. Then, glue together the maps. First, let $y \in Y$ and choose $V \ni y$ s.t. $\widetilde{H}$ extends on $V \times I$. This is possible since we can consider $I := \bigcup_{1 \le m \le n} [\frac{m-1}{n}, \frac{m}{n}]$ and write $\{y\} \times I = \bigcup_{1 \le m \le n} \{y\} \times [\frac{m-1}{n}, \frac{m}{n}]$ and choose n sufficiently large. Then H on each of these disjoint sets is contained in some evenly covered subset U_m. So there is a $V_m \ni y$ s.t. $H(V_m \times [\frac{m-1}{n}, \frac{m}{n}]) \subseteq U_m$ and let $V := \bigcap_{m=1}^n V_m$. So, $H(V \times [\frac{m-1}{n}, \frac{m}{n}]) \subseteq U_m$. Then define $\widetilde{H}$ via the preceding paragraph.

Next, if Y is a point, $\widetilde{H}$ is unique. This follows by breaking I into pieces.

For general Y, $\widetilde{H}$ is unique. Indeed, if $\widetilde{H_1}, \widetilde{H_2 2}$ ar elifts, we have

$$\widetilde{H_1}|_{\{y\} \times I} = \widetilde{H_2}|_{\{y\} \times I}$$

for all $y \in I$.

Finally, $\widetilde{H}$ itself exists. For each $y \in Y$, there exists $V_y \ni y$ s.t. $\widetilde{H_y} : V_y \times I \to \widetilde{X}$ exists. For all y, y', we have $\widetilde{H_y}|_{(V_y \cap V_{y'}) \times I} = \widetilde{H_{y'}}|_{(V_y \cap V_{y'}) \times I}$ by the preceding paragraph. Now, glue these maps together to get the desired homotopy lifting which is unique. □

5. Lecture 5: October 5th, 2021

Abstract. In this lecture, we prove that $\pi_1(S^1) \cong \mathbb{Z}$ through using the fact that $\mathbb{R}^1$ is a universal covering space of S^1. We also discuss the induced homomorphisms on fundamental groups which one obtains via the functor $\pi_1(-)$. This functor is well-behaved and commutes with the change-of-basepoint homomorphism.

Recall that we described that a covering space with covering map gives a local homeomorphism over neighborhoods of the space we cover. Furthermore, we showed that covering maps have the Unique Homotopy Lifting Property (UHLP) and then derived a some corollaries of this fact.

Corollary 5.0.1. For any $p : \widetilde{X} \to X$ covering map, any $x \in A$, take $\widetilde{a} \in p^{-1}(a)$. Then

1. given any path $\gamma : I \to X$ with $r(0) = 0$, there exists has a unique lift of γ s.t. $\widetilde{\gamma} : I \to \widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{a}$;
2. if $\gamma \simeq \gamma'$, then $\widetilde{\gamma} \simeq \widetilde{\gamma}$. In particular, $\widetilde{\gamma}(1)\widetilde{\gamma}'(1)$.
3. If $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$, then $\gamma \simeq \gamma'$.

Proof. We proved (1) and (2) last lecture, so we prove (3). If $\widetilde{X}$ is simply connected, we know $\widetilde{\gamma}(0) = \widetilde{\gamma}'(0)$ and $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$. Since $\widetilde{X}$ is simply connected, $\widetilde{\gamma} \simeq \widetilde{\gamma}'$. Then,

$$p \circ \widetilde{\gamma} \simeq p \circ \widetilde{\gamma}' \implies \gamma \simeq \gamma'.$$

□

Definition 5.1. A covering space $(\widetilde{X}, p)$ with $\widetilde{X}$ simply connected is called a **universal covering space**.

Proposition 5.2. The covering space $\mathbb{R}^1$ of S^1 is a universal covering space.

Proof $\pi_1(X, x_0) \cong \mathbb{Z}$. We now return to showing that $\pi_1(S^1, x_0) \cong \mathbb{Z}$. WLOG, set $x_0 = 1$. Recall $p : \mathbb{R}^1 \to S^1$ given by $\theta \mapsto e^{2\pi i \theta}$ was the covering map. Apply the previous corollary (3) to uniquely lift $[\gamma] \in \pi_1(S^1, x_0)$ to $[\widetilde{\gamma}] : I \to \mathbb{R}$ with $\widetilde{\gamma}(0) = 0$. Then $\widetilde{\gamma}(1) \in p^{-1}(0) = \mathbb{Z} \subseteq \mathbb{R}$. From the corollary, $\gamma \simeq \gamma'$ iff $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1) \in \mathbb{Z}$. Then this defines a map

$$\varphi : \pi_1(S^1, 1) \to \mathbb{Z} \qquad [\gamma] \mapsto \widetilde{\gamma}(1).$$

The integer that the classes of path are mapped to is known as the **winding number**. The map φ is injective from the corollary and one can check that it is a group homomorphism. Indeed,

$$\varphi[\gamma \cdot \eta] = \widetilde{\gamma \cdot \eta}(1) = \widetilde{\gamma}(1) + \widetilde{\eta}(1) = \varphi[\gamma] + \varphi[\eta]$$

because φ gives the number of times $[\gamma \cdot \eta]$ goes around the circle and the number of times it goes around the circle is the number of times one moves via $[\gamma]$ and then $[\eta]$.

Furthermore, it is surjective since we can define $\gamma_n : I \to S^1$ by $\theta \mapsto e^{2\pi i n\theta}$ which then has $\widetilde{\gamma_n}(1) = n$ and hence, $\varphi([\gamma_n]) = n$. So φ is an isomorphism and $\pi_1(S^1, 1) \cong \mathbb{Z}$. □

The map $\pi_1(-)$ is a functor from the **category of based spaces** to the **category of groups**. The objects on the left are (X, x_0) with morphisms $(X, x_0) \xrightarrow{f} (Y, y_0)$ which gets sent to $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0)$ where $y_0 = f(x_0)$. We define the map f_* explicitly bellowing.

Definition 5.3. Let $f : X \to Y$ with $f(x_0) = y_0$. If $\gamma : Z \to X$ is a loop based at x_0, then $f \circ \gamma$ is a loop based at $y_0 = f(x_0)$.

One can check that $\gamma_1 \simeq \gamma_2$ implies $f \circ \gamma_1 \simeq f \circ \gamma_2$.

Then define $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $[\gamma] \mapsto [f \circ \gamma]$. This f_* is called the **induced homomorphism** of π_1.

Proposition 5.4. In the setting of the definition above,

- f_* is a group homomorphism;
- $(f \circ g)_* = f_* \circ g_*$;
- $(\mathrm{id})_* = \mathrm{id}$;
- (constant map)$_* = 0$;
- if γ is a path from x_0 to x_1, we get a commutative diagram

$$\begin{tikzcd}
\pi_1(X, x_1) \arrow[r, "f_*"] \arrow[d, "\gamma_\#"] & \pi_1(Y, f(x_1)) \arrow[d, "(f \circ r)_\#"] \\
\pi_1(X, x_0) \arrow[r, "f_*"] & \pi_1(Y, f(x_0))
\end{tikzcd}.$$

Another question is what happens to $(f_0)_*$ and $(f_1)_*$ if $f_0, f_1 : X \to Y$ are homotopic? We have

$$(f_0)_* : \pi_1(X, x_0) \to \pi_1(Y, f_0(x_0))$$
$$(f_1)_* : \pi_1(X, x_0) \to \pi_1(Y, f_1(x_0)).$$

Now since $f_0 \simeq f_1$, the points $f_0(x_0)$ and $f_1(x_0)$ are in the same path component. Given a homotopy $H : X \times I \to Y$ from f_0 to f_1. We get a path $h : I \to Y$ from $f_0(x_0)$ to $f_1(x_0)$ given by $t \mapsto H(x_0, t)$. Using this, we the following theorem.

Theorem 5.5. The map $h_\#$ is an isomorphism and the following diagram commutes:

$$\begin{tikzcd}
 & \pi_1(Y, f_0(x_0)) \\
\pi_1(X, x_0) \arrow[ur, "(f_0)_*"] \arrow[dr, "(f_1)_*"'] & \\
 & \pi_1(Y, f_1(x_0)) \arrow[uu, "h_\#"']
\end{tikzcd}$$

Proof from lecture. Take $[\gamma] \in \pi_1(X, x_0)$. We show

$$[f_0 \circ \gamma] = [h \cdot (f_1 \circ f) \cdot \overline{h}]$$

For any $s \in 1$, take a path $\alpha_s : I \to I \times I$ given by $\alpha_s(t) = (s, t)$. Then we have maps

$$I \xrightarrow{\alpha_s} I \times I \xrightarrow{\gamma \times \mathrm{id}_I} X \times I \xrightarrow{H} Y.$$

If $s = 0$, this is $f_0 \circ \gamma$. While if $s = 1$, this is $h \cdot (f_1 \circ \gamma) \cdot \overline{h}$. $\square$

Hatcher's proof. Let $[\gamma] \in \pi_1(X, x_0)$. We show $[f_0 \circ \gamma] = [h \cdot (f_1 \circ f) \cdot \overline{h}]$ by exhibiting a homotopy. Define $h_t(s) := h(ts)$ is a path in Y. Essentially, it is a restriction of h to $[0, t]$. Then $h_t \cdot (H(\gamma(-), t)) \cdot \overline{h_t}$ is the desired homotopy because

$$t = 0 \implies f_0 \circ \gamma(-) \qquad \& \qquad t = 1 \implies h \cdot (f_1 \circ \gamma) \cdot \overline{h}.$$

$\square$

6. Lecture 6: October 6th, 2021

Abstract. In this lecture, we defined retracts, deformation retracts, and strong deformation retracts. We use this to show that S^1 is not a deformation retract of D^2 and Brouwer's fixed point theorem for $n = 2$. Our computation of the fundamental group also lends itself to another proof of the fundamental theorem of algebra.

Lemma 6.1. If $f_0 \simeq f_1 : X \to Y$ and $g_0 \simeq g_1 : Y \to Z$, then $g_0 \circ f_0 \simeq g_1 \circ f_1 : X \to Z$.

Proof. Assume $g_0 = g_1 = g$. Let $H : X \times I \to Y$ be a homotopy from f_0 to f_1, then $g \circ H : X \times I \to Z$ is a homotopy from $g \circ f_0$ to $g \circ f_1$. Similarly, assume $f_0 = f_1 = f$ and $H' : Y \times I \to Z$ is a homotopy from g_0 to g_1. Then

$$X \times I \xrightarrow{f \times \mathrm{id}_I} Y \times Z \xrightarrow{H'} Z$$

is a homotopy from $g_0 \circ f$ to $g_1 \circ f$. So in the general case, $g_0 \circ f_0 \simeq g_1 \circ f_0 \simeq g_1 \circ f_1$. $\square$

Lemma 6.2. (1) If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism for all $x_0 \in X$.
(2) If $f : X \to Y$ is null-homotopic, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is zero for any $x_0 \in X$.

Proof. We only prove (1). Assume $X = Y$. Then $f \simeq \mathrm{id}_X$ and

$$\begin{tikzcd}
 & \pi_1(X, f(x_0)) \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "\mathrm{id}_*"'] & \\
 & \pi_1(X, x_0) \arrow[uu, "\cong"']
\end{tikzcd}$$

and the commutativity of the diagram means f_* is an isomorphism. In the general case, $f : X \to Y$ being a homotopy equivalence means there is a $g : Y \to X$ s.t. $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Then $f_* \circ g_* = \mathrm{id}$ and $g_* \circ f_* = \mathrm{id}$. Then f_* is an isomorphism. $\square$

Corollary 6.2.1. If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Corollary 6.2.2. If $X \simeq \{*\}$ is contractible, then $\pi_1(X) = 0$.

Definition 6.3. Let A be a subspace of X. A **retraction from** X **to** A is a map $r : X \to A$ s.t. $r(a) = a$ for all $\in A$. We say A is a **retract of** X if there exists a retraction from X to A.

A **deformation retraction** form X to A is a map $H : X \times I \to X$ s.t.

(1) $H(x, 0) = x$ for all $x \in X$,
(2) $H(x, 1) = A$ for all $x \in A$,
(3) $H(a, 1) = a$ for all $a \in A$.

We say A is a **deformation retract** of X if there exists a deformation retraction from X to A. Essentially a deformation retraction is a homotopy between id_X and a retract $r : X \to A$.

A **strong deformation retraction** is a deformation retraction satisfying

(1) $H(a, t) = a$ for all $a \in A, t \in T$.

Lemma 6.4. A deformation retract A of X has embedding $i : A \hookrightarrow X$ that is a homotopy equivalence.

Proof. If $H : X \times I \to X$ is a deformation retract, define $r : X \to A$ by $x \mapsto H(x, 1) \in A$. Then $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_X$ (by using H, that is, observe that $i \circ r = H(x, 1) : X \to X$ and using H, this is homotopic to $H(x, 0) = \mathrm{id}_X$.). □

Example 6.5. Let $A := \{*\}$. Then A is always a retract of X since $X \to \{*\}$. A point is a deformation retract iff X is contractible.

Example 6.6. Let $A = S^1 \subseteq \mathbb{C}$ and $X = \mathbb{C} - \{z_0\}$ with $|z_0| < 1$, then A is a deformation retraction of X. In particular, if $z_0 = 0$, we can use the map $r : X \to A$ given by $r(z) = \frac{z}{\|z\|}$.

There are many other maps to get a deformation retract when $|z_0| < 1$.

If $|z_0| > 1$, then A is a not a retract of X. Indeed, if we use the next proposition, and note $\pi_1(X) \cong \mathbb{Z}$, then we must have $i_* : \pi_1(A) \hookrightarrow \pi_1(X)$. But a loop in A is homotopic to the constant loop in X when $|z_0| > 1$. Therefore, $i_*(\pi_1(A)) = 0$ which means it is not injective.

Proposition 6.7. If $i : A \hookrightarrow X$, then we have an induced map $i_* : \pi_1(A, a) \to \pi_1(X, a)$. Then,

(1) if A is a retract of X then i_* is injective;
(2) if A is a deformation retract of X, then i_* is an isomorphism.

Proof.
(1) Let $r : X \to A$ be a retract and so $r \circ i = \mathrm{id}_A$ which means $r_* \circ i_* = (\mathrm{id}_A)_*$. So i_* is injective.
(2) There is a retract $r : X \to A$ s.t. $r_* = (\mathrm{id}_X)_*$. But then, $(r \circ i)_* = (\mathrm{id}_A)_* = (\mathrm{id}_X \circ i)_* = i_*$. So, $i_* = (\mathrm{id}_A)_*$. That is, i_* is an isomorphism.

□

Corollary 6.7.1. Despite S^1 being the boundary of the disk D^2, S^1 is not a retract of D^2.

Proof. We have $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(D^2) = 0$ so if S^1 were a retract, then $i_* : \mathbb{Z} \to 0$ would be injective. This is absurd. □

Theorem 6.8 (Brouwer Fixed Point Theorem for $n = 2$)**.** Every continuous map $f : D^2 \to D^2$ has a fixed point.

Proof. Assume $f(x) \neq x$ for all $x \in D^2$. Then define $r : D^2 \to S^1$ via the ray projection starting from $f(x)$ to x and eventually crossing S^1 (take the point crossing S^1 as $r(x)$). Then $r|_{S^1} = \mathrm{id}_{S^1}$. We know r is well-defined since $f(x) \neq x$ for all $x \in D^2$.

Then r is actually a retraction from D^2 to S^1. Indeed, $r(x) = x$ for all $x \in S^1$ and the projection itself is continuous. However, the proposition implies $i_* : \pi_1(S^1) \hookrightarrow \pi_1(D^2)$ which is absurd. Contradiction. $\square$

Remark 2. This proof cannot generalize to higher dimensions. We will need to use "better" functors in place of π_1 to do the job.

Theorem 6.9 (Fundamental Theorem of Algebra)**.** Any non-constant polynomial has a root in $\mathbb{C}$.

Proof. Let $p(z) := z^n + \cdots + a_{n-1}z^{n-1} + \cdots + a_0 \in \mathbb{C}[z]$ for $n \geq 1$. Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $p : \mathbb{C} \to \mathbb{C} - \{0\}$. Let $R > 0$, then define $f_R : S^1 \to \mathbb{C} - \{0\}$ given by $z \mapsto p(Rz)$. Then $f_R \simeq 0$ since we can extend f_R to the disk D^2. Now take $R > 1 + |a_{n-1}| + \cdots + |a_0|$.

We claim that $f_R \simeq (Rz)^n \not\simeq 0$. This can be shown using a linear homotopy and the fact that

$$|Rz|^n > |a_{n-1}(Rz)^{n-1} + \cdots + a_0|.$$

Indeed,

$$H(z,t) = (Rz)^n + t(a_{n-1}(Rz)^{n-1} + \cdots + a_1 Rz + a_0)$$

works. Define

$$g_n := (Rz)^n : S^1 \to \mathbb{C} - \{0\} \overset{z \mapsto \frac{z}{|z|}}{\to} S^1.$$

Then

$$(g_n)_* : \pi_1(S^1) \overset{1 \mapsto n}{\to} \pi_1(\mathbb{C} - \{0\}) \overset{\cong}{\to} \pi_1(S^1) \qquad \text{going from } \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}.$$

This is a contradiction because this shows $(Rz)^n \not\simeq 0$ and $0 \simeq f_R \simeq (Rz)^n \not\simeq 0$ which is absurd. $\square$

7. Lecture 7: October 8th, 2021

Abstract. This lecture discusses certain categorical constructions. A review of group presentations is also provided.

We begin by discussing products and coproducts from category theory.

Definition 7.1. Let $\mathcal{C}$ be a category and X_1, X_2 objects in $\mathcal{C}$. A **product** of X_1 and X_2 denoted by $X_1 \times X_2$ is the following data

(1) there are maps $p_i : X_1 \times X_2 \to X_i$,
(2) for maps $f_i : Y \to X_i$ s.t. the big triangle in the following following diagram commutes

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists! f"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & X_2 \end{tikzcd}$$

there exists a uniqe $f : Y \to X_1 \times X_2$ making the entire diagram commute.

Example 7.2. (1) In the category of sets, a product is $X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$ and the p_i are the usual projection maps.

(2) In the category of topological spaces, the product is $X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$ with the product topology.
(3) In the category of pointed topological spaces, the product is $(X_1 \times X_2, (x_1, x_2))$.
(4) In the category of groups, $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G, g_2 \in G\}$ with a group structure.

Definition 7.3. Let $\mathcal{C}$ be a category and X_1, X_2 objects in $\mathcal{C}$. A **coproduct** of X_1 and X_2, denoted by $X_1 \coprod X_2$ (or $X_1 \sqcup X_2$), is the following data

(1) there are maps $p_i : X_i \to X_1 \sqcup X_2$,
(2) for maps $f_i : X_i \to Y$ s.t. the big triangle in the following following diagram commutes

$$\begin{tikzcd} & Y & \\ X_1 \arrow[ur, "f_1"] & X_1 \coprod X_2 \arrow[l, "p_1"'] \arrow[u, dashed, "\exists! f"] \arrow[r, "p_2"'] & X_2 \arrow[ul, "f_2"'] \end{tikzcd}$$

there exists a unique $f : X_1 \sqcup X_2 \to Y$ making the entire diagram commute. That is, it satisfies a universal property.

Example 7.4. (1) In the category of sets, the coproduct $X_1 \sqcup X_2$ is a disjoint union.
(2) In the category of topological spaces, it is a disjoint union of spaces.
(3) In the category of pointed topological spaces, the coproduct is $(X_1 \sqcup X_2)/(x_1 \sim x_2)$. Essentially, we have $X_1 \vee X_2$ where we identified the base points.
(4) In the category of abelian groups, $G_1 \oplus G_2$ is the coproduct of two groups.
(5) In the category of groups, the coproduct is the free product $G_1 * G_2$ which we define later this lecture.

Proposition 7.5. We have $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. The universal property says that the maps (f_1, f_2) are in bijection with f and $(f_1/\simeq, f_2/\simeq)$ is in bijection with $f/\simeq$ where $f/\simeq$ indicates the homotopy class of the map. Thus,

$$\pi_1(X, x_0) \times \pi_1(Y, y_0) \cong \pi_1(X \times Y, (x_0, y_0)).$$

$$\begin{tikzcd} & (S, *) \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists! f"] \arrow[dr, "f_2"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "p_1"] \arrow[r, "p_2"'] & (Y, y_0) \end{tikzcd}$$

□

Example 7.6. Since $\pi_1(S^1) \cong \mathbb{Z}$, we deduce that $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$. Similarly, $\pi_1(S^1 \times S^1 \times S^1 \times \dots) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \dots$ for some finite number of copies of S^1.

Definition 7.7. Let G, H be a groups. A **word** in G and H is a product $s_1 s_2 \dots s_n$ with $s_i \in G$ or $s_i \in H$. Such a word can be **reduced**

(1) remove any identity elements e_G, e_H,
(2) replace pairs $g_1 g_2$ by their product in G and do the same for H.

Thus, every reduced word has form similar to $g_1 h_1 g_2 \dots g_k h_k$ (and $g_1 = e$ or $h_k = e$ could occur, but we do not indicate this here).

The **free product** of G and H, denoted by $G * H$, is defined as the group of reduced words with the usual multiplication. The identity is $e_G = e_H = e$ and $(g_1h_1g_2h_2)^{-1} = h_2^{-1}g_2^{-1}h_1^{-1}g_1^{-1}$. The product is well-defined.

The **abelianization** of G is $G/[G,G] = \mathrm{Ab}(G)$ where $[G,G]$ is generated by elements of the form $g_1g_2g_1^{-1}g_2^{-1}$.

Lemma 7.8. We have $\mathrm{Ab}(G * H) \cong \mathrm{Ab}(G) \oplus \mathrm{Ab}(H)$.

Definition 7.9. Let S be a set. The **free group generated by** S denoted by F_S is

$$\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$$

where a copy of $\mathbb{Z}$ appears for each $s \in S$. That is, $F_S = *_{s\in S}\mathbb{Z}$.

Definition 7.10. Let $R \subseteq F_S$ and the **group generated by** S **with relations** R, defined $\langle S|R\rangle$ is defined as follows. Let N_R be the normal subgroup F_S generated by R. That is,

$$N_R := \{\text{products of } \alpha^{-1}r^{\pm 1}\alpha : \alpha \in F_S, r \in R\}$$

Then $\langle S|R\rangle := F_S/N_R$ and we call S a set of generators and R a set of relations. Essentially, it is obtained from F_S by imposing the relation $r = e$ for all $r \in R$.

Example 7.11.
(1) $\langle a,b|aba^{-1}b^{-1}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$,
(2) $\langle a,b|aba^{-1}b^{-1}, a^2, b^2\rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
(3) $\langle a,b|aba, ba\rangle = \{e\} = \langle a|a\rangle$.
(4) $\langle S|R\rangle$ is called a **presentation** of a group G if $G \cong \langle S|R\rangle$.

Definition 7.12. The **rank** of $G \cong \langle S|R\rangle$ is the minimal size of S and if $\mathrm{rank}(G) < \infty$, we say G is finitely generated. If both S, R are finite sets, then G is **finitely presented**.

Remark 3. The **word problem** asks if one can determine when two presentations give isomorphic groups. It turns out that it is unsolvable.

Theorem 7.13 (Boone-Rogers)**.** The word problem is unsolvable. There is no uniform algorithm that determines whether $\langle S|R\rangle$ is trivial or not for arbitrary S, R.

8. Lecture 8: October 11, 2021

Abstract. We state Van Kampen's theorem, and sketch a proof of the theorem.

Theorem 8.1. Suppose X is path-connected, $\bigcup_{\alpha\in S} A_\alpha$ is an open cover of X s.t. $x_0 \in X$ is a basepoint, $x_0 \in A_\alpha$ for all $\alpha \in S$, and let us denote the induced inclusions by

$$i_\alpha : \pi_1(A_\alpha, x_0) \hookrightarrow \pi_1(X, x_0) \qquad \& \qquad i_{\alpha\beta}\pi_1(A_\alpha \cap A_\beta, x_0) \hookrightarrow \pi_1(A_\alpha, x_0).$$

(1) Suppose $A_\alpha \cap A_\beta$ is always path connected. Then

$$\Phi : *_{\alpha\in S}\pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

is surjective.
(2) Suppose $A_\alpha \cap A_\beta \cap A_\beta$ is always path-connected. Then

$$\ker(\Phi) := \text{ normal subgroup generated by } \{i_{\alpha\beta}(w)[i_{\beta\alpha}(w)]^{-1} : \alpha, \beta \in S, w \in \pi_1(A_\alpha \cap A_\beta, x_0)\}$$

If (1) and (2) hold, then $\pi_1(X, x_0) \cong *_{\alpha\in S}\pi_1(A_\alpha, x_0)/N$.

Proof sketch. Ignore x_0 in the notation. Given $h \in \pi_1(X)$, we define a factorization of h to be a sequence $(h_1, \dots, h_n)$ s.t. $h_i \in \pi_1(A_{\alpha_i})$ and

$$i_{\alpha_1}(h_1) \dots i_{\alpha_n}(h_n) = h.$$

Define an equivalence relation on all factorizations by saying that two factorizations are equivalent if they are related by a series of the following transformation (and their inverses):

(1) If $\alpha_i = \alpha_{i+1}$, then $(h_1, \dots, h_i, h_{i+1}, \dots, h_n) \sim (h_1, \dots, h_i \cdot h_{i+1}, \dots, h_n)$.
(2) If $h_i = i_{\alpha_i \beta_i}(w)$ for some $w \in \pi_1(A_{\alpha_i} \cap A_{\beta_i})$, then set

$$h_i' := i_{\beta_i \alpha_i}(w).$$

Then we define $(h_1, \dots, h_i, \dots, h_n) \sim (h_1, \dots, h_i', \dots, h_n)$.

We make three claims:

(1) Two equivalent factorizations give rise to the same h.
(2) Any h has factorization.
(3) Any two factorizations of h are $\sim$.

Any factorization gives an element in the free product. An element $h \in \pi_1(X)$ has a factorization iff $h \in \mathrm{im}(\Phi)$. So the quotient obtained from the group group is just adding relations which give a way to identify two factorizations are equivalent.

The proof of claim (1) is relatively easy to see. So, we focus on proving claim (2). The idea is that if we have a loop $h : I \to X$, we can decompose the unit interval $0 = t_0 < t_1 < \cdots < t_n = 1$ s.t. each $f([t_i, t_{i+1}]) \subseteq A_{\alpha_i}$ for all i. Now, for all i, $f(t_i) \in A_{\alpha_i} \cap A_{\alpha_i + 1}$, pick a path p_i that goes from x_0 to $f(t_1)$. Then,

$$f|_{[t_0,t_1]} \cdot \overline{p_1} \cdot p_1 \cdot f|_{[t_1,t_2]} \cdot \overline{p_2} \cdot p_2 \dots f|_{[t_{n-1},t_n]}.$$

Now, for each $f|_{[t_i,t_{i+1}]} \cdot \overline{p_i} \cdot p_i \in \pi_1(A_{\alpha_i})$.

For the proof of claim (3), see Hatcher's text. The idea is that if we have two factorizations, there is a homotopy $H : I \times I \to X$ between them. The $I \times I$ is a square and the idea is to cut up the square and disturb each square slightly. $\square$

Remark 4. Each of the hypotheses of Van Kampen's theorem are as general as they can be.

9. Lecture 9: October 13, 2021

Abstract. We derive some consequences of the Van Kampen's theorem. For instance, we shall show that fundamental group of the wedge sum is just the free product of individual groups provided our spaces are nice.

Corollary 9.0.1. Let $X = \bigcup_{\alpha \in S} A_\alpha$ s.t. A_α is open and simply connected for all α, $\bigcap_{\alpha \in S} A_\alpha \neq \emptyset$, $A_\alpha \cap A_\beta$ is path-connected for all $\alpha, \beta \in S$. Then, X is simply connected.

Proof. Choose a basepoint in $\bigcap_{\alpha \in S} A_\alpha$. Part (1) of van Kampen's theorem is fulfilled and the surjection implies $\pi_1(X) = 0$. $\square$

Remark 5. The philosophy of Van Kampen's theorem is we can build up the fundamental group of a space via the subspaces. However, we cannot use it to compute $\pi_1(S^1) \cong \mathbb{Z}$ because we would need a subspace with nontrivial fundamental group.

Lemma 9.1. For all $a \in S^n$, $S^n - \{a\}$ is homeomorphic to $\mathbb{R}^n$.

Proof. Rotate to assume $a = (1, 0, \dots, 0)$. Then, let $p : S^n - \{a\} \to \mathbb{R}^n$ be given by stereographic projection and it is a known fact that this is a homeomorphism of $S^n - \{a\}$ to $\mathbb{R}^n$. $\square$

Corollary 9.1.1. For $n \geq 2$, S^n is simply connected.

Proof. Let $a \neq b$ be points on S^n. Then $A_1 = S^n - \{a\}$ and $A_2 = S^n - \{b\}$ are an open cover of $S = A_1 \cup A_2$. Furthermore, $A_1 \cap A_2 = S^n - \{a, b\} \cong \mathbb{R}^n - \{\text{point}\}$ is path-connected when $n \geq 2$. By the lemma, $\pi_1(A_i) = 0$. So, Van Kampen's theorem says $\pi_1(S^n) = 0$. $\square$

Definition 9.2. Recall that a pointed space is a pair (X, x_0) with $x_0 \in X$ called a base point of X. Let $\{(X_\alpha, x_\alpha)\}_{\alpha \in S}$ be a collection of pointed spaces. We defined the **wedge** of the spaces to be

$$\bigvee_{\alpha \in S} X_\alpha := \left(\coprod_{\alpha \in S} X_\alpha \right) \Big/ \sim$$

where $x_\alpha \sim x_\beta$ for all $\alpha, \beta \in S$.

Our goal is to show that $\pi_1(\bigvee_\alpha X_\alpha) = *_\alpha \pi_1(X_\alpha)$. However, if we apply Van Kampen's theorem with $A_\alpha = X_\alpha$, we cannot guarantee that A_α is always open. This motivates the next definition.

Definition 9.3. A base point x_0 is **nice** if there exists an open neighborhood $U \ni x_0$ in X s.t. U strongly deformation retracts to x_0. That is, there exists a homotopy $H : U \times I \to x_0$ s.t.

$$H(x, 0) = x \ \forall x \in U, \ H(x, 1) = x_0 \ \forall x \in U, \ H(x_0, t) = x_0 \ \forall t \in I.$$

Proposition 9.4. Let $X = \bigvee_{\alpha \in S} X_\alpha$. Suppose x_α is nice for all $\alpha \in S$. Then $\pi_1(\bigvee_{\alpha \in S} X_\alpha) \cong *_{\alpha \in S} \pi_1(X_\alpha)$.

Remark 6. The nice condition is wild. It is satisfied by a large class of spaces. For instance, CW-complexes which will be introduced in the next lecture.

Proof. Let $A_\beta := X_\beta \vee \left(\bigvee_{\alpha \in S, \alpha \neq \beta} U_\alpha \right)$ where each of the U_α are the nice open neighborhood of x_β in X_α. Then $X = \bigcup_{\alpha \in S} A_\alpha$. Furthermore, if $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = \bigvee_{\alpha \in S} U_\alpha$ is simply connected.

By Van Kapmpen's theorem, $\pi_1(\bigvee_{\alpha \in S} X_\alpha\}) \cong *_{\alpha \in S} \pi_1(A_\alpha)$. But $\pi_1(A_\alpha) \cong \pi_1(X_\alpha)$ since A_α strongly deformation retracts to X_α. So we get the desired isomorphism. $\square$

Example 9.5. We have $\pi_1(\bigvee_{n=1}^m S^1) \cong *_{n=1}^m \mathbb{Z}$.

Example 9.6. Let $X := \mathbb{R}^2 \setminus \{p_1, \dots, p_n\}$. Then X is homotopic to the wedge of n copies of S^1. Thus, $\pi_1(X) \cong *_{m=1}^n \mathbb{Z}$.

Example 9.7. Let $X := \mathbb{R}^2 - \{p_1, p_2, \dots, p_n, \dots\}$. There are two cases.

If $\{p_i\}$ does not converge, then $\pi_1(X) = \pi_1(\bigvee_{n=1}^\infty S^1) = *_{n=1}^\infty \mathbb{Z}$.

If $\{p_i\}$ converges, say for instance we have the points $p_i = \left(\frac{1}{i}, 0\right)$, then we get what is called the Hawaiian earring. In this case, $H := X \not\simeq \bigwedge_{n=1}^\infty S^1$. The Hawaiian earring is not nice since we can define $f_n : H \to C_n$ which collapses all other circles to C_n. This induces a map $(f_n)_* : \pi_1(H) \to \pi_1(C_n)$. Then, $\prod (f_n)_* : \pi_1(H) \to \prod_{n=1}^\infty \mathbb{Z}$.

We claim that $\prod (f_n)*$ is surjective. Indeed, we can define a loop $\gamma : I \to H$ s.t. $\gamma(0) = \gamma(1) = \gamma(1/2) = \cdots = 0$ and $|\gamma|_{[\frac{1}{n+1}, \frac{1}{n}]}$ goes around C_n each a_n-times. That is, we map $[\gamma] \mapsto (a_1, a_2, \dots)$. Therefore, $\pi_1(H) \to \prod_{n=1}^{\infty} \mathbb{Z}$ is surjective and the number of loops is uncountable because $|\prod_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}| = 2^{\mathbb{N}}$.

10. Lecture 10: October 15, 2021

Abstract. This lecture introduces the concept of a CW complex and discusses a more general construction called a "attaching along a map". The process of constructing a CW is just a special case of this more general construction.

The idea of CW complexes is to build up spaces by gluing cells. Cells shall be the building blocks for certain spaces. We fix notation

$$D^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$
$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

To discuss CW complexes, we begin with a general construction. Let $A \subseteq B$ be denoted by (B, A). Given a map $f : A \to X$, we can define

$$X \cup_f B := X \sqcup B / \sim$$

where $\sim$ is generated by $x \sim f(x)$ for all $x \in A$. We call $X \cup_f B$ the space obtained by attaching B to X along A via the attaching map f.

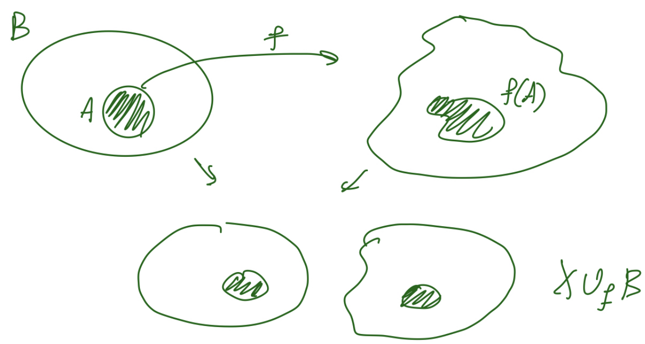


Figure 3. Method of gluing one space to another.

There three types of points in the resulting space:

- for all $x \in X \setminus \operatorname{im} f$, then x is in an equivalence class by itself,
- for all $b \in B \setminus A$, then b is in an equivalence class by itself,
- for all $x \in \operatorname{im} f$, then $\{x\} \sim \{f^{-1}(x)\}$ is the equivalenc class. Note that $f^{-1}(x)$ may not be a singleton set.

A special case is pictured in figure 4.

The construction gives a diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, hook, "\text{inclusion}"] \\ B \arrow[r, hook, "\text{inclusion}"'] & X \cup_f B \end{tikzcd}$$

which is called a **pushout diagram**. The pushout construction satisfies a universal property.

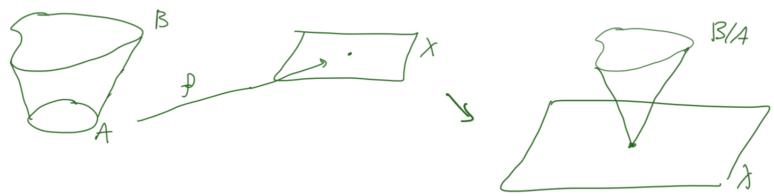


FIGURE 4. Gluing B to X by collapsing A.

Proposition 10.1. Suppose there are maps $j : X \to Y$ and $g : B \to Y$ s.t. $j \circ f = g \circ i$. Then there exists a unique map $X \cup_f B \to Y$ making the whole diagram commute.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, hook, "\text{inclusion}"] \arrow[ddr, bend left, "j"] & \\
B \arrow[r, hook, "\text{inclusion}"'] \arrow[drr, bend right, "g"'] & X \cup_f B \arrow[dr, dotted, "\exists!"] & \\
 & & Y
\end{tikzcd}$$

PROOF. Trivial. □

Example 10.2.
- If X is a point, then $X \cup_f B = B/A$.
- If $X = \emptyset$, then $X \cup_f B = B \coprod X$.
- If A is a point, then $X \cup_f B = B \wedge X$.
- If $(B, A) = (D^n, S^{n-1})$, then $X \cup_f D^n$ is called attaching an n-cell to X along f.

Example 10.3. Recall that the **torus** can be constructed using a gluing diagram of the square $I \times I$. Here, we show how to obtain the torus via gluing cells.

Let X_0 be a single point. Attach two 1-cells to the point to get a figure eight. Give a label of the figure eight as in figure 5.

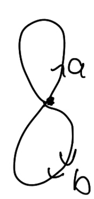


FIGURE 5. The figure eight with an orientation.

Now, we attach a single 2-cell (D^2, S^1) via $S^1 \mapsto aba^{-1}b^{-1}$ and from the diagram

$$\begin{tikzcd}
S^1 \arrow[r, "f"] \arrow[d] & X_1 \arrow[d] \\
D^2 \arrow[r] & X_2 = T^2
\end{tikzcd}$$

The map $S^1 \mapsto aba^{-1}b^{-1}$ glues S^1 along the path a, b, a^{-1}, b^{-1}.

Definition 10.4. We say X is a CW-complex if there exists a sequence of subspaces where $\emptyset = \mathrm{sk}_{-1} X$ (here "sk" stands for skeleton)

$$\emptyset = \mathrm{sk}_{-1} X \subseteq \mathrm{sk}_0 X \subseteq \mathrm{sk}_1 X \subseteq \cdots \subseteq X$$

s.t.

(1) $X = \cup_n \mathrm{sk}_n X$,
(2) U is open (resp. closed) in X iff $U \cap \mathrm{sk}_n(X)$ is open (resp. closed). in $\mathrm{sk}_n(X)$,
(3) for all $n \geq 0$, $\mathrm{sk}_n X$ fits into a pushout diagram

$$\begin{tikzcd} \sqcup_{\alpha \in A_n} S^{n-1}_\alpha \arrow[r, "f_n"] \arrow[d] & \mathrm{sk}_{n-1} X \arrow[d] \\ \sqcup_{\alpha \in A_n} D^n_\alpha \arrow[r, "g_n"'] & \mathrm{sk}^X_n \end{tikzcd}.$$

That is, $\mathrm{sk}_n X$ is obtained from $\mathrm{sk}_{n-1} X$ by attaching n-cells. Here, $\{D^n_\alpha\}$ is indexed by A_n. Here, $f_n = \sqcup f_{n,\alpha}$ is given by $f_{n,\alpha} : S^{n-1}_\alpha \to \mathrm{sk}_{n-1} X$ is called **attaching maps for n-cells** D^n_α.

If $X = X_n$ for some n, then we say X is an **n-dimensional CW-complex** (so dimension is a topological invariant).

Example 10.5. We showed $X = T^2 = S^1 \times S^1$ is a CW-complex with $\mathrm{sk}_0 X$ a point, $\mathrm{sk}_1 X = S^1 \vee S^1$, and $\mathrm{sk}_2 X = X$.

Definition 10.6. A CW complex X is **finite-dimensional** if $X = \mathrm{sk}_n X$ for some n. It is **of finite type** if each A_α is finite (finite may cells in each dimension). It is **finite** if it is finite dimensional and of finite type.

Proposition 10.7. (1) Any CW complex is Hausdorff.
(2) Any CW complex is compact iff it is finite.
(3) Any smooth manifold admits a CW structure.

11. Lecture 11: October 18, 2021

Abstract. We compute the fundamental group of CW-complexes and classify one and two dimensional complexes.

Let us classify the carious CW-complexes. For 0-dimensional CW complexes, we simply have a disjoint union of points. So, $X = \mathrm{sk}_0 X = \coprod\{*\}$ is a discrete topological space.

The situation for 1-dimensional CW complexes is more interesting. These are graphs.

Definition 11.1. A graph is called a **tree** if it contains no "loop" i.e. it contains no cycles. A tree is contractible if it is connected.

A sub-graph T of X is a **maximal tree** if

(1) T is a tree,
(2) T contains all vertices of X,
(3) if we an edge $e \notin X \setminus T$, then $e \cup T$ is not a tree.

Proposition 11.2. A connected graph has a maximal tree.

Theorem 11.3. If T is a maximal tree of a connected graph X and S parameterizes the edges in $X \setminus T$, then $\pi_1(X) \cong F_S$ where F_S is the free group generated by S.

Proof. For each edge $\alpha \in S$, pick p_α in the interior of the corresponding edge e_α.

Define $U_\alpha = X - \{p_\beta\}_{\beta \in S - \{\alpha\}}$. Then U_α is open and deformation retracts to S^1. If $\alpha \neq \alpha'$, then $U - \alpha \cap U_{\alpha'} = X - \{p_\alpha\}_{\alpha \in S}$ is contractible.

By Van Kampen's theorem, $\Phi : *\pi_1(U_\alpha) \to \pi_1(X)$ and $*\pi_1(U_\alpha) \cong F_S$. This is an isomorphism because $i_{\alpha\beta}(w) i_{\alpha\beta}(w)^{-1}$ is trivial for all $w \in \pi_1(U_\alpha \cap U_\beta) = 0$. □

Now we consider the case of 2-dimensional CW-complexes. In the following theorem, we do not concern ourselves with basepoints for our fundamental groups.

Theorem 11.4. Consider the pushout diagram

$$\begin{tikzcd} \coprod S^1_\alpha \arrow[r, "\coprod f_\alpha"] \arrow[d, hook] & Y \arrow[d, hook] \\ \coprod D^2_\alpha \arrow[r] & X \end{tikzcd}$$

where X is connected. The maps f_α determine a loop in Y and therefore an element in $\pi_1(Y)$. Let N be the normal subgroup of $\pi_1(Y)$ generated by $[f_\alpha]_{\alpha \in S}$.

The inclusion $Y \to X$ induces a surjective map $\pi_1(Y) \to \pi_1(X)$ and the kernel is N. In particular, $\pi_1(X) \cong \pi_1(Y)/N$.

Proof. We start with simple case of a single D^2. Then $X = Y \cup_f D^2$. See figure 6 for a visual.

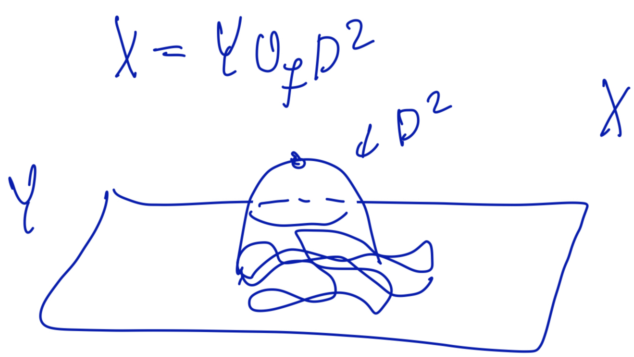


Figure 6. The squiggles represent the image $f(S^1)$ one which D^2 is attached.

Let $U_1 := X \setminus \{*\}$ where $*$ is the center of D^2. Then U_1 deformation retracts to Y. Let $U_2 = \operatorname{Int} D^2$. Then U_2 is contractible.

By Van Kampen's theorem,

$$\Phi : \pi_1(Y) \cong \pi_1(U_1) * \pi_1(U_2) \to \pi_1(X)$$

is a surjective map with kernel generated by $[f]$. But then we get $\pi_1(Y)/N \cong \pi_1(X)$.

This finishes the proof for a single 2-cell. It is not hard to prove the result for finitely many 2-cells. Furthermore, it is a *fact* that any compact subset of a CW complex can only intersect finitely many cells. This can be used for the case of infinitely many cells. □

Theorem 11.5. Suppose X is obtained from Y by attaching cells of dimension dim ≥ 3. Then the inclusion $Y \hookrightarrow X$ induces an isomorphism on π_1.

The proof of the theorem is essentially the same as the last.

Corollary 11.5.1. One has $\pi_1 X \cong \pi_1(\operatorname{sk}_2 X)$ for a CW-complex X.

12. Lecture 12: October 20, 2021

Abstract. We do some computations using CW-complexes and demonstrate an example on how to construct and think about the universal covering space.

Let us demonstrate some examples of computations with cell complexes.

Example 12.1. Let K be the Klein bottle. It has gluing pattern as in figure 7. The cell

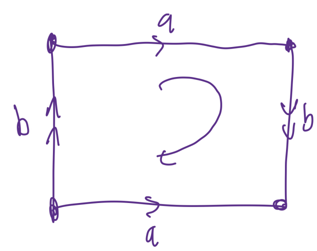


Figure 7. We identify a pair of edges in the obvious way. We identify the remaining pair of edges in the "opposite".

structure is constructed as follows. First, $\operatorname{sk}_0 X = \{a\}$ is a single point because all four points are identified. Let $\operatorname{sk}_1 X = S^1 \vee S^1$ be a figure eight with each of them labeled a, b. Then, glue D^2 onto $\operatorname{sk}_1 X$ via $S^1 \to aba^{-1}b$. Then the presentation of $\pi_1(K)$ is

$$\pi_1(K) \cong \langle a, b \mid aba^{-1}b \rangle.$$

Example 12.2. Consider the sphere S^2. It has a cell structure with 2 0-cells, 2 1-cells, and 2 2-cells. It is easy to show via the cell structure that $\pi_1(S^2) = 0$.

Let $X = S^2/(x \sim -x)$ be the quotient space after we identify antipodal points. Then we get a CW-structure on X that has 1 0-cell, 1 1-cell, and 1 2-cell. We call X the **real projective plane** and denote it $\mathbb{R}P^2$. The construction of the cell complex is as follows.

First, $\operatorname{sk}_0 \mathbb{R}P^2 = \{*\}$ is a singleton. Then $\operatorname{sk}_1 \mathbb{R}P^2 \simeq S^1$ attaches a single 1-cell. Then $\operatorname{sk}_2 \mathbb{R}P^2 = \mathbb{R}P^2$ attaches a single 2-cell which goes around the circle twice. In this case, one finds

$$\pi_1(\mathbb{R}P^2) \cong \langle a \mid a^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Example 12.3. The higher dimension sphere S^n has a CW-structure consisting of 2 i-cells for $0 \leq i \leq n$. The quotient where we identify antipodal points is $S^n/(x \sim -x) \simeq \mathbb{R}P^2$ which has a CW-structure consisting of 1 i-cell for $0 \leq i \leq n$. Also, $\operatorname{sk}_m \mathbb{R}P^n = \mathbb{R}P^m$ for $n \geq m$ and for $n \geq 2$,

$$\pi_1(\mathbb{R}P^n) \cong \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}.$$

We now begin discussing π_1 *in more detail and covering spaces.* We recall some definitions.

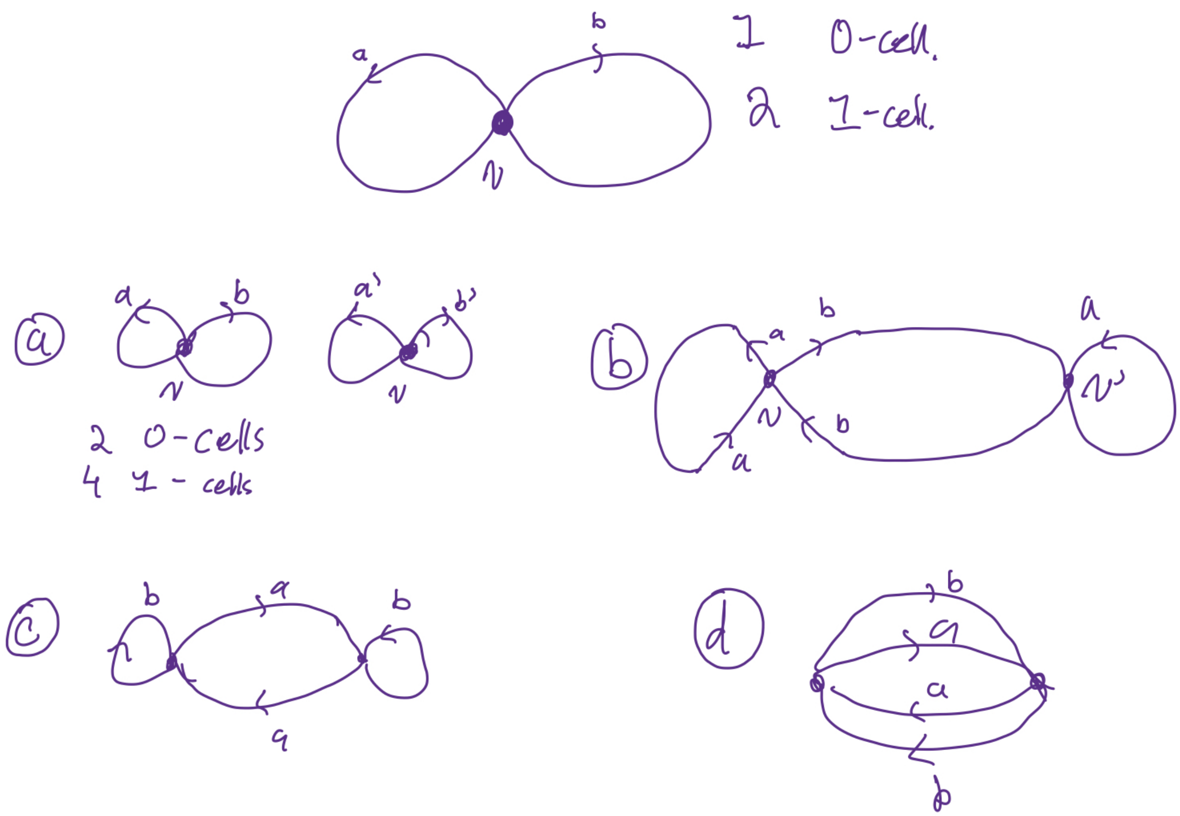


FIGURE 8. The covering spaces map each vertex onto the vertex v of $S^1 \vee S^1$. Observe that each vertex has a and b each coming in and out exactly once.

Definition 12.4. A map $p : \widetilde{X} \to X$ is a **covering map** if for all $x \in X$, there exists a neighborhood $U \ni x$ s.t.

$$p^{-1}(U) = \coprod_{\alpha} U_\alpha$$

and each U_α is open with $p|_{U_\alpha}$ a homeomorphism. Each U_α is called a *sheet* of $\widetilde{X}$ over U. We called $\widetilde{X}$ a covering space of U. The cardinality of $p^{-1}(x)$ is the number of sheets over any evenly covered $U \ni x$. We have a natural function $X \to \mathbb{Z} \cup \{+\infty\}$ given by $x \mapsto |p^{-1}(x)|$. This function is locally constant so if X is connected, it is constant.

Example 12.5. Let $\mathbb{R}^1 \to S^1$ be given by $\theta \mapsto e^{2\pi i n\theta}$. Then $\mathbb{R}^1$ is a **universal cover** of S^1 i.e. it is a simply connected cover and this particular map is an n-fold cover.

Example 12.6. Let $S^n \to \mathbb{R}P^n$ be the map onto the quotient with $n \geq 2$. This is a 2-fold cover and $\pi_1(S^n) = 0$. So, S_n is a universal cover for $\mathbb{R}P^n$.

Let us discuss covering spaces for CW-complexes.

Example 12.7. Let $X = S^1 \vee S^1$ be as pictured in figure 8

The key idea is that locally around the point, we have a going into the point and out of the point once each and the same goes for b.

The next few lectures will be focused on discussing covering spaces. The question one may have is: what is the universal cover of $S^1 \vee S^1$? Well locally, we want something as in figure 9.

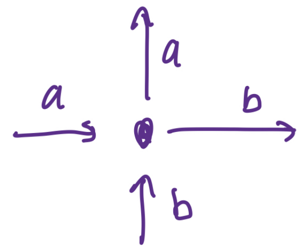


FIGURE 9. We want a and b coming in and out once.

If one continues this picture in each direction by adding vertex points at the ends of each arrows, we get an object as in figure 10.

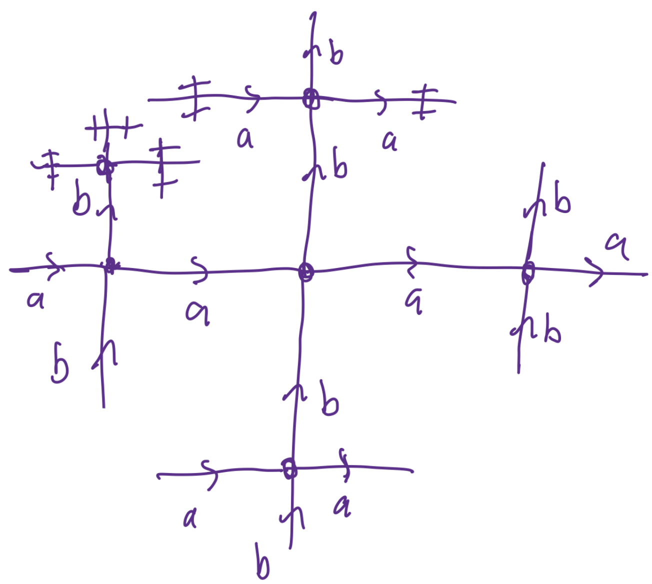


FIGURE 10. The universal cover of $S^1 \vee S^1$. It contains all of the other possible covers of $S^1 \vee S^1$ and is simply connected since we can deformation retract the entirety of the space to a single point.

13. Lecture 13: October 22, 2021

ABSTRACT. To discuss covering spaces in more detail, we prove some basic results using the Unique Homotopy Lifting Property. In particular, the importance of the covering space is due to the lifting property and *not* the fact that they have a local homeomorphism onto the space X.

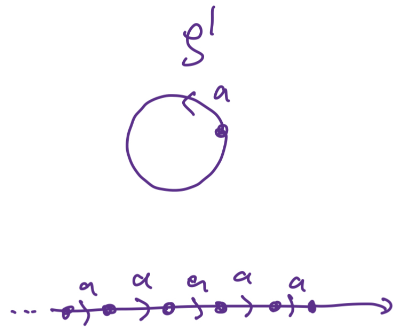


Figure 11. The universal cover of S^1 following the construction for that of $S^1 \vee S^1$.

We begin by recalling some facts. A lifting of $f : Y \to X$ was a map $\widetilde{f} : Y \to \widetilde{X}$ s.t.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes. We can also talk about the lifting of paths.

Theorem 13.1 (Unique Homotopy Lifting Property)**.** Given $f : Y \to X$ and $H : Y \times I \to X$ a homotopy with $H|_{Y\times\{0\}} = f$ and a lift $\widetilde{f} : Y \to \widetilde{X}$, then there exists a unique homotopy $\widetilde{H} : Y \times I \to \widetilde{X}$ that lifts H i.e.

$$\begin{tikzcd} Y \arrow[r, "\widetilde{f}"] \arrow[d, hook, "i"'] & \widetilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & X \end{tikzcd}$$

Corollary 13.1.1. If $f : Y \to X$ has a lift $\widetilde{f} : Y \to \widetilde{X}$ with Y connected, then f is null-homotopic iff $\widetilde{f}$ is null-homotopic.

Proof. Fix $p : \widetilde{X} \to X$ a covering. If $\widetilde{f}$ is null-homotopic, then the composite from the lifting diagram implies f is null-homotopic.

Suppose f is null-homotopic and choose $H : Y \times I \to X$ a homotopy with $H|_{Y\times\{1\}}$ constant and $H|_{Y\times\{0\}} = f$. By the UHLP, there exists $\widetilde{H} : Y \times I \to \widetilde{X}$ with $\widetilde{H}|_{Y\times\{0\}} = \widetilde{f}$, $\widetilde{H}|_{Y\times\{1\}}$ constant by the following lemma. Thereby $\widetilde{f}$ is homotopic to a constant map and so it is null-homotopic. $\square$

Lemma 13.2. If Y is connected, any lift $\widetilde{c} : Y \to \widetilde{X}$ of a constant map $c : Y \to X$ is also constant.

Proof. Let $c(y) = x_0$. Then $\widetilde{c}(y) \in p^{-1}(x_0)$. But $p^{-1}(x_0)$ is a discrete topological space and Y being connected implies that $p^{-1}(x_0)$ is singleton. $\square$

We also proved some results on lifting paths in previous lectures.

Theorem 13.3. Let $x_0 \in X$, $p : \widetilde{X} \to X$ be a covering, and $\widetilde{x_0} \in p^{-1}(x_0)$. Then,

(1) for any path γ with starting point x_0, there exists a unique lift $\widetilde{\gamma} : I \to \widetilde{X}$ s.t. $\widetilde{\gamma}(0) = \widetilde{x_0}$,

(2) give paths γ, γ' in X with $\gamma(0) = \gamma'(0) = x_0$, their lists $\widetilde{\gamma}, \widetilde{\gamma}'$ satisfy

$$\gamma \simeq \gamma' \iff \widetilde{\gamma} \simeq \widetilde{\gamma}'.$$

So, the lifts have $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$.

Corollary 13.3.1. In the situation above, consider $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$. Then

(1) p_* is injective,

(2) $\mathrm{im}(p_*) = \{[\gamma] \in \pi_1(X, x_0) : \gamma$ can be lifted to a loop based at $\widetilde{\gamma}$ based at $\widetilde{x_0}\}$.

(3) if $\widetilde{X}$ is connected, then $\operatorname{im} p_* \leq \pi_1(X, x_0)$ with index $|p^{-1}(x_0)|$.

Proof. (1) Assume $p_*([\widetilde{\gamma_1}]) = p_*([\widetilde{\gamma_2}])$. Then this means $\gamma_1 \simeq \gamma_2$. But the unique lifting property for paths implies $\widetilde{\gamma_1} \simeq \widetilde{\gamma_2}$. Therefore,

$$p \circ \widetilde{\gamma_1} \simeq p \circ \widetilde{\gamma_2} \implies \widetilde{\gamma_1} \simeq \widetilde{\gamma_2} \implies p_* \text{ injective.}$$

(2) Suppose $\gamma \in \pi_1(X, x_0)$ can be lifted to a loop. Then $[\gamma] = p_*([\widetilde{\gamma}]) \in p_*(\pi_1(\widetilde{X}, x_0))$.

Suppose $[\gamma] \in p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$. Then choose $\gamma \simeq \gamma'$ s.t. γ' satisfies $\gamma' = p \circ \widetilde{\gamma}'$ for a $\widetilde{\gamma}'$ loop based at $\widetilde{x_0}$. Let $\widetilde{\gamma}$ be the lift of γ with $\widetilde{\gamma}(0) = \widetilde{x_0}$. Then $\gamma \simeq \gamma'$ which means $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1) = \widetilde{x_0}$. So $\widetilde{\gamma}$ is a loop.

(3) Define $\Phi : \pi_1(X, x_0)/\operatorname{im} p_* \to |p^{-1}(x_0)|$ by $[\gamma] \operatorname{im} p_* \to \widetilde{\gamma}(1)$ where $\widetilde{\gamma}$ is a lift of γ with $\widetilde{\gamma}(0) = \widetilde{x_0}$. We will show that this is a bijection.

First Φ is well-defined. If $\gamma' = \gamma \cdot h$ for $[h] \in \operatorname{im} p_*$, then $\widetilde{h}$ is a loop based at $\widetilde{x_0}$. Then $\widetilde{\gamma}' = \widetilde{\gamma}\widetilde{h}$ implies $\widetilde{\gamma}'(1) = \widetilde{\gamma(1)}$.

Next, Φ is surjective. Given $\widetilde{x_0'} \in p^{-1}(x_0)$, take a path $\widetilde{\gamma} : I \to X$ form $\widetilde{x_0}$ to $\widetilde{x_0'}$. Then $\gamma = p \circ \widetilde{\gamma}$ and $\Phi([\gamma] \operatorname{im} p_*) = \widetilde{x_0'}$.

Third, Φ is injective. If $\Phi([\gamma] \operatorname{im} p_*) = \Phi([\gamma'] \operatorname{im} p_*)$, then $\widetilde{\gamma_1}, \widetilde{\gamma_2}$ are paths from $\widetilde{x_0}$ to the same endpoint. But then $\widetilde{\gamma_1}\widetilde{\gamma_2}^{-1}$ is a loop based at $\widetilde{x_0}$. Then,

$$p_*([\widetilde{\gamma_1} \cdot \widetilde{\gamma_2}']) \cdot [\gamma_2] = [\gamma_1].$$

□

Example 13.4. See figure 12.

Definition 13.5. A space is **locally path-connected** if for any $y \in Y$, any any open neighborhood $U \ni y$, there exists a small open neighborhood V with $y \in V \subseteq U$ and V path-connected.

Example 13.6. Any CW-complex is locally contractible and hence, locally path-connected. See Proposition A.4 of Hatcher.

Proposition 13.7. If Y is locally path-connected, then Y is connected iff Y is path-connected.

Proof. □

Example 13.8. Topology's sine curve is path-connected, but not locally path-connected. See figure 13.

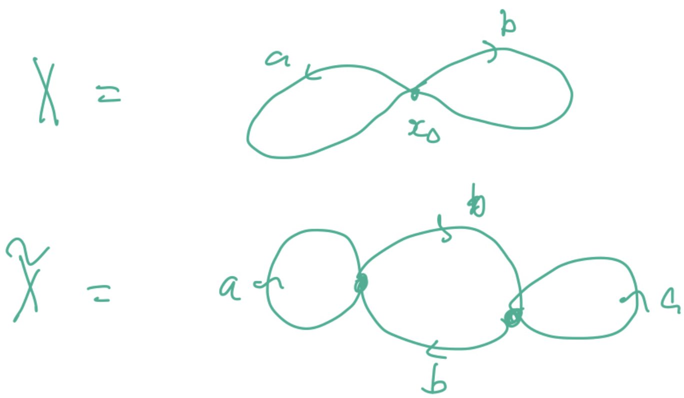


FIGURE 12. The space $\widetilde{X}$ is a 2-fold cover of X. We have fundamental groups $\pi_1(X, x_0) = \mathbb{Z} * \mathbb{Z} = \langle a, b\rangle$ and $\pi_1(\widetilde{X}, \widetilde{x_0}) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \langle a, b^2, bab\rangle$. This is an index 2 subgroup of of $\pi_1(X, x_0)$.

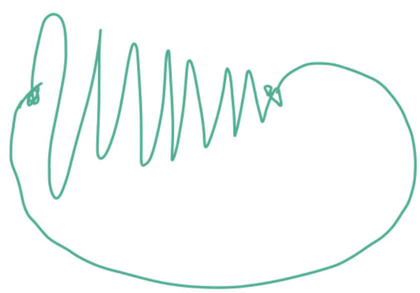


FIGURE 13. The space is path-connected since we can always use the larger arc to connect the origin with points on the sine curve. However, it is not locally path-connected because there is no smaller open neighborhood about the origin that is path-connected.

Theorem 13.9. Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a covering map and $f : (Y, y_0) \to (X, x_0)$ with Y connected and locally path-connected. Then f has a lift

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"] \arrow[ur, dotted, "\widetilde{f}"] & X \end{tikzcd}$$

iff $f_*(\pi_1(X, x_0)) \hookrightarrow p_*)\pi_1(\widetilde{X}, \widetilde{x_0}))$.

PROOF. ($\Longrightarrow$) is obvious so we show the other direction. We shall define $\widetilde{f}$ explicitly. Given $y \in Y$, take a path $\gamma : I \to Y$ from y_0 to y. Then $f \circ \gamma$ is a path in X that starts at x_0. Let $\widetilde{f \circ \gamma}$ be the lift of $f \circ \gamma$ that starts with $\widetilde{x_0}$. Define $\widetilde{f}(y) = \widetilde{f \circ \gamma}(1)$.

We check $\widetilde{f}$ is well-defined. Let γ' be another path from y_0 to y. Then $[\gamma \cdot \overline{\gamma'}] \in \pi_1(Y, y_0)$ and $f_*([\gamma \cdot \overline{\gamma'}] \in \operatorname{im}(f_*) \subseteq p_*$. But $f_*([\gamma \cdot \overline{\gamma'}] = (f \circ \gamma)(\overline{f \circ \gamma})$ which means $(f \circ \gamma)(\overline{f \circ \gamma})$ can be lifted to a loop based at $\widetilde{x_0}$. Therefore,

$$\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \gamma'}(1).$$

We check that $\widetilde{f}$ is continuous. Given $y \in Y$. Choose a neighborhood $y \in V$ and choose an evenly covered set $U \supseteq f(V)$. Then $\widetilde{f}|_V = p^{-1} \circ f$ (and p^{-1} is well-defined because $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism). $\square$

14. Lecture 14: October 25, 2021

Abstract. In this lecture, we work on constructing the universal covering space for arbitrary topological spaces satisfying certain "nice" conditions.

Definition 14.1. A space X is **semilocally simply connected** if for any $x \in X$ and open neighborhood $U \ni x$, there exists an open neighborhood V s.t. $x \in V \subseteq U$ and

$$\pi_1(V, x) \to \pi_1(X, x)$$

is trivial.

Example 14.2. All CW-complexes are semilocally simply connected (as they are locally contractible).

OTOH, the Hawaiian earring is *not* semilocally simply connected. Any arbitrarily small neighborhood of the basepoint contains an essential loop in X.

Recall that a covering space $(\widetilde{X}, p)$ is the universal covering space if $\widetilde{X}$ is simply connected. There is a necessary condition it must satisfy.

Proposition 14.3. If X has a universal covering space, $p : \widetilde{X} \to X$, then X is semilocally simply connected.

Proof. For any $x \in X$, let U' be an evenly covered neighborhood of x. Then $p_\alpha := p|_{U'_\alpha} : U'_\alpha \to U$ is a homeomorphism. For any open neighborhood U of x, let $V = U \cap U'$. Then we have a commutative diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ V \arrow[ur, "\widetilde{i}"] \arrow[r, "i"] & X \end{tikzcd}$$

with $\widetilde{i}(x) = p_\alpha^{-1}(x)$. Then i_* is trivial because $i_* = \widetilde{i_*} \circ p_*$ and $\pi_1(\widetilde{X}) = 0$. $\square$

Example 14.4. The Hawaiian earring has no universal covering space.

Theorem 14.5. If X is path connected, locally path connected, and semilocally path connected, then X has a universal covering space.

Some motivation is warranted. Pretend that $\widetilde{X}$ did exist. How might we construct $\widetilde{X}$ using points and paths in X? There are two defining properties we know.

- Any path γ in X with $\gamma(0) = 0$ has a unique lift $\widetilde{\gamma}$ s.t. $\widetilde{\gamma}(0) = \widetilde{x_0}$.
- $\gamma \simeq \gamma'$ iff $\widetilde{\gamma}(1) = \widetilde{\gamma'}(1)$ by virtue of $\pi_1(\widetilde{X}) = 0$.

We shall show that there is a well-defined bijection

$$\varphi : \overline{X} := \{[\gamma] \mid \gamma : I \to X, \gamma(0) = x_0\} \to \widetilde{X}$$

given by $[\gamma] \mapsto \widetilde{\gamma}(1)$.

We can define $\overline{X}$ for any X. Our goal is to then recover $\widetilde{X}$ from $\overline{X}$ by giving $\overline{X}$ a topology.

Suppose $x_1 = \gamma(1)$. Choose an evenly covered neighborhood U of x_1. By shrinking U a little, we may assume U is path-connected. The $U_\alpha \subseteq \widetilde{X}$ are sheets which cover U and choose U_α specifically so that it contains $\widetilde{\gamma}(1)$.

How do we describe $\varphi^{-1}(U_\alpha)$ (as a subset in $\overline{X}$)? Simply take

$$\varphi^{-1}(U_\alpha) = \{[\gamma \cdot \eta] \mid \eta \text{ is a path starting at } x_1 \text{ in } U\}$$

and require $\varphi^{-1}(U)$ to be open.

Definition 14.6. Given any path γ in X starting at x_0 and any open neighborhood U of $\gamma(1)$. We define

$$\langle U, [\gamma]\rangle := \{[\gamma \cdot \eta] \mid \eta : I \to U,\ \eta(0) = \gamma(1)\}.$$

We define a topology on $\overline{X}$ by specifying a basis of open sets

$$\mathcal{U} := \{\langle U, [\gamma]\rangle \mid \gamma : I \to X, \gamma(0) = x_0,\ U \text{ open neighborhood of } \gamma(1)\}.$$

Lemma 14.7. This is in fact a topology. One needs to show $\langle U, [\gamma]\rangle \cap \langle U', [\gamma']\rangle$ is open.

Theorem 14.8. Assume X is path-connected, locally path-connected, and semilocally simply connected. Then $(\overline{X}, p)$ is the universal covering space where p is defined by $p([\gamma]) = \gamma(1)$.

Proof sketch. For $x \in X$, choose an open neighborhood $U \ni x$ s.t. U is path-connected and $\pi_1(U, x) \to \pi_1(X, x)$ is trivial Then

$$p^{-1}(U) = \coprod \langle U, [\gamma]\rangle$$

where the coproduct is over the homotopy classes of paths from x_0 to x.

First, one can check that $\langle U, [\gamma]\rangle \cap \langle U, [\gamma']\rangle = \emptyset$.

Second, $p|_{\langle U,[\gamma]\rangle} : \langle U, [\gamma]\rangle \to U$ is a homeomorphism. One needs to construct an inverse $p^{-1} : U \to \langle U, [\gamma]\rangle$. Simply take the map that sends x' to $[\gamma \cdot \eta]$ where $\eta : I \to U$ is a path from x to x'.

So, U is evenly covered and p is a covering map.

Third, we check that $\overline{X}$ is path-connected. For all $[\gamma] \in \overline{X}$, we define $\gamma_t : I \to X$ by $\gamma_t(s) = \gamma(ts)$. Then $t \mapsto [\gamma_t]$ is a path in $\overline{X}$ from $[\gamma_0] = [C_{x_0}]$ to $[\gamma_1] = [\gamma]$. So every point $[\gamma]$ in $\overline{X}$ can be connected to $[C_{x_0}]$.

Fourth, $\overline{X}$ is simply-connected. The map

$$p_* : \pi_1(\overline{X}, [C_{x_0}]) \to \pi_1(X, x_0)$$

is injective. So we need to show that the image is zero in $\pi_1(X, x_0)$. Take any path $\gamma \in p_*(\pi_1(\overline{X}, [C_{x_0}]))$ There exists a unique lift $\widetilde{\gamma} : I \to \overline{X}$ that starts at $[C_{x_0}]$. We can construct it explicitly by $\widetilde{\gamma}(t) = [\gamma_t]$. Since $\gamma \in p_*(\pi_1(\overline{X}, [C_{x_0}]))$, we know $\widetilde{\gamma}$ is a loop. But then $[\gamma]$ is a loop which corresponds to $\widetilde{\gamma}(1) = \widetilde{\gamma}(0)$ which has a path to $[C_{x_0}]$ as points in $\overline{X}$. So, $\gamma \simeq C_{x_0}$. So $[\gamma] = 0$ in $p_*(\pi_1(\overline{X}, [C_{x_0}]))$. ∎

15. Lecture 15: October 27, 2021

ABSTRACT. This lecture proves certain nice "algebraic" properties of the universal covering space. In general, there is a sort of "Galois correspondence" between connected covering spaces and subgroups of the fundamental group.

Recall that a simply connected covering space is called a *universal covering space*. We recall the proof of the following theorem.

Theorem 15.1. Let X be a connected, locally path connected, semilocally simply connected, topological space. Then X has a universal cover. The condition that X is connected and locally path connected occurs iff it is semilocally simply connected.

PROOF SKETCH. We defined $\overline{X} := \langle [\gamma] \mid \gamma : I \to X, \gamma(0) = x_0 \rangle$ with $p : \overline{X} \to X$ given by $p([\gamma]) = \gamma(1)$. We gave it a topology which was the on generated by the

$$\langle U, [\gamma] \rangle := \langle [\gamma \cdot \eta] \mid \eta : I \to U, \eta(0) = \gamma(1) \rangle$$

for every $[\gamma] \in \overline{X}$ with open neighborhood $U \ni \gamma(1)$. Essentially, $U \subseteq \overline{X}$ is open iff A is the union of the sets above i.e. $A = \bigcup_\alpha \langle U_\alpha, [\gamma_\alpha] \rangle$. □

We know that $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$ was injective and the theorem of last time gave a partial converse, That is, universal covers correspond to $p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$ trivial.

For the rest of this lecture, fix X connected, locally path connected, and semilocally simply connected.

Theorem 15.2. For all $H \leq \pi_1(X, x_0)$, there exists a covering space $p : X_H \to X$ s.t. $p_*(\pi_1(X_H, \widetilde{x_0})) = H$ and $\widetilde{x_0} \in p^{-1}(x_0)$.

PROOF. The proof is similar to the proof of existence of a universal cover. Take $\overline{X} := \langle [\gamma] \mid \gamma : I \to X, \gamma(0) = x_0 \rangle$ and this is simply connected. Define an equivalence relation $\sim_H$ on $\overline{H}$ by $[\gamma_1] \sim_H [\gamma_2]$ iff $\gamma_1(1) = \gamma_2(2)$ and $[\gamma_1 \overline{\gamma_2}] \in H \subseteq \pi_1(X, x_0)$. This is an equivalence relation.

Define $X_H := \overline{X}/\sim_H$. One can check that p is a covering map and that $p_*(\pi_1(X_H, \widetilde{x_0})) = H$ where $\widetilde{x_0}$ is the constant path at x_0.

For all $[\gamma] \in \pi_1(X, x_0)$, we know that γ has unique lift $\widetilde{\gamma} \in X_H$ starting at $\widetilde{x_0}$. But $\widetilde{\gamma}$ can be explicitly constructed as a map $I \to X_H$ that sends $s \mapsto [\gamma_s]$ where $\gamma_s : I \to X$ is $\gamma_s(t) = \gamma(st)$. Uniqueness ensures this is correct.

Note that $\widetilde{\gamma}(1) = [\gamma]$ and so,

$$[\gamma] \in p_*(\pi_1(X_H, \widetilde{x_0})) \iff \widetilde{\gamma}(1) = \widetilde{x_0} \in X_H \iff [\gamma] \sim_H [\text{constant map}] \iff [\gamma] \in H.$$

□

Example 15.3. We know $\pi_1(S^1) = \mathbb{Z} \supseteq \{e\}$ which occurs iff $\pi_1(\mathbb{R}^1)$. What are the corresponding covering maps for $n\mathbb{Z} \subseteq \mathbb{Z}$?

Well, $n\mathbb{Z} \cong \mathbb{Z}$ and this should give an n-fold cover. Realize that $S^1 \to S^1$ given by $z \mapsto z^n$ yields the desired covering.

For $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$, we know the universal cover is $\pi_1(\mathbb{R}^2)$. Meanwhile, each subgroup $a\mathbb{Z} \times b\mathbb{Z}$ corresponds to coverings given by $(z_1, z_2) \mapsto (z_1^a, z_2^b)$ which is an ab-fold cover.

Definition 15.4. Two covering spaces $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ are **isomorphic** if there exists a homeomorphism $h : X_1 \to X_2$ s.t. $p_1 = p_2 \circ h$.

Theorem 15.5. Given two connected covering spaces $p_1 : X_1 \to X$, $p_2 : X_2 \to X$ with $x_0 \in X$, $\widetilde{x_1} \in p_1^{-1}(x_0)$, $\widetilde{x_2} \in p_2^{-1}(x_0)$. Then (X_1, p_1) is isomorphic to (X_2, p_2) iff

$$p_{1*}(\pi_1(X_1, \widetilde{x_1})) \qquad \& \qquad p_{2*}(\pi_1(X_2, \widetilde{x_2}))$$

are conjugate subgroups.

Remark 7. In the theorem, is worth noting that changing base points from $\widetilde{x_1} \in p^{-1}(x_0)$ to $\widetilde{x_2} \in p^{-1}(x_0)$ corresponds to conjugating H by an element $[\gamma]$ where $[\gamma]$ lifts to a path from $\widetilde{x_0}$ to $\widetilde{x_1}$.

To prove the theorem, we shall need a lemma.

Lemma 15.6. Let $f : Y \to X$, $f(y_1) = f(y_2) = x_0$, $\widetilde{\gamma} : I \to X$ the lift with $\widetilde{\gamma}(0) = y_1$ and $\widetilde{\gamma}(1) = y_2$. Then

$$f_*(\pi_1(Y, y_2)) = [\gamma]^{-1} f_*(\pi_1(Y, y_1))[\gamma].$$

Proof. Use the change of basepoint diagram

$$\begin{tikzcd} \pi_1(Y, y_2) \arrow[r, "\widetilde{\gamma}_\#"] \arrow[d, "f_*"] & \pi_1(Y, y_1) \arrow[d, "f_*"] \\ \pi_1(X, x_0) \arrow[r, "\gamma_\#"] & \pi_1(X, x_0) \end{tikzcd}.$$

Following the diagram,

$$\gamma_\#([\beta]) = [\gamma]^{-1}[\beta][\gamma]$$

for all $[\beta] \in \pi_1(X, x_0)$. □

Lemma 15.7. Given $f : Y \to X$ and $p : \widetilde{X} \to X$ a cover. Let $\widetilde{f_1}, \widetilde{f_2} : Y \to \widetilde{X}$ be two lifts of f and assume Y is connected. If $\widetilde{f_1}(y_0) = \widetilde{f_2}(y_0)$ for some y_0, then $\widetilde{f_1} = \widetilde{f_2}$.

Proof. The trick is to show the set A where they are equal is both open and closed. Then as it is nonempty, Y being connected implies it is the whole space.

First, A is a closed set because $\widetilde{f_1}, \widetilde{f_2}$ are continuous.

Second, let U_α be a sheet over A containing $\widetilde{f_1}(y) = \widetilde{f_2}(y)$ by choosing an evenly covered neighborhood of $f(y)$. Now $\widetilde{f_1}, \widetilde{f_2}$ being continuous means there is a $V \ni y$ neighborhood s.t. for all $y' \in V$, $\widetilde{f_1}(y') \in U_\alpha$ and $\widetilde{f_2}(y') \in U_\alpha$. But then

$$p(\widetilde{f_1}(y')) = p(\widetilde{f_2}(y')) = f(y)$$

(as the lifts are equal). But $p|_{U_\alpha}$ is injective which means $\widetilde{f_1}(y') = \widetilde{f_2}(y')$. So $V \subseteq A$ which means A is open. □

Proof of Theorem. There is an induced diagram

$$\begin{tikzcd} \pi_1(X, \widetilde{x_1}) \arrow[r, "h_*"] \arrow[d, "p_{1*}"] & \pi_1(X, h(\widetilde{x_1})) \arrow[dl, "p_{2*}"] \\ \pi_1(X, x_0) & \end{tikzcd}$$

and commutativity means $p_{1*}(\pi_1(X_1, \widetilde{x_1})) = p_{2*}(\pi_1(X_2, \widetilde{h}(x_2))$. The first lemma implies $p_{2*}(\pi_1(X_2, h(\widetilde{x_2}))$ is conjugate to $p_{2*}(\pi_1(X_2, \widetilde{x_2}))$.

For the other direction, assume $p_{1*}(\pi_1(X_1, \widetilde{x_1})) = [\gamma]p_*(\pi_1(X_2, \widetilde{x_2}))[\gamma]^{-1}$. Let $\widetilde{\gamma} : I \to \widetilde{X_2}$ be a lift of γ starting at $\widetilde{x_2}$. Then

$$[\gamma]p_*(\pi_1(X_2, \widetilde{x_2}))[\gamma]^{-1} = p_{2*}(\pi_1(X_2, \widetilde{\gamma}(1)).$$

By replacing $\widetilde{x_1}$ by $\widetilde{\gamma}(1)$, assume that

$$p_{1*}(\pi_1(X_1, \widetilde{x_1})) = p_{2*}(\pi_1(X_2, \widetilde{x_2})).$$

We have a diagram

$$\begin{tikzcd} X_1 \arrow[r, "\widetilde{p_1}"] \arrow[dr, "p_1"'] & X_2 \arrow[r, "\widetilde{p_2}"] \arrow[d, "p_2"] & X_1 \arrow[dl, "p_1"] \\ & X & \end{tikzcd}$$

and the composite $\widetilde{p_2} \circ \widetilde{p_1} : X_1 \to X_1$ lifts o_1 w.r.t. p_1 and $\widetilde{p_1} \circ \widetilde{p_2} : X_2 \to X_2$ lifts p_2 w.r.t p_2. So the second lemma says $\widetilde{p_2} \circ \widetilde{p_1}$ and $\widetilde{p_2} \circ \widetilde{p_1}$ are the identity. Indeed, So, $\widetilde{p_1}$ is a homeomorphism. $\square$

To summarize, if X is connected, locally path connected, and semilocally simply connected, there is an equivalence

$$\left\{\frac{\text{connected covering spaces of } X}{\text{isomorphisms}}\right\} \leftrightarrow \left\{\frac{\text{subgroups of } \pi_1(X)}{\text{conjugation}}\right\}$$

The correspondence is given by

$$\left[(\widetilde{X}, p)\right] \leftrightarrow \left[p_*(\pi_1(\widetilde{X}))\right].$$

This correspondence is similar to the Galois correspondence.

16. Lecture 16: October 29, 2021

Abstract. This lecture discusses the group of covering transformations on a covering space which help classify the covering spaces. If we view the group of deck transformations as sort of action on the covering space, we may consider the orbit space which is a useful tool for constructing covering spaces and classifying possible subgroups of the fundamental group.

Assume X is path-connected, locally path-connected, and semilocally simply-connected. Recall that if $X_1, X_2 \to X$ were two covering spaces, an isomorphism h between (X_1, p_1) and (X_2, p_2) is a homeomorphism h s.t. $p_2 \circ h = p_1$.

Assume X_1, X_2 are connected throughout.

Theorem 16.1. Given $x_0 \in X$, $\widetilde{x_1} \in p_1^{-1}(x_0)$, and $\widetilde{x_2} \in p_2^{-1}(x_0)$, then there exists an isomorphism $h : X_1 \to X_2$ with $\widetilde{x_1} \xrightarrow{h} \widetilde{x_2}$ iff

$$p_{1*}(\pi_1(X_1, \widetilde{x_1})) = p_{2*}(\pi_1(X_2, \widetilde{x_2}))$$

as subgroups of $\pi_1(X, x_0)$.

Definition 16.2. A **deck transformation** (or **covering transformation**) on a covering space $(\widetilde{X}, p)$ is an isomorphism $h : (\widetilde{X}, p) \to (\widetilde{X}, p)$ s.t. $p \circ h = p$. Let $G(\widetilde{X})$ denote the group deck transformations.

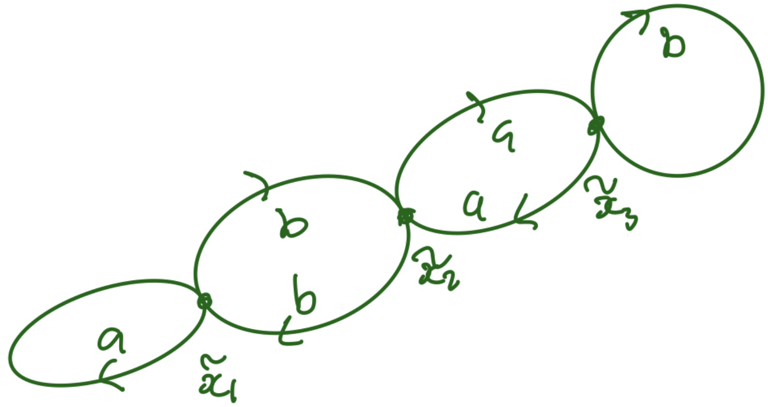


FIGURE 14. Cover of $S^1 \vee S^1$.

Example 16.3. Consider the covering $p_n : S^1 \to S^1$ given by $z \mapsto z^n$. Then the deck transformations are rotations and $h(z) = e^{2\pi i/n}z$ being the rotation by $\frac{2\pi}{n}$ generates $G(\widetilde{X})$. So, $G(\widetilde{X}) \cong \mathbb{Z}_n$. Furthermore, $G(\widetilde{X}) = \pi_1(X)/\pi_1(\widetilde{X})$.

Example 16.4. Let $\widetilde{X} \to S^1$ be the universal covering space. Then the deck transformation h sending x to $x+1$ generates. Therefore, $G(\widetilde{X}) \cong \mathbb{Z}$. So, $G(\widetilde{X}) \cong \pi_1(X)/\pi_1(\widetilde{X})$ and $\pi_1(\widetilde{X}) = 0$.

Example 16.5. Let $\widetilde{X}$ be $\mathbb{R}$ with loops attached at each integer. The edge from n to $n = 1$ are labeled by b and the loops are labeled by a. This gives a covering $p : \widetilde{X} \to S^1 \vee S^1$. The situation is the same as above since the deck transformation $h(x) = x + 1$ generates $G(\widetilde{X}) \cong \mathbb{Z}$.

Example 16.6. Consider the cover pictured in figure 14. A deck transformation must send points to points which preserve the paths around them. However, each point is distinct in this sense because the loop a in $S^1 \vee S^1$ can lift to a loop a at $\widetilde{x_1}$, but not at $\widetilde{x_2}, \widetilde{x_3}$. The same applies for b with $\widetilde{x_3}$. So the only deck transformation is the identity.

The idea here is that a reflection would send $\widetilde{a}$ to $\widetilde{b}$, but pushing down to $S^1 \vee S^1$ would require $a = b$ which is not true. So, $G(\widetilde{X}) = \{e\}$.

Definition 16.7. A covering map is **normal** or **regular** if for every $\widetilde{x_1}, \widetilde{x_2} \in \widetilde{X}$ with $p(\widetilde{x_1}) = p(\widetilde{x_2})$, there exists $h \in G(\widetilde{X})$ s.t. $h(\widetilde{x_1}) = \widetilde{x_2}$.

Example 16.8. The first example from before is normal. The second and third are as well. However, the fourth is not!

An interesting question to ask is if we can determine if a covering map is normal and how to compute deck transformations in more generality.

Remark 8 (Group Theory)**.** Denote S^G for the **normal closure** of $S \leq G$. Let $N_G(S)$ denote the normalizer of S. Then $S \trianglelefteq G$ iff $S^G = S$ iff $N_G(S) = G$.

Theorem 16.9. If $p : \widetilde{X} \to X$ is a connected covering space, $H = p_*(\pi_1(\widetilde{X}, \widetilde{x_0})) \subseteq \pi_1(X, x_0)$, then

(1) $G(\widetilde{X}) \cong N_G(H)/H$,
(2) $\widetilde{X}$ is a normal covering space iff $H \trianglelefteq \pi_1(X, x_0)$.

In particular, a universal cover $\widetilde{X}$ is always normal and $G(\widetilde{X}) \cong \pi_1(X)$.

Proof. See Hatcher. □

An alternative way to think about normal covering spaces is now given. Let G be a **topological group** i.e. it is a group with a topology s.t. its group operations are continuous as maps $G \times G \to G$. A **G-action** on Y is a continuous map $\rho : G \times Y \to Y$ s.t. $\rho(e, -) : Y \to Y = \mathrm{id}_Y$, $\rho(g_1, -) \circ \rho(g_2, -) = \rho(g_1 g_2, -)$, and each $\rho(g, -)$ is a homeomorphism. We will write for shorthand $\rho(g, y) = g \cdot y = g(y) \in Y$.

Define an equivalence $\sim_G$ on Y by $g_1 \sim_Y y_2$ iff there exists $g \in G$ s.t. $g(y_1) = y_2$.

Denote by Y/G the space $Y/\sim_G$ and this is called the **orbit space**.

Definition 16.10. An action is a **covering space action** if for all $y \in Y$, there exists a neighborhood $U \ni y$ s.t.

$$g_1(U) \cap g_2(U) = \emptyset \qquad \forall g_1 \neq g_2 \in G.$$

The LHS condition is similar to requiring that our space Y be a covering space. Note that if G is finite and Y is Hausdorff, then G is covering space action iff the action is free.

We could also rewrite the condition to mean $U \cap g(U) = \emptyset$ iff $g = e \in G$.

Example 16.11. We give an example of space with a free action but which does not give a covering space action. Let $\mathbb{Z}$ act on S^1 by rotation by an angle α where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then this is free because $n(x) = x$ for $x \in S^1, n \in \mathbb{Z}$ iff n is the identity. However, this is not a covering space action due to density. Indeed, if $U \ni y$, then we can always find a $g \in \mathbb{Z}$ s.t. $g(U) \cap U \neq \emptyset$. This has to do with how the orbits $\mathbb{Z}x$ are dense in S^1.

Theorem 16.12. Let G be a covering space action of Y. Then,

(1) $Y \to Y/G$ is a normal covering,
(2) if Y is path-connected, then $G = G(Y)$ is the group of deck transformations of $Y \to Y/G$,
(3) if Y is path-connected and locally path-connected, then $G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$.

Proof. See Hatcher. □

Example 16.13. The group $G = \mathbb{Z}_2$ acts freely on S^n via the antipodal action. Indeed, $\overline{1}(y) = -y$ has order two. The orbit space is $S^n/G \cong \mathbb{R}P^n$ and the theorem says $S^n \to S^n/G$ is a covering space. This is a universal covering space when $n \geq 2$.

17. Lecture 17: November 1, 2021

Abstract. This lecture discusses some applications of covering space actions and the orbit space. There is also a basic introduction to topological manifolds which will prepare for the classification of closed manifolds of dimension 2. Additionally, we discuss some basic ways to obtain examples of manifolds such as by the **Inverse Function Theorem**.

Recall that G is a **topological group** if it is a group with a topology s.t. the group operations are continuous. Let Y be a space. Then it is a **space with G-action** if there is a $\rho : G \times Y \to Y$ defined s.t. $\rho(g, -) : Y \to Y$ is always a homeomorphism. We define a relation $\sim_G$ on Y by $y_1 \sim_G y_2$ s.t. there exists $g \in G$ s.t. $g(y_1) = y_2$. We denoted the quotient space by Y/G and call it the **orbit space**.

Recall that we also defined a **covering space action**. It was an action G on Y s.t. for all $y \in Y$, there exists an open $U \ni y$ s.t. $U \cap g(U) = \emptyset$ for all $g \in G$ not the identity. Note that if G is finite and Y is Hausdorff, then we have a covering space action iff it is a free action.

Example 17.1. Let $\mathbb{Z} \oplus \mathbb{Z}$ act on $\mathbb{R}^2$ by $(a, b)(x, y) = (x + a, y + b)$. This is a covering space action (choose balls of radius $< 1/3$). The quotient is $\mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \cong S^1 \times S^1$. So, $\mathbb{R}^2$ is the universal covering space of T^2 and $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$.

Example 17.2. Let $p, q \in \mathbb{Z} - \{0\}$. Let $G = \mathbb{Z}/p\mathbb{Z}$ and G act on $S^3 := \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$ as follows. Let $l \in \mathbb{Z}/p\mathbb{Z}$ and

$$l(z_1, z_2) = \left(e^{\frac{2\pi i}{p}} lqz_1, e^{\frac{2\pi i}{p} l} z_2\right).$$

So, we just have rotation by $\frac{2\pi i}{p} ql$ in the $(x_1, y_1) = z_1$ plane and $e^{\frac{2\pi i}{p} l}$ in the $(x_2, y_2) = z_2$ plane. One can check that if $(p, q) = 1$, the action is free. The quotient space is $L(p, q) = S^3/G$ which is the **lens space**. The quotient is has a covering map with S^3 as its universal covering space. By construction, $\pi_1(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z}$. A partciular case is $L(2, 1) \cong \mathbb{R}P^3$.

Example 17.3. See Example 1.41 in Hatcher. We have a $\mathbb{Z}/5\mathbb{Z}$ action on M_{11} s.t. $M_{11}/(\mathbb{Z}/5\mathbb{Z}) = M_3$. In which case, $\pi_1(M_3)$ contains $\pi_1(M_{11})$ as a normal subgroup of index 5. Here, we observe that $\pi_1(M_3) \cong \mathbb{Z}^7$ while $\pi_1(M_{11}) \cong \mathbb{Z}^{23}$.

We now discuss manifolds and surfaces. I will be very quick in the presentation in these notes since this overlaps with the differential geometry course.

Definition 17.4. Let X be Hasudorff and $n \geq 1$. Then X is an n**-dimensional topological manifold** if it is locally Euclidean of dimension n (here, there is no assumption on second countability).

It is said to be a topological manifold with boundary if it is locally homeomorphic to $\mathbb{R}^n_+$ and there is some open subset that must be homeomorphic to an open subset of $\mathbb{R}^n_+$ that intersection $\{x_n = 0\} \subseteq \mathbb{R}^n_+$.

By convention, a 0-dimensional manifold is just a discrete space and the empty manifold is a manifold of any dimension.

The easiest examples of n-manifolds are S^n, D^n. The boundary ∂X consists of the boundary points of X i.e. those points whose local homeomorphisms has to intersect $\{x_n = 0\}$. The boundary is an $(n-1)$-manifold. The space $S^1 \vee S^1$ is not a manifold due to a singularity at the base-point of the wedge.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞-function s.t. $f = (f_1, \dots, f_m)$. Assume $n \geq m$. We say $x \in \mathbb{R}^n$ is a **regular point** if rank $\left(\frac{\partial f_i}{\partial x_j}\right) = m$ at x and $y \in \mathbb{R}^m$ is a **regular value** if $f^{-1}(y)$ consists of only regular points.

Theorem 17.5 (Inverse Function Theorem)**.** Let y be a regular value of f. Then $f^{-1}(y)$ is an $(n-1)$-dimensional manifold.

Example 17.6. Let $\mathbb{R}^3 \to \mathbb{R}$ given by $(x, y, z) \mapsto x^2 + y^2 - z^2$. One can check that the regular values are $a \neq 0$. In this case, $f^{-1}(a)$ is a 2-manifold when $a \neq 0$.

When $a > 0$, we get a hyperboloid of one sheet. At $a = 0$, we get an elliptic cone which has a singularity at the origin. When $a < 0$, we get a hyperboloid of two sheet.

18. Lecture 18: November 3, 2021

ABSTRACT. This lecture is meant to prepare for the classification of closed 2-manifolds. Here, we define the Euler characteristic and orientability which classify all closed 2-manifolds up to homeomorphism. Additionally, we define the connected sum which is used to state the classification theorem itself.

Let us start by reviewing definitions covered last lecture. Recall that a **manifold** was a Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$. A **manifold with boundary** had the condition replaced by $\mathbb{R}^n_+$.

Definition 18.1. A **closed manifold** is a compact manifold without boundary i.e. a manifold without boundary whose underlying topological space is compact.

A 2-manifold is called a **surface**.

Example 18.2. The disk D^2 is compact but has boundary. Meanwhile, $\operatorname{int} D$ is not compact and has no boundary. The sphere S^2 is a closed surface. The space $S^1 \times I$ has boundary (which are the two boundary circles). The space $S^1 \times \mathbb{R}$ has no boundary but is not compact. Finally, $\mathbb{R}^2$ is not compact.

The goal for today is to discuss the classification of closed surfaces. We begin with the situation for 1-dimensional manifolds.

Proposition 18.3. If X is a connected 1-manifold with or without boundary, then X is homeomorphic to one of

$$\emptyset, \mathbb{R}, S^1, [0,1], [0,1).$$

The proof of this proposition is elementary. For the situation with surfaces, we have some examples. For instance, T^2 the torus is one. Also the surfaces M_g of genus g. Meanwhile, the Möbius band has boundary which is the boundary circle. Interestingly, the Möbius band is also nonorientable (which we will not define precisely here).

For 2-manifolds, if one picks an orientation p, then it determines the orientation in a small neighborhood about p. So any path $\gamma : I \to X$ from p to another p' defines an orientation O_p which we extend along a path to get an orientation at O_p'. We denote said orientation by $\gamma_\#(O_p)$.

If $\gamma(0) = \gamma(1) = p$, then $\gamma_\#(O_p)$ is another orientation at p. There is no guarantee they will be the same. Indeed, γ is an **orientation preserving loop** if $\gamma_\#(O_p) = O_p$ and reversing otherwise. We will say that a 2-manifold is **nonorientable** if there exists an orientation reversing loop. For instance, S^2, D^2 do not have such curves.

Definition 18.4. A surface Σ is **orientable** if all loops are orientation preserving. A surface Σ is **non-orientable** if an orientation reversing loop exists.

Example 18.5. The surfaces $\mathbb{R}^2, D^2, S^2, T^2$ are all orientable. The surface $\mathbb{R}P^2$ is nonorientable because it contains the Möbius band.

Definition 18.6. Let X_1, X_2 be closed manifolds. We define the **connected sum** $X_1 \# X_2$ as follows.

Take $X_1 - U_1, X_2 - U_2$ for U_1, U_2 disks. These new surfaces are surfaces with boundary and the boundary is homeomorphic to S^1. Identify the S^1's to obtain $X_1 \# X_2$. This operation is symmetric. Essentially,

$$X_1 \# X_2 = \frac{(X_1 - U_1) \coprod (X_2 - U_2)}{x \sim f(x)}$$

where $f : \partial U_1 \to \partial U_2$ is a homeomorphism. One can easily check that the connected sum of closed 2-manifolds is a closed 2-manifold.

Example 18.7. One has $S^2 \# X \cong X$.

Theorem 18.8 (Classification of Closed Surfaces)**.** If Σ is a connected closed surface, then Σ is homeomorphic to one of $S^2, \#_{i=1}^n T^2, \#_{i=1}^n \mathbb{R}P^2$. The first two of which are orientable and the last is nonorientable. Moreover, the surfaces are pairwise nonhomeomorphic. The numbers n, m are called the **genus** of Σ and denoted by $g(\Sigma)$.

Corollary 18.8.1. A connected closed surface is determined by its orientability and genus.

Consider a CW structure on Σ. The **Euler characteristic** is

$$\chi(\Sigma) = \sum_n (-1)^n \#\{n\text{-cells}\}.$$

For surfaces, we only need to go up to $n = 2$. We shall see later that this is actually a homotopy invariant. For now, we have a lemma.

Lemma 18.9. One has $\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2$.

Proof. Observe $\chi(\Sigma_1 - \operatorname{int} D^2) + \chi(D^2) - \chi(S^1) = \chi(\Sigma_1 - D^2) + 1$. The connected sum has

$$\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1 - \operatorname{int} D^2) + \chi(\Sigma_2 - \operatorname{int} D^2) - \chi(S^1) = \chi(\Sigma_1) - 1 + \chi(\Sigma_2) - 1 = \chi(\Sigma_1) + \chi(\Sigma_2) - 2.$$

□

Corollary 18.9.1.

$$\chi(\Sigma) = \begin{cases} 2 - 2g(\Sigma) & \Sigma \text{ is orientable} \\ 2 - g(\Sigma) & \Sigma \text{ is nonorientable} \end{cases}.$$

Proof. Observe $\chi(T^2) = 0$, $\chi(S^2) = 2$, and $\chi(\mathbb{R}P^2) = 1$. The proof then follow from above and the classification theorem. □

Example 18.10. The Klein bottle K has $\chi(K) = 0$ and is nonorientable. So, $K \cong \mathbb{R}P^2 \# \mathbb{R}P^2$ by the classification theorem.

Proposition 18.11 (Exercise for the reader)**.** Let M_g denote an orientable closed surface of genus g. Let N_h denote a non-orientable closed surface of genus h.

Then the following are true

$$M_g \# M_h \cong M_{g+h} \qquad \& \qquad N_g \# N_h \cong N_{g+h} \cong M_g \# N_g \cong N_{2g+h}.$$

19. Lecture 19: November 5, 2021

Abstract. We discuss the classification of closed connected 2-manifolds and its consequences. We explain how to construct such surfaces, compute their fundamental group, explain what their universal covers look like, and discuss the orientable double cover.

Recall from last lecture the classification theorem for connected closed surfaces.

Theorem 19.1. Any connected closed surface is homeomorphic to one of $S^2, \#_{i=1}^n T^2, \#_{j=1}^m \mathbb{R}P^2$. The first two are orientable and the last is nonorientable.

We observed from last lecture that closed connected surfaces are determined up to homeomorphism by orientability and genus (the numbers n, m in the theorem). The method of proof for the above theorem would consist of using cuts along certain curves. Another proof would use the methods of triangulation. For this course, we do not prove this result, but merely use it as needed.

Recall from last lecture that we also defined the Euler characteristic. We also proved a formula $\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2$. Using this, we could deduce that $\chi(\Sigma) = 2 - g(\Sigma)$ for Σ orientable and $\chi(\Sigma) = 2 - g(\Sigma)$ for Σ non-orientable. Using this, one can deduce that $K = \mathbb{R}P^2 \# \mathbb{R}P^2$¿

We now discuss CW-structures on certain closed surfaces. We begin with the orientable surfaces. Let P be a polygon with $2L$ edges. If we label each edge (in pairs and with a letter and an orientation), we can then define an equivalence relation on P by declaring $x \sim x'$ if they correspond to points on the indicated edges. Then, $\Sigma := P/\sim$ is a 2-dimensional CW-complex (so Σ from P is obtained by gluing edges). Here, $\mathrm{sk}_0 \Sigma$ are the vertices modulo equivalence, and $\mathrm{sk}_1 \Sigma$ is just the polygon (without the face) modulo equivalence. There are L 1-cells. Also, $\mathrm{sk}_2 \Sigma = \Sigma$ and the 2-cell is obtained by gluing along ∂P. It turns out every closed connected surface can be obtained this way.

From this construction, we may deduce the following lemma.

Lemma 19.2. If Σ is a closed connected surface, then Σ is orientable iff every pair of identified edges consists of a single edge going clockwise.

The proof of the lemma follows by noting that if two edges went in the same direction, then we could obtain a subspace which is homeomorphic to a Möbius band.

There are canonical ways to obtain $\#_{i=1}^n T^2$ and $\#_{i=1}^m \mathbb{R}P^2$. For the former, we take a $2n$-gon and glue along the edges clockwise from some vertex given by

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1} = [a_1, b_1] \dots [a_n, b_n].$$

This gives the desired space since this is the same as attaching $\partial D \to \bigvee_{i=1}^{2n} S^1$ via the indicated gluing. Alternatively, this construction as one 0-cell, $2n$ 1-cell, and one 2-cell so the genus is $2 - 2n$ (and what we obtain is oriented, closed, and connected).

For $\#_{i=1}^m \mathbb{R}P^2$, we take an m-gon. Then label the edges counterclockwise as $a_1 a_1 a_2 a_2 \dots a_m a_m$. This has 1 0-cell, m 1-cells, and 1 2-cell. This also is a nonorientable surface by the lemma. In this way, the classification theorem says we have the desired surface.

Corollary 19.2.1. One has

$$\pi_1(\#_{i=1}^n T^2) = \langle a_1, \dots, a_n, b_1, \dots, b_n \mid [a_1, b_1] \dots [a_n, b_n] \rangle$$

and

$$\pi_1(\#_{i=1}^n \mathbb{R}P^2) = \langle a_1, \dots, a_m \mid a_1^2 \dots a_m^2 \rangle.$$

The proof essentially follows by applying Van Kampen's theorem.

We now give some closing remarks on covering spaces of closed surfaces. If X is a k-dimensional CW-complex and $\widetilde{X}$ is an m-fold cover, then $\widetilde{X}$ is also a k-dimensional CW-complex. Furthermore, $\chi(\widetilde{X}) = m\chi(X)$ for $m < \infty$. In particular, any covering space of closed surfaces is a surface (and it is closed iff $m < \infty$).

Theorem 19.3. If Σ is a connected closed surface, then there exists a universal cover $\widetilde{\Sigma}$. They are

- $\widetilde{\Sigma} = S^2$ if $\chi(\Sigma) > 0$ (as this corresponds to $S^2, \mathbb{R}P^2$),
- $\widetilde{\Sigma} = \mathbb{R}^2$ if $\chi(\Sigma) = 0$ (corresponding to T^2, K),
- $\widetilde{\Sigma} = \operatorname{int} D^2$ if $\chi(\Sigma) < 0$ (for all other cases).

The reason why we distinguish the last two cases (despite the interior of a disk being homeomorphic to $\mathbb{R}^2$) is because they can be distinguished if given extra structure (for instance, a smooth manifold structure).

As a concluding remark, there is a particular cover of interest called the **orientable double cover** of a closed surface. Recall that if Σ is a nonorientable surface, then there exists a loop $\gamma : I \to \Sigma$ that is *not* orientation preserving. We can define $H_0 := \{[\gamma] \in \pi_1(X, x_0) \mid \gamma \text{ orientation preserving}\}$.

Theorem 19.4. If Σ is non-orientable, then $H_0 \trianglelefteq \pi_1(X, x_0)$ with index 2.

There exists a connected covering space $p : \widetilde{\Sigma} \to \Sigma$ corresponding to H_0. We have $\widetilde{\Sigma}$ orientable iff $p_*(\pi_1(\widetilde{\Sigma})) \subseteq H_0$.

In particular, H_0 itself is a 2-fold cover which is the orientable double cover. For instance, the orientable double cover of a Möbius band M can be obtained by gluing two copies of M so that we get a cylinder $S^1 \times I$. For the Klein bottle, we can glue the two squares to get the gluing pattern of T^2 (and so $T^2 \to K$ is the orientable double cover). For $\mathbb{R}P^2$, the orientable is just S^2.

At last, we now come to the end of our chapter studying of covering spaces and the fundamental group. Starting next lecture, we shall being discussing homology theories and the origin of homology came from Poincaré's seminal *Analysis Situs* (and within it he introduced simplicial homology, the fundamental group, stated the Poincaé conjecture, defined the Euler Characteristic for chain complexes, and stated an early formulation of Poincaé duality).

20. Lecture 20: November 8, 2021

Abstract. This lecture begins our march toward singular homology. We define the nth singular homology groups and observe that it is a functor from the category of topological spaces.

From here on, we shall follow Haynes Miller's notes more closely than Hatcher's textbook. Recall the classification of closed connected surfaces and how we found the fundamental groups. It is easy to see that they are nonisomorphic if we consider their abelianizations (we write $H_1(-) = \pi_1(-)/[\pi_1(-), \pi_1(-)]$ for the **abelianization of** π_1 which will turn out to be the first homology group)

$$H_1(S^2) = 0, \qquad H_1(\#_{i=1}^n T^2) = \mathbb{Z}^{*2n},$$
$$H_1(\#_{i=1}^m \mathbb{R}P^2) = \left(\bigoplus_{i=1}^m \mathbb{Z}\right) \setminus \{(2, 2, \ldots, 2)\mathbb{Z}\} \cong \mathbb{Z}^{m-1} \oplus \mathbb{Z}/2.$$

As the abelianizations are different, the groups are different. It turns out that $\pi_1(X) = [S^1, X]$ is in general a group and it is determined by the set of homotopy classes of maps $S^1 \to X$. So a generalization is $\pi_n(X) = [S^n, X]$ and these are always abelian groups. Though easy to define, they are difficult to compute. Homology is the alternative.

Definition 20.1. Let $n \geq 0$. An n**-simplex** Δ^n is the convex hull of the standard basis of $\mathbb{R}^{n+1}$ i.e.

$$\Delta^n = \left\{ \sum_{i=0}^{n} t_i e_i \mid \sum_{i=0}^{n} t_i = 1, t_i \in [0,1] \right\} \subseteq \mathbb{R}^{n+1}.$$

The $(t_0, \dots, t_n)$ are called **barycentric coordinates**. Define **face maps** (or face inclusions) $d^i : \Delta^{n-1} \to \Delta^n$ by sending vertices to vertices in order except with e_i omitted. In general, there are $n+1$ face maps $\Delta^{n-1} \to \Delta^n$.

Definition 20.2. Let X be a topological space. A **singular** n**-simplex** in X is a continuous map $\sigma : \Delta^n \to X$. Let $\mathrm{Sin}_n(X)$ be the set of singular n-simplices in X. Define

$$d^i : \Delta^{n-1} \to \Delta^n \text{ by } \sigma \mapsto \sigma \circ d^i.$$

Putting the faces together, we obtain the boundary of a simplex inside X. One may observe that a closed loop will correspond to a 1-simplex σ with $d_1\sigma = d_0\sigma$.

Definition 20.3. We define the abelian group $S_n(X) := \mathbb{Z}\,\mathrm{Sin}_n(X)$ of **singular** n**-chains** as the free abelian group generated by $\mathrm{Sin}_n(X)$. If $n < 0$, $\mathrm{Sin}_n(X) = \emptyset$ and so $S_n(X) = 0$. The **boundary operator** is defined via $d : \mathrm{Sin}_n(X) \to S_{n-1}(X)$ by $\sigma \mapsto \sum_{i=0}^{n} (-1)^{i-1} d_i \sigma$ and then extended to a group homomorphism $d : S_n(X) \to S_{n-1}(X)$.

Definition 20.4. An n**-chain** (i.e. an element of S_n) in X is **cyclic** if $dc = 0$. An n-chain b is a **boundary** if $d \in \operatorname{im} d$. Let $Z_n(X) = \ker(d : S_n(X) \to S_{n-1}(X))$ and $B_n(X) = \operatorname{im}(d : S_{n+1}(X) \to S_n(X))$ be the group of n-cycles and n-boundaries.

Theorem 20.5. Any boundary is a cycle i.e. $d^2 = 0$.

Definition 20.6. A **graded abelian group** is a sequence of abelian groups index by $\mathbb{Z}$. A **chain complex** is a graded abelian group $\{A_n\}_{n\in\mathbb{Z}}$ together with group homomorphisms $d : A_n \to A_{n-1}$ s.t. $d^2 = 0$.

In this way, $(S_n(X), d)$ is a chain complex.

Definition 20.7. The n**th singular homology group** of X is

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{\ker d}{\operatorname{im} d}.$$

The set $\{H_n(X)\}_{n\in\mathbb{Z}}$ is a graded abelian group (these are not necessarily free).

When X is nice (for instance connected compact manifolds), $H_n(X)$ is finitely generated.

Later, we shall show $H_n(-)$ is a functor from the category of topological spaces to the category of graded abelian groups. Also, we shall see $H_n(S^n) = \mathbb{Z}$ and that if X is contractible, then $H_m(X) = 0$ for all m.

21. Lecture 21: November 10th, 2021

Abstract. This lecture shall explains the differential maps and provides some basic computations for singular homology. Additionally, we define the induced map on singular homology obtained from a continuos map on topological spaces.

Recall last we defined the standard n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ and the singular n-simplex in X as $\sigma : \Delta^n \to X$. Then, we defined $\mathrm{Sin}_n(X) := \{$singular $n-$ simplices in X. For each of $0 \leq i \leq n$, we had specified maps $d^i : \Delta^{n-1} \to \Delta^n$ which we called the face maps.

These induced maps $d_i : S_n(X) \to S_{n-1}(X)$ by sending $\sigma : \Delta^n \to X$ to $\sigma \circ d^i$. where $S_n(X) := \mathbb{Z}\,\mathrm{Sin}_n(X)$ was the set of n-singular chains. Afterwards, we defined boundary maps $d : S_n(X) \to S_{n-1}(X)$ (sometimes denoted ∂) which mapped

$$d\sigma = \sum_{i=0}^{n} (-1)^i d_i \sigma.$$

One can check that $d^2 = 0$ and then one defines the nth singular homology group

$$H_n(X) := \frac{Z_n(X)}{B_n(X)} = \frac{\ker(d)}{\mathrm{im}(d)}.$$

One can also check that the d^i satisfy a quadratic relation $d^i d^j = d^{j+1} d^i$ for $i \le j$ and so $d_j d_i = d_i d_{j+1}$.

Definition 21.1. A collection $\{K_n\}_{n \ge 0}$ with maps $d_n : K_n \to K_{n-1}$ s.t. $d_j d_i = d_i d_{j+1}$ for $i \le j$ is called a **semi-simplicial set**. This definition is motivated by $\mathrm{Sin}_*(X)$ which is a semi-simplicial set.

Categorically, we have the following set up

$$\begin{tikzcd}
\{\text{spaces}\} \arrow[r, "H_*"] \arrow[d, "\mathrm{Sin}_*"] & \{\text{graded abelian groups}\} \\
\{\text{semi-simplicial sets}\} \arrow[d, "\mathbb{Z}"] & \\
\{\text{semi-simplicial abelian groups}\} & \{\text{chain complexes}\} \arrow[uu, "\text{take homology of complexes}"']
\end{tikzcd}$$

Example 21.2. Let $\sigma : \Delta^1 \to X$ which is a path in X. Define $\phi : \Delta^1 \to \Delta^1$ which sends $(t, 1-t) \mapsto (1-t, t)$ (i.e. it reflects the interval and changes the orientation). Then $\overline{\sigma} = \sigma \circ \phi : \Delta^1 \to X$ is a 1-singular chain. However, $\overline{\sigma} \neq -\sigma$ as abstract objects in $S_1(X) = \mathbb{Z}\,\mathrm{Sin}(X)$. Indeed, both $\overline{\sigma}, \sigma$ are generators.

One can amend this by observing that $\overline{\sigma} \equiv -\sigma \bmod B_1(X)$ i.e. there exists a 2-chain $\tau \in S_2(X)$ s.t. $d\tau = \overline{\sigma} + \sigma$. So, if $d_0\sigma = d_1\sigma$ (i.e. $\sigma \in Z_1(X)$), then $[\overline{\sigma}] = -[\sigma]$ in $H_1(X) = Z_1(X)/B_1(X)$.

Let $\pi : \Delta^2 \to \Delta^1$ be the projection sending $e_0 \mapsto e_0$, $e_1 \mapsto e_1$, $e_2 \mapsto e_0$.

Fix $C^n_x : \Delta^n \to X$ the constant map with value $x \in X$.

Observe that

$$\begin{aligned}
d(\sigma \circ \pi) &= \sigma\pi d^0 - \sigma\pi d^1 + \sigma\pi d^2 \\
&= \overline{\sigma} - C^1_{\sigma(0} + \sigma \\
d(C^2_{\sigma(0)}) &= C^1_{\sigma(0)} - C^1_{\sigma(0)} + C^1_{\sigma(0)} = C^1_{\sigma(0)}.
\end{aligned}$$

Therefore,

$$d(\sigma \circ \pi + C^2_{\sigma(0)}) = \overline{\sigma} + \sigma.$$

Example 21.3. Let $X = \emptyset$. Then, $\mathrm{Sin}_n(X) = \emptyset$, $S_n(\emptyset) = 0$, so that means $Z_n(\emptyset) = 0$, $B_n(\emptyset) = 0$, and thereby $H_n(\emptyset) = 0$.

Example 21.4. Let $X = \{*\}$. Then $\mathrm{Sin}(X) = \{C_*^n\}$ and so $S_n(X) = \mathbb{Z}$ when $n \geq 0$ and $S_n(X) = 0$ when $n \leq -1$.

For all i, we have $d_i C_*^n = C_*^{n-1}$ and

$$d(C_*^n) = \sum_{i=0}^{n} (-1)^i C_*^{i-1} = \begin{cases} C_*^{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

So, we have a complex

$$\ldots \longrightarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

from which, we compute that $H_0(X) = \mathbb{Z}$ and $H_i(X) = 0$ for all $i \geq 1$.

Let $f : X \to Y$ be continuous and $f_* : \mathrm{Sin}_n(X) \to \mathrm{Sin}_n(Y)$ be the induced map. In particular, $\sigma \mapsto f \circ \sigma$. Now, f_* is a map of semi-simplicial sets i.e.

$$\begin{tikzcd} \mathrm{Sin}_n(X) \arrow[r, "f_*"] \arrow[d, "d"] & \mathrm{Sin}_n(Y) \arrow[d, "d"] \\ \mathrm{Sin}_{n-1}(X) \arrow[r, "f_*"] & \mathrm{Sin}_{n-1}(Y) \end{tikzcd}$$

is a commutative diagram. Indeed,

$$\begin{aligned} d_n f_* \sigma &= (f_* \sigma) \circ d^n = f \circ \sigma \circ d^n \\ f_* d_n \sigma &= (f_* d_n) \sigma = f \circ \sigma \circ d^n \end{aligned}$$

which implies $f_* : S_*(X) \to S_*(Y)$.

Definition 21.5. Let C_*, D_* be chain complexes. Then a chain map is a collection of maps $f_n : C_n \to D_n$ in which $df_n = f_{n-1}d$.

Example 21.6. If $f : X \to Y$ is continuous, then f_* is a chain map $S_*(X) \to S_*(Y)$.

Proposition 21.7. If f is continuous, there is an induced map $f_* : H_*(C) \to H_*(D)$ i.e. $f_n : H_n(C) \to H_n(D)$ for all n.

Proof. One shows f_* sends $Z_n(C) \to Z_n(D)$ and $B_n(C) \to B_n(D)$. In which case, $f_* : H_n(C) \to H_n(D)$ is obtained. The proof is trivial. □

22. Lecture 22: November 12th, 2021

Abstract. In this lecture, we put into context what singular homology exactly is. We define basic categorical objects such as groupoids, the fundamental groupoids, and the simplex category which have made quite a bit of use of in the previous lectures.

Last time, we defined chain maps between chain complexes. An example was the induced chain map $f_* : S_*(X) \to S_*(Y)$ from a continuous function $f : X \to Y$. Indeed, f_* is a map of semi-simplicial sets $\mathrm{Sin}_*(X) \to \mathrm{Sin}_*(Y)$ as it commutes with face maps and this gives the induced maps $f_* : S_*(X) \to S_*(Y)$ by linearity.

Whenever a chain map $f_* : C_* \to D_*$ of chain complexes exist, one can check that it maps boundaries to boundaries and cycles to cycles which means it induces a map on homology $f_* : H_*(C) \to H_*(D)$.

We now briefly review category theory. Most of this can be found in Haynes Miller's Lectures Notes. We call a category $\mathcal{C}$ **small** if $\mathrm{obj}(\mathcal{C})$ is just a set.

A set M is a **monoid** if there exists a composition law $M \times M \to M$ that is associative and there is an identity w.r.t. this law on M. For example, $(\mathbb{N}_0, +)$ is a monoid. Alternatively, a monoid is a category with one object. If all morphisms were to be required to be isomorphism, we get a group structure on $\mathcal{C}(X, X)$.

Definition 22.1. A **groupoid** is a category $\mathcal{C}$ s.t. all morphisms are isomorphisms.

Example 22.2. Groups are groupoids with one object.

If X is a topological space, then its **fundamental groupoid** $\pi_1(X)$ is the category where $\mathrm{obj}(\pi_1(X)) = \{\text{points in } X\}$ and

$$\pi_1(X)(x, y) = \{\text{homotopy classes of paths from } x \text{ to } y\}.$$

One can check that this is indeed a groupoid. In a sense, the fundamental groupoid is a basepoint free version of $\pi_1(X, x_0)$ and is much larger. See May's A Concise Course in Topology for a version of the Van Kampen's Theorem for the fundamental groupoid.

Example 22.3. The **simplex category** Δ has $[n] := \{0, 1, \dots, n\}$ (for $n \geq 0$) as its objects and maps which are **weakly order preserving** (i.e. $n \leq m$ implies $f(n) \leq f(m)$).

Example 22.4. A **semi-simplex category** Δ_{inj} has the same objects are for the simplex category, but the morphisms are required to be injective as well.

Finally, we give some examples of **functors** between categories.

Example 22.5. The following are all functors:

- $\pi_1 : \mathrm{Top}_* \to \mathrm{Grps}$
- $\mathrm{Sin}_n : \mathrm{Top} \to \mathrm{Sets}$,
- $\mathrm{Sin}_* : \mathrm{Top} \to \text{Semi-Simp}$,
- $S_n : \mathrm{Top} \to \mathrm{Ab}$,
- $S_* : \mathrm{Top} \to \mathrm{Ch}$, and
- $H_n : \mathrm{Top} \to \mathrm{Ab}$.

23. Lecture 23: November 15th, 2021

Abstract. In this lecture, we define some more categorical notions and state the Yoneda Lemma, a foundational theorem of category theory that has widespread applications. Additionally, we define split epimorphisms and split monomorphisms which are used to help us define the reduced homology groups of a space.

Recall last time that we discussed categories and functors. Our main examples were $\mathrm{Sin}_n(-) : \mathrm{Top} \to \mathrm{Set}$, $\mathbb{Z}(-) : \mathrm{Set} \to \mathrm{Ab}$, $S_n := \mathbb{Z}\,\mathrm{Sin}_n(-) : \mathrm{Top} \to \mathrm{Ab}$, and $H_n(-) : \mathrm{Top} \to \mathrm{Ab}$. We now define a natural "functor" map between functors. These are called **natural transformations**.

Definition 23.1. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** between F, G is a collection $\theta_X : F(X) \to G(X)$ of maps for all objects $X \in \mathrm{obj}(\mathcal{C})$ s.t. for all $f : X \to Y$ in $\mathcal{C}(X, Y)$, the following diagram commutes

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "G(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"] & G(Y) \end{tikzcd}$$

Example 23.2. There is a natural transformation between $S_n, S_{n-1} : \mathrm{Top} \to \mathrm{Ab}$ given by $d : S_n(X) \to S_{n-1}(X)$. Indeed, this follows from the fact that if $f : X \to Y$, then $f_* \circ d = d \circ f_*$.

Definition 23.3. If $\mathcal{C}, \mathcal{D}$ are categories with $\mathcal{C}$ small, then the **category of functors** (or **functor category**) $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is defined as follows. Its objects are functors $\mathcal{C} \to \mathcal{D}$. Its morphisms are natural trnasformations between functors i.e. $\mathrm{Fun}(\mathcal{C}, \mathcal{D})(F, G)$ is the set of natural transformations from F to G. By requiring $\mathcal{C}$ small, these collections of natural transformations are sets.

Example 23.4. Let G be a group viewed as a category with one object. Consider $F \in \mathrm{Fun}(G, \mathrm{Top})$. Then F sends the only object of G to some topological space and the morphisms to morphisms from that topological space X to itself. So, the image is equivalent to having a space with a G-action. Given another functor F', then a natural transformation between F and F' induces a G-map from X to X'.

Example 23.5. Let Vec_k be the cateogry of vector spaces over a field k. Then the dual functor $V \mapsto V^*$ where $V^* = \mathrm{Hom}(V, k)$ is a **contravariant functor** (i.e. it satisfies all of the functor axioms except the arrows are reversed). All functors from before are called a **covariant functor**. A contravariant functor is nothing more than a functor from a category $\mathcal{C}^{op} \to \mathcal{D}$ where $\mathcal{C}^{op}$ is the category $\mathcal{C}$ with all of its arrows reversed.

Definition 23.6. Let $Y \in \mathcal{C}$. The functor $\mathcal{C}(Y, -) : \mathcal{C} \to \mathrm{Set}$ is covariant and called a **corepresentable functor** (functors naturally isomorphic to it are also called corepresentable). The functor $\mathcal{C}(-, Y) : \mathcal{C} \to \mathrm{Set}$ is called a **representable functor**.

Theorem 23.7 (Yoneda Lemma)**.** Let $F : \mathcal{C} \to \mathrm{Set}$ be a functor and $X \in \mathcal{C} : \mathcal{C} \to \mathrm{Set}$. Then

$$\mathrm{Nat}(\mathcal{C}(X, -), F) \cong F(X)$$

where $\mathrm{Nat}(F, G)$ is the set of natural transformations of the functor F to G.

Corollary 23.7.1. Take $F = \mathcal{C}(Y, -)$. Then,

$$\mathrm{Nat}(\mathcal{C}(X, -), \mathcal{C}(Y, -)) \cong \mathcal{C}(Y, X).$$

Recall that a simplex category Δ had objects $\mathrm{obj}(\Delta) = [n]$ its morphisms were the weakly order-preserving map. We had face maps $d^i : [n-1] \to [n]$ which skips the element i and the degeneracy maps $s^i : [n+1] \to [n]$ which repeats the element i.

Proposition 23.8. All weakly order-preserving maps are a compositions of face and degeneracy maps.

Definition 23.9. Let $\mathcal{C}$ be a category. Then a **simplicial object** in $\mathcal{C}$ is a functor $K : \Delta^{op} \to \mathcal{C}$ and a **semi-simplicial object** is a functor $K : \Delta^{op}_{inj} \to \mathcal{C}$.

So essentially, the simplicial sets are the simplicial objects in the category Set.

Example 23.10. Let X be a space. Then $\mathrm{Sin}_* : \Delta^{op} \to \mathrm{Set}$ is given by $[n] \mapsto \mathrm{Sin}_n(X) = \mathrm{Top}(\Delta^n, X)$. The maps $d^i : [n-1] \to [n]$ are sent to $\mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$. So, $\mathrm{Sin}_n(X)$ is actually a simplicial set.

Definition 23.11. Let sSet denote the **category of simplicial sets**. The objects are simplicial sets which are functors $\Delta^{op} \to \mathrm{Set}$. The morphisms are natural transformations between these functors.

Definition 23.12. A morphism $f : X \to Y$ in $\mathcal{C}$ is a **split epimorphism** if there exists a $g : Y \to X$ s.t. $g \circ f = \mathrm{id}_Y$.

A morphism $g : Y \to X$ is a **split monomorphism** if there exists $f : X \to Y$ s.t. $f \circ g = \mathrm{id}_Y$.

Lemma 23.13. A morphism is an isomorphism iff it is a split epimorphism and a split monomorphism.

Lemma 23.14. Any functor will send split epimorphisms to split epimorphisms and split monomorphisms to split monomorphisms.

Proposition 23.15. Let $\mathcal{C} =:$ Ab. Assume $f : A \to B$ is a split epimorphism. Then there exists $g : B \to A$ s.t. $i : \ker f \to A$ and it gives an isomorphism

$$\ker f \oplus B \stackrel{i \oplus g}{\to} A.$$

Example 23.16. The map $\mathbb{Z} \to \mathbb{Z}/2$ is an epimorphism but *not a split epimorphism* while $\mathbb{Z}/6 \to \mathbb{Z}/2$ *is a split epimorphism.*

Example 23.17. For every space X, there exists a unique map $X \to \{*\}$ and $\{*\}$ serves as a terminal object in the category of topological spaces. It induces a map $\epsilon_* : H_*(X) \to H_*(\{*\}) := \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases}$.

Let $c = \sum a_i x_i$ be a 0-cycle in $S_0(X)$ (so $a_i \in \mathbb{Z}, x_i \in X$). Then $S_0(X) \to S_0(\{*\})$ which maps $c \mapsto \sum a_i$. This induces a map

$$\epsilon : H_*(X) \to H_*(\{*\}).$$

Let $X \neq \emptyset$, then the map $X \to *$ is a split epimorphism. So, the map $\epsilon_* : H_*(X) \to H_*(\{*\})$ is a split epimorphism as well. This then induces an isomorphism

$$H_*(X) \cong \ker(\epsilon_*) \oplus H_*(\{*\}) \cong \widetilde{H}_*(X) \oplus \mathbb{Z}.$$

The groups $\widetilde{H}_*(X)$ are called the **reduced homology** of X. The individual groups are called the **reduced homology group** of X .

Formally, one thinks of the $\mathbb{Z}$ in the augmented complex

$$\ldots \to S_2(X) \to S_1(X) \to S_0(X) \to \mathbb{Z} \to 0$$

as being generated by the unique map $[\emptyset] \to X$ of the empty simplex into X. The augmentation is then the usual boundary map.

24. Lecture 24: November 17th, 2021

Abstract. In this lecture, we prepare for the proof of homotopy invariance of homology. If this statement were false, then singular homology would be totally useless for us in studying homotopy. The idea rests up the definition of a chain homotopy and reducing the problem to showing the existence of a chain homotopy.

Last time, we had discussed the categorical language and reduced homology. For any space X, there exists a unique map $X \to \{*\}$ and this gives an induced map on homology groups that is a split epimorphism. So if $X \neq \emptyset$, then $H_*(X) \cong \widetilde{H}_*(X) \oplus \mathbb{Z}$ and we called $\widetilde{H}_*(X)$ the **reduced homology group**. In particular, $\widetilde{H}_n(*) = 0$ for all $n \in \mathbb{Z}$.

Some natural questions are:

(1) For which X is $\widetilde{H}_*(X) = 0$?
(2) For which X is $H_*(X) \to H_*(\{*\})$ an isomorphism?
(3) When do the two maps $1_X : X \to X$ and $f : X \to * \to X$ induce the same homology?
(4) When do the two maps $f_1, f_2 : X \to Y$ induce the same maps on homology?

The idea is that homology is a discrete invariant and so it should be unchanged by deformation. We shall prove the following theorem.

Theorem 24.1 (Homotopy Invariance of Homology)**. Homotopy Invariance of Homology** is true. That is, if $f_0 \simeq f_1$, then $H_*(f_0) = H_*(f_1)$.

Recall that $f_0, f_1 : X \to Y$ are homotopic if there exists a map $h : X \times I \to Y$ s.t. $h|_{X\times\{0\}} = f_0$ and $h|_{X\times\{1\}} = f_1$. This was denoted by $f_0 \simeq f_1$. Recall also that homotopy was an equivalence relation on $\mathrm{Top}(X, Y)$ and we shall write $[X, Y]$ for the set of homotopy classes of maps $X \to Y$. For instance, we know $[S^1, S^1] \cong \mathbb{Z}$.

If $f_0 \simeq f_1$ are homotopic maps $X \to Y$, h is their homotopy, and $g : Y \to Z$, then $g \circ h$ is a homotopy for $g \circ f_0 \simeq g \simeq f_1$. Similarly, if $K : W \to X$, then $h \circ k : f_0 \circ k \simeq f_2 \circ k$ is a homotopy. So categorically, we get a commutative diagram

$$\begin{tikzcd} \mathrm{Top}(X,Y) \times \mathrm{Top}(Y,Z) \arrow[r] \arrow[d] & \mathrm{Top}(X,Z) \arrow[d] \\ {[X,Y] \times [Y,Z]} \arrow[r, dotted] & {[X,Z]} \end{tikzcd}$$

Definition 24.2. The **homotopy category** hTop of topological spaces is the category with $\mathrm{obj}(\mathrm{hTop}) = \mathrm{obj}(\mathrm{Top})$ and whose morphisms are the homotopy classes of maps i.e. $\mathrm{hTop}(X, Y) = [X, Y] = \mathrm{Top}(X, Y)/\simeq$.

The homology functors $H_n : \mathrm{Top} \to \mathrm{Ab}$ then factors through $\mathrm{Top} \to \mathrm{hTop} \to \mathrm{Ab}$ by homotopy invariance.

Recall that $f : X \to Y$ is called a **homotopy equivalence** if $[f] \in [X, Y]$ is an isomorphism (or before, we said that there exists a $g : Y \to X$ s.t. $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$). This definition in terms of the homotopy category sheds light on why our original definition was the way it was.

Corollary 24.2.1. By homotopy invariance, homotopy equivalence induces an isomorphism in homology.

Recall that a space X is **contractible** if $X \to \{*\}$ is a homotopy equivalence. Then this next corollary follows by invariance.

Corollary 24.2.2. If X is contractible, then $\epsilon : H_*(X) \to \mathbb{Z}$ is an isomorphism.

Our goal now is to prove homotopy invariance. However, the proof itself is not at all straightforward. So, we being with a special case and start to understand homotopy on the level of chain complexes.

Definition 24.3. Let $f_0, f_1 : C_* \to D_*$ be chain maps. A **chain homotopy** $h : f_0 \simeq f_1$ is a collection of homomorphisms $h : C_n \to D_{n+1}$ s.t.

$$dh + hd = f_1 - f_0.$$

This is an equivalence relation because if $f_0 \simeq f_1$, we can use $-h$ in place of h to get $f_1 \simeq f_0$. If $h : f_0 \simeq f_1$, $h' : f_1 \simeq f_2$, then $h + h' : f_0 \simeq f_2$. If we want $f_0 \simeq f_0$, we just take $h = 0$.

If there exists a chain homotopy between f_0 and f_1, we say f_0, f_1 are **chain homotopic**.

Lemma 24.4. If $f_0 \simeq f_1$, then they induce the same maps on homology.

PROOF. We need to show that if $c \in Z_n(C_*)$, then $f_1 c - f_0 c$ is a boundary. But c is a cycle which means

$$f_1 c - f_0 c = (dh + hd)c = dhc + hdc = dhc + 0 = d(hc) \in \operatorname{im} d.$$

□

Homotopy invariance then follows from the following proposition.

Proposition 24.5. If $f_0, f_1 : X \to Y$ are homotopy, then $f_{0*}, f_{1*} : S_*(X) \to S_*(Y)$ are chain homotopic.

Again, this is still quite complicated to show. So, we deal with the simpler case of star shaped regions.

Definition 24.6. A subspace $X \subseteq \mathbb{R}^n$ is **star-shaped** w.r.t $b \in X$ if for all $x \in X$, the interval $\{tb + (1-t)x : t \in I\}$ lies in X.

Proposition 24.7. Any nonempty convex region is star-shaped. It is not necessarily true that any star-shaped region is convex. Also, any star-shaped region is contractible.

Now we state the theorem in the case that X is star-shaped.

Proposition 24.8. Let X be star-shaped. Then $\epsilon : S_*(X) \to \mathbb{Z}$, and $\eta : \mathbb{Z} \to S_*(X)$ (given by $1 \mapsto c_b^0$ the constant simplex at b) are two maps which satisfy $\epsilon\eta = \mathrm{id}_{\mathbb{Z}}$. They also satisfy $\eta\epsilon \simeq \mathrm{id} : S_*(X) \to S_*(X)$.

PROOF. Only the last part is hard. To do this, note that $\eta\epsilon : S_n(X) \to \mathbb{Z} \to S_n(X)$ is given by

$$\sigma \mapsto \begin{cases} 0 & n \geq 1 \\ c_b^0 & n = 0 \end{cases}.$$

We want to construct a chain homotopy $h : S_q(X) \to S_{q+1}(X)$ s.t. $dh + hd = 1 - \eta\epsilon$. The proof is based on a **cone construction**. We define

$$h(\sigma) := b * \sigma : \Delta^{q+1} \to X \qquad \text{where} \qquad \sigma : \Delta^q \to X$$

by having

$$b * \sigma(t_0, \ldots, t_{q+1}) = \begin{cases} t_0 b + (1-t_0)\sigma\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_{q+1}}{1-t_0}\right) & \text{if } t_0 \neq 1 \\ b & \text{if } t_0 = 1 \end{cases}.$$

Pictorially, we have a cone from our simplex to our point as in figure 15.

Now, setting $t_0 = 0$, we have $d_0(b * \sigma) = \sigma$ and setting $t_i = 0$ for $i \geq 1$, one has

$$d_i(b * \sigma) = (b * \sigma)d^i \overset{(1)}{=} b * (\sigma \circ d^{i-1}) = b * d_{i-1}\sigma$$

The equality at (1) may not be obvious at first, but it follows by checking what input for σ is omitted in the definition of $h(\sigma)$. So,

$$d(b * \sigma) = \sum_{i \leq q+1} (-1)^i d_i(b * \sigma) = \sigma - b * d\sigma$$

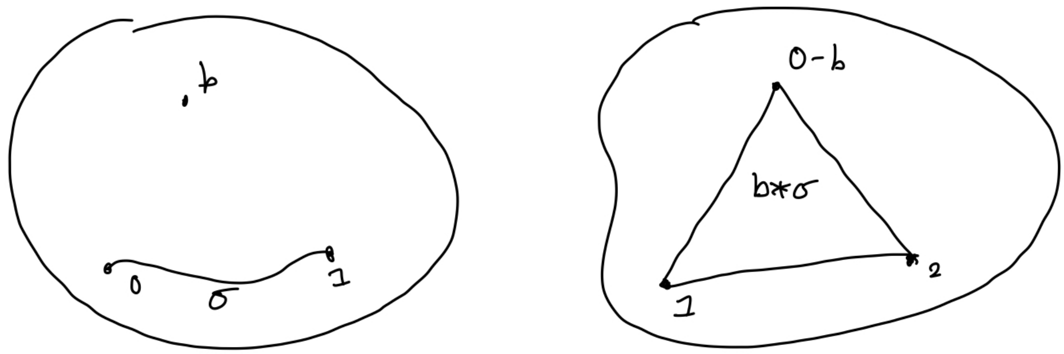


FIGURE 15. Picture of the cone in the case of $q = 1$. Here, we are given $\sigma : \Delta^1 \to X$. Then the cone is given by $b*\sigma(t_0, t_1, t_2) = t_0 b + (1-t_0)\sigma\left(\frac{(t_0,t_1,t_2)}{1-t_0}\right)$.

(as the second equality follows by bring in the d_i) and

$$d(b * \sigma) + b * d\sigma = \sigma \qquad (q \neq 0).$$

For instance, if $q = 0$, then $d(b * \sigma) = \sigma - c_b^0$. Therefore,

$$dh\sigma + hd\sigma = d(b * \sigma) + b * (d\sigma) = \sigma - c_b^0 = \sigma - \eta\epsilon\sigma$$

which means $dh + hd = 1 - \eta\epsilon$. □

Remark 9. This proposition shows that the homology of a star-shaped region is trivial. This is because

$$\ldots \to 0 \to \mathbb{Z} \to 0 \to 0 \to \ldots$$

has trivial homology.

25. Lecture 25: November 19th, 2021

ABSTRACT. The notes for this lecture continues from p. 14 of Miller's notes and proceeds until the end of Theorem 6.2. Please refer to Miller's notes if the presentation here is not sufficiently clear.

Recall we defined a chain homotopy between two chain maps $h : f_0 \simeq f_1 : C_* \to D_*$ as a map h that satisfies

$$dh + hd = f_1 - f_0.$$

We also observed that if $h : f_0 \simeq f_1 : C_* \to D_*$ is a chain homotopy, then $f_{0*} \cong f_{1*}$ as maps $H_*(C) \to H_*(D)$. Additionally, the last lecture proved the following result.

Proposition 25.1. *If X is a star-shaped space, then $S_*(X) \xrightarrow{\epsilon} \mathbb{Z}$ is a chain homotopy equivalence where $\mathbb{Z}$ denotes the chain complex*

$$\ldots \to 0 \to 0 \to \mathbb{Z} \to 0 \to .$$

Recall that a chain homotopy equivalence would mean there is a map $\eta : \mathbb{Z} \to S_*(X)$ s.t. $\epsilon \circ \eta \simeq \mathrm{id}_{\mathbb{Z}}$ and $\eta \circ \epsilon \mathrm{id}_{S_*(X)}$. So in particular, $H_*(X) \to \mathbb{Z}$ is an isomorphism. Additionally, $\widehat{H}_*(X) = 0$. Our goal now is to prove the full theorem.

Theorem 25.2 (Homotopy Invariance of Homology)**.** *If $h : f_0 \simeq f_1$ is a homotopy of maps, then there exists a chain homotopy $f_{0*} \simeq f_{1*} : S_*(X) \to S_*(Y)$. This implies the induced maps on homology groups are isomorphic.*

Recall the fact that homotopy is nice w.r.t. composition. That is, if $f_0 \simeq f_1 : X \to Y$ and $g : Y \to Z$, then $g \circ f_0 \simeq g \circ f_1$. This extends to chain complexes in the following lemma.

Lemma 25.3. Let $f_0, f_1 : C_* \to D_*$ be chain homotopic maps and $g : D_* \to E_*$. Then $g \circ f_0$ and $g \circ f_1$ are chain homotopic.

The proof of the lemma is essentially from definition. This lemma will be useful because when we have a homotopy $h : f_0 \simeq f_1$, there is a diagram

$$\begin{tikzcd}
X \times \{0\} \arrow[dr, hook, "\iota_0"] & & \\
& X \times I \arrow[r, "h"] & Y \\
X \times \{1\} \arrow[ur, hook, "\iota_1"] & &
\end{tikzcd}$$

and $h \circ \iota_1 = f_1, h \circ \iota_0 = f_0$. We then can show that ι_0, ι_1 induce equivalent chain maps $S_*(X) \to S_*(X \times I)$. Then post composing with h gives the result. So, we show there exists a $k : S_n(X) \to S_{n+1}(X \times I)$ s.t.

$$df + kd = \iota_{1*} - \iota_{0*}.$$

The idea is to define $S_n(X) \to S_{n+1}(X \times I)$ on generators $\sigma : \Delta^n \to X$. So, we need to get a map $\Delta^{n+1} \to X \times I$. We may identify I with Δ^1 to reduce to needing a map

$$\Delta^{n+1} \to \Delta^n \times \Delta^1 \to X \times I.$$

The question is then how to define the map $\Delta^{n+1} \to \Delta^n \times \Delta^1$. The following theorem plays an essential role.

Theorem 25.4. There exists a map $\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$ called the **cross product**. Given a pair of chains $\sigma \in S_p(X)$, $\tau \in S_q(Y)$, we shall write $\sigma \times \tau$ for the image. The cross product satisfies the following conditions.

(1) It is functorial. That is, if $f : X \to X'$ and $g : Y \to Y'$, then we have a commutative diagram

$$\begin{tikzcd}
S_p(X) \times S_q(Y) \arrow[r, "\times"] \arrow[d, "(f_*, g_*)"'] & S_{p+q}(X \times Y) \arrow[d, "(f \times g)_*"] \\
S_p(X') \times S_q(Y') \arrow[r, "\times"] & S_{p+q}(X' \times Y')
\end{tikzcd}$$

i.e. for all $a \in S_p(X), b \in S_q(Y)$, we have $f_*(a) \times g_*(b) = (f \times g)_*(a \times b)$.

(2) The map $\times$ is bilinear i.e.

$$(a + a') \times b = a \times b + a' \times b \qquad \& \qquad a \times (b + b') = a \times b + a \times b'.$$

(3) The map satisfies the **Leibniz rule** i.e.

$$d(a \times b) = da \times b + (-1)^p a \times db$$

for $a \in S_p(X)$.

(4) It is **normalized**. Let $x \in X, y \in Y, b \in S_q(Y)$, and $a \in S_p(X)$. Let $j_x : Y \to X \times Y$ map $y \mapsto (x, y)$ and $j_{x*}b = c_x^0 \times b$. So, we get $S_0(X) \times S_q(Y) \to S_q(X \times Y)$ where we start by fixing a generating element in $S_0(X)$.

If we fix $y \in Y$, we get $i_y : X \to X \times Y$ defined by $x \mapsto (x, y)$. This gives $i_{y*}a = a \times c_y^0$.

Essentially, we have maps

$$(c_x^0, b) \in S_0(X) \times S_q(Y) \qquad \mapsto \qquad (j_x)_* b \in S_q(X \times Y)$$

and

$$(a, c_y^0) \in S_p(X) \times S_0(X) \mapsto (i_y)_* a \in S_p(X \times Y).$$

The proof of the theorem will follow by induction on $p + q$. It is clear that the normalization property gives the result for when $p + q \in \{0, 1\}$. The idea of proof is to examine some of the basic properties the cross product must satisfy.

Proof. Let $\sigma : \Delta^p \to X$ and $\tau : \Delta^q \to Y$. We want to define $\sigma \times \tau : \Delta^{p+q} \to X \times Y$. Take $i_p = \mathrm{id}_{\Delta^p}$ and $i_q = \mathrm{id}_{\Delta^q}$.

Then, $\sigma := \sigma \circ \mathrm{id}_{\Delta^p} = \sigma_*(i_p)$ which gives a map

$$\sigma_* : \mathrm{Sin}_p(\Delta^p) \to \mathrm{Sin}_p(X) \qquad i_p \mapsto \sigma.$$

In part, we need to define $i_p \times i_q : \Delta^{p+q} \to \Delta^p \times \Delta^q$ and this is enough as $\sigma \times \tau := (\sigma \times \tau)_*(i_p \times i_q)$ gives

$$\Delta^{p+q} \to \Delta^p \times \Delta^q \overset{\sigma\times\tau}{\to} X \times Y.$$

Assume $p \geq 1, q \geq 1$. By the Leibniz rule, we know that

$$d(i_p \times i_q) = di_p \times i_q + (-1)^p i_p \times i_q \in S_{p+q-1}(\Delta^p + \Delta^q).$$

So, $d^2 = 0$ and it is necessary for $d((-1)^p i_p \times i_q \in) = 0$. Writing this out,

$$d((-1)^p i_p \times i_q \in) = d^2 i_p \times di_q + (-1)^{p-1} di_p \times di_q + (-1)^p di_p \times di_q + (-1)^p(-1)^p i_p \times d^2 i_q = 0$$

as the terms cancel. Note that this is the reason why the $(-1)^p$ factor must appear. Now, $\Delta^p \times \Delta^q \subseteq \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ is convex which means that it is star-shaped. So, $H_{p+q-1}(\Delta^p \times H^q) = 0$ for $p \geq 1, q \geq 1$. So,

$$d(i_p \times i_q) \in Z_{p+q-1}(\Delta^p \times \Delta^q) = B_{p+q-1}(\Delta^p \times \Delta^q).$$

So choose any element in $S_{p+q}(\Delta^p \times \Delta^q)$ with boundary (i.e. its image under d)

$$di_p + i_q + (-1)^p i_p + di_q.$$

This will do for a choice. Next lecture, we shall see how to make an explicit choice. Indeed, the this procedure constructs a non-unique cross product and depends on choice of the chain $i_p \times i_q$ for each (p, q) pair with $p + q > 1$. $\square$

Now, homotopy invariance follows quite naturally. We need to define a chain homotopy $h : S_n(X) \to S_{n+1}(X \times I)$ s.t. $dh + hd = \iota_{1*} - \iota_{0*}$. Pick $i : \Delta^1 \to I$ which sends $0 \to 0$ and $1 \to 1$. So, $d_0 c = c_1^0$ and $d_1 c = c_0^0$. So, $h\sigma := (-1)^n \sigma \times i \in S_{n+1}(X \times I)$ will do as a map from $\mathrm{Sin}_n(X) \to S_{n+1}(X \times I)$. Indeed, extend this linearly to a map from $S_n(X)$. Now,

$$
\begin{aligned}
(dh + hd)\sigma &= dh\sigma + hd\sigma \\
&= d((-1)^n \sigma \times i)) + (-1)^{n-1} d\sigma \times i \\
&= (-1)^n d_0 \times i + (-1)^n (-1)^n \sigma \times di + (-1)^{n-1} d\sigma \times i \\
&= \sigma \times di \\
&= \sigma \times (c_1^0 - c_0^0) \\
&= \sigma \times c_1^0 - \sigma \times c_0^0 \\
&= i_{1*}\sigma - i_{0*}\sigma.
\end{aligned}
$$

Here,the last equality follows by the normalization property for the cross product. So, $i_{0*} \simeq i_{1*}$ which means homology is a homotopy invariant.

Though we already stated this above, we shall detail it here again. We now have the following diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & S_{n+1}(X) \arrow[r, "d"] \arrow[d] \arrow[dl] & S_n(X) \arrow[r, "d"] \arrow[d, "i_{0*}"', "i_{1*}"] \arrow[dl, "h"'] & S_{n-1}(X) \arrow[r] \arrow[d] \arrow[dl, "h"'] & \cdots \\
\cdots \arrow[r] & S_{n+1}(X \times I) \arrow[r, "d"'] & S_{n+1}(X \times I) \arrow[r, "d"'] & S_{n-1}(X \times I) \arrow[r] & \cdots
\end{tikzcd}
$$

which tells us that $i_{0*}, i_{1*} : H_n(X) \to H_n(X \times I)$ are the same. Pre compose by H_* where H is the homotopy $H : X \times I \to Y$ between f_0, f_1, we get $H_* \circ i_{0*}, H_* \circ i_{1*}$ induce the same maps on homology due to chain homotopy being nice with composition. But functoriality says that

$$(H \circ i_0)_* = H_* \circ i_{0*} = H_* \circ i_{1*} = (H \circ i_1)_*$$

as maps.

26. Lecture 26: November 22nd, 2021

Abstract. This lecture explains a more concrete construction of the cross product. The construction is based on the Eilenberg-Zilber chain which is shown to exist.

Let us begin by reviewing what we did last time. We had already proven the following theorem.

Theorem 26.1. If $f_0 \simeq f_1 : X \to Y$ are homotopic, then $f_{0*} \cong f_{1*} : H_*(X) \to H_*(Y)$. IP, homotopy equivalent spaces have the same homology.

The idea of proof was as follows. First, one can show that if $g_0 \simeq g_1 : C_* \to D_*$ are chain homotopic chain maps, then $g_{0*} \cong g_{1*} : H_*(C) \to H_*(D)$.

Now if $f_0 \simeq f_1 : X \to Y$ are homotopic, then $f_{0*} \simeq f_{1*} : S_*(X) \to S_*(Y)$ are chain homotopic. Now a homotopy implies that we have a diagram

$$\begin{tikzcd} X \times \{0\} \arrow[dr, "i_0"] & & \\ & X \times I \arrow[r, "h"] & Y \\ X \times \{1\} \arrow[ur, "i_1"] & & \end{tikzcd}$$

so that $f_0 = h \circ i_0$ and $f_1 = h \circ i_1$. It then suffices, as homotopy is nice w.r.t. composition, to show that $i_{0*} \simeq i_{1*} : S_*(X) \to S_*(X \times I)_*$. That is, we show that there is a chain homotopy $k : S_n(X) \to S_{n+1}(X \times I)$ s.t. $dk + kd = i_{1*} - i_{0*}$.

If $\sigma \in \mathrm{Sin}_n(X)$, we defined

$$k\sigma = (-1)^n \sigma \times i$$

where $i : \Delta^1 \to I$ is the natural identification and $\sigma \times i$ is the cross product. To construct the cross product, we had the following theorem.

Theorem 26.2. There exists a map $\times : S_q(X) \times S_q(Y) \to S_{p+q}(X \times Y)$ called the **cross product** which was natural, bilinear, satisfied the Leibniz rule, and was normalized.

We prove that such a map existed via a nonconstructive method. That is, we implicitly constructed a map $i_p \times i_q$ for $i_p = \mathrm{id}_{\Delta^p} : \Delta^p \to \Delta^p$ and $i_q = \mathrm{id}_{\Delta^q} : \Delta^q \to \Delta^q$ and $i_p \times i_q : \Delta^{p+q} \to \Delta^p \times \Delta^q$. We did not prove this explicitly and only relied on the convexity of $\Delta^p \times \Delta^q$ in $\mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ to deduce that it was star-shaped and therefore $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$ when $p + q \geq 2$.

The question to ask now is if there is a more explicit/geometric construction to $i_p \times i_q$? Yes, there is and it is called the **Eilenberg-Zilber construction**. The idea is triangulate a prism.

Consider the ordered set $[p] = \{0, 1, \ldots, p\}$ and $[p] \times [q] = \{(a, b) : a \in [p], b \in [q]\}$ with the **lexicographic order** i.e. $(a, b) \leq (a', b')$ iff $a < a'$ or both $a = a'$ and $b < b'$. Fix this ordering for $[p] \times [q]$.

Let $\mathcal{O}$ denote the set of order preserving maps $w : [p+q] \to [p] \times [q]$. The elements of $\mathcal{O}$ can be visualized as staircases in $[0, p] \times [0, q]$. See figure 16.

Definition 26.3. Let $A : \mathcal{O} \to \mathbb{Z}$ be the area under the **staircase**. In figure 16, we have $A(\omega_0) = 0$, $A(\omega_1) = 1$, $A(\omega_2) = 0$. Now extend A linearly to $A : \mathbb{Z}\mathcal{O} \to \mathbb{Z}$.

An element $\omega \in \mathcal{O}$ determines $w : \Delta^{p+q} \to \Delta^p \times \Delta^q$. Recall that $\Delta^p := \{\sum_{i=0}^p t_i e_i \in \mathbb{R}^{p+1} : 0 \leq t_i \leq 1, \sum_{i=0}^p t_i = 1\}$.

If $w(j) = (k, l)$, then $w(e_j) = e_k \times e_l$ which means $j \in [p+q], k \in [p], l \in [q]$. Extend linearly. See figure 17. A better picture is in 18.

Definition 26.4. We define $i_p \times i_q = \sum_{w \in \mathcal{O}} (-1)^{A(w)} w$ where the sum is over all order-preserving maps. We may decompose $\Delta^p \times \Delta^q$ into pieces and each piece is homeomorphic to Δ^{p+q}. So, $i_p \times i_q \in S_{p+q}(\Delta^p \times \Delta^q)$.

Proposition 26.5. The cross product defined by the **Eilenberg-Zilber chain** (i.e. $i_p \times i_q = \sum (-1)^{A(w)} w$) satisfies the Leibniz rule.

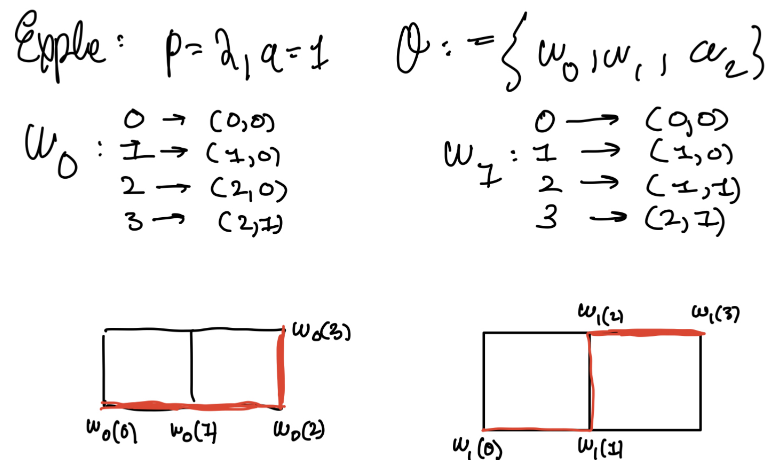


FIGURE 16. An example in the case of $p = 2$ and $q = 1$.

PROOF. This is straightforward and just a combinatorial proof. $\square$

This then gives an explicit map $\beta_{X,Y} : S_*(X) \times S_*(Y) \to S_*(X \times U)$ that is associative and commutative. That is, the following diagram commutes

$$\begin{array}{ccc} S_*(X) \times S_*(Y) \times S_*(Z) & \longrightarrow & S_*(X \times Y) \times S_*(Z) \\ \downarrow & & \downarrow \\ S_*(X) \times S_*(Y \times Z) & \longrightarrow & S_*(X \times Y \times Z) \end{array}$$

and so does the following (where $T : X \times Y \to Y \times X$ switches coordinates and is normalized i.e. $T(a, b) = (-1)^{pq}(b, a)$ for all $(a, b) \in X \times Y$)

$$\begin{array}{ccc} S_*(X) \times S_*(Y) & \xrightarrow{\beta_{X,Y}} & S_*(X \times Y) \\ \downarrow & & \downarrow T_* \\ S_*(Y) \times S_*(X) & \xrightarrow{\beta_{Y,X}} & S_*(Y \times X) \end{array}$$

In this way, the cross product gives a map $\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$.

Let A_*, B_*, C_* be chain complexes and $\times : A_* \times B_* \to C_*$ be a bilinear map that satisfies the Leibniz rule. It turns out that these properties determine a bilinear map on the homology.

Lemma 26.6. The data above determines a bilinear map

$$\times : H_*(A) \times H_*(B) \to H_*(C)$$

i.e.

$$\frac{Z_*(A)}{B_*(A)} \times \frac{Z_*(B)}{B_*(B)} \to \frac{Z_*(C)}{B_*(C)}.$$

PROOF. Let $a \in Z_p(A), b \in Z_q(B)$. Set $[a] \times [b] := [a \times b]$. Now, the Leibniz rule gives $d[a \times b] = da \times b + (-1)^p a \times db = 0$. So $a \times b$ is a cycle.

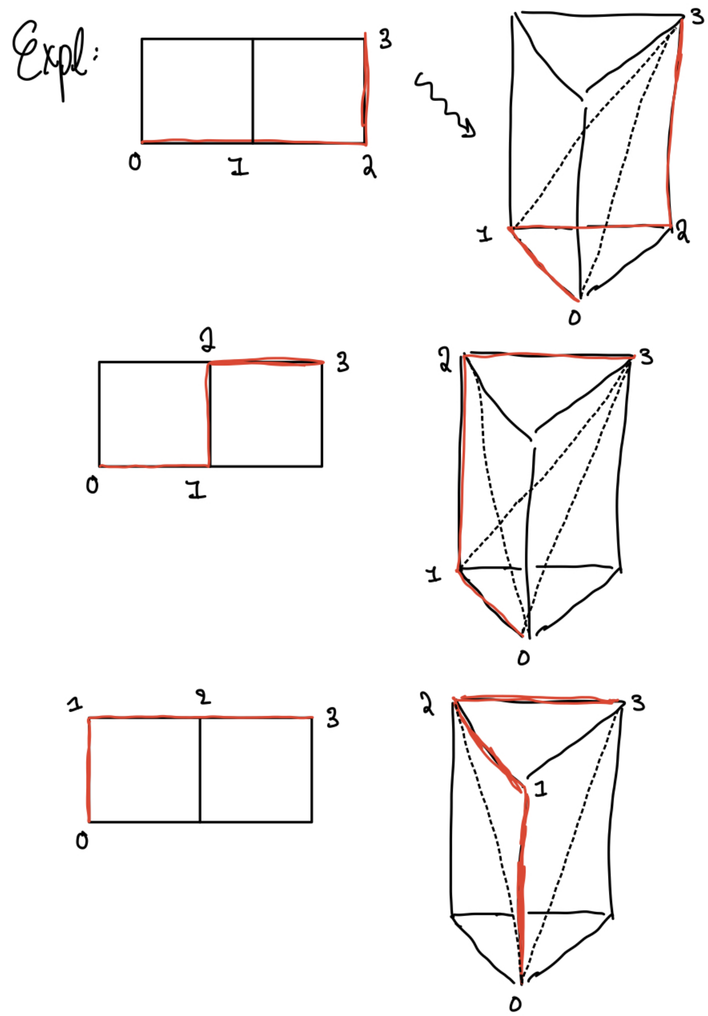


FIGURE 17. Examples of the corresponding prisms.

We now check that this is well-defined. Let $a' = a + d\overline{a}$ and $b' = b + d\overline{b}$. Then, we shall show that $a' \times b' - a \times b$ is also a boundary. We know

$$a' \times b' - a \times b = (a + d\overline{a}) \times (b + d\overline{b}) - a \times b$$
$$a \times d\overline{b} + b \times d\overline{a} + d\overline{a} \times d\overline{b}.$$

We can compute the following using the fact that $da = 0$, $db = 0$ and $d^2 = 0$ respectively

$$d(a \times \overline{b}) = (-1)^p \times d\overline{b}$$
$$d(b \times \overline{a})(-1)^q \times d\overline{a}$$
$$d(\overline{a} \times d\overline{b}) = d\overline{a} \times d\overline{b}.$$

So,

$$a' \times b' - a \times b = d((-1)^p a \times b + (-1)^q b \times a + a \times db)$$

is a boundary. □

$$\iota_p \times \iota_q = \sum (-1)^{A(\omega)} \omega \,.$$

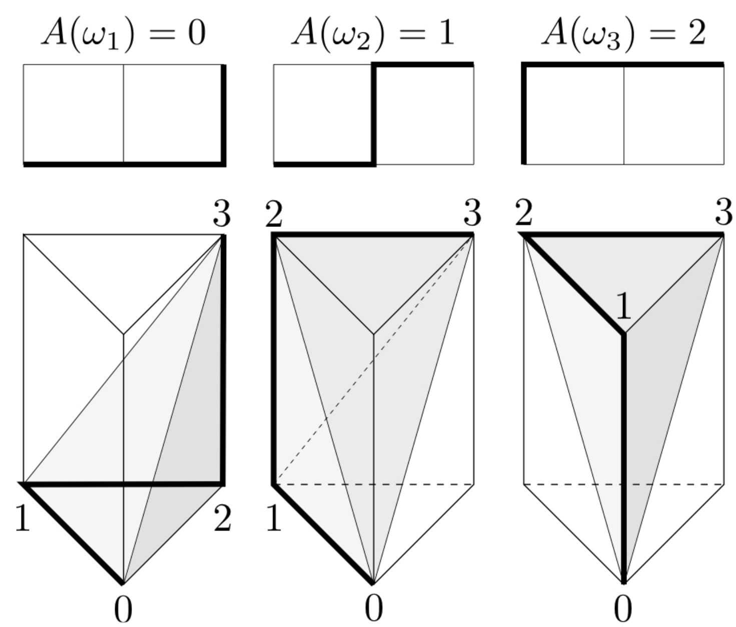


FIGURE 18. A drawing for the **Eilenberg-Zilber chain** which is taken from Miller's lecture notes.

Theorem 26.7. There exists a natural bilinear normalized associative and commutative map

$$\times : H_p(X) \times H_q(Y) \to H_{p+q}(X \times Y).$$

So, although chain level cross product is *not unique*, the homology level cross product *is unique*.

27. Lecture 27: November 24th, 2021

ABSTRACT. In this lecture, we work towards more computational tools. So far, we have avoided computing the homology of any really complicated spaces because we did not have enough weapons to attack such problems. For instance, we have not computed the singular homology of the all of the closed 2-manifolds. To make computations easier, we shall prove the Mayer-Vietoris Theorem which relates the homology of a union of two spaces. We also define relative homology which hels us determine the homology of a space with knowledge of the homology of a subspace.

The goal of the next few lectures will be to provide some actual computational tools for computing homology. In the previous lectures, we had only focused on the theoretical construction of **singular homology** and not so much on computations.

The goal of algebraic topology is to understand $[X, Y]$ i.e. the homotopy classes of maps from X to Y. This is not as strong as studying the classes of homeomorphisms between X and Y, but this already gives more than sufficient information to solve many problems. Homology can be viewed as an additive approximation to these goals. In particular, we may expect

- if $A \subseteq X$ is a subspace, then $H_*(X)$ can be related to a combination of $H_*(A)$ and $H_*(X \setminus A)$,
- and that $H_*(A \cup B)$ is related to A, B in some way "like" $H_*(A) + H_*(B) - H_*(A \cap B)$.

The formal notions for the first idea is the idea of the **long exact sequence** (LES) and the latter is formalized by the **Mayer-Vietoris Theorem**. The last few lectures of this quarter will be focused on these topics.

The goal of today is to discuss **relative homology**. First off, it will not be the case that $H_*(X/A) \cong H_*(X)/H_*(A)$ for all spaces, but we can get something quite close.

Definition 27.1. A **sequence of abelian groups** is a diagram of abelian groups

$$\dots \to C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \to \dots$$

s.t. $\mathrm{im}(f_{n+1}) \subseteq \ker(f_n)$. It is **exact** if this is actually an equality for all n. We say it is **exact at** C_n if it is exact for n.

Remark 10. The difference between chain complexes and sequences of abelian groups is minor. Here, something like $0 \to A \to B \to C \to 0$ can be called a sequence of abelian groups while for a chain complex, we would require it to be extended to the left and right indefinitely:

$$\dots \to 0 \to A \to B \to C \to 0 \to 0 \to \dots.$$

Also, observe that a sequence is exact at C_n iff $H_n(C_*) = 0$ which means that homology measures failure of exactness.

Example 27.2. A sequence $0 \to A \xrightarrow{f} B$ is exact at A iff f is injective. OTOH, $A \xrightarrow{g} B \to 0$ is exact iff g is surjective.

Definition 27.3. A **short exact sequence** (SES) is an exact sequence of form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. Note that we sometimes call a general exact sequence a **long exact sequence** (LES).

Example 27.4. Some examples of SESs are

$$\begin{aligned} 0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0 \\ 0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0 \\ 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0. \end{aligned}$$

It is left to the reader to specify the maps. OTOH,

$$0 \to \mathbb{Z}/2 \to \mathbb{Z} \to \mathbb{Z} \to 0$$

can be made to be a nonexample. In fact, it is impossible for there to be a pair of maps that makes this sequence exact.

Definition 27.5. Let B_* be a chain complex. A **subcomplex** A_* of B_* is a chain complex A_* with injective chain maps $i_* : A_* \to B_*$

$$\begin{tikzcd} \dots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & B_{n+1} \arrow[r] & B_n \arrow[r] & B_{n-1} \arrow[r] & \dots \end{tikzcd}$$

Example 27.6. If $A \subseteq X$ is a subspace, then $S_*(A)$ is a subcomplex of $S_*(X)$ and the map $i : A \hookrightarrow X$ induces the desired chain map $i_* : S_*(A) \to S_*(X)$.

Definition 27.7. If A_* is a subcomplex of a chain complex B_*, we may define the **quotient chain comple** $(B/A)_*$ by

(1) $(B/A)_k = B_k/A_k$,
(2) and if $[a] \in (B/A)_k$, the differential map $d[a] := [da]$ is well-defined and satisfies $d^2 = 0$.

Lemma 27.8. The differential d on $(B/A)_*$ is well-defined.

PROOF. The proof is easy. If $[a] = [a']$ in $(B/A)k$, write $a = a' + \overline{a}$ for $\overline{a} \in A_k$. Then applying d gives

$$da = da' + d\overline{a}.$$

But because A_* is a subcomplex, $d\overline{a} \in A_{k-1}$ which means $[da] = [da']$ as elements in $(B/A)_{k-1}$. Also, $d^2 = 0$ follows immediately from the equation. ☐

Essentially, we have the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & B_{n+1} & \longrightarrow & B_n & \longrightarrow & B_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & B_{n+1}/A_{n+1} & \longrightarrow & B_n/A_n & \longrightarrow & B_{n-1}/A_{n-1} & \longrightarrow & \cdots
\end{array}$$

Definition 27.9. Let $A \subseteq X$ be a subspace. We define the **relative homology** $H_*(X, A)$ as the homology of the quotient complex $S_*(X)/S_*(A)$. So, if $A = \emptyset$, then $H_*(X, A) = H_*(X)$. So in a sense, the relative homology generalizes the usual notion of singular homology.

We shall state the following theorem which shall be proved later. It is a powerful tool for computations.

Theorem 27.10. If A is a subcomplex of the CW-complex X, then $H_*(X, A) \cong H_*(X/A)$.

Let us try to understand the relative homology categorically. First, we define a category Top_2 Top_2 which has objects being pairs (X, A) s.t. $A \subseteq X$ and morphisms $f : (X, A) \to (Y, B)$ being morphisms $f : X \to Y$ with $f(A) \subseteq B$. There are some obvious functors from Top_2 to Top that relate them. Consider $\mathrm{Top} \mapsto \mathrm{Top}_2$ defined by $X \mapsto (X, \emptyset)$ and $\mathrm{Top} \to \mathrm{Top}_2$ defined by $X \mapsto (X, X)$. OTOH, there is $\mathrm{Top}_2 \to \mathrm{Top}$ defined by $(X, A) \to X$ and $\mathrm{Top}_2 \to \mathrm{Top}$ defined by $(X, A) \to A$.

Proposition 27.11. Relative homology is a functor $H_n(-,-) : \mathrm{Top}_2 \to \mathrm{Ab}$ given by $(X, A) \mapsto H_n(X, A)$.

PROOF. Let $f : (X, A) \to (Y, B)$. We show it induces a map $f_* : H_*(X, A) \to H_*(Y, B)$. There is a diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\uparrow & & \uparrow \\
A & \longrightarrow & B
\end{array}$$

which induces another diagram

$$\begin{array}{ccc} S_*(X) & \xrightarrow{f} & S_*(Y) \\ \uparrow & & \uparrow \\ S_*(A) & \longrightarrow & S_*(B) \end{array}.$$

So, the diagram shows that we have an induced map $S_*(X)/S_*(A) \to S_*(Y)/S_*(B)$ (which is actually a chain map). Taking homology induces a map $f_* : H_*(X, A) \to H_*(Y, B)$. □

Remark 11. Observe that $S_n(X)$ and $S_n(A)$ were defined to be free abelian groups. By the identification of $S_n(A) \leq S_n(X)$ as a subgroup, one can actually show (think about it) that $S_n(X)/S_n(A)$ is a few group whose generators are the cosets of singular n-chains in X that do not lay entirely in A.

Example 27.12. Consider Δ^n and how $\partial\Delta^n = \bigcup_{i=0}^n \mathrm{im}(d_i : \Delta^n \to \Delta^{n-1})$. Consider i_n : $S_n(\Delta^n) \to S_n(\Delta^n)$ given by the identity map. Then

$$d(i_n) = \sum_j (-1)^j (d : \Delta^{n-1} \to \Delta^n) \neq 0$$

so i_n is not a cycle, but it does land in $S_{n-1}(\Delta^n)$. So, $d(i_n) = 0$ as an element of $S_{n-1}(\Delta^n)/S_{n-1}(\partial\Delta^n)$. So, i_n is a relative n-cycle and it represents an element $[i_n] \in H - n(\Delta^n, \partial\Delta^{n-1})$. We will later show that $H_n(\Delta^n, \Delta^{n-1}) \cong \mathbb{Z}$ and it is generated by $[i_n]$.

28. Lecture 28: November 29th, 2021

Abstract. In this lecture, we complete the construction of the long exact sequence of homology which is a powerful computational tool for relative homology.

Recall that (X, A) denotes a pair where X is a topological space and A is a topological subspace with induced topology $A \subseteq X$. We defined the **relative homology**. The idea was that if A_* is a subcomplex of B_*, then one can defined the quotient complex $(B/A)_* := B_*/A_*$. Indeed, take $S_*(A) \subseteq S_*(X)$ to obtain $S_*(X)/A_*(A)$ and the homology of this complex is $H_*(X, A) := H_*(\frac{S_*(X)}{S_*(A)})$. This may be viewed as a functor $\mathrm{Top}_2 \to \mathrm{Ab}$.

Another way to understand this is to consider the SES

$$0 \to S_*(A) \to S_*(X) \to S_*(X)/S_*(A) \to 0$$

where all of the maps are chain maps. Taking homology, we get a sequence $H_*(A) \to H_*X) \to H_*(X, A)$ which we may ask whether or not it is exact. The answer is yes, but is generally not a SES. The proof itself follows by a diagram chase. In general, we have a long exact sequence of homology which follows from a more general theorem.

Theorem 28.1 (The **Long Exact Sequence of Homology**). If $0 \to A_* \to B_* \to C_* \to 0$ is a SES of chain complexes, then there exists a natural homomorphism $\partial : H_n(C_*) \to H_{n-1}(A_*)$ s.t. the sequence

$$\ldots \to H_{n+1}(C) \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to \ldots$$

is exact (we omit the $*$ if the complexes are understood).

Proof. Apply the Snake Lemma (or do a diagram chase). □

Corollary 28.1.1. Let (X, A) be a pair and then there exists a LES

$$\ldots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \ldots$$

Corollary 28.1.2. If $0 \to \widetilde{S}_*(A) \to \widetilde{S}_*(A) \to S_*(X, A) \to 0$ is a SES (where $\widehat{S}_*$ indicates the reduced complex), then there is a LES

$$\ldots \to H_n(X, A) \to \widetilde{H_{n-1}}(A) \to \widetilde{H_{n-1}}(X) \to H_{n-1}(X, A) \to \ldots$$

Example 28.2. Consider $(X, A) := (D^n, S^{n-1})$. We know $\widetilde{H_*}(D^n) = 0$ and so there is sequence

$$0 \to H_n(X, A) \to \widetilde{H_{n-1}}(A) \to 0 \to H_{n-1}(X, A) \to \widetilde{H_{n-2}}(A) \to \ldots$$

and so, $\partial : H_q(D^n, S^{n-1}) \to H_{q-1}(S^{n-1})$ is an isomorphism.

Lemma 28.3 (5-lemma)**.** Given a diagram

$$\begin{tikzcd}
A_4 \arrow[r, "d"] \arrow[d, "f_4"] & A_3 \arrow[r, "d"] \arrow[d, "f_3"] & A_2 \arrow[r, "d"] \arrow[d, "f_2"] & A_1 \arrow[r, "d"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] \\
B_4 \arrow[r, "d"] & B_3 \arrow[r, "d"] & B_2 \arrow[r, "d"] & B_1 \arrow[r, "d"] & B_0
\end{tikzcd}$$

where $d^2 = 0$ and everything commutes. Then,

(1) if f_0 is injective, f_1 and f_3 are surjective, then f_2 is surjective;
(2) if f_4 is surjective, f_1 and f_3 are injective, then f_2 is injective;
(3) if f_0, f_1, f_2, f_4 are isomorphisms, then $f)2$ is an isomorphism.

PROOF. (3) follows immediately from (1) and (2), but the other parts are via diagram chases. □

Corollary 28.3.1. Given map of SESs of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r] \arrow[d] & B_* \arrow[r] \arrow[d] & C_* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'_* \arrow[r] & B'_* \arrow[r] & C'_* \arrow[r] & 0
\end{tikzcd}$$

in which two of three of the vertical maps induce isomorphisms on homology. Then the third map induces an isomorphism on homology.

PROOF. Write out the LES of homology and the chain map between the two LESs. Then apply the 5-lemma. □

Proposition 28.4. If $f : (X, A) \to (Y, B)$ is a map in Top_2 and two of the three maps $X \to Y, A \to B, (X, A) \to (Y, B)$ induce isomorphisms on homology, then the third map induces an isomorphism on homology.

PROOF. Apply the preceding result. □

29. Lecture 29: December 1st, 2021

Recall that we have been discussing relative homology $H_*(X, A)$ for $A \subseteq X$ which was constructed from the quotient complex $S_*(X)/S_*(A)$. A tool for computing the relative homology was the LES of homology

$$\ldots \to H_{n+1}(X) \to H_{n+1}(X, A) \to H_n(A) \to H_n(X) \to H_n(X, A) \to \ldots$$

which was obtained via an application of the snake lemma (though we only implicitly used it). To forget more information, we may consider the relatively homology $H_*(X) \cong \widetilde{H}_*(X) \oplus \mathbb{Z}$ where our $\mathbb{Z}$ is concentrated in degree 0. Then, $H_*(X, \{*\}) \cong \widetilde{H}_*(X)$ via the LES and $H_*(\{*\}) = \mathbb{Z}$. Two crucial properties we have obtained are the homotopy invariance of homology and the existence of relative homology. In a sense, $H_*(X, A)$ gives information for "$X - A$" which we now make rigorous.

Definition 29.1. A triple (X, A, U) of sets $U \subseteq A \subseteq X$ is called **excisive** if $\overline{U} \subseteq \operatorname{int}(A)$. The inclusion $(X - U, A - U) \subseteq (X, A)$ is called an **excision**. We have essentially excised A our of our space.

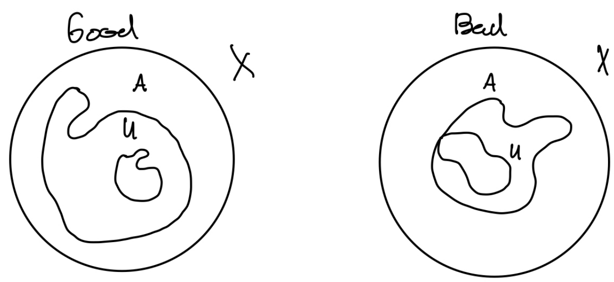


Figure 19. Examples of what counts for an excision. The RHS does not work since U touches the boundary of A.

Theorem 29.2. If (X, A, U) is excisive, then the excision induces an isomorphism on homology groups i.e.

$$H_*(X - A, A - U) \xrightarrow{\cong} H_*(X, A).$$

Remark 12. Intuitively, relative homology depends only on $X - A$ so removing U does not change it. The proof of the theorem would take a few lectures so we leave this for next quarter.

Given a pair (X, A), we have a **collapsing map** $(X, A) \to (X/A, A/A = \{*\})$ obtained by quotienting by A.

Corollary 29.2.1. Assume there exists $B \subseteq X$ s.t. $\overline{A} \subseteq \operatorname{int}(B)$ and $A \to B$ a deformation retraction. Then there is an isomorphism $H_*(X, A) \xrightarrow{\cong} H_*(X/A, \{*\})$.

Proof. There is a commutative diagram

$$\begin{tikzcd}
(X, A) \arrow[r, hook, "i"] \arrow[d] & (X, B) \arrow[d] & (X - A, B - A) \arrow[l, hook', "j"'] \arrow[d, "k"] \\
(X/A, \{*\}) \arrow[r, hook, "\bar{i}"] & (X/A, B/A) & (X/A - \{*\}, B/A - \{*\}) \arrow[l, hook', "\bar{j}"']
\end{tikzcd}$$

and all of the vertical maps are collapsing maps. All horizontal maps are inclusions. To show that $(X, A) \to (X/A, \{*\})$ induces an isomorphism on homology, it suffices to show that $i, j, k, \overline{j}, \overline{i}$ do by commutativity of the diagram.

We know $j, \overline{j}$ do by the excision result. We know k does because it is a homeomorphism (intuitively, $X - A$ has A removed so quotienting by A first and removing the point A collapses to is the same i.e. $X - A \cong X/A - \{*\})$). Since $i : (X, A) \to (X, B)$ has $A \hookrightarrow B$ a deformation retraction, homotopy invariance gives us what we want. Now look at the LES

$$\begin{tikzcd}
\ldots \arrow[r] & H_*(A) \arrow[r] \arrow[d, "\cong"] & H_*(X) \arrow[r] \arrow[d, "\cong"] & H_*(X, A) \arrow[r] \arrow[d] & H_{*-1}(A) \arrow[r] \arrow[d, "\cong"] & H_*(A) \arrow[r] \arrow[d, "\cong"] & \ldots \\
\ldots \arrow[r] & H_*(B) \arrow[r] & H_*(X) \arrow[r] & H_*(X, B) \arrow[r] & H_{*-1}(B) \arrow[r] & H_*(X) \arrow[r] & \ldots
\end{tikzcd}$$

and the isomorphisms were all justified above. By the 5-lemma, $H_*(X/A, \{*\}) \to H_*(X/A, B/A)$ is an isomorphism. □

Corollary 29.2.2. If $A \subseteq B \subseteq X$ with B deformation retracting to A and $\overline{A} \subseteq \operatorname{int}(B)$, then $H_n(X, A) \cong \widetilde{H_n}(X/A)$ (this holds in the particular case where X is a CW-complex).

Corollary 29.2.3. For $n \geq 1$, $H_k(D^n, S^{n-1}) \cong \widetilde{H_k}(S^n)$.

Proof. Let $X = D^n$ and $A = S^{n-1}$. Set $B := D^n - \{0\}$ and use $D^n/S^{n-1} \cong S^n$. □

Lemma 29.3. If X is contractible, then $H_k(X, A) \cong \widetilde{H_{k-1}}(X)$ for all k.

Proof. Fomr the LES,

$$\to \widetilde{H_k}(X) \to H_k(X, A) \to \widetilde{H_{k-1}}(A) \to \widetilde{H_{k-1}}(X) \to \ldots$$

and since the objects on the LHS and RHS are zero, there is an isomorphism in the middle. □

Corollary 29.3.1. One has $H_k(D^n, S^{n-1}) \cong \widetilde{H_{k-1}}(S^{n-1})$.

The above corollary follows by the preceding result. Now $\widetilde{H_{k-1}}(S^{n-1}) \cong \widetilde{H_k}(S^n)$. So we know $\widetilde{H_*}(S^0) = \mathbb{Z}$ in degree 0 (this is clear by S^0 consisting of two points). So it tells us that

$$\widetilde{H_*}(S^n) \cong \mathbb{Z}$$

concentrated in degree zero. It follows that th usual homology is

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 0, k = 0 \\ 0 & n = 0, k > 0 \\ \mathbb{Z} & n \geq 1, k = 0, n \\ 0 & n \geq 1, k \neq 0, n \end{cases}.$$

Also,

$$H_k(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases}$$

Now $(\Delta, \partial \Delta^n) \cong (D^n, S^{n-1})$ so we know what the generator of the copy of $\mathbb{Z}$ is.

Corollary 29.3.2. By homotopy invariance, if $m \neq n > 0$, then S^m is not homotopy equivalent to S^n.

Corollary 29.3.3. For all $n \geq 0$, S^n is not contractible.

Corollary 29.3.4. For $m \neq n$, $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$.

PROOF. Remove a point and use the homotopy equivalence $\mathbb{R}^m \setminus \{*\} \simeq S^{m+1}$. $\square$

Corollary 29.3.5. For $n \geq 1$, S^{n-1} is not a retract of D^n.

PROOF. Suppose not and there exists $r : D^n \to S^{n-1}$ s.t. $r \circ i = \mathrm{id}_{S^{n-1}}$ for $i : S^{n-1} \hookrightarrow D^n$ the inclusion. Apply the homology functor to get

$$\widetilde{H_{n-1}}(S^{n-1}) \to \widetilde{H_{n-1}}(D^n) \to \widetilde{H_{n-1}}(S^{n-1})$$

which is just a composite $\mathbb{Z} \to 0 \to \mathbb{Z}$. Clearly there is not way for the composite to be an isomorphism. $\square$

Corollary 29.3.6 (Brouwer Fixed Point Theorem). Any continuous map $f : D^n \to D^n$ has a fixed point.

PROOF. If not, define $g(x)$ as the boundary point that is on the ray from $f(x)$ to x. This is a retract of D^n to S^{n-1}. Contradiction. $\square$

30. Lecture 30: December 3rd, 2021

This final lecture is simply a quick review of everything that was covered this quarter. The quarter itself can be split into two parts: the first part covered π_1 and covering spaces and the second part covered singular homology which is a functor $H_* : \mathrm{Top} \to \mathrm{Ab}$ that is homotopy invariant.

At the start of the course, we defined a homotopy between maps $f \simeq g : X \to Y$ and used this to define a homotopy equivalence between two spaces $X \simeq Y$. This is the notion of equivalence that is of interest in algebraic topology and is weaker (though not too weak) than homeomorphisms. We defined $\pi_1(X, x_0) = [(S^1, *), (X, x_0)]$ which was the fundamental group and there is an equivalent way to define it as the group of homotopy classes of loops. We showed that if $X \simeq Y$, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

To begin using the fundamental group, we need to know some basic computational examples. The easier nontrivial example is $\pi_1(S^1)$ which was computed using covering space theory. Additionally, we need properties of $\pi_1(-)$ like functoriality and that it is a functor from Top_* to Ab. One could also observe that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

A major computation tool is Van Kampen's Theorem. It roughly says that if $X = \bigcup_{s \in S} A_s$ and there exists $x_0 \in A_s$ for all $s \in S$, then $\pi_1(X, x_0) \cong *_{s \in S} \pi_1(A_s, x_0)/N$ where N is the normal subgroup generated by the elements $i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1}$ for $\alpha, \beta \in S$, $w \in \pi_1(A_\alpha \cap A_\beta, x_0)$ and $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta, x_0) \to \pi_1(A_\alpha, x_0)$.

For $n \geq 2$, Van Kampen's theorem shows $\pi_1(S^n) = 0$. Additionally, the theorem shows how we can compute π_1 for CW-complexes $X = \bigcup_{n \in \mathbb{N}_0} \mathrm{sk}_n X$. Here, recall that push-out diagram construction of $\mathrm{sk}_n X$. By Van Kampen's Theorem, $\pi_1(X) \cong \pi_1(\mathrm{sk}_2(X))$. Using this, if $\mathrm{sk}_0 X = \{*\}$ is a singleton set, then attaching 1-cells gives generators for $\pi_1(-)$ while attaching $\pi_2(-)$ gives relations. So every group arises as the fundamental group of some topological space.

Covering spaces are useful for studying the fundamental group. Recall that a covering space of X is a map $p : \widetilde{X} \to X$ that is locally a homeomorphism. It has lifting properties. For instance, if there is an $f : Y \to X$, then we can lift $\widetilde{f} : Y \to \widetilde{X}$ iff $f_* \pi_1(Y) \subseteq p_* \pi_1(\widetilde{X})$. Also, one can show that $\pi_1(\widetilde{X}) \leq \pi_1(X)$ from injectivity of p_*. If $\pi_1(\widetilde{X}) = 0$, then $\widetilde{X}$ is called a universal covering space (and these are unique up to covering space isomorphism).

The correspondence related connected covering spaces with subgroups of $\pi_1(X)$ under some conditions on X. Indeed, there is one-to-one correspondence of connected covering spaces modulo covering space isomorphisms with the subgroups of $\pi_1(X)$ modulo conjugation.

In our study of covering spaces, we wanted some examples to work with. We computed $S^1 \vee S^1$'s double-sheeted covers. Additionally, we stated the classification theorem of closed surfaces: any such space Σ is homeomorphic to the orientable $S^2, \#_n T^2$ surfaces or the non-orientable surfaces $\#_m \mathbb{R}P^2$. Using the Euler characteristic

$$\chi(\Sigma) = \sum_n (-1)^n \#\{n\text{-cells} = \begin{cases} 2-2g & \text{if } \Sigma \text{ is orientable} \\ 2-g & \text{if } \Sigma \text{ is nonorientable} \end{cases}.$$

If $p : \widetilde{\Sigma} \to \Sigma$ is an n-sheeted cover of a closed surface, then $\chi(\widetilde{\Sigma}) = n\chi(\Sigma)$. For instance, we gave an example of a 5-fold cover of a genus 3 surface by a genus 11 surface. Van Kampen's Theorem can also tell us the fundamental groups of these closed surfaces:

$$\pi_1(\#_n T^2) = \langle a_1, \ldots, a_n, b_1, \ldots, b_n \mid \prod_{i=1}^n [a_i, b_i] \rangle$$
$$\pi_1(\#_m \mathbb{R}P^2) = \langle a_1, \ldots, a_m \mid a_1^2 \ldots a_m^2 \rangle.$$

For the second part of the course, we developed the theory of singular homology. We have not been able to do too many concrete computations (which is for next quarter). Recall we started with $\mathrm{Sin}_n(X)$ which is the set of continuous maps $\Delta^n \to X$. Then we defined singular chain complex $S_n(X) = \mathbb{Z}\,\mathrm{Sin}_n(X)$ with the differentials $d_i : \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$ given by precomposing by $d^i : \Delta^{n-1} \to \Delta^n$. It follows easily that $d^2 = 0$. Then singular homology is defined as the homology of the singular chain complex.

The most important properties of $H_n(-)$ are the fact that it is a functor and homotopy invariant. Proving the latter property required defining chain homotopies of chain complexes. Afterwards, we defined the relative homology $H_*(X, A) := H_*(S_*(X)/S_*(A))$ and the most important property of the relative homology was the LES

$$\ldots \to H_*(A) \to H_*(X) \to H_*(X, A) \to H_{*-1}(A) \to \ldots$$

The LES is useful for computations and is constructed from a diagram chase (used to prove the Snake Lemma). Some practice in diagram chasing can be obtained by proving the 5-lemma.

CHAPTER 2

Math 290B - UCSD

Abstract. This course is a continuation of the previous Math 290A course taught by Zhouli Xu. The course is being taught asynchronously. There is no single official textbook though we will mainly use Miller's Lecture Notes. References for this course are listed as follows. Algebraic Topology, by Allen Hatcher: here. A Concise Course in Algebraic Topology, by J.P. May: here. Lectures on Algebraic Topology, by H. Miller: here.

1. Lecture 1: January 3rd, 2022

Abstract. This first lecture sets up the logistics of the course and describes our rough plan. Most of the material here is in Section 1.11 of Miller's notes.

The course grading scheme is the same as before (80% homework and 20% exam). The final is on Friday March 16th.

For this quarter, we shall cover most of the equivalent of Hatcher's Chapters 2 and 3. In particular, we shall continue following Miller's lecture notes. We will deal with homology theories like singular homology, simplicial homology, and cellular homology as well as cohomology.

Definition 1.1. A **homology theory** on Top is

- a sequence of functors $h_n : \mathrm{Top}_2 \to \mathrm{Ab}$ for $n \in \mathbb{Z}$,
- a sequence of natural transformations $\partial : h_n(X, A) \to h_{n-1}(A, \emptyset)$

s.t.

- (**Homotopy invariance**): if $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then
$$f_{0*} = f_{1*} : h_n(X, A) \to h_n(Y, B).$$
- (**Excision**) Excisions induce isomorphisms (recall excisions are $(X - U, A - U) \hookrightarrow (X, A)$ for $\overline{U} \subseteq \mathrm{Int}(A)$).
- (**Long Exact Sequence**) For any pair (X, A), there is a sequence
$$\to h_{q+1}(X, A) \xrightarrow{\partial} h_q(A) = h_q(A, \emptyset) \to h_q(X) \to h_q(X, A) \xrightarrow{\partial} \dots$$
that is exact.
- (**Dimension axiom**) The group $h_n(*)$ is nonzero only for $n = 0$. Note for singular homology, $H_0(*) = \mathbb{Z}$, $H_q(*) = 0$ if $q \neq 0$.
- (**Milnor axiom**) Let I be set and for each $i \in I$, let X_i be a space. Consider $X_i \to \coprod X_i$. Then this induces $h_n(X_i) \to h_n(\coprod X_i)$. Then there is a map $\bigoplus_{i \in I} h_n(X_i) \to h_n(\coprod X_i)$. The Milnor axiom asserts that
$$\bigoplus_{i \in I} h_n(X_i) \to h_n\left(\coprod_{i \in I} X_i\right)$$
is an isomorphism i.e. the homology functors preserve coproducts.

Observe that singular homology satisfies all of these axioms. The dimension axiom is similar to the parallel postulate. It is obvious but omitting it leads to many interesting alternatives. If the dimension axiom is in place, one has “ordinary homology”.

Definition 1.2. Let X be a topological space. A family $\mathcal{A}$ of subsets of X is a **cover** if X is the union of the interiors of elements of $\mathcal{A}$.

Definition 1.3. Let $\mathcal{A}$ be a cover of X.

An n-simplex σ is **$\mathcal{A}$-small** if there exists $A \in \mathcal{A}$ s.t. Image(σ) is entirely in A.

Example 1.4. Let $X = \mathbb{R}$ and $\mathcal{A} = \{(-\infty, 1), (0, +\infty)\}$. Consider the 1-simplex $\sigma : \Delta^1 \to \mathbb{R}$ whose image is $[-1, 2]$. This is *not* $\mathcal{A}$-small.

OTOH, if $\sigma' : \Delta^1 \to \mathbb{R}$ has image $[1, 2]$, then it is $\mathcal{A}$-small since $[1, 2] \subseteq (0, +\infty)$.

If σ_1 has image $[-1, \frac{1}{2}]$ and σ_2 has image $[\frac{1}{2}, 2]$, then they are both $\mathcal{A}$-small.

Example 1.5. Note that if $\sigma : \Delta^n \to X$ is $\mathcal{A}$-small, then so is $d_i\sigma$.

Let $\mathrm{Sin}_*^{\mathcal{A}}(X)$ denote the space of $\mathcal{A}$-small simplices which is a subgroup of $\mathrm{Sin}_*(X)$. Let $S_*^{\mathcal{A}}(X) := \mathbb{Z}\,\mathrm{Sin}_*^{\mathcal{A}}(X)$ and this is a subgroup of $S_*(X) = \mathbb{Z}\,\mathrm{Sin}_*(X)$.

The work in the previous example shows that $S_*^{\mathcal{A}}$ is a chain complex (and in particular, a sub-chain complex of $S_*(X)$. Therefore, we can take homology and denote the homology groups by $H_*^{\mathcal{A}}(X)$.

Theorem 1.6 (The Locality Principle). The inclusion $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ induces an isomorphism in homology

$$H_*^{\mathcal{A}}(X) \xrightarrow{\cong} H_*(X).$$

The proof of the locality principle is quite long, but it implies excision.

Remark 13. We shall show that locality implies excision. Let $\overline{U} \subseteq \mathrm{Int}\, 9A)$ and so $U \subseteq A \subseteq X$. Let $B := X - U$ and take $\mathcal{A} := \{A, B\}$ as the cover of X. Now,

$$(B, A \cap B) = (X - U, A - U) \hookrightarrow (X, A).$$

We want to show that $S_*(B, A \cap B) \to S_*(X, A)$ induces an isomorphism on H_*. We have a diagram of chain complexes with exact rows (with $\Rightarrow$ indicating they are the same)

$$\begin{tikzcd}
0 \arrow[r] & S_*(A) \arrow[r] \arrow[d, Rightarrow] & S_*(X)^{\mathcal{A}} \arrow[r] \arrow[d] & S_*^{\mathcal{A}}(X)/S_*(\mathcal{A}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_*(A) \arrow[r] & S_*(X) \arrow[r] & S_*(X, A) \arrow[r] & 0
\end{tikzcd}$$

This induces a morphism of exact sequences

$$\begin{tikzcd}
\dots \arrow[r] & H_n(A) \arrow[r] \arrow[d, Rightarrow] & H_n^{\mathcal{A}}(X) \arrow[r] \arrow[d, "\cong"] & H_n(S_*^{\mathcal{A}}(X)/S_*(A)) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d, Rightarrow] & H_{n-1}^{\mathcal{A}}(X) \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & \dots
\end{tikzcd}$$

and the indicated isomorphisms $\cong$ follow from the locality principle. By the 5-lemma, $H_n(S_*^{\mathcal{A}}(X)/S_*(A)) \cong H_n(X, A)$.

Now we observe that

$$\frac{S_*^{\mathcal{A}}(X)}{S^*(A)} \cong \frac{S_*(A) + S_*(B)}{S_*(A)} \cong \frac{S_*(B)}{S_*(A) \cap S_*(B)} \cong \frac{S_*(B)}{S_*(A \cap B)}.$$

Then,

$$H_n(S_*^{\mathcal{A}}(X)/S_*(A)) = H_n(B, A\cap B).$$

So for all n,

$$H_*(X,A) \cong H_n(S_*^{\mathcal{A}}(X)/S_*(A)) \cong H_n(B, A\cap B).$$

We now discuss the **Mayer-Vietoris sequence** (abbreviated MV-sequence). Let $\mathcal{A} = \{A, B\}$ be a cover of X. We have a diagram

$$\begin{tikzcd} A\cap B \arrow[r, hook, "j_1"] \arrow[d, hook, "j_2"] & A \arrow[d, hook, "i_1"] \\ B \arrow[r, hook, "i_2"] & X \end{tikzcd}$$

Theorem 1.7 (**MV-sequence**)**.** Let $\mathcal{A} := \{A, B\}$ be a cover of X. Then there exists natural maps

$$\partial : H_n(X) \to H_{n-1}(A\cap B)$$

s.t. the following sequence is exact

$$\begin{tikzcd} & \dots \arrow[r] & H_{n+1}(X) \arrow[dll, "\partial"'] \\ H_n(A\cap B) \arrow[r, "\alpha"] & H_n(A)\oplus H_n(B) \arrow[r, "\beta"] & H_n(X) \arrow[dll] \\ H_{n-1}(A\cap B) \arrow[r] & \dots & \end{tikzcd}$$

where $\alpha = \begin{pmatrix} j_{1*} \\ -j_{2*}\end{pmatrix}$ and $\beta = \begin{pmatrix} i_{1*} & i_{2*}\end{pmatrix}$.

Proof. There is a SES of chain complexes

$$0 \to S_*(A\cap B) \xrightarrow{\alpha} S_*(A)\oplus S_*(B) \xrightarrow{\beta} S_*^{\mathcal{A}}(X) \to 0$$

given by $\alpha : c \mapsto (c, -c)$ and $\beta : (d, e) \mapsto d + e$. This then induces the desired LES on H_* if we use the locality principle. $\square$

2. Lecture 2: January 5th, 2022

Abstract. This lecture discussions that subdivision map which is needed to prove that locality implies excision.

Last time, we discussing homology theory in general (and some examples are/will be singular, simplicial, and cellular homology). A homology theory was the data of functors $h_n : \mathrm{Top}_2 \to \mathrm{Ab}$ and natural transformations $\partial : h_n(X,A) \to h_{n-1}(A,\emptyset)$ s.t. it is homotopy invariant, has a LES, satisfies the Milnor axiom, has excision, and satisfies the dimension axiom. We also discussed $\mathcal{A}$-small chains. Let $\mathcal{A}$ be a cover of X. Then $\sigma : \Delta^n \to X$ is $\mathcal{A}$-small if its image lands entirely in some A for $A \in \mathcal{A}$.

Let $S_*^{\mathcal{A}}(X)$ be the subgroup of $S_*(X)$ generated by $\mathcal{A}$-small chains (i.e. they are generated generated by $\mathcal{A}$-small simplices $\sigma : \Delta^n \to X$). The locality principle says that $H_*^{\mathcal{A}}(X) \to H_*(X)$ is an isomorphism.

We showed that locality implies excision and the Mayer-Vietoris sequence

$$\ldots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \ldots$$

Our goal not is to prove the locality principle itself. For this, we need to define subdivision (pictures can be found in Miller's notes or Zhouli's lecture notes).

Recall the cone construction which was a map $b_* : \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n+1}(X)$. It is defined as follows. If X is a star-shaped region in $\mathbb{R}^n$ w.r.t. $b \in X$, then $h_*(X) \to S_{n+1}(X)$ sends σ to $b * \sigma$ where $b * \sigma$ is the cone connected to σ with tip b. We also showed that b_* is a chain homotopy between id and $\eta_b \epsilon$. Here,

$$\eta_b : \mathbb{Z} \to S_*(X)$$

sends 1 to the constant zero chain C_b^0. Also,

$$\epsilon : S_*(X) \to \mathbb{Z}$$

is the augmentation map on degree 0. Being chain homotopy equivalent through b_* means

$$db_* + b_* d = \mathrm{id} - \eta_b \epsilon.$$

Definition 2.1. An **affine simplex** is the **convex hull** of a finite set of points in Eulcidean space.

The affine simplex is non-generated it the points $v_0, \ldots, v_n$ has $\{v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0\}$ forming a linearly independent set.

The **barycenter** of this affine simplex is the center of mass of the vertices: $b = \frac{1}{n+1} \sum_i v_i$.

Definition 2.2. Subdivision $\$: S_n(X) \to S_n(X)$ is the natural transformation defined as follows.

First, define $\$(\iota_n)$ for $\iota_n : \Delta^n \to \Delta$ the identity. Then, we define $\$(\sigma) = \sigma_*(\$(\iota_n))$ and extend linearly.

For $n = 0$, set $\$\iota_0 = \iota_0$. Then inductively, $n > 0$ has $\$\iota_n = b_n * \$d\iota_n$.

Proposition 2.3. $\$$ is a chain map $S_*(X) \to S_*(X)$ that is naturally chain homotopy to the identity.

Proof. Do it yourself or see Proposition 12.1 in Miller's notes. It is just a long but straightforward proof. Then we introduction triangulations of spaces and use the geometric realization to do this. □

3. Lecture 3: January 7th, 2022

Abstract. We finish the proof of the locality principle for singular homology.

Last time, we discussed subdivision $\$: S_*(X) \to S_*(X)$. We showed it is a chain map $\$d = d\$$ and that it is chain homotopy to the identity i.e. there exists a T homotopy s.t. $dT + Td = \$ - \mathrm{id}$. Geometrically, we defined $\$\iota_n = b_n * \$d\iota_n$ for b_n the barycenter of Δ^n and set $\$\sigma = \sigma_* \$\iota_n$.

Now we proceed to our goal: show the locality principle of singular homology. We shall need some facts about $\mathbb{R}^n$ and metric spaces.

Recall that if X is a metric space, $\mathrm{diam}(X) := \sup\{d(x,y) : x, y \in X\}$ is the **diameter** of X.

Lemma 3.1. If σ is an affine n-simplex, then the diameter exists and its maximum is achieved. As σ is convex, it is the maximum distance between its pairs of vertices.

Let τ be the image of an n-simplex in $\$\sigma$. Then it is another affine n-simplex and one can show $\operatorname{diam}(\tau) \leq \frac{n}{n+1} \operatorname{diam}(\sigma)$. For a proof, see Miller's notes.

Lemma 3.2 (Lebesgue Covering Lemma)**.** If M is a compact metric space and $\mathcal{U}$ an open cover, then there exists an $\epsilon > 0$ s.t. if $x \in M$, there exists $U \in \mathcal{U}$ s.t. $B_\epsilon(x) \subseteq U$.

Lemma 3.3. For any singular chain c, some iterate $\$^k$ of the subdivision operator sends c to an $\mathcal{A}$-small chain.

Proof. It is intuitively clear – do it yourself. ☐

Lemma 3.4. For all $k \geq 1$, we have $\$^k \simeq \mathrm{id} : S_*(X) \to S_*(X)$.

Proof. A straightforward computation. ☐

Theorem 3.5 (Locality Principle)**.** If $\mathcal{A}$ is a cover of X, then $S_*^{\mathcal{A}}(X) \subseteq S_*X()$ induces an isomorphism

$$H_*^{\mathcal{A}}(X) \xrightarrow{\cong} H_*(X).$$

Proof. Prove injectivity and surjectivity. Both of these are straightfoward from the definition and what we did above (see Zhouli's notes or Miller's notes). ☐

We now begin discussing simplicial homology.

Definition 3.6. A **simplicial complex** is a set V (a set of vertices which may be infinite), a subset $F \subseteq 2^V$ s.t. $\sigma \in F$ and $\tau \subseteq \sigma$ implies $\tau \in F$.

Example 3.7. If $V = \{0,1,2,3\}$, then $F = \{\emptyset, \{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{1,2\}, \{0,2\}, \{1,3\}, \{2,3\}\}$ gives a simplicial set. This generalizes to arbitrary n instead of $n = 3$.

Definition 3.8. If $K = (V, F)$ is a simplicial complex, its **geometric realization** $|K|$ is constructed as follows.

Elements in V are vertices and attach a Δ^n to any $v_0, \ldots, v_n$ for which $\{v_0, \ldots, v_n\} \in F$.

Example 3.9. Try drawing out the picture for our $K = (V, F)$ above. In the $F = 2^V$ and $V = \{0, 1, \ldots, n\}$, we get $|K| = B^n$.

Definition 3.10. A **triangulation** of a space X is a homeomorphism $X \xrightarrow{\cong} |K|$ for some K. X is called **triangulable** in this case.

Remark 14. There exists spaces that are not triangulable. Indeed, all smooth manifolds are triangulable but this is not true for topological manifolds. Be some recent work, it was show that if the dimension is > 4, there always exists topological manifolds that are no triangulable.

Remark 15. It is known the minimal number of triangles one needs to triangulate orientable surfaces: Minimal triangulations on orientable surfaces by Jungerman and Rigel.

4. Lecture 4: January 10th, 2022

Abstract. The goal of this lecture is to introduce the concept of mapping degrees. Hopf's famous theorem explains that it determines a map between spheres up to homology. The mapping degree is also a more general concept which can be defined for manifolds.

We review what we did last time. First, we showed the locality principle for singular homology. Then, we began working toward simplicial homology. Towards this, we defined a **simplicial complex**, constructed its **geometric realization**, and used it to define **triangulation** of a topological space. Not all spaces are triangulable e.g. the topologist sine curve. It is a known fact that any smooth manifold is triangulable, but there always exists nontriangulable manifolds in any dimension > 5. Our goal now is to define a sort of combinatorial homology for simplicial complexes.

Definition 4.1. Let K be a simplicial complex. Fix a total order "$<$" on V. Let $F_n := \{\sigma \in F : \sigma \text{ has } n+1\}$ elements.Define

$$S_n^\Delta := \mathbb{Z}F_n := \{\Sigma_i a_i \sigma_i : \sigma_i \in F_n, a_i \in \mathbb{Z}\}$$

be the free abelian group generated on F_n. Define $d : S_n^\Delta \to S_{n-1}^\Delta$ as follows. First, if $\sigma = \{v_0, \dots, v_n : v_0 < \dots < v_n\}$, then

$$d\sigma := \sum_{i=0}^{n} (-1)^i \sigma_i$$

for $\sigma_i := \sigma \setminus \{v_i\}$.

Lemma 4.2. $d^2\sigma = 0$.

The proof of the lemma is just as before (or see Hatcher's Chapter 2). In any case, (S_n^Δ, d) defines a chain complex and then nth **simplicial homology** of K is $H_n^\Delta(X) := \frac{\ker d}{\operatorname{im} d}$.

Theorem 4.3. For any simplicial complex K, there is an isomorphism

$$H_n^\Delta(X) \xrightarrow{\cong} H_n(|K|).$$

PROOF. Simplicial homology is "special case" of cellular homology and cellular homology $\cong$ singular homology in this case. □

Now define morphisms between simplicial complexes K_i. A map $f : K_1 \to K_2$ is a **map of simplicial complexes** if $f : V_1 \to V_2$ is the a map which induces a map $f : F_1 \to F_2$. We let $|f| : |K_1| \to |K_2|$ be the continuous map on geometric realizations it induces.

Note that not all maps $|K_1| \to |K_2|$ need to come from a **simplicial map** f (maps as defined above). However, there is a version of subdivision \$ that takes K to \$$K$ s.t. the geometric realizations are the same.

Theorem 4.4 (Simplicial approximation). If $f : |K_1| \to |K_2|$ is continuous, there is an n s.t. $g : \$^n K_1 \to K_2$ induces a a map $|g| \simeq f$.

We do not prove this result, but assume it as a fact (we have not even constructed the map \$).

Any simplicial map $f : K_1 \to K_2$ induces a chain map $S_*^\Delta(K_1) \to S_*^\Delta(K_2)$ and hence, a map on homology $H_*^\Delta(K_1) \to H_*^\Delta(K_2)$.

Denote by SimpComp the category of simplicial complexes with morphisms being simplicial maps. Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{SimpComp} \arrow[r, "|\cdot|"] \arrow[d, "H_*^\Delta(\cdot)"'] & \mathrm{Top} \arrow[dl, "H_*(\cdot)"] \\
\mathrm{Ab} &
\end{tikzcd}$$

We now discuss the concept of a **mapping degree**. Recall that $\widetilde{H_n(S^n)} \cong \mathbb{Z}$ and is zero in other degrees. Then any continuous $f : S^n \to S^n$ induces a map $f_* : \mathbb{Z} \to \mathbb{Z}$ on homology. Define $\deg(f) := f_*(1)$ as the **mapping degree** of f. By homotopy invariance, $f_1 \simeq f_2$ implies $\deg(f_1) = \deg(f_2)$. In Math 290C, we shall prove the converse called **Hopf's Theorem**:

Theorem 4.5 (Hopf)**.** If $f_1, f_2 : S^n \to S^n$ are continuous, then

$$f_1 \simeq f_2 \Leftrightarrow \deg(f_1) = \deg(f_2).$$

Corollary 4.5.1. $[S^n, S^n] \cong \mathbb{Z}$.

Let $r_n : S^n \to S^n$ be the map that sends $(x_0, x_1, x_2, \dots, x_{n+1}) \to (-x_0, x_1, x_2, \dots, x_{n+1})$.

Proposition 4.6. $\deg(r_n) = -1$ for all $n \geq 0$.

Proof. In the case $n = 0$, we know r_0 sends $r_0(-1) = 1$ and $r_0(1) = -1$. Recall that $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by $[c_+^0]$ and $[c_{-1}^0]$. Then,

$$\widetilde{H_0}(S^0) = \ker(H_0(S^0) \to H_0(\mathrm{pt})) = \langle [c_{+1}^0] - [c_{-1}^0] \rangle.$$

So, $r_0([c_1^0] - [c_{-1}^0]) = [c_{-1}^0] - [c_1^0]$. So, $\deg(r_0) = -1$.

Now we use induction. Assume $\deg(r_k) = -1$. Decompose $S^{k+1} = S_+^{k+1} \cup_{S^k} S_-^{k+1}$ into its northern and southern hemispheres. Then $S_+^{k+1} \cong S_-^{k+1} \cong D^{k+1}$ are contractible spaces. It follows that their reduced homology are zero. So the Mayer-Vietoris sequence gives

$$0 \to \widetilde{H_{k+1}}(S^{k+1}) \xrightarrow{\partial} \widetilde{H_k}(S^k) \to 0$$

which means ∂ is an isomorphism. But r_{k+1} preserves the the decomposition and restricts to r_k on the equator. By naturality,

$$\begin{array}{ccc} \widetilde{H_{k+1}}(S^{k+1}) & \xrightarrow{\partial} & \widetilde{H_k}(S^k) \\ \downarrow{\scriptstyle (r_{k+1})_*} & & \downarrow{\scriptstyle (r_k)_*} \\ \widetilde{H_{k+1}}(S^{k+1}) & \xrightarrow{\partial} & \widetilde{H_k}(S^k) \end{array}$$

with horizontal maps being isomorphisms. That means $\deg(r_{k+1}) = \deg(r_k) = -1$. $\square$

5. Lecture 5: January 12th, 2022

Abstract. For this lecture, we provide some applications of simplicial homology and show how it can be used to study topological spaces.

Last time, we discussed simplicial homology. Recall that if X is a CW-complex, there is a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$$

and we set $\mathrm{sk}_i\, X = X_i$ for the ith skeleton. So in particular X_0 is the set of discrete points we inductively construct X_{n+1} as the pushout diagram

$$\begin{array}{ccc} \coprod_{\alpha \in A} S_\alpha^n & \xrightarrow{f_n} & X_n \\ \downarrow & & \downarrow \\ \coprod_{\alpha \in A} D_\alpha^{n+1} & \xrightarrow{g_n} & X_{n+1} \end{array}$$

and the maps g_n are called **characteristic maps**. Therefore, X_{n+1} is obtained by attaching $(n+1)$-cells $\{D^{n+1}_\alpha\}$ for $\alpha \in A_{n+1}$ the indexing set to X_n along $\partial D^{n+1}_\alpha = S^n_\alpha$ via the attaching maps f_n.

Note that g_n itself may not be injective. Therefore, D^{n+1}_α cannot always be viewed as subspaces. However, $g_n|_{\mathrm{Int}(D^{n+1}_\alpha)}$ is injective. Thus, $\mathrm{Int}(D^{n+1}_\alpha)$ can always be viewed as a subspace. These are called the **open** $(n+1)$**-cells**. Note that different open cells do not intersect each other. Therefore, there is a decomposition

$$X = \bigcup_{n\geq 0}\bigcup_{\alpha\in A_n} \mathrm{Int}(D^n_\alpha)$$

of X.

Definition 5.1. Let X be a CW-complex and a **subcomplex** of X is a $Y \subseteq X$ s.t. for all n, there exists $B_n \subseteq A_n$ s.t. $\mathrm{sk}_n Y = Y \cap \mathrm{sk}_n X$ which provides Y with a CW-structure.

Example 5.2. If we let X be the unit interval with the usual CW-complex structure, then $Y = [0,1)$ is *not* a subcomplex. This is because Y does not have a CW-complex structure. Another non-example is $(0,1)$.

We shall take the following statement as fact.

Theorem 5.3. The union and intersection of subcomplexes is again a subcomplex.

To construct S^n, we could form it by taking $\mathrm{sk}_0 = \{*\}$ and $\mathrm{sk}_1 S^n = \cdots = \mathrm{sk}_{n-1} S^n = \{*\}$ and set $\mathrm{sk}_n S^n = \{*\} \cup_f D^n$ via $\partial D^n \to \{*\}$.

We can also construct S^n by have two cells in dimensions $0,1,\dots,n$. The construction itself is as follows.

Set $S^n := \{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} : \sum_i x_i^2 = 1\}$. Then,

$$\mathrm{sk}_\ell(S^n) := \{(x_0,\dots,x_l,0,\dots,0) : \sum_i x_i^2 = 1\} \simeq S^l.$$

Now take $D^\ell_+ \cup_{\mathrm{sk}_{\ell-1}} D^\ell_-$ where D^ℓ_+ and D^ℓ_- are the subsets of $\mathrm{sk}_\ell(S^n)$ in which $x_\ell \geq 0$ and $x_\ell < 0$. Then, we have a pushout diagram

$$\begin{tikzcd}
S^{\ell-1}_+ \coprod S^{\ell-1}_- \arrow[r, "f"] \arrow[d] & \mathrm{sk}_{\ell-1} S^n = S^{\ell-1} \arrow[d] \\
D^\ell_+ \coprod D^\ell_- \arrow[r] & \mathrm{sk}_\ell S^n = S^\ell
\end{tikzcd}$$

and here, $f|_{S^{\ell-1}_+} : S^{\ell-1} \to S^{\ell-1}$ is the identity.

Now using this construction, we can consider S^∞. Indeed, we take S^∞ to be the CW-complex with 2 cells in each dimension. That is, $S^\infty = \bigcup_{n\geq 0} S^n$ and explicitly,

$$S^\infty = \{(x_0,x_1,\dots) \in \bigoplus_{i=0}^\infty \mathbb{R} =: \bigoplus_\infty \mathbb{R} : \sum x_i^2 = 1\}$$

and the condition of being the direct sum ensures the RHS sum is always finite.

We know that S^n is not contractible because $H_n(S^n) \cong \mathbb{Z}$. However, S^∞ is special in the fact that it is contractible.

Proposition 5.4. S^∞ is contractible.

PROOF. Let $T : S^\infty \to S^\infty$ be defined via $(x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots)$. This is well-defined. We shall show that $T \simeq \mathrm{id}_{S^\infty}$. Indeed, define the homotopy by

$$h(x,t) := \frac{tx + (1-t)Tx}{|tx + (1-t)Tx|} \in S^\infty$$

and when $t = 0$, we get Tx while we get id_{S^∞} when $t = 1$. The above formula is well-defined because the denominator is nonzero.

Now we show T is homotopy to the constant map $S^\infty \to (1, 0, 0, \dots)$. Indeed, this is done via the linear homotopy

$$h(x,t) := \frac{t(1,0,0,\dots) + (1-t)Tx}{|t(1,0,0,\dots) + (1-t)Tx|}.$$

This implies $\mathrm{id}_{S^\infty} \simeq$ constant map which implies S^∞ is contractible. $\square$

We know that $\mathbb{Z}/2$ acts on S^n by the antipodal map. Therefore, $f : D^n_+ \to D^n_-$ and this yields a CW-structure on $\mathbb{R}P^2$. The same ideal can be done for $\mathbb{R}P^\infty$ and we observe that $\mathrm{sk}_\ell(\mathbb{R}P^\infty) = \mathbb{R}P^\ell$. Furthermore,

$$\begin{tikzcd} S^{k-1} \arrow[r, "f"] \arrow[d] & \mathbb{R}P^{k-1} \arrow[d] \\ D^k \arrow[r] & \mathbb{R}P^k \end{tikzcd}$$

Now we study **complex projective space** $\mathbb{C}P^n$. It is the set of all lines through the origin of $\mathbb{C}^{n+1}$. Observe that

$$\mathbb{CP}^n = S^{2n+1}/S^1$$

where S^{2n+1} is the unit sphere in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Now define $e^{i\theta}(z_0, \dots, z_n) = (e^{i\theta}z_0, \dots, e^{i\theta}z_n)$. So, define a quotient map

$$h_n : S^{2n+1} \to \mathbb{C}P^n$$

which is called the **Hopf fibration**. There is a CW-structure on $\mathbb{C}P^n$ given by

$$\emptyset \subseteq \mathbb{C}P^0 \subseteq \mathbb{C}P^1 \subseteq \mathbb{C}P^2 \subseteq \cdots \subseteq \mathbb{C}P^n.$$

Here, $\mathbb{C}P^\ell := \{[z_0, \dots, z_\ell, 0, 0, \dots, 0] \in \mathbb{C}P^n\}$. Then we have a diagram

$$\begin{tikzcd} S^{2\ell-1} \arrow[r] \arrow[d, "h^{\ell-1}"] & D^{2\ell} \arrow[d, "g^{\ell-1}"] \\ \mathbb{C}P^{\ell-1} \arrow[r] & \mathbb{C}P^\ell \end{tikzcd}$$

where

$$g_{\ell-1}(z_0, \dots, z_{\ell-1}) = [z_0, \dots, z_{\ell-1}, 1 - \sum_{i=0}^{\ell-1} |z_i|^2] \in \mathbb{C}P^\ell.$$

This means we may obtain $\mathbb{C}P^\ell$ from $\mathbb{C}P^{\ell-1}$ by attaching a 2ℓ-cell $D^{2\ll}$ with attaching map equal to the Hopf map. So, $\mathbb{C}P^n$ has a CW-structure with a single cell in each even dimension $2n$.

6. Lecture 6: January 14th, 2022

Abstract. In this lecture, we begin discussing cellular homology which is more convenient for studying topological spaces that are CW-complexes. Using this theory, we will later show that any group can be realized as a homology group of some space.

Recall that a CW-complex is a space with a sequence of subspaces

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$$

s.t. $X = \bigcup_{k \geq -1} X_k$ where X_k denotes the k-skeleton. Then, there is a pushout diagram

$$\begin{tikzcd}
\coprod_{\alpha\in A_k} S^{k-1}_\alpha \arrow[r, hook] \arrow[d, "f_k"] & \coprod_{\alpha\in A_k} D^k_\alpha \arrow[r] \arrow[d, "g_k"] & \bigvee_{\alpha\in A_k} S^k_\alpha \arrow[d, "\cong\text{homeomorphism}"] \\
X_{k-1} \arrow[r] & X_k \arrow[r] & X_k/X_{k-1}
\end{tikzcd}$$

and here, f_k the attaching map and $\bigvee_{\alpha\in A_k} S^k_\alpha$ is a bouquet of k-spheres.

Recall that a CW-complex is of **finite type** if there are finitely many cells attached at each step. By excision we have an isomorphism

$$H_q(X_k, X_{k-1}) \xrightarrow{\cong} H_q(X_k/X_{k-1}, \{*\}).$$

In fact, X_{k-1} is a deformation retract of $X_k - \{\text{center of each } k\text{-cell}\}$. Therefore,

$$H_q(X_k/X_{k-1}, \{*\}) = H_q(\bigvee_{\alpha\in A_k} S^k_\alpha, \{*\}) = \begin{cases} \mathbb{Z}\langle A_k\rangle & q = k \\ 0 & q \neq k \end{cases}.$$

Here, $\mathbb{Z}\langle A_k\rangle$ denotes the free abelian group generated by A_k.

Thus, $H_k(X_k/X_{k-1}, *) \cong H_k(X, X_{k-1}) \cong \mathbb{Z}\langle A_k\rangle$ so the relative homology keeps track of the k-cells in X in terms of the rank.

Definition 6.1. The k**th cellular chain group** of a CW-complex X is

$$C_k(X) := H_k(X_k, X_{k-1}) \cong \mathbb{Z}\langle A_k\rangle.$$

The cellular homology is better controlled than singular homology. Looking at the LES,

$$\begin{tikzcd}
{} \arrow[r] & H_{k+1}(X_{k-1}) \arrow[r] & H_{k+1}(X_k) \arrow[r] & H_{k+1}(X_k, X_{k-1}) = 0 \arrow[dll] \\
& H_k(X_{k-1}) \arrow[r] & H_k(X_k) \arrow[r] & H_k(X_k, X_{k-1}) \arrow[dll] \\
& H_{k-1}(X_{k-1}) \arrow[r] & H_{k-1}(X_k) \arrow[r] & H_{k-1}(X_k, X_{k-1}) = 0 \arrow[r] & {}
\end{tikzcd}$$

we see the that LES splits shorter sequences. On closer inspection, we see that for all $q \neq k, k-1$ there is an isomorphism $H_q(X_{k-1}) \to H_q(X_k)$. So cellular homology detects when we add k-cells. That is, attaching k-cells only affects H_{k-1} and H_k.

Fix $q > 0$. Then the sequence

$$H_q(X_0) = 0 \xrightarrow{\cong} H_q(X_1) = 0 \xrightarrow{\cong} H_q(X_2) \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_q(X_{q-1}) \to H_q(X_q) \to H_q(X_{q+1}) \xrightarrow{\cong} \ldots \to H_q(X)$$

and there are only two maps that may possibly be a nonisomorphism. We have thus shown the following:

Proposition 6.2. Fix $k > 0, q > 0$. Then the following are true:

- $H_q(X_k) = 0$ for $q \geq k+1$ (so if X is of dimension k, then $H_q(X) = 0$ for $q \geq k+1$ and $X = X_K$),
- $H_q(X_k) \to H_q(X)$ is an isomorphism for $k \geq q+1$.

The n**th cellular chain group** is defined to be

$$C_n(X) := H_n(X_n, X_{n-1}) \cong \mathbb{Z}\langle A_n \rangle$$

where the RHS is the free abelian group generated by the set of n-cells. We shall show that these abelian groups are related i.e. they form a chain complex.

First off, from the pushout diagram we have

$$\begin{tikzcd} D^n \arrow[r] & X & \\ S^{n-1} \arrow[u] \arrow[r] & X_{n-1} \arrow[u] \arrow[r, two heads] & X_{n-1}/X_{n-2} \cong \bigvee_{\alpha \in A_{n-1}} S^{n-1}_\alpha \end{tikzcd}$$

and we define the differential as

$$d : C_n(X) = H_n(X_n, X_{n-1}) \overset{\partial_{n-1}}{\to} H_{n-1}(X_{n-1}) \overset{j_{n-1}}{\to} H_{n-1}(X_{n-1}, X_{n-2}) = C_{n-1}(X)$$

and here, j_{n-1} is the quotient map.

Theorem 6.3. $d^2 = 0$.

Proof. Given the pairs $(X_{n+1}, X_n), (X_n, X_{n-1}), (X_{n-1}, X_{n-2})$, we have a diagram

$$\begin{tikzcd} & C_{n+1}(X) := H_{n+1}(X_{n+1}, X_n) \arrow[d, "\partial_n"] \arrow[dr, "d"] & & 0 = H_{n-1}(X_{n-2}) \arrow[d] \\ H_n(X_{n-1}) = 0 \arrow[r] & H_n(X_n) \arrow[r, "j_n"] \arrow[d, "\phi"] & C_n(X) = H_n(X_n, X_{n-1}) \arrow[r, "\partial_{n-1}"] \arrow[dr, "d"] & H_{n-1}(X_{n-1}) \arrow[d, "j_{n-1}"] \\ & H_n(X_{n+1}) \arrow[d] & & C_{n-1}(X) = H_{n-1}(X_{n-1}, X_{n-2}) \\ & H_n(X_{n+1}, X_n) & & \end{tikzcd}$$

and observe that $d^2 = 0$ because $\partial_n \circ j_n = 0$ as it sits inside a LES.

Therefore, there is a cellular chain complex $(C_n(X), d)$ and cellular homology is the homology of this chain.

All of the j_n, j_{n-1} maps are injective maps. Thus, $\ker d = \ker j_{n-1}\partial_{n-1} = \ker \partial_{n-1} = \operatorname{im} j_n = H_n(X_n)$. OTOH, $\operatorname{im} d = j_n(\operatorname{im} \partial_n) \cong \operatorname{im} \partial_n \subseteq H_n(X_n)$. So,

$$H_n(C_*(X), d) = \ker d / \operatorname{im} d \cong H_n(X_n)/\operatorname{im} \partial_n \cong H_n(X_n)/\ker \phi \cong H_n(X_{n+1})$$

and this is preserved as adding higher order cells does not change it. So, $H_n(X_{n+1}) \cong H_n(X)$. $\square$

Theorem 6.4. Let X be a CW-complex, then $H_*(C_*(X), d) \cong H_*(X)$ i.e. cellular homology groups are isomorphic to singular homology groups.

This allows for easier computations. For instance, we have the next corollary.

Corollary 6.4.1. Let X be a CW-complex with only even dimension cells. Then $H_*(X) \cong C_*(X)$ i.e. $H_n(X)$ is always a free abelian group that is zero for odd n and whose rank is equal the number of cells of dimension n when n is even.

PROOF. Our chain $C_*(X)$ looks like

$$0 \leftarrow \mathbb{Z}\langle A_0\rangle \leftarrow 0 \leftarrow \mathbb{Z}\langle A_2\rangle \leftarrow 0 \leftarrow \dots$$

□

Example 6.5. As $\mathbb{C}P^n$ has a 1-cell in each even dimension up to $2n$,

$$H_q(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & 0 \leq q \leq 2n \text{ even} \\ 0 & \text{else} \end{cases}.$$

7. Lecture 7: January 19th, 2022

ABSTRACT. We prove that cellular homology and singular homology coincide for CW-complexes. Furthermore, this provides a more efficient computation of certain topological spaces.

Last time, we saw that if X is a CW-complex, then we could define a chain complex $C_*(X)$ which gave rise to the cellular homology $H_*(C_*(X))$ of X. The filtration

$$X: \qquad \emptyset = \operatorname{sk}_{-1} X \subseteq \operatorname{sk}_0 X \subseteq \cdots \subseteq X$$

then gave rise to the nth cellular chain group

$$C_n(X) := H_n(X_n, X_{n-1}) \cong H_n(X_n/X_{n-1}, \{*\}) \cong H_n(\bigvee_{\alpha\in A_n} S^n, \{*\}) \cong \mathbb{Z}\langle A_\rangle.$$

Then, we defined $d : C_n(X) \to C_{n-1}(X)$ via

$$\mathbb{Z}\langle A_n\rangle = H_n(X_n, X_{n-1}) \overset{\partial_{n-1}}{\to} H_{n-1}(X_{n-1}) \overset{j_{n-1}}{\to} H_{n-1}(X_{n-1}, X_{n-2}) = \mathbb{Z}\langle A_{n-1}\rangle$$

and this map is defined on the generators. We saw that $d^2 = 0$ and that

Theorem 7.1. $H_*(C_*(X)) \cong H_*(X)$.

Let's discuss the map d a bit more. We need to examine the map on the n-dimensional cells.

Let S^{n-1}_α be the boundary of an n-cell D^n_α for each $\alpha \in A_n$. Then we have a diagram

$$\begin{tikzcd} S^{n-1}_\alpha \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & X_{n-1}/X_{n-2} \arrow[r] & \bigvee_{\beta\in A_{n-1}} S^{n-1}_\beta \arrow[r, "\text{quotient}"] & S^{n-1}_\beta \\ D^{n-1}_\alpha \arrow[r] & X_n \end{tikzcd}$$

which gives a map $f_{\alpha\beta} : S^{n-1}_\alpha \to S^{n-1}_\beta$ as a composite of the top maps.

So for every $\alpha \in A_n$ and $\beta \in A_{n-1}$, we have a map

$$f_{\alpha\beta *} : H_{n-1}(S^{n-1}_\alpha) \to H_{n-1}(S^{n-1}_\beta)$$

given by $\mathbb{Z} \mapsto \deg(f_{\alpha\beta}) \cdot \mathbb{Z}$. So, $d : C_n(X) \to C_{n-1}(X)$ is determined by $\deg(f_{\alpha\beta})$ for varying α, β. This shows that d is determined by a matrix with coordinates α, β.

We remark that cellular homology only sees the first order attaching map structure of a CW-complex. That is, the attaching map $\coprod S^{k-1} \to X_{k-1}$ determines everything. So if we should look at $S^{k-1} \to X^{k-1} \to X^{k-1}/X^{k-2} \to S^{k-1}$.

Example 7.2. Let us compute the homology of $\mathbb{R}P^n$. It has a CW-structure where $\mathrm{sk}_k\, \mathbb{R}P^n = \mathbb{R}P^k$ and it has 1 cell in each dimemsion $0 \le k \le n$. The attaching map $S^n \to \mathbb{R}P^n = S^n/(x \sim -x)$ is the double cover map. The cellular chain complex looks like

$$\begin{tikzcd}
0 & C_0(\mathbb{R}P^n) \arrow[l] \arrow[d, Rightarrow, no head] & C_1(\mathbb{R}P^n) \arrow[l] \arrow[d, Rightarrow, no head] & \dots \arrow[l] & C_n(\mathbb{R}P^n) \arrow[l] \arrow[d, Rightarrow, no head] & 0 \arrow[l] \\
0 & \mathbb{Z} \arrow[l, "d_0"'] & \mathbb{Z} \arrow[l, "d_1"'] & \dots \arrow[l] & \mathbb{Z} \arrow[l, "d_n"'] & 0 \arrow[l]
\end{tikzcd}$$

and $d_{k+1} : S^k \to \mathbb{R}P^k \to \mathbb{R}P^k/\mathbb{R}P^{k-1} \cong S^k$ is exactly the degree of this. In particular,

$$\begin{tikzcd}
S^{k-1} \arrow[r] \arrow[d, "D.C"] & S^k \arrow[r] \arrow[d, "D.C"] & S^k/S^{k-1} \cong S^k \vee S^k \arrow[d, "D.C"] \\
\mathbb{R}P^{k-1} \arrow[r] & \mathbb{R}P^k \arrow[r] & \mathbb{R}P^k/\mathbb{R}P^{k-1} \cong S^k
\end{tikzcd}$$

and the map on the RHS is $\mathrm{id} \vee \alpha$ for $\alpha : S^k \to S^k$ the antipodal map.

The maps $D.C$ in the diagram and just the double cover maps.

Recall from before that the reflection map has degree one. The antipodal map α is a reflection composed $k+1$ times. So the degree of the antipodal map is $(-1)^{k+1}$. Hence, $\deg(S^k \to S^k) = 1 + (-1)^{k+1}$ which means $\deg(d_{k+1}) = 1 + (-1)^{k+1}$.

Therefore, the cellular chain complex of $\mathbb{R}P^n$ is

$$0 \longleftarrow \mathbb{Z} \xleftarrow[0]{} \mathbb{Z} \xleftarrow[2]{} \mathbb{Z} \xleftarrow[0]{} \dots \longleftarrow \mathbb{Z} \xleftarrow[1+(-1)^n]{} \mathbb{Z} \longleftarrow 0$$

It follows that we have the following table for the singular homology of $\mathbb{R}P^n$.

$\mathbb{R}P^n$	H_0	H_1	H_2	H_3	H_4	H_5	$\dots$	
$\mathbb{R}P^0 = \{*\}$	$\mathbb{Z}$	0	0	0	0	0	$\dots$	
$\mathbb{R}P^1 = S^1$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	$\dots$		
$\mathbb{R}P^2$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	$\dots$		
$\mathbb{R}P^3$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	$\dots$		
$\mathbb{R}P^4$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	$\dots$			
$\mathbb{R}P^5$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	$\dots$

We have shown

Proposition 7.3. The singular homology of projective space is

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = n \ \&\ n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd and } 1 \le k \le n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 16. If the student wishes to see more details for these computations, please see pp. 51-52 of Miller's notes.

8. Lecture 8: January 21st, 2022

Abstract. In this lecture, we discuss an important invariant of topological spaces – the Betti number and the Euler characteristic. We show how to compute them via homology and that the Euler characteristic coincides with the usual definition for CW-complexes.

Last time, we learned how to compute the cellular homology of $\mathbb{R}P^n$. Now we return to an old promise from last quarter and discuss the Euler characteristic.

Let X be a finite type CW-complex. Then let $a_n := \#$ of n-cells $= |A_n| < \infty$. Now set $\beta :=$ rank of $H_n(X)$ which is called the **Betti number**.

Definition 8.1. The Euler characteristic is $\chi(X) := \sum_k (-1)^k a_k = \sum_k (-1)^k \beta_k$ and we shall soon prove the second equality. This second inequality implies that the Euler characteristic is homotopy invariant of X. In particular, it does not depend on the CW-complex structure of X.

Remark 17. Originally, Euler showed that if v, e, f are the number of vertices, edges, and faces of a convex polyhedron, then $v - e + f = 2$. Of course, any such surface is homotopy equivalent to S^2 and Euler's original theorem follows from the homotopy invariance. Indeed,

$$H_*(S^2) = \begin{cases} \mathbb{Z} & * = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

so the alternating sum is exactly 2.

Recall from abstract algebra that any finitely generated abelian group A has form

$$A \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_t\mathbb{Z}$$

where $n|n_2| \dots |n_t$. We call r the **rank of** A.

If X is a finite CW-complex, then $H_k(X)$ is always finitely generated because $C_k(X)$ is finitely generated. Indeed,

$$Z_k(X)/B_k(X) \leftarrow Z_k(X) \hookrightarrow C_k(X) \implies !H_k(X) \text{ is f.g..}$$

So now we assume $H_k(X) = \mathbb{Z}^{r(k)} \oplus \bigoplus_{i=1}^{t(k)} \mathbb{Z}/n_i(k)$ and $n_1(k)| \dots n_{t(k)}(k)$ and $r(k)$ is then the rank of $H_k(X)$.

Lemma 8.2. If $0 \to A \to B \to C \to 0$ is a SES of f.g. abelian groups, then $\operatorname{rank}(A) - \operatorname{rank}(B) + \operatorname{rank}(C) = 0$.

Proof. Apply $- \otimes \mathbb{Q}$ and use rank-nullity. $\square$

Proof of the Euler characteristic formula. We know there is a SES

$$0 \to Z_k \to C_k \xrightarrow{d} B_{k-1} \to 0$$

and another sequence

$$0 \to B_k \to Z_k \to H_k \to 0.$$

Then

$$\begin{aligned} \sum (-1)^k a_k &= \sum_k (-1)^k \operatorname{rank} C_k = \sum_k (-1)^k \operatorname{rank} Z_k + \operatorname{rank} B_{k-1} \\ &= \sum_k (-1)^k \operatorname{rank} B_k + \sum_k (-1) \operatorname{rank} H_k + \sum_k (-1)^k \operatorname{rank} B_{k-1} = \sum_k (-1)^k \operatorname{rank} H_k. \end{aligned}$$

$\square$

Now a natural question to ask is if we can give a lower bound on the number of k-cells in terms of $H_*(X)$.

Let

$$H_k(X) = \mathbb{Z}^{r(k)} \oplus \bigoplus_{i=1}^{t(k)} \mathbb{Z}/n_i(k)$$

for $n_1(k)| \dots |n_{t(k)}(k)$. Divide $H_k(X)$ into its torsionfree and torsion parts. For $H_k = \mathbb{Z}^r$, and $H_q = 0$ for all $q \neq k$, we can consider the chain complex

$$\dots \to 0 \to \mathbb{Z}^r \to 0 \to \dots$$

concentrated in dimension k. Then $\widetilde{H_*}(\bigvee_r S^k)$. So our choice is $\bigvee_r S^k$.

For $H_k = \mathbb{Z}/n$ and $H_q = 0$ where $q \neq k$, then the minimal chain complex of free abelian groups is

$$\dots 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0 \to \dots$$

where the first $\mathbb{Z}$ appears in dimension $k+1$. Then the CW-complex is

$$\begin{tikzcd} S^k \arrow[r] \arrow[d, "n"] & D^{k+1} \arrow[d, hook] \\ S^k \arrow[r] & M(\mathbb{Z}/n, k) \end{tikzcd}$$

where we write $M(\mathbb{Z}/n, k)$ to mean the Moore space. It has 1 zero-cell, 1 k-cell, and 1 $k+1$-cell.

As an example, $M(\mathbb{Z}/2, 1)$ is $\mathbb{R}P^2$. It follows now that the lower bound on the number of k-cells is

$$r(k) + t(k) + t(k-1).$$

Proposition 8.3. For all graded abelian group $A_* = \bigoplus_k A_k$ with $A_k = 0$ for $k \leq 0$, there exists a connected CW-complex X s.t.

$$\widetilde{H_*(X)} = A_*.$$

Proof. We use the method due to Moore. For A_k, write $A_k = \mathbb{Z}^{r(k)} \oplus \bigoplus_{i=1}^{t(k)} \mathbb{Z}/n_i(k)$ with $n_1(k)| \dots |n_{t(k)}(k)$ and $r(k)$ the rank of A_k. Realize it as $M(A_k, k) = \bigvee_{r(k)} S^k \vee \bigvee_i M(\mathbb{Z}/n_i(k), k)$. Then realize A_* via

$$\bigvee_k M(A_k, k).$$

$\square$

Theorem 8.4. If X is a simply connected CW-complex of finite type, there exists a CW-complex Y with

$$r(k) + t(k) + t(k-1)$$

k-cells and a map $Y \to X$ that is a homotopy equivalence.

9. Lecture 9: January 24th, 2022

ABSTRACT. In tis lecture, we continue the study of homology by introducing homology with coefficients. These will provide an array of functors which can then be used to study topological spaces. For instance, studying homology groups with $\mathbb{F}_2$ coefficients often reduces the computations.

Recall from last time our discussion of the Euler characteristic. Fix X a finite CW-complex. The Euler characteristic could be defined via two sums

$$\chi(X) := \sum_k (-1)^k a_k = \sum_k (-1)^k \beta_k$$

where a_n is the number of n-cells of X and β_n is the rank of $H_n(X)$. This second sum shows that the Euler characteristic is defined up to homotopy type due to the invariance of $H_*(X)$ up to homotopy. This implies Euler's classical result o convex polyhedrons. Another problem was to realize every graded f.g. abelian group A_* as the homology of some space. We had the following result:

Proposition 9.1. If $A_* := \bigoplus_k A_k$ with $A_k = 0$ for $k \leq 0$, then there exists a connected CW-complex X s.t.

$$\widetilde{H_*}(X) \cong A_*.$$

SKETCH. Write $A \cong \mathbb{Z}^r \bigoplus_{i=1}^k \mathbb{Z}/n_i\mathbb{Z}$ for some abelian group A that is f.g. and realize it as the homology of a space that is the wedge sum of Moore spaces. Taking the wedge sum of these spaces given A_*. The construction was as follows.

If $\mathbb{Z}$ is concentrated in dimension n, the chain complex $\ldots \to 0 \to \mathbb{Z} \to 0 \to \ldots$ has $\mathbb{Z}$ in dimension n and the reduced homology of S^n has this form. For $\mathbb{Z}/k$ in dimension n, $\ldots \to \mathbb{Z} \xrightarrow{k} \mathbb{Z} \to 0 \to \ldots$ with $\mathbb{Z} \to \mathbb{Z}$ starting in dimension $n+1$ has homology that of the Moore space $M(\mathbb{Z}/k, n)$. Now take the wedge sum over all of these spaces. $\square$

Example 9.2. $M(\mathbb{Z}/2, 1) \cong \mathbb{R}P^2$.

Today, we shall discuss homology group coefficients. So far, singular homology H_* is a functor from topological spaces to graded abelian groups. The question now is whether or not we can map to vector spaces over some field k or R-modules over some commutative ring R with unity.

We begin with the case of a commutative ring. Fix R a commutative ring with unity. Define $\mathrm{Sin}_n(X) := \{\sigma : \Delta^n \to X\}$ and let

$$S_n(X;R) := \text{ free } R\text{-module generated by } \mathrm{Sin}_n(X) = R(\mathrm{Sin}_n(X)) = \left\{ \sum_{i=1}^m r_i\sigma_i \ : \ r_i \in R, \sigma_i : \Delta^n \to X \right\}.$$

We can define the differentials as before i.e. $d : S_n(XlR) \to S_{n-1}(X;R)$ is given by $d\sigma = \sum_{i=0}^n (-1)^i d^i\sigma$. Then $d^2 = 0$ as before and we get a chain complex

$$\ldots \to S_{n+1}(X;R) \xrightarrow{d} S_n(X;R) \xrightarrow{d} S_{n-1}(X;R) \to \ldots.$$

Taking the homology gives $H_n(X;R)$ which is an R-module. The usual theory of singular homology takes $R := \mathbb{Z}$.

Homology with coefficients in R satisfies many familiar properties. We list them below.

- $H_*(-,R) : \mathbf{Top} \to R - \mathbf{Mods}$,

- $f \simeq g : X \to Y$, then $f_* = g_* : H_*(X; R) \to H_*(Y; R)$,
- $H_*(-; R)$ satisfies all the Eilenberg-Steenrod axioms except the dimension axiom,
- $H_n(X; R) = \widetilde{H}_n(X; R)$ if $n \geq 0$ while $H_n(X; R) = \widetilde{H}_n(X; R) \oplus R$ for $n = 0$,
- $H_*(\{*\}; R) = R$ in degree zero and $\widetilde{H}_*(\{*\}, R) = 0$,
- $\widetilde{H}_*(S^n; R) = R$ concentrated in degree n,
- for CW-complexes X, we can define the cellular chain complexes in R by
$$C_n(X; R) = \widetilde{H}_n(X_n/X_{n-1}; R) = R\langle A_n\rangle$$
with A_n the number of cells in dimension n,
- $d : C_n(X; R) \to C_{n-1}(X; R)$ is defined similarly and we obtain the cellular homology
$$H_*^C(X; R) \cong H_*(X; R).$$

Example 9.3. We consider $\mathbb{R}P^n$ in different coefficients. Recall $\mathbb{R}P^n$ a single cell in each dimension $0 \leq k \leq n$.

Take $R := \mathbb{F}_2$. Then the cellular chain complex looks
$$0 \leftarrow \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \xleftarrow{2} \mathbb{F}_2 \xleftarrow{0} \cdots \leftarrow \mathbb{F}_2 \leftarrow \mathbb{F}_2 \leftarrow 0$$
and so all the maps are zero. Since d is zero, the homology is
$$H_k(\mathbb{R}P^n; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$
which is very different from the usual singular homology:
$$H_k(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{k=0} \\ \mathbb{Z} & \text{if } k = 0, n, \text{ and } n \text{ odd} \\ \mathbb{Z}/2 & \text{if } k \text{ odd and } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}.$$
We have a similar answer for k with characteristic 2 i.e. any $\mathbb{F}_{2^k}$ or $\overline{\mathbb{F}_2}$.

For $\mathbb{F}_3$, $H_*(\mathbb{R}P^n, \mathbb{F}_3)$ is the homology of
$$0 \leftarrow \mathbb{F}_3 \xleftarrow{0} \mathbb{F}_3 \xleftarrow{2} \mathbb{F}_3 \xleftarrow{0} \cdots \leftarrow \mathbb{F}_3 \leftarrow \mathbb{F}_3 \leftarrow 0$$
and in $\mathbb{F}_3$, the element 2 is a unit. So the maps for multiplication by 2 are isomorphisms. Thus, the homology is simpler if n is even,
$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{F}_3 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \cong H_*(\{*\}).$$
If n were odd, we get
$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{F}_3 & k = 0 \text{ and } k = n \\ 0 & \text{otherwise} \end{cases} \cong H_*(S^n).$$
Similar answers occur when 2 is a unit (such as when $R := \mathbb{Q}, \mathbb{Z}[1/2], \mathbb{R}$).

We leave it to the reader to doe the same problem with $R := \mathbb{Z}/4\mathbb{Z}$.

A natural question to ask now is whether or not we can define homology with coefficients in some module. Additionally, if we are given information about $H_*(X; R)$, can we deduce information about $H_*(X; M)$ for an R-module M? The answer to these two questions are the tensor product and the **Universal Coefficient Theorem** respectively.

10. Lecture 10: January 26th, 2022

Abstract. Today, we build towards a valuable tool for computing homology with coefficients called the Universal Coefficient Theorem. We start by reviewing some properties of tensor products and then explain the construction of homology with coefficients once more to describe how one can take coefficients in a module M.

Last time, we defined singular homology with coefficients in a commutative ring with unity R (we shall not repeat these hypotheses when speaking of a commutative ring ever again). At the end of the lecture, two natural questions arose. For today, we shall apply $\otimes$ tensor products to define $H_*(X; M)$ which aid in the study of bilinear maps.

A good portion of this lecture discusses tensor products, but due to their ubiquity in mathematics, we refer the reader of these notes to any textbook on Abstract Algebra.

A few fundamental facts about tensor products we shall use repeatedly are listed:

(1) $R \otimes_R M \cong M$ with $R \times M \to M$ defined by $(r, m) \to rm$,
(2) $L \otimes (M \otimes N) \cong (L \otimes M) \otimes N)$ via $L \times (M \times N) \to (L \times M) \times N$,
(3) $M \otimes N \cong N \otimes M$ via $M \times N \cong N \times M$,
(4) $\bigoplus_\alpha (M \otimes N_\alpha) \cong M \otimes (\bigoplus_\alpha N_\alpha)$,
(5) if $f : M \to M'$ and $g_0, g_1 : N \to N'$, then $f \otimes (g_0 + g_1) \cong f \otimes g_0 + f \otimes g_1$ and $f \otimes rg_0 = r(f \otimes g_0$ as maps $M \otimes N \to M' \otimes N'$ and any $r \in R$.

Example 10.1. One has $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}$. This is an easy exercise for the reader.

Now we define homology with coefficients in M for M an R-module. Define $S_n(X; M) := S_n(X; R) \otimes_R M$ and $d := d_R \otimes 1 : S_n(X; M) \to S_{n-1}(X; M)$. The maps d are R-linear and $d^2 = 0$. This gives a chain complex

$$\dots S_{n+1}(X; M) \to S_n(X; M0 \to S_{n-1}(X; M) \to \dots$$

and taking homology, we obtain $H_n(X; M)$. For any $A \subseteq X$, the map $S_n(A; R) \to S_n(X; R)$ is a split homomorphism of free R-modules. Then we have a SES

$$0 \to S_n(A; R) \to S_n(X; R) \to S_n(X, A; R) \to 0$$

where the RHS R-module is defined as the quotient. Applying $- \otimes_R M$, we obtain a SES

$$0 \to S_n(A; M) \to S_n(X; M) \to S_n(X, A; M) \to 0.$$

Then we define $H_n(X, A; M) = H_n(S_*(X, A; M))$ which is the relative singular homology with coefficients in M. Additionally, there is a LES

$$\dots \to H_n(A; M) \to H_n(X; M) \to H_n(X, A; M) \to H_{n-1}(A; M) \to \dots$$

of relative homology.

Given an R-module homomorphism $M \to M'$, we get a map $S_*(X; M) \to S_*(X; M')$. This induces a map $H_*(X; M) \to H_*(X; M')$ on homology. Using this, a SES $0 \to M' \to M \to M'' \to 0$ gives rise to a SES $0 \to S_*(X, M') \to S_*(X, M) \to S_*(X, M'') \to 0$ of chain complexes. Then the Snake Lemma induces a LES

$$\dots \to H_n(X, M') \to H_n(X, M) \to H_n(X, M'') \to H_{n-1}(X, M') \to \dots$$

A natural question to ask now is how to compute $H_*(X, M)$ in terms of $H_*(X)$. The answer is not easy because in general

$$H_*(X; M) \ncong H_*(X) \otimes M.$$

We already say such an example with $\mathbb{R}P^n$ in different coefficients.

11. Lecture 11: January 28th, 2022

Abstract. Today, we construct the derived functors $\operatorname{Tor}_i^R(-, N)$ and show that they are well-defined. In particular, we shall state some useful facts about them to reduce future computations.

Last time, we constructed the tensor product. It is a common fact that the universal property determines the tensor product and one always considers $M \otimes_R N$ as a pair with the map $\otimes : M \times N \to M \otimes_R N$. We observed a number of properties about $\otimes_R$ and constructed homology with coefficients in M.

Today, our goal is to relate $H_*(X; M)$ with $H_*(X) \otimes_R M$. First, note that the functor $M \otimes_R -$ from the category of R-modules to itself is only a **right exact functor**. Left exactness fails with the example $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ and tensoring by $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} -$. However, left exactness may be preserved if M is a free R-module (or more generally a flat R-module). This leads to homological algebra which is a tool to measure failure of exactness.

We will need to define **free resolutions**. For now, fix R to be a PID. Then a standard fact of algebra is that any submodule of a free R-module is also free.

Fix M to be an R-module. Then find a free module F_0 s.t. $F_0 \xrightarrow{\phi_0} M \to 0$ is exact. Then take the kernel of this to get a SES $0 \to F_1 \to F_0 \to M \to 0$ with F_1, F_0 free R-modules. This is an example of a free resolution, but it need notbe unqiue.

Now $- \otimes_R N$ is right exact and applying it gives a sequence $F_1 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0$ that is exact. The kernel of the LHS map is called $\operatorname{Tor}_1^R(M, N)$ and fits into a sequence

$$0 \to \operatorname{Tor}_1^R(M, N) \to F_1 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0.$$

We shall observe that Tor is independent of the resolution taken.

Example 11.1. Fix M free. Then $0 \to M \to M \to 0$ is a free resolution and hence, $\operatorname{Tor}_1^R(M, N) = 0$.

Example 11.2. If $R = \mathbb{Z}$ and $M = \mathbb{Z}/n$. Then $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a free resolution. Computing Tor, we find that $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, N) = \ker(N \xrightarrow{n} N)$ which is the set of n-torsion elements in N.

Example 11.3. $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$ and $\operatorname{Tor}_1^N(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and more generally, $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/\gcd(m, n)\mathbb{Z}$.

Our goal not is to general from the case of a PID to any commutative ring. Let R be a commutative ring and M an R-module. Then there is a diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d"] \arrow[dr] & F_3 \arrow[r, "d"] \arrow[dr, two heads] & F_2 \arrow[r, "d"] \arrow[dr, two heads] & F_1 \arrow[r, "d"] \arrow[dr, two heads] & F_0 \arrow[dr, two heads] & & \\
\cdots & K_3 \arrow[u] \arrow[dr] & K_2 \arrow[u] \arrow[dr] & K_1 \arrow[u] \arrow[dr] & K_0 \arrow[u] \arrow[dr] & M \arrow[dr] & \\
0 \arrow[ur] & 0 \arrow[ur] & 0 \arrow[ur] & 0 \arrow[ur] & 0 & 0 & 0
\end{tikzcd}$$

It is constructed as follows. Take a free module surjecting onto M via $F_0 \to M \to 0$. Let K_0 be the kernel of this. Then take a free module F_1 surjecting onto F_0. This gives maps $F_1 \to K_0 \to F_0$ which we compose to call d. Then repeat this process inductively.

Observe that $d^2 = 0$ because there is a SES $0 \to K_i \to F_i \to K_{i-1} \to 0$ which appears in the composite. So there is a chain complex (F_*, d) called the free resolution of M

$$\dots F_3 \xrightarrow{d} F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \to 0.$$

Now there is a diagram

$$\begin{tikzcd}
\dots \arrow[r] & F_3 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & 0 \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

which is a chain map. This induces an isomorphism on homology: $\operatorname{im}(F_{n+1} \to F_n) = \operatorname{im}(K_n \to F_n)$ and $\ker(F_{n+1} \to F_n) = \ker(F_n \to K_{n-1})$ whenever $N \geq 1$. We see that

$$H_n((F_*, d)) = 0 \text{ for } n \geq 1 \qquad \& \qquad H_0((F_*, d)) \cong F_0/\operatorname{im}(K_0 \to F_0) \cong M.$$

So the free resolution of M is a "replacement" of M with the same homology. Now $F_* \otimes N$ gives a chain complex

$$\dots F_2 \otimes N \xrightarrow{d\otimes 1} F_1 \otimes N \xrightarrow{d\otimes 1} F_0 \otimes N \to 0$$

with potentially nontrivial homology. We define

$$\operatorname{Tor}_n^R(M, N) := H_n(F_* \otimes N)$$

for F_* a free resolution of M.

We shall later show that this is well-defined (i.e. independent of free resolution) and that the Tor-functors are functorial.

Remark 18. If M is free, $\operatorname{Tor}_n^R(M, N) = 0$ for $n \geq 1$.

If R is a PID, $\operatorname{Tor}_n^R(M, N) = 0$ for $n \geq 2$.

One has $\operatorname{Tor}_0^R(M, N) = M \otimes N$ and the higher index n are called the **derived functors of** $\otimes_R N$.

12. Lecture 12: January 31st, 2022

Abstract. In this lecture, we focus on categorical constructions. It is useful to know which categorical constructions our derived functors preserve. In particular, whether or not they preserve limits.

Last time, we discussed $\operatorname{Tor}_n^R(M, N)$ and free resolutions. We always take R to be a commutative ring with unity. If M is an R-module, there is a free resolution F_* of M and the construction relied on properties of free modules. Then, $H_n((F_*, d)) = M$ concentrated in degree zero. We then defined $\operatorname{Tor}_n^R(M, N) := H_n(F_* \otimes_R N)$.

We observed that higher tor functors vanished when M is free and $\operatorname{Tor}_n^R(M, N)$ vanished for M a PID and $n \geq 2$.

In this lecture, we proved the following theorem.

Theorem 12.1 (Fundamental Theorem of Homological Algebra)**.** If M, N are free R-modules and E_*, F_* are their respective free resolutions, any R-module map $f : M \to N$ lifts to a chain map $f_* : M_* \to N_*$.

Proof. See Miller's notes on Weibel's Introduction to Homological Algebra. We take the Lang approach to homological algebra and suggest the reader of these notes to attempt the proof themselves. □

Definition 12.2. One might observe that the proof of the fundamental theorem only required the property of projective modules.

That is, we call P a projective module if any map $P \to N$ facts through a surjection i.e. if $M \to N \to 0$ is exact, then there exists a lift $P \to M$.

Every free module is projective so one could consider projective resolutions instead of free resolutions.

A direct summand of a projective module is projective. Any projective module is a direct summand of a free module. Over any PID, projective modules are free.

Corollary 12.2.1. $\mathrm{Tor}_n^R(-, N)$ is well-defined for any R-module N.

Proof. We can always lift the identity map $\mathrm{id}_M : M \to M$ two a map of free resolutions $f_* : F_* \to F'_*$. By uniqueness up to chain homotopy, one sees that f_* an isomorphism and so gives an isomorphism on homology after tensoring $- \otimes N$. □

We state a few facts. First, any homomorphism $f : M' \to M$ with free resolutions F'_*, F_* induces a lift $F'_* \otimes N \to F_* \otimes N$ in which the map of homology is independent of the choice of lifts of f.

Proof. Given f'_* another lift, $f'_* \simeq f_*$ so they induce the same map on homology. Then $f_* \otimes 1 \simeq f'_* \otimes 1$. □

This proof also shows functoriality in M and N for the Tor-functor.

Remark 19. It is true, but difficult to show that $M \otimes N \cong N \otimes M$ implies $\mathrm{Tor}_n^R(M, N) \cong \mathrm{Tor}_n^R(N, M)$. The proof uses **double complexes**.

13. Lecture 13: February 2nd, 2022

Abstract. For this lecture, we do some computations and explain how to prove the universal Coefficient Theorem.

Last time, we define the Tor-functors as the homology of $F_* \otimes_R N$ where F_* is a free resolution of a module M. To show that it is well-defined, we proved the Fundamental Theorem of Homological Algebra which asserts that any R-module homomorphism $f : M \to N$ can lift to a chain map $f_* : F_* \to E_*$ of the respective free resolutions.

Definition 13.1. Let M, N be R-modules and define

$$\mathrm{Hom}_R(M, N) := \{f : M \to N : f \text{ is an } R\text{-linear map}\}.$$

This is an abelian group and an R-module if we define $(rf)(m) = f(rm) = rf(m)$.

Proposition 13.2. There is a natural isomorphism of R-modules

$$\mathrm{Hom}_R(L \otimes_R M, N) \cong \mathrm{Hom}_R(L, \mathrm{Hom}(M, N))$$

for M, N, L all R-modules.

The proof of the proposition is a routine. This shows that the functors $- \otimes_R M$ and $\mathrm{Hom}(M, -)$ are **adjoint functors**. Recall that if $\mathcal{C}, \mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors, we say F is left adjoint to G if there is a natural isomorphism

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$$

for any objects $X \in \mathrm{obj}(\mathcal{C})$ and $Y \in \mathrm{obj}(\mathcal{D})$.

It follows from right exactness of $- \otimes_R M$ and right adjointness of $\mathrm{Hom}_R(M-)$ that $\mathrm{Hom}_R(M, -)$ is a left exact functor.

Definition 13.3. Let I be a small category. Let $X : I \to \mathcal{C}$ be a functor. A **cone under** X is a natural transformation

$$\eta : X \to \mathrm{const}_Y$$

where $\mathrm{const}_Y : I \to \mathcal{C}$ is the functor mapping all objects of I to some $Y \in \mathrm{obj}(\mathcal{C})$ and all maps of I to the identity id_Y. In this way, we may view I as a sort of indexing set.

That is, for all $i \in I$, and $i \to j$ map in I, there exists $\eta_i : X_i \to Y$ s.t.

$$\begin{tikzcd}
X_i \arrow[dd] \arrow[dr, "\eta_i"] & \\
& Y \\
X_j \arrow[ur, "\eta_j"] &
\end{tikzcd}$$

commutes.

Definition 13.4. A **colimit** of X is an initial cone under X. That is, $\mathrm{colim}_I\, X$ is an object of X such satisfying the universal property indicated by the diagram

$$\begin{tikzcd}
X_i \arrow[dd] \arrow[dr] \arrow[drr, "\eta_i"] & & \\
& X \arrow[r, dotted, "\exists!"] & Y \\
X_j \arrow[ur, "\eta_j"] \arrow[urr] & &
\end{tikzcd}$$

Example 13.5. If $I = \emptyset$, then $\mathrm{colim}_I\, X$ is the initial object of $\mathcal{C}$. It may or may not exist.

Example 13.6. Let I be discrete i.e. there are no morphisms other than the identity morphism. Then $\mathrm{colim}_I\, X$ is the coproduct. In the category of sets, this is the disjoint union. In the category of abelian groups, it is the direct sum and of groups, it is the free product. In the category of topological spaces, it is the disjoint union. In the category of pointed spaces, it is the wedge sum.

Example 13.7. Suppose I is a **directed system** i.e. it is a partially ordered set s.t. for all $i, j \in I$ there exists $k \in I$ s.t. $i \leq k$ and $j \leq k$. Additionally, for all $i \leq j$, $I(i, j)$ is a singleton set $\{f_{ij}\}$ s.t. $f_{ii} = \mathrm{id}_i$ and $f_{ik} = f_{jk} \circ f_{ij}$. This is usually referred to as a **directed set**.

In this case, $\operatorname{colim}_I X$ is the direct limit. One writes this as $\varinjlim$ and explicitly in the category of sets,

$$\varinjlim_{i \in I} A_i \cong \coprod_{i \in I} A_i / \sim$$

where $x_i \in A_i$ is equivalent to $x_j \in A_j$ if there exists $k \in I$ s.t. $f_{ik}(x_i) = f_{jk}(x_j)$.

Example 13.8. Consider the case where $I \cong \mathbb{N}$ is the directed set. Consider

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{\times 4} \mathbb{Z} \xrightarrow{\times 5} \ldots.$$

Then the colimit is $\mathbb{Q}$. Indeed, there is a diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r, "\times 3"] \arrow[d, "1/2!"] & \mathbb{Z} \arrow[r, "\times 4"] \arrow[d, "1/3!"] & \ldots \\
\mathbb{Q} \arrow[r, "1"] & \mathbb{Q} \arrow[r, "1"] & \mathbb{Q} \arrow[r, "1"] & \ldots
\end{tikzcd}$$

so one deduces that the direct limit here must be $\mathbb{Q}$.

Example 13.9. Let I be the index set given by $A \to B$ and $A \to C$. This is no longer a directed set. However, the colimit should be very familiar. It is the pushout. In the category of topoloogical spaces, $\operatorname{colim}_I = B \cup_A C = B \coprod C / \sim$ where $f(a) \sim g(a)$ for all $a \in A$.

Example 13.10. Let I be the indexing set $f : A \to B, g : A \to B$. Then colim_I is the **coequalizer**. In the category of abelian groups, if $g = 0$ then the colimit is $\operatorname{coker}(f)$.

Example 13.11. Let G be a group. View G as a groupoid (i.e. a category with 1 object whose maps are given by elements in G). If there is a functor from G to some space, then this is given equivalently by an action of G on a topological space. The colimit with G as the index set is the orbit space.

The concept of a limit is defined dually to that of a colimit. One takes a terminal object instead of an initial object. One has products instead of coporducts, inverse limits instead of direct limits, pullbacks instead of pushforwards, equalizers instead of coequalizers, and fixed points instead of orbits.

One may view colimits and limits as functors $\mathcal{C}^I \to \mathcal{C}$ from the functor category.

Definition 13.12. A category is **complete** if it has all limits. It is **cocomplete** if it has all colimits.

Theorem 13.13. A category if (co)complete iff all (co)equalizers and (co)products exist.

14. Lecture 14: February 4th, 2022

Abstract. Today, we employ the UCT to compute some examples. We then work toward the Künneth Theorem and do so by constructing the tensor product. The Künneth Theorem provides a way to relate homology of two spaces with their product.

Last time, we introduced the Hom-functor: $\operatorname{Hom}_R(-, N)$ where $\operatorname{Hom}_R(M, N)$ is the set of R-module homomorphisms $M \to N$. It is an R-module defined by $(rf)(m) = f(rm) = rf(m)$. Additionally, $\operatorname{Hom}_R(-, N)$ defines a functor from R-modules to R-modules and it is right adjoint to the tensor product

$$\operatorname{Hom}_R(L \otimes_R M, N) \cong \operatorname{Hom}_R(L, \operatorname{Hom}_R(M, N))$$

i.e. $-\otimes_R M$ is left adjoint to $\mathrm{Hom}_R(M,-)$. We also defined **cones**, **colimits**, and **limits**. We also stated what a **complete category** was and gave a criterion for a category to be complete or cocomplete.

Example 14.1. Let $\mathcal{C}^I \to \mathcal{C}$ be the colimit functor. Let const : $\mathcal{C} \to \mathcal{C}^I$ be the **constant functor**. Then there is an adjoint pair

$$\mathcal{C}^I(X, \mathrm{const}_Y) \cong \mathcal{C}(\mathrm{colim}_I X, Y)$$

and similarly, there is an adjoint pair for limits

$$\mathcal{C}^I(\mathrm{const}_Y, X) = \mathcal{C}(Y, \lim_I X).$$

This shows that colim is a left adjoint and that lim is a right adjoint. This also gives an example of a functor with both a left and right adjoint.

Theorem 14.2. Left adjoints preserve colimits (sometimes referred to as **cocontinuous**). Right adjoints preserve limits (sometimes called **continuous**).

The proof of the above theorem can be done via the **Yoneda Lemma**, but this course is not about category theory.

Example 14.3. Recall that Hom $-\otimes$-adjunction. We now know from the theorem that $-\otimes_R M$ is a right exact functor while $\mathrm{Hom}_R(M,-)$ is a left exact functor. Furthermore, we know $-\otimes_R M$ preserves colimits while $\mathrm{Hom}_R(M,-)$ preserves limits.

As direct limits are colimits, $-\otimes_R M$ preserves direct limits i.e.

$$\varinjlim M_i \otimes_R N \cong \varinjlim (M_i \otimes_R N).$$

Theorem 14.4 (Facts about limits)**.** Limits commute with limits and similarly for colimits. Limits and colimits do not always comute.

Example 14.5. Direct limits in the category of abelian groups is right exact. OTOH, inverse limits are left exact.

Theorem 14.6 (Fact about direct limits)**.** Direct limits in abelian groups are left exact functors but are not generally right exact.

Proposition 14.7. Let I be a direct system. Let X_i, Y_i, Z_i be in the category of abelian groups and assume $X_i \to Y_i \to Z_i$ is exact for all $i \in I$. Then

$$\varinjlim_I X_i \to \varinjlim Y_i \to \varinjlim Z_i$$

is exact.

Proof. Omitted. □

Corollary 14.7.1. Let $I \to \{\text{chain complexes}\}$ be given by $i \to C_{*i}$ where each C_{*i} is a chain complex. Then

$$\varinjlim_I H_*(C_{*i}) \cong H_*(\varinjlim_I C_{*i})$$

Example 14.8. Suppose $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n$ be a filtration for X_i subspaces of X_n. View $\bigcup_i X_i = \varinjlim X_i$ as a colimit. Then,

$$H_*(X) \cong \varinjlim_I H_*(X_i).$$

Example 14.9. One has a sequence of isomorphisms

$$H_*(X;\mathbb{Q}) \cong H_*(S_*(X)\otimes\mathbb{Q}) \cong H_*(S_*(X)\otimes \varinjlim_n \mathbb{Z}) \cong H_*(\varinjlim_n (S_*(X)\otimes\mathbb{Z}))$$
$$\cong \varinjlim H_*(S_*(X)\otimes\mathbb{Z}) \cong \varinjlim H_*(S_*(X))$$

because $-\otimes_{\mathbb{Z}}\mathbb{Z} \cong H_*(X)\otimes\mathbb{Q}$ for the last isomorphism. We conclude that

$$H_*(X;\mathbb{Q}) \cong H_*(X)\otimes_{\mathbb{Z}}\mathbb{Q}$$

and this also holds when $H_*(X)$ is a f.g. abelian group (this is also an easy computation).

The above isomorphism is not true if $\mathbb{Q}$ is replaced by some other field $\mathbb{F}$.

15. Lecture 15: February 7th, 2022

Abstract. To prove the Künneth Theorem, we study models and acyclic objects. These become important tools for homological algebra and play an essential role in proving the Eilenber-Zilber Theorem.

Let R be a commutative ring and M, N be a R-modules. We defined $\operatorname{Tor}_n^R(M,N)$ as the derived functors of $-\otimes_R N$. We know $\operatorname{Tor}_n^R(M,N) = 0$ for $n \geq 2$ whenever R is a PID.

Recall we had asked the following question: *Can we compute/deduce $H_*(X;M)$ from knowledge of $H_*(X;R)$?*

Example 15.1. Consider $\mathbb{R}P^2, S^2$ with the map $\mathbb{R}P^2 \to S^2$ that quotients out the 1-skeleton of $\mathbb{R}P^2$. We explicitly know that

$$H_k(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2\mathbb{Z} & k=1 \\ 0 & k \text{ otherwise} \end{cases}.$$

But $C_*(\mathbb{R}P^2)\otimes\mathbb{Z}/2\mathbb{Z}$ gives singular homology

$$H_k(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2 & k=0,1,2 \\ 0 & k \text{ otherwise} \end{cases}.$$

OTOH,

$$H_k(S^2;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2 & k=0,2 \\ 0 & k \text{ otherwise} \end{cases}.$$

Examining the above, the quotient map $p : \mathbb{R}P^2 \to S^2$ gives corresponding maps on homology. In particular,

$$H_0(-) : \mathbb{Z} \xrightarrow{1} \mathbb{Z} \qquad \& \qquad H_1(-) : \mathbb{Z}/2 \xrightarrow{0} 0, \qquad \& \qquad H_2(-) : 0 \xrightarrow{0} \mathbb{Z}$$

while

$$H_0(-;\mathbb{Z}/2\mathbb{Z}) : \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \qquad \& \qquad H_1(-;\mathbb{Z}/2\mathbb{Z}) : \mathbb{Z}/2 \to 0 \qquad \& \qquad H_2(-;\mathbb{Z}/2) : \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2.$$

The computation of the $H_2(-;\mathbb{Z}/2)$ map follows from the collapsing of the 1-skeleton of $\mathbb{R}P^2$. Also, $H_2(-)$ and $H_2(-;\mathbb{Z}/2)$ are different maps and because $H_2(-)$, we can deduce that $H_*(-;\mathbb{Z}/2\mathbb{Z})$ is not functorially determined by $H_2(-)$.

Though there is failure of functoriality, we can still obtain a natural map. This is done algebraically. Let C_* be a chain complex of modules over R. There exists a natural map

$$\alpha : H_n(C_*)\otimes M \to H_n(C_*\otimes M)$$

via $[z] \otimes m \mapsto [z \otimes m]$ because $d(z) = 0$ implies $d \otimes 1(z \otimes m) = d(z) \otimes m = 0$. Now, we consider the following diagram

$$\begin{tikzcd}
C_* \otimes M \arrow[r, Rightarrow, "\cong"] & C_* \otimes M \\
Z_n(C_*) \otimes M \arrow[u, "\text{not necessarily injective}"] \arrow[r] & Z_n(C_* \otimes M) \arrow[u, hook]
\end{tikzcd}$$

and the horizontal map is given by $z \otimes m \to z \otimes m$. If $Z_n(C_*) \otimes M \to C_* \otimes M$ is injective, an easy diagram chase shows $Z_n(C_*) \otimes M \to Z_n(C_* \otimes M)$.

Now there is a diagram

$$\begin{tikzcd}
0 & 0 \\
H_n(C_*) \otimes M \arrow[u] \arrow[r, "\alpha"] & H_n(C_* \otimes M) \arrow[u] \\
Z_n(C_*) \otimes M \arrow[u, two heads] & Z_n(C_* \otimes M) \arrow[u, two heads] \\
C_{n+1} \otimes M \arrow[u] \arrow[r, Rightarrow] & C_{n+1} \otimes M \arrow[u, "d\otimes 1"] \\
& 0 \arrow[u]
\end{tikzcd}$$

If the middle map is injective, then α is also injective.

Combining our two claims on injectivity, $Z_n(C_*) \otimes M \hookrightarrow C_* \otimes M$ being injective implies $H_n(C_* \otimes M) \hookrightarrow H_n(C_* \otimes M)$.

Suppose R were a PID and C_n were free over R (this will be the case with the cellular and singular chain complexes). There is a diagram

$$0 \to Z_n(C_*) \to C_n \xrightarrow{d} B_{n-1}(C_*) \subseteq C_{n-1} \to 0.$$

Because C_{n-1} is free, $B_{n-1}(C_*)$ is free. So there exists a splitting (i.e. $Z_n(C_*) \hookrightarrow C_n$ is a split monomorphism) $C_n \cong B_{n-1}(C_*) \oplus Z_n(C_*)$. So, $Z_n(C_*) \otimes M \to C_n \otimes M$ is a split monomorphism

$$C_{n+1} \otimes M \cong (Z_n(C_*)) \otimes M \oplus (B_n(C_*) \otimes M).$$

This fits into a diagram

$$\begin{tikzcd}
C_{n+1} \otimes M \cong \arrow[d, "d\otimes 1"] & Z_n(C_*) \otimes M & \oplus(B_n(C_*) \otimes M) \arrow[dl] \\
C_n \otimes M \cong & Z_n(C_*) \otimes M & \oplus(B_{n-1}(C_*) \otimes M)
\end{tikzcd}$$

where that diagonal map need not be injective. Taking homology, we see that

$$H_n(C_* \otimes M) \cong (H_n(C_*) \otimes M) \oplus (?)$$

where (?) is some module we do not know yet. This shows that $\alpha : H_n(C_*) \otimes M \to H_n(C_* \otimes M)$ is a split monomorphism thus far. Our goal to explicitly determine (?).

If $M \cong R$, then $(?) = 0$ since α would be an isomorphism. So the idea shall be that it is easier to work with free modules. Replace M be its free resolution and if R is a PID (as it will be in virtually all of our applications),

$$0 \to F_1 \to F_0 \to M \to 0.$$

Tensoring by C_* gives

$$0 \to C_* \otimes F_1 \to C_* \otimes F_0 \to C_* \otimes M \to 0$$

and exactness follows from C_* a complex of free modules. Then there is a LES of homology

$$\dots \to H_n(C_* \otimes F_1) \to H_n(C_* \otimes F_0) \to H_n(C_* \otimes M) \to H_{n-1}(C_* \otimes F_1) \to \dots$$

From our work above,

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\operatorname{coker}(H_n(C_* \otimes F_1) \to H_n(C_* \otimes F_0)) \arrow[r, "="] \arrow[d] & \operatorname{coker}(H_n(C_*) \otimes F_1 \to H_n(C_*) \otimes F_0) \arrow[d] \\
H_n(C_* \otimes M) \arrow[r, "="] \arrow[d] & H_n(C_* \otimes M) \arrow[d] \\
\ker(H_{n-1}(C_* \otimes F_1) \to H_{n-1}(C_* \otimes F_0)) \arrow[r, "="] \arrow[d] & \ker(H_{n-1}(C) \otimes F_1 \to H_{n-1}(C) \otimes F_0) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Now the cokernel in the diagram is $H_n(C_*) \otimes M$ because tensor products preseve cokernels. OTOH, kernel is $\operatorname{Tor}_1^R(M, H_{n-1}(C_*)) = \operatorname{Tor}_1^R(H_{n-1}(C_*), M)$ because we restricted our attention to the sequence

$$\operatorname{Tor}_1^R(M, H_{n-1}(C_*)) \to F_1 \otimes H_{n-1}(C_*) \to F_0 \otimes H_{n-1}(C_*) \to M \otimes H_{n-1}(C_*) \to 0.$$

So it follows that the splitting is

$$H_n(C_* \otimes M) \cong (H_{n-1}^R(C_*) \otimes M) \oplus \operatorname{Tor}_1^R(H_{n-1}(C_*), M).$$

Theorem 15.2 (**Universal Coefficient Theorem**)**.** If R is a PID, C_* is a chain complex over R s.t. each C_n is a free R-module, then there a SES of R-modules

$$0 \to H_n(C_*) \otimes M \to H_n(C_* \otimes M) \to \operatorname{Tor}_1^R(H_{n-1}(C_*), M) \to 0$$

that splits but the splitting is not functorial.

Corollary 15.2.1. Let X be a topological space. Let R be a PID and M an R-module. Then there exists a nonfunctorial split SES

$$0 \to H_n(X; R) \otimes M \to H_n(X; M) \to \operatorname{Tor}_1^R(H_{n-1}(X; R), M) \to 0$$

and so

$$H_n(X; M) \cong (H_n(X; R) \otimes M) \oplus \operatorname{Tor}_1^R(H_{n-1}(X; R), M).$$

Example 15.3. From before, we know

$$H_n(X;\mathbb{Q}) \cong H_n(X) \otimes \mathbb{Q}$$

from a colimit computation. But from the Universal Coefficient theorem, we get the isomorphism from $\mathrm{Tor}_1^R(H_{n-1}(X), \mathbb{Q}) = 0$ as $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

16. Lecture 16: February 9th, 2022

Last time, we proved the universal coefficient theorem which says that if R is a PID, C_* is a chain complex with each C_n being free R-modules, then there is a SES of R-modules

$$0 \to H_n(C_*;R) \otimes M \to H_n(C_*;M) \to \mathrm{Tor}_1^R(H_{n-1}(C_*;R), M) \to 0$$

which splits noncanonically. As corollary, we had:

Corollary 16.0.1. If X is a topological space and M a module over a PID R, then there is a noncanonical split SES

$$0 \to H_n(X;R) \otimes M \to H_n(X;M) \to \mathrm{Tor}_1^R(H_{n-1}(X;R), M) \to 0.$$

Some immediate consequences are that $H_n(X;\mathbb{Q}) \cong H_n(X) \otimes \mathbb{Q}$ and

$$H_n(\mathbb{R}P^2;\mathbb{Z}) = \begin{cases} \mathbb{Z}/2 \oplus 0 & n=0 \\ \mathbb{Z}/2 \oplus 0 & n=1 \\ 0 \oplus \mathbb{Z}/2 & n=2 \\ 0 & \text{otherwise} \end{cases}.$$

Indeed, this computation follows from some basic facts about $\mathrm{Tor}_1^{\mathbb{Z}}$ and the splitting

$$H_n(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) \cong H_n(\mathbb{R}P^2) \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathbb{R}P^2), \mathbb{Z}/2).$$

Theorem 16.1 (Basic Facts about $\mathrm{Tor}_1^{\mathbb{Z}}$)**.** $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/\gcd(m,n)$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m) = 0$ for all $m, n \in \mathbb{Z}$.

In a similar vein, one can show that

$$H_n(\mathbb{R}P^2;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3 \oplus 0 & n=0 \\ 0 & \text{otherwise} \end{cases}$$

Example 16.2. Suppose we are given a space with

$$H_n(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2 & n=1 \\ 0 & n=2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/3 & n=4 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & n=50 \quad \text{otherwise} \end{cases}$$

We now from previous lectures that there will exist such a space (the construction used Moore spaces and wedge sums with spheres). Now the Universal Coefficient Theorem can help us compute

$$H_n(X;\mathbb{Z}/2) \cong H_n(X)\otimes\mathbb{Z}/2\oplus \mathrm{Tor}_1^R(H_{n-1}(X,\mathbb{Z}/2),\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & n=0\\ \mathbb{Z}/2 & n=1\\ 0\oplus\mathbb{Z}/2 & n=2\\ \mathbb{Z}/2 & n=3\\ \mathbb{Z}/2 & n=4\\ (\mathbb{Z}/2\oplus\mathbb{Z}/2)\oplus\mathbb{Z}/2 & n=5\\ 0\oplus\mathbb{Z}/20 & \text{otherwise}\end{cases}$$

A general heuristic is that the Tor factor of the universal coefficient theorem only ever contributes to the next line.

Another example is

$$H_n(X;\mathbb{Z}/3) \cong H_n(X)\otimes\mathbb{Z}/3\oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X),\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3 & n=0\\ 0 & n=1\\ 0 & n=2\\ 0 & n=3\\ \mathbb{Z}/3 & n=4\\ \mathbb{Z}/3\oplus\mathbb{Z}/3 & n=5\\ 0 & \text{otherwise}\end{cases}.$$

Let us now discuss the **Künneth Theorem**. Recall that the cross product $\times : S_p(X)\times S_q(X)\to S_{p+q}(X\times Y)$ is a bilinear map and that it satisfied a Leibniz formula

$$d(x\times y) = (dx)\times y + (-1)^p x\times(dy).$$

From the universal property of the tensor product, we get an induced cross product map (by abuse of notation also called $\times$)

$$\times : S_p(X)\times S_q(Y)\to S_{p+q}(X\times Y)$$

where $d(x\times y) = (dx)\otimes y + (-1)^p x\otimes (dy)$.

Definition 16.3. Given two chain complexes C_*, D_*. We can define the **tensor product of chain complexes** by

$$(C_*\otimes D_*)_n = \bigoplus_{p+q=n} C_p\otimes C_q \qquad \& \qquad d:(C_*\otimes D_*)_n\to (C_*\otimes D_*)_{n-1}$$

and $d(x\otimes y) = (dx)\otimes y \oplus (-1)^p x\otimes dy$ Observe that $d^2=0$ by the same reasoning that $d^2(x\times y)=0$. Now $C_p\otimes D_q\to C_{p-1}\otimes D_q + C_p\otimes D_{q-1}$ is how the map acts on each summand.

Remark 20. The cross product is actually a map of chain complexes $S_*(X)\otimes S_*(Y)\to S_*(X\times Y)$.

There are some natural questions to ask. Does the cross product induce an isomorphism on homology? The answer is actually yes:

$$H_*(S_*(X)\otimes S_*(Y))\cong H_*(S_*(X\times Y))\cong H_*(X\times Y)$$

and the first isomorphism is called the **Eilenberg-Zilber Theorem**.

Theorem 16.4 (Eilenberg-Zilber Theorem)**. Eilenberg-Zilber Theorem** Take coefficients in a commutative ring R. There exist unique chain homotopy classes of natural chain maps

$$S_*(X)\otimes_R S_*(Y) \rightleftarrows S_*(X\times Y)$$

covering the usual isomorphisms $H_0(X) \otimes_R H_0(Y) \cong H_0(X \times Y)$ and any such pair are natural chain homotopy inverses.

Corollary 16.4.1. There exists canonical natural isomorphism $H_*(S_*(X) \otimes_R S_*(Y)) \cong H_*(X \times Y)$.

Another question is whether or not $H_*(C_* \otimes D_*)$ can be determined from $H_*(C_*) \otimes H_*(D_*)$? The answer is a bit technical, but in the case of D_* being concetrated in degree zero we already have an answer – the Universal Coefficient Theorem!

The full answer is known as the **Algebraic Künneth Theorem**.

Theorem 16.5 (**Algebraic Künneth Theorem**)**.** Let R be a PID and C_*, D_* chain complexes of modules over R with C_n free for all n. Then there is a SES that splits nonfunctorially

$$0 \to \bigoplus_{p+q=n} H_p(C^*) \otimes H_q(D^*) \to H_n(C_* \otimes D_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_*), H_q(D_*)) \to 0$$

Corollary 16.5.1. Let R be a PID and X, Y topological spaces. Then there is a SES that splits nonfunctorially

$$0 \to \bigoplus_{p+q=n} H_p(X;R) \otimes H_q(Y;R) \to H_n(X \times Y;R) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X;R), H_q(X;R)) \to 0$$

and note that the middle module being $H_n(X \times Y;R)$ is due to the Eilenberg-Zilber Theorem.

Example 16.6. If $k = R$ is a field, then Tor_1^k is always zero. Then there is a **Künneth isomorphism** from the first map

$$\times : H_*(X;K) \otimes H_*(Y;K) := \bigoplus_{p+q=n} H_p(X;k) \otimes H_*(Y;k) \to H_*(X \times Y;k).$$

Example 16.7. Let us compute $H_*(\mathbb{R}P^3 \times \mathbb{R}P^3; \mathbb{Z}/3)$. We know that

$$H_n(\mathbb{R}P^3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2 & n=1 \\ 0 & n=2 \\ \mathbb{Z} & n=30 \quad \text{otherwise} \end{cases}$$

and from the Universal Coefficient Theorem,

$$H_n(\mathbb{R}P^3;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3 & n=0 \\ 0 & n=1 \\ 0 & n=2 \\ \mathbb{Z}/3 & n=30 \quad \text{otherwise} \end{cases}$$

Applying the Künneth formula,

$$H_*(\mathbb{R}P^3 \times \mathbb{R}P^3;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3 & n=0 \\ 0 & n=1 \\ 0 & n=2 \\ \mathbb{Z}/3 \oplus \mathbb{Z}/3 & n=3 \\ 0 & n=4 \\ 0 & n=5 \\ \mathbb{Z}/3 & n=60 \quad \text{otherwise} \end{cases}.$$

Additionally, one also finds

$$H_n(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & n = 0 \\ (\mathbb{Z}/2)^2 & n = 1 \\ (\mathbb{Z}/2)^3 & n = 2 \\ (\mathbb{Z}/2)^4 & n = 3 \\ (\mathbb{Z}/2)^3 & n = 4 \\ (\mathbb{Z}/2)^2 & n = 5 \\ (\mathbb{Z}/2) & n = 6 \\ 0 & \text{otherwise} \end{cases}.$$

For these computations, it is often useful to use a table. For instnace, if we want to compute $H_n(\mathbb{R}P^3 \times \mathbb{R}P^3, \mathbb{Z})$, the table

$\otimes$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
0	0	0	0	0
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$

and a computation of the Tor functors tells us that we have $\mathbb{Z}, \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}$ for degrees $0, 1, 2, 3, 4$ and 5 and zero for all else.

17. Lecture 17: February 11th, 2022

The goal for today's lecture is to prove $H_*(S_*(X) \otimes_R S_*(Y)) \cong H_*(S_*(X \times Y)) \cong H_*(X \times Y)$ to obtain Künneth's Theorem for topological spaces. Recall that the cross product $\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$ induces a linear map $\times : S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y)$ which gives a chain map with $d(x \otimes y) = (dx) \otimes y + (-1)^p x \otimes (dy)$.

To do this, we must discuss acyclic modules.

Definition 17.1. Let $\mathcal{C}$ be any category and $\mathcal{M}$ a fixed set of objects (called **models**) in the category. A functor $F : \mathcal{C} \to \mathrm{Ab}$ is **$\mathcal{M}$-free** if F can be factored

$$\mathcal{C} \to \mathrm{Set} \to \mathrm{Ab}$$

where the first map is a coproduct of functors $\coprod \mathcal{C}(M_i, -)$ for $M_i \in \mathcal{M}$ and the second map is the functor that takes sets to the free abelian group generated on that set.

Example 17.2. Let $\mathcal{C}$ be Top and then $S_n(X) = \mathbb{Z}\,\mathrm{Sin}_n(X) = \mathbb{Z}\,\mathrm{Top}(\Delta^n, X)$. So we can take $\mathcal{M} = \{\Delta^n : n \geq 0\}$ which means $S_n(-)$ is an $\mathcal{M}$-free functor.

Example 17.3. Let $\mathcal{C} = \mathrm{Top} \times \mathrm{Top}$. Let $\mathcal{M} := \{(\Delta^p, \Delta^q) : p, q \geq 0\}$. Consider $S_n(X \times Y) := \mathbb{Z}\,\mathrm{Sin}_n(X \times Y) = \mathbb{Z}\,\mathrm{Top}(\Delta^n, X \times Y)$. This can be identified with $\mathbb{Z}(\mathrm{Top} \times \mathrm{Top})((\Delta^n, \Delta^n), (X \times Y))$ and this is $\mathcal{M}$-free on a single model.

Now $(S_*(X) \otimes S_*(Y))_n = \bigoplus_{p+q=n} S_p(X) \otimes S_q(X) : \mathrm{Top} \times \mathrm{Top} \to \mathrm{Ab}$ equals

$$\bigoplus_{p+q=n} \mathbb{Z}\,\mathrm{Sin}_p(X) \otimes \mathbb{Z}\,\mathrm{Sin}_q(Y) = \bigoplus_{p+q=n} \mathbb{Z}\langle \mathrm{Sin}_p(X) \times \mathrm{Sin}_q(X)\rangle = \mathbb{Z}\left\langle \coprod_{p+q=n} \mathrm{Sin}_p(X) \otimes \mathrm{Sin}_q(Y) \right\rangle$$
$$= \mathbb{Z}\left\langle \coprod_{p+q=n} \mathrm{Top}(\Delta^p, X) \times \mathrm{Top}(\Delta^q, Y) \right\rangle$$
$$= \mathbb{Z}\left\langle \coprod_{p+q=n} \mathrm{Top} \times \mathrm{Top}((\Delta^p, \Delta^q), (X, Y)) \right\rangle$$

is also $\mathcal{M}$-free with models (Δ^p, Δ^q). Essentially we showed both $S_n(-)$ and $(S_*(-)\otimes S_*(-))_n$ are $\mathcal{M}$-free.

Definition 17.4. A natural transformation of functors $\theta : F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathrm{Ab})$ is an **$\mathcal{M}$-epimorphism** if $\theta_M : F(M) \to G(M)$ is surjective in Ab for all $M \in \mathcal{M}$.

A sequence of natural transformations $G' \to G \to G''$ is $\mathcal{M}$-exact if $G'(M) \to G(M) \to G''(M)$ is exact for all $M \in \mathcal{M}$.

Example 17.5. The sequence

$$\ldots \to S_n(X \times Y) \to S_{n-1}(X \times Y) \to \ldots \to S_0(X \times Y) \to H_0(X \times Y) \to 0$$

is $\mathcal{M}$-exact for $\mathcal{M} = \{(\Delta^p, \Delta^q) : \forall p, q \geq 0\}$. Take $X = \Delta^p$ and $Y = \Delta^q$ and we would know $\Delta^p \times \Delta^q$ is contractible. Hence, we get exactness.

Example 17.6. Consider the sequence

$$\ldots (S_*(X)\otimes S_*(Y))_n \to (S_*(X)\otimes S_*(Y))_{n-1} \to \ldots \to S_0(X)\otimes S_0(Y) \to H_0(X)\otimes H_0(Y) \to 0$$

which is $\mathcal{M}$ exact for the same choices of $\mathcal{M}$.

Lemma 17.7. Let R be a PID and $C'_* \to C_*$ and $D'_* \to D_*$ induce isomorphisms on homology. Assume $(C_*)_n, (C'_*)_n$ are always free. Then $C'_* \otimes D'_* \to C_* \otimes D_*$ induces an isomorphism on homology.

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & H_*(C'_*) \otimes H_n(D'_*) \arrow[r] \arrow[d, "\cong"] & H_*(C'_* \otimes D'_*) \arrow[r] \arrow[d] & \mathrm{Tor}_1^R * H_*(C'_*), H_*(D'_*)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H_*(C_*) \otimes H_*(D_*) \arrow[r] & H_*(C_* \otimes D_*) \arrow[r] & \mathrm{Tor}_1^R(H_*(C_*), H_*(D_*)) \arrow[r] & 0
\end{tikzcd}$$

and apply the 5-lemma. □

Lemma 17.8. Let $\mathcal{C}$ be a category with a set of models $\mathcal{M}$. Let F, G, G' be functors $\mathcal{C} \to \mathrm{Ab}$ where F is $\mathcal{M}$-free and $G' \to G$ is an $\mathcal{M}$-epimorphism and $f : F \to G$ is a natural transformation. Then there exists a lift

$$\begin{tikzcd}
 & G' \arrow[d, two heads] \\
F \arrow[r, "f"] \arrow[ur, dotted, "\overline{f}"] & G
\end{tikzcd}$$

PROOF. Since F is $\mathcal{M}$-free, we know $F(X) = \coprod \mathbb{Z}\mathcal{C}(M, X)$ for $M \in \mathcal{M}$. So we only need to consider $F(X) = \mathbb{Z}\mathcal{C}(M, X)$. Take $X = M$ and consider $1_M \in \mathbb{Z}\mathcal{C}(M, M)$. By $\mathcal{M}$-epimorphism, we can choose a $c_M \in G'(M)$ s.t. c_M maps to $f_M(1_M)$. Define $\widetilde{f_M}(1_M) = c_M$. Then for any $\varphi : M \to X$, define $\overline{f_X}$ by $\overline{f_X}(\varphi) = \varphi_*(c_M)$. That is,

$$\begin{tikzcd}
\varphi & \mathcal{C}(M,X) \arrow[r, "\overline{f_X}"] & G'(X) & \varphi_*(c_M) \\
1_M \arrow[u] & \mathcal{C}(M,M) \arrow[u, "\varphi"] \arrow[r, "\overline{f_M}"] & G'(M) \arrow[u, "\varphi_*"'] & c_M \arrow[u]
\end{tikzcd}$$

is the diagram for a natural transformation. □

Theorem 17.9 (Acyclic Models)**. Acyclic Models** Let $\mathcal{C}$ be a category with a set of models $\mathcal{M}$. Let $\theta : F \to G$ be a natural transformation of functors $\mathcal{C} \to$ Ab. Let F_*, G_* be functors $\mathcal{C} \to$ ChainComplex with natural transformations $F_0 \to F$, $G_0 \to G$. Assume the F_n are all $\mathcal{M}$-free. Also assume that $G_* \to G \to)$ is $\mathcal{M}$-exact.

Then there exists a chain map $F_* \to G_*$ covering θ that is unique up to chain homotopy

$$\begin{tikzcd}
\dots \arrow[r] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "\theta"] & 0 \\
\dots \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

Corollary 17.9.1. If θ is a natural isomorphism and G_n were also $\mathcal{M}$-free with $F_* \to F \to 0$ is exact, then any chain map $F_* \to G_*$ covering θ is a chain homotopy equivalence.

PROOF. Similar to the fundamental theorem of homological algebra. □

Theorem 17.10 (Eilenberg-Zilber Theorem)**.** Let R be a commutative ring. Then there exists unique chain homotopy classes of natural chain maps

$$S_*(X) \otimes S_*(Y) \to S_*(X \times Y) \qquad \& \qquad S_*(X) \otimes S_*(Y) \leftarrow S_*(X \times Y)$$

covering $H_0(X) \times H_0(Y) \cong H_0(X \times Y)$ that are natural chain homotopy inverses.

PROOF. Work with $\mathcal{C} = \text{Top} \times \text{Top}$ and $\mathcal{M} = \{(\Delta^p, \Delta^q) : p, q \geq 0\}$. Then the previous examples show that $S_*(X) \otimes S_*(Y)$ and $S_*(X \times Y)$ are $\mathcal{M}$ free and that $S_*(X) \otimes S_*(Y) \to H_0(X) \otimes H_0(Y)$ and $S_*(X \times Y) \to H_0(X \times Y)$ are $\mathcal{M}$ exact. Now apply the acyclic model theorem. □

Corollary 17.10.1. For any R coefficient in place of $\mathbb{Z}$

$$H_*(S_*(X) \otimes S_*(Y)) \cong H_*(S_*(X \times Y)) \cong H_*(X \times Y).$$

As a consequence, we obtain the Künneth theorem.

Theorem 17.11. Let R be a PID. Then there exists a SES that splits nonfunctorially

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X), H_q(Y)) \to 0.$$

18. Lecture 18: February 14th, 2022

Abstract. In today's lectures, we describe what sort of algebraic objects singular homology groups form. Then, we construct the dual theory called cohomology. It turns out that the cohomology groups have "more" algebraic structure and this warrants their study.

Recall that the **Eilenberg-Zilber Theorem** states that for any commutative ring R, there exists a chain homotopy equivalence

$$S_*(X;R)\otimes S_*(Y;R)\to S_*(X\times Y;R)$$

between the tensor product of two singular chain complexes and the singular chain complex of the Cartesian product. Then, it was possible to compute $H_*(X\times Y)$ via the **Künneth Theorem**. In sufficiently nice cases, one could have **Künneth Isomorphism**

$$\times : H_*(X;R)\otimes H_*(Y;R)\to H_*(X\times Y;R)$$

such as when all Tor-functors vanish in the the Künneth Theorem computation.

We now proceed to discuss the case of $X\times X$. Let $\Delta : X\to X\times X$ denote the **diagonal map**. This induces a map on homology $\Delta : H_*(X;R)\to H_*(X\times X;R)\cong H_*(X;R)\otimes_R H_*(X;R)$ if R is a PID and $H_*(X;R)$ is free over R. This map is called a coproduct. More generally, we have:

Definition 18.1. Let C be a commutative ring. A **graded coalgebra** over R is a graded R-module M with a **comultiplication map** $\Delta : M\to M\otimes_R M$ and a **counit map** $\epsilon : M\to R$ s.t. the following diagrams commutes

$$\begin{tikzcd} & M \arrow[dl, Rightarrow] \arrow[d, "\Delta"] \arrow[dr, Rightarrow] & \\ R\otimes_R M & M\otimes_R M \arrow[l] \arrow[r] & M\otimes_R R \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"] & M\otimes_R M \arrow[d, "\Delta\otimes 1"] \\ M\otimes_R M \arrow[r, "\Delta\otimes 1"] & M\otimes_R M\otimes_R M \end{tikzcd}$$

Also, it is **cocommuative** if the following diagram commutes

$$\begin{tikzcd} & M \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ M\otimes_R M \arrow[rr, "\tau"] & & M\otimes_R M \end{tikzcd}$$

where $\tau(x\otimes y)=(-1)^{|x||y|}y\otimes x$.

At the level of chain comlexes and using acyclic models, we know

$$\begin{tikzcd} S_*(X)\otimes S_*(Y)\otimes S_*(Z) \arrow[r, "\times\otimes 1"] \arrow[d, "1\otimes\times"] & S_*(X\times Y)\otimes S_*(Z) \arrow[d, "\times"] \\ S_*(X)\otimes S_*(Y\times Z) \arrow[r, "\times"] & S_*(X\times Y\times Z) \end{tikzcd}$$

commutes up to chain homotopy. Similarly, if $T(x,y)=(y,x) : X\times Y\to Y\times X$, the diagram

$$\begin{tikzcd} S_*(X)\otimes S_*(Y) \arrow[r, "\times"] \arrow[d, "\tau"] & S_*(X\times Y) \arrow[d, "T_*"] \\ S_*(Y)\otimes S_*(X) \arrow[r] & S_*(X\times Y) \end{tikzcd}$$

commutes up to homotopy. In other words, $\times$ is **associative** and **commutative**. Similar diagrams apply to the functor H_* of the tensor products of chain complexes.

Theorem 18.2. Let R be a PID and $H_*(X;R)$ be free over R. Then $H_*(X;R)$ is a cocommutative graded coalgebra over R.

Proof. This follows from our discussion above. $\square$

In some sense, coalgebras are dual to algebras. Then homology is dual to cohomology. For the rest of the lecture, we begin to introduce cohomology.

Definition 18.3. Let N be an abelian group. A **singular n-chain** on X with values in N is a function

$$\mathrm{Sin}_n(X) = \mathrm{Top}(\Delta^n, X) \to N$$

which we can extend $\mathbb{Z}$-linearly $S_n(X) = \mathbb{Z}\,\mathrm{Sin}_n(X) \to N$. If N is an R-module, we can extend R-linearly to get $S_n(X;R) = S_n(X) \otimes_{\mathbb{Z}} R : R\,\mathrm{Sin}_n(X) \to N$.

Notation. We shall write $S^n(X;N) = \mathrm{Set}(S_n(X), N) = \mathrm{Hom}_R(S_n(X;R), N)$. This RHS object is an R-module if N is an R-module.

This is a contravariant functor. Now recall that $d : S_{n+1}(X;R) \to S_n(X;R)$ was a differential. It induces a map $d : S^n(X;N) \to S^{n+1}(X;N)$ by $f \to df$ where

$$df(\sigma) = (-1)^{n+1} f(d\sigma)$$

for any $\sigma \in \mathrm{Sin}_{n+1}(X)$ and $d\sigma \in S_n(X;R)$. The factor $(-1)^{n+1}$ is important for later.

Proposition 18.4. $d^2 f = 0$ for all $f \in S^n(X;N)$.

Proof. $d^2 f(\sigma) = \pm d(f(d\sigma)) = \pm f(d^2\sigma) = \pm f(0) = 0$. $\square$

The proposition tells us that $S^n(X;N)$ is a **cochain complex** with differentials increasing by degree 1.

Definition 18.5. The nth **singular cohomology group** is the cohomology of this cochain complex i.e. it is the **singular cohomology group of X with coefficients in** N given as

$$H^n(X;N) = \ker(d : S^n(X,N) \to S^{n+1}(X;N)) / \operatorname{im}(d : S^{n-1}(X,N) \to S^n(X;N)).$$

So if N is an R-module, then $H^n(X;N)$ is an R-module.

Remark 21. Consider $H^0(X;N)$. The 0-chains are just functions $f : \mathrm{Sin}_0(X) \to N$ i.e. $f : X \to N$. Then $df \in \mathrm{Sin}^1(X)$ has

$$df(\sigma) = -f(d\sigma) = -f(\sigma(e_1) - \sigma(e_0) = f(\sigma(e_0)) - f(\sigma(e_1))$$

for $\sigma : \Delta^1 \to N$. Now f is a cocycle iff this is zero. Equivalently, this is iff f is constant on path components of X.

Lemma 18.6. One has $H^0(X;N) = \mathrm{Map}(\pi_0(X), N)$ where $\pi_0(X) = \{\text{path components of } X\}$.

Warning. $S^n(X;\mathbb{Z}) = \mathrm{Map}(\mathrm{Sin}_n(X);\mathbb{Z}) = \prod_{\mathrm{Sin}_n(X)} \mathbb{Z}$ is a direct product and **not** a direct sum. So this is **not** a free abelian group. This is very different from homology because $\bigoplus \mathbb{Z} \subsetneq \prod \mathbb{Z}$.

Notation. Let $S^v_{-n}(X;N) = S^n(X;N)$. Then the diagram

$$\begin{tikzcd} S^n(X;N) \arrow[r] \arrow[d] & S^{n+1}(X;N) \arrow[d] \\ S^v_{-n}(X;N) \arrow[r] & S^v_{-n-1}(X;N) \end{tikzcd}$$

In this way,

$$(S^v_*(X;N) \otimes S_*(X))_n = \bigoplus_{p+q=n} S^v_p(X;N) \otimes S_q(X).$$

Then an evaluation map $\langle -,- \rangle : S^v_* * X;N) \otimes S_*(X) \to N$ is defined by $(f,\sigma) \to \langle f,\sigma \rangle$. We would like this to be a chain map to N concentrated in degree zero. So

$$0 = d\langle f,\sigma \rangle = \langle df,\sigma \rangle + (-1)^{|f|}\langle f, d\sigma \rangle$$

gives

$$df(\sigma) = \langle df,\sigma \rangle = -(-1)^{|f|} f(d\sigma) = (-1)^{|f|+1} f(d\sigma).$$

This explains the sign and so we get a chain map

$$H_{-n}(S^v_*(X;N)) \otimes H_n(S_*(X)) \to H_0(S^v_*(X;N)) \otimes S_*(X) \to N.$$

We call this the **Kronecker pairing** $\langle -,- \rangle : H^n(X;N) \otimes H_n(X) \to N$. We can develop the properties of cohomology in analogy with properties of homology using this.

Let $A \subseteq X$. There exists a restriction map $S^n(X;N) \to S^n(A;N)$ induced by $\mathrm{Sin}_n(A) \hookrightarrow \mathrm{Sin}_n(X)$ and any function $\mathrm{Sin}_n(A) \to N$ extends to a function $\mathrm{Sin}_n(X) \to N$. Hence, $S^n(X;N) \to S^n(A;N)$ is surjective.

Definition 18.7. The **relative nth chain group with coefficient in** N is $S^n(X,A;N) = \ker(S^n(X;N) \to S^n(A;N))$. This defines a sub-chain complex of $S^*(X;N)$. We set

$$H^n(X,A;N) = H^n(S^*(X,A;N)).$$

So there exists a SES of cochain complexes

$$0 \to S^*(X,A;N) \to S^*(X;N) \to S^*(A;N) \to 0$$

(note that it is in the other direction compared to homology). However, the LES of homology is a covariant operation so it gives

$$\ldots \to H^n(X,A;N) \to H^n(X;N) \to H^n(A;N) \to H^{n+1}(X,A;N) \to \ldots.$$

19. Lecture 19: February 16th, 2022

Abstract. Last time, we had describe various algebraic structures that cohomology satisfies. We combined cohomology and homology to define the Kronecker pairing. Now, we import some results about homology to cohomology by beginning with the Universal Coefficient Theorem.

Last time, we had discussed coalgebras and cohomology. Let us briefly review what was covered.

Fix R a PID and $H_*(X;R)$ all free over R. Then $H_*(X;R)$ is a cocommutative graded algebra over R. We can take the dual of coalgebras to obtain algebras and of homology to get cohomology.

Let us define cohomology. Let $S^n(X;N)$ be the nth cochain group with coefficients in N. It is defined to be $\mathrm{Hom}_R(S_n(X;R),N)$. It has a coboundary map $d : S^n(X;N) \to$

$S^{n+1}(X;N)$ that maps $f \to df$ which acts as $df(\sigma) := (-1)^{|f|+1} f(d\sigma)$. This gives a cochain complex $S^*(X;N)$ and taking homology gives the chomology groups

$$H^*(X;N) = \frac{\ker d}{\operatorname{im} d}.$$

Now one can regrade to get a chain complex by $S^v_{-n}(X;N) := S^n(X;N)$. With chain complexes, we can take the tensor product with $S_*(X)$ to get $S^v_*(X;N) \otimes S_*(X)$. The evaluation map gives a chain map

$$\langle -, - \rangle : S^v_*(X;N) \otimes S_*(X) \to N$$

which sends $f \otimes \sigma \to \langle f, \sigma \rangle = f(\sigma)$.

This is a chain map due to the modification by a sign. Indeed (noting that we view N as being in degree zero),

$$0 = d\langle f, \sigma \rangle = \langle df, \sigma \rangle + (-1)^n \langle f, d\sigma \rangle$$

which forces

$$df(\sigma) = \langle df, \sigma \rangle = (-1)^{n+1} f(d\sigma)$$

which explains the sign we had from earlier.

Furthermore, we get the **Kronecker pairing**

$$\langle -, - \rangle : H^n(X;N) \otimes H_n(X) \to N$$

which is a composition

$$H_{-n}(S^v_*(X;N)) \otimes H_n(S_*(X)) \xrightarrow{\mu} H_0(S^v_*(X;N) \otimes S_*(X)) \to N.$$

Indeed, μ is just the identification of $H_{-n}(S^v_*(X;N)) \otimes H_n(S_*(X))$ with the LHS of the Kronecker pairing to get a mapping $H_0(S^v_*(X;N)) \otimes S_*(X)) \to N$.

For today, we study the Kronecker pairing some more. It has an adjoint property and it is adjoint to the map

$$\beta : H^n(X;N) \to \operatorname{Hom}_R(H_n(X), N).$$

We can view this map as an estimate of H^* in terms of H_* of X. In particular, we get the following theorem,

Theorem 19.1. Let R be a PID, N an R-module, and C_* a chain complex of free R-modules. Then there is a SES or R-modules

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(C_*), N) \to H^n(\operatorname{Hom}_R(C_*, N)) \to \operatorname{Hom}_R(H_n(C_*), N) \to 0$$

which is natural in C_* and N which splits nonfunctorially.

Example 19.2. Let $C_* := S_*(X;R)$. This gives the sequence

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(X);N) \to H^n(X;N) \xrightarrow{\beta} \operatorname{Hom}_R(H_n(X), N) \to 0.$$

Now what is Ext? Recall that $\operatorname{Hom}_R(M, -)$ is a left exact covariant functor while $\operatorname{Hom}_R(-, N)$ is a left exact contravariant functor. We shall define $\operatorname{Ext}^*_R(-, N)$ as the left derived functors of $\operatorname{Hom}_R(-, N)$ (and note that this requires one to use a projective resolution instead).

Definition 19.3. $\operatorname{Ext}^n_R(M, N) = H^n(\operatorname{Hom}_R(F_*, N))$ where $0 \leftarrow M \leftarrow F_*$ is a projective resolution of M. It is well-defined by the Fundamental Theorem of Homological Algebra.

Remark 22. If M is free or projective over R, then $\mathrm{Ext}^n_R(M,N)=0$ for all $n\geq 1$. If R is a PID, then $\mathrm{Ext}^n_R(M,N)=0$ for all $n\geq 2$. If R is a field, $\mathrm{Ext}^n_R(M,N)=0$ for all $n\geq 1$. Also, $\mathrm{Ext}^0_R(M,N)=\mathrm{Hom}_R(M,N)$.

Example 19.4. One easily computes from the two entry projective resolutions of $\mathbb{Z}/k$ that

$$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/k,N)=\ker(N\xrightarrow{\times k}N)\qquad\&\qquad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k,N)=\mathrm{coker}(N\xrightarrow{\times k}N)=N/kN.$$

One finds that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m,\mathbb{Z}/n)=\mathbb{Z}/\gcd(m,n)\mathbb{Z}$ and $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m,\mathbb{Z}/n)=\mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

Example 19.5. Let us compute $H^*(\mathbb{R}P^3,\mathbb{Z}/2)$. We know the homology of $\mathbb{R}P^3$ in $\mathbb{Z}$ very well. It is $\mathbb{Z},\mathbb{Z}/2,0,\mathbb{Z}$. Using the Universal Coefficient Theorem, we deduce that

$$H^n(\mathbb{R}P^3,\mathbb{Z}/2):=\begin{cases}\mathbb{Z}/2 & n=3\\ 0\oplus\mathbb{Z}/2 & n=2\\ \mathbb{Z}/2 & n=1\\ \mathbb{Z}/2 & n=0\end{cases}=H_n(\mathbb{R}P^3,\mathbb{Z}/2).$$

But observe that

$$0\to\mathrm{Ext}^1_{\mathbb{Z}/2}(H_{n-1}(\mathbb{R}P^3;\mathbb{Z}/2),\mathbb{Z}/2)\to H^n(\mathbb{R}P^3;\mathbb{Z}/2)\to\mathrm{Hom}_{\mathbb{Z}/2}(H_n(\mathbb{R}P^3;\mathbb{Z}/2),\mathbb{Z}/2)\to 0$$

has LHS equal to zero so we get an isomorphism for the second map. This is the cohomological analogue of the Künneth isomorphism.

Proof of UCT for H^*. The proof itself is not identical to the homological version. Some work is needed. An n-cocycle is a map $C_n\to N$ s.t. $C_{n+1}\xrightarrow{d}C\xrightarrow{f}N$ is zero i.e. $d(f)=f(d\sigma)=0$ i.e. it is zero on B_n. So,

$$Z^n(\mathrm{Hom}_R(C_*,N)\cong\mathrm{Hom}_R(C_n/B_n,N).$$

Then, $H_n(C_*)$ is a submodule of C_n/B_n which gives a SES

$$0\to H_n(C_*)\to C_n/B_n\to B_{n-1}\to 0.$$

Sine B_n is free, the sequence splits to give $C_n/B_n\cong H_n(C_*)\oplus B_{n-1}$. Apply $\mathrm{Hom}_R(-.N)$ to get (the top) SES (and we have a full diagram from our work above)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(B_{n-1},N) \arrow[r] & \mathrm{Hom}_R(C_n/B_n,N) \arrow[r] & \mathrm{Hom}_R(H_n(C_*),N) \arrow[r] & 0\\
0 \arrow[r] & B^n(\mathrm{Hom}_R(C_*,N)) \arrow[r] \arrow[u, "?"] & Z^n\,\mathrm{Hom}_R((C_*,N)) \arrow[r] \arrow[u, "\cong"] & H^n(\mathrm{Hom}_R(C_*,N)) \arrow[r] \arrow[u, "\beta"] & 0
\end{tikzcd}$$

For the nth coboundary, note that $C_n\to N$ factors through $C_n\xrightarrow{d}C_{n-1}\to N$. So the SES $0\to Z_{n-1}\to C_{n-1}\to B_{n-2}\to 0$ gives (as B_{n-2} is free) a factorization through $C_n\xrightarrow{d}Z_{n-1}\to N$. Extend $Z_{n-1}\to N$ to $C_{n-1}\to N$ to deduce that $B^n(\mathrm{Hom}_R(C_*,N))=\mathrm{Hom}_R(Z_{n-1},N)$. Apply the Snake Lemma to get

$$\begin{aligned}\ker\beta&\cong\mathrm{coker}(B^n(\mathrm{Hom}_R(C_*,N))\to\mathrm{Hom}_R(B_{n-1},N))\\&=\mathrm{coker}(\mathrm{Hom}_R(Z_{n-1},N)\to\mathrm{Hom}_R(B_{n-1},N))\\&=\mathrm{Ext}^1_R(H_{n-1}(C_*),N))\end{aligned}$$

which proves the theorem. ∎

Remark 23. The reason why Ext is named so is due to its relationships to extensions of M by N. An extension by M by N is an L s.t. $0 \to N \to L \to M \to 0$ is a SES. The general theory can be found in Weibel's "Introduction to Homological Algebra".

Remark 24. The Universal Coefficient Theorem an be used to transfer properties of homology to cohomology. IT satisfies the appropriate Eilenberg-Steenrod analogies.

It is homotopy invariant. It satisfies excision i.e. if $U \subseteq A \subseteq X$ s.t. $\overline{U} \subseteq \mathrm{Int}(A)$, then $H^*(X, A; N) \cong H^*(X - U, A - U; N)$. It satisfies the Milnor axiom i.e. $H^*(\coprod_\alpha X_\alpha; N) = \prod_\alpha H^*(X_\alpha; N)$. Consequently, it has a Mayer-Vietoris sequence i.e. if $A, B \subseteq X$ cover X, then there is a LES

$$\ldots \to H^{n-1}(A \cap B; N) \to H^n(X; N) \to H^n(A; N) \oplus H^n(B; N) \to H^n(A \cap B; N) \to H^{n+1}(X; N) \to \ldots$$

20. Lecture 20: February 18th, 2022

Abstract. We would like to import an analogue of the Künneth Theorem for the singular homology groups to cohomology. We construct the cohomology cross product via the Alexander-Whitney map and by using the affine extension maps. We also get more because we can give cohomology a ring structure via the cup product. It is a natural extension of the fact that the cohomology cross product exists and that the diagonal map goes in the "right" direction.

Recall from last time that we defined the Ext-functors $\mathrm{Ext}^n_R(-, N)$ as the derived functors of $\mathrm{Hom}_R(-, N)$. The definition was natural and well-defined. With the definition, we also saw some properties of $\mathrm{Ext}^n_R(-, N)$. Recall the Universal Coefficient Theorem for Cohomology.

Theorem 20.1. Let R be a PID and C_* be a chain complex. Let N be an R-module. Then the SES

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*), N) \to H^n(\mathrm{Hom}_R(C_*, N)) \to \mathrm{Hom}_R(H_n(C_*), N) \to 0$$

splits nonnaturally. Let $C_* := S_*(X; R)$. Then

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X; R), N) \to H^n(X; N) \to \mathrm{Hom}_R(H_n(X; R), N) \to 0.$$

Today we do products for cohomology. Then H_* has a coalgebra structure but no we can consider H^*'s algebra structure. Recall the coalgebra structure of H_* came from the cross product. Then

$$\times : S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y) \;\Rightarrow\; \times : H_p(X) \otimes H_q(Y) \to H_{p+q}(X \times Y).$$

The Künneth Theorem tells us that this is an isomorphism in nice cases i.e. if R is a PID and $H_p(X)$ is R-free. Another is R being a field. The diagonal map $\Delta : X \to X \times X$ induces a map. In nice cases, $H_*(X) \xrightarrow{\Delta_*} H_*(X \times X) \xleftarrow{\times}$. So $H_*(X)$ has a coproduct. In genera, the map is not natural and we would like to $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.

When the isomorphism exists, it is natural.

Recall the Eilenberg-Zilber Theorem: $S_*(X) \otimes S_*(Y) \xrightarrow{\times} S_*(X \times Y)$ gives a homotopy equivalence. The map in other direction is called Alexander-Whitney map. Call it α :

$S_{p+q}(X \times Y) \to S_p(X) \otimes S_q(Y)$. Now there is a diagram

$$\begin{tikzcd}
& & X \\
\Delta^{p+q} \arrow[urr, "\sigma_1"] \arrow[r, "\sigma"] \arrow[drr, "\sigma_2"'] & X \times Y \arrow[ur, "pr_1"'] \arrow[dr, "pr_2"] & \\
& & Y
\end{tikzcd}$$

that is commutative. It now suffices to make a map $\Delta^p \to \Delta^{p+q}$. Though naive, the following **front face** and **back face** maps work: $\alpha_p : [p] \to [p+q]$ defined by $i \to i$ for $0 \leq i \leq p$ and $\omega_q : [q] \to [p+q]$ defined by $j \to j+p$ for $0 \leq j \leq q$. These **affine extensions** of $\alpha_p : \Delta^p \to \Delta^{p+q}$ and $\omega_q : \Delta^q \to \Delta^{p+q}$ is

$$\alpha : S_{p+q}(X \times Y) \to S_p(X) \otimes S_q(Y)$$

defined by $\sigma \to (\sigma_1 \circ \alpha_p) \otimes (\sigma_2 \circ \beta_q)$.

Theorem 20.2. This is a chain map and works over any R.

We shall not prove this theorem. There is an algebraic analogue. If C_*, D_* are two chain complexes, then $\mathrm{Hom}_R(C_*, R)$ and $\mathrm{Hom}_R(D_*, R)$ are dual chain complexes. Now

$$\mathrm{Hom}_R(C_*, R) \otimes_R \mathrm{Hom}_R(D_*, R) \to \mathrm{Hom}_R(C_* \otimes D_*, R)$$

is defined by

$$f \otimes g \to \begin{cases} x \otimes y \to (-1)^{pq} f(x)g(y) & \text{if } |x| = |f| = p \text{ and } |y| = |g| = q \\ 0 & \text{if not} \end{cases}.$$

Theorem 20.3. This is a chain map. Now

$$\begin{aligned} S^p(X) \otimes S^q(Y) = \mathrm{Hom}_R(S_q(X), R) \otimes \mathrm{Hom}_R(S_q(Y), R) &\to \mathrm{Hom}_R(S_p(X) \otimes_R S_q(Y), R) \\ &\xrightarrow{AW} \mathrm{Hom}_R(S_{p+q}(X \times Y), R) = S^{p+q}(X \times Y) \end{aligned}$$

is a cochain map. So, we have defined a cochain level cross product. So it induces a map on cohomology

$$\times : H^*(X) \otimes H^*(Y) \xrightarrow{\mu} H^*(S^*(X) \otimes S^*(Y)) \to H^*(X \times Y).$$

We call this the **cohomology cross product**.

Definition 20.4. Set $X = Y$. Let $\Delta : X \to X \times X$ be the diagonal map. The preceding theorem gives the cup product

$$\smile : H^p(X) \otimes H^q(X) \xrightarrow{\times} H^{p+q}(X \times X) \xrightarrow{\Delta^*} H^{p+q}(X).$$

Remark 25. The identity for the cup product is the element $H^0(X)$ that takes value 1 on every path component $\pi_0(X) \to R$.

Theorem 20.5. The cup product is associative on the chain level.

PROOF. The diagonal map is associative. If we prove that the Alexander-Whitney cross product is also associative, we are done. This is the next theorem. ☐

Theorem 20.6. Let $f \in S^p(X), g \in S^q(Y), h \in S^r(Z)$, $\sigma : \Delta^{p+q+r} \to X \times Y \times Z$. Then

$$((f \times g) \times h)(\sigma) = (f \times (g \times h))(\sigma).$$

Proof. Let $\sigma_{12} := (X \times Y \times Z \to X \times Y) \circ \sigma$ and use similar notatin for σ_{ij}. Then

$$\begin{aligned}((f \times g) \times h)(\sigma) &= (-1)^{(p+q)r}(f \times g)(\sigma_{12} \circ \alpha_{p+q})h(\sigma_3 \circ \omega_r) \\ &= (-1)^{(p+q)r}(-1)^{pq} f(\sigma_1 \circ \alpha_p)g(\sigma_2 \circ \mu_q)h(\sigma_3 \circ \omega_r)\end{aligned}$$

where μ_q is the **middle face map**. Similarly,

$$(f \times (g \times h))(\sigma) = (-1)^{p(q+r)}(-1)^{qr} f(\sigma_1 \circ \alpha_p)g(\sigma_2 \circ \omega_q)h(\sigma_3 \circ \omega_r).$$

The powers of (-1) are the same. $\square$

The cross product on S^* gives an associative cross product on H^*. This gives a cup product that is associative on H^*. So,

$$(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$$

for all $\alpha, \beta, \gamma \in H^*(X; R)$.

21. Lecture 21: February 23rd, 2022

Abstract. We describe the algebraic structure the cohomology ring has. Furthermore, we show that that the cohomology cross product preserves said structure. We then use the cup product to compute the effects a continuous map induce on cohomology in the case of spheres.

Recall from last time that we had the Alexander-Whitney map and the cohomology cross product. We had also discussed the cup product and its associativity. Let us briefly review how this was done. The Eilenberg-Zilber map gave a chain homotopy map $S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$. Meanwhile, the Alexander-Whitney map gave a chain homotopy in the other direction which induces a chain homotopy equivalence of complexes.

By definition, the Alexander-Whitney map $\alpha : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ was defined by sending

$$\sigma : \Delta^{p+q} \to X \times Y \qquad \to \qquad (\sigma_1 \circ \alpha_p) \otimes (\sigma_2 \circ \omega_q)$$

for $\sigma_1 : \Delta^{p+q} \to X \times Y \to X$ and respectively $\sigma_2 : \Delta^{p+q} \to X \times Y \to Y$. The map α_p was the front face map while ω_q was the back face map.

To define the cohomological cross product on S^*, we had

$$\begin{aligned}S^p(X) \otimes S^q(Y) &= \mathrm{Hom}_R(S_p(X), R) \otimes \mathrm{Hom}_R(S_q(Y), R) \\ &\to \mathrm{Hom}_R(S_p(X) \otimes S_q(Y), R) \\ &\xrightarrow{\alpha} \mathrm{Hom}_R(S_{p+q}(X \times Y), R) = S^{p+q}(X \times Y).\end{aligned}$$

This composite gives a cochain map. The composite is called the cross product on cohomology

$$H^*(X) \otimes H^*(Y) \to H^*(S^*(X) \otimes S^*(Y)) \to H^*(X \times Y).$$

Moreover, the cup product is defined as the composite

$$\smile : H^p(X) \otimes H^q(X) \xrightarrow{\times} H^{p+q}(X \times X) \xrightarrow{\Delta^*} H^{p+q}(X).$$

We had proved last lecture that $\smile$ is associative on the cochain level (first show that the cross product is associative on H^* and then the cup product is associative on $H^*(X; R)$).

Definition 21.1. Fix a commutative ring R. A **graded R-algebra** is a graded R-module $\{A_n\}_{n\in\mathbb{Z}}$ equipped with a map $A_p \otimes_R A_q \to A_{p+q}$ and a map $\eta : R \to A_0$ s.t. the following diagrams commute

$$\begin{tikzcd} A_p \otimes_R R \arrow[r, "1\otimes\eta"] \arrow[dr, "="'] & A_p \otimes_R A_0 \arrow[d] \\ & A_p \end{tikzcd} \qquad \begin{tikzcd} A_0 \otimes_R A_p \arrow[d] & R \otimes_R A_p \arrow[l, "\eta\otimes 1"'] \arrow[dl, "="] \\ A_p & \end{tikzcd}$$

$$\begin{tikzcd} A_p \otimes_R A_q \otimes_R A_r \arrow[r] \arrow[d] & A_p \otimes_R A_{q+r} \arrow[d] \\ A_{p+q} \otimes A_r \arrow[r] & A_{p+q+r} \end{tikzcd}$$

A graded R-algebra A is commutative if for $\tau(x \otimes y) = (-1)^{pq}(y \otimes x)$, the following diagram commutes

$$\begin{tikzcd} A_p \otimes A_q \arrow[r, "\tau"] \arrow[dr] & A_q \otimes A_p \arrow[d] \\ & A_{p+q} \end{tikzcd}$$

Theorem 21.2. $H^*(X; R)$ is a commutative graded R-algebra with product given by the cup product $\smile$.

Proof. It is clearly a graded R-algebra. Also, $H^0(X; R) \cong \mathrm{Map}(\pi_0(X); R)$ as R-algebras (the maps are defined by ending r to the constant r-valued function).

The commutativity is nontrivial! We have the following lemma.

Lemma 21.3. The diagram

$$\begin{tikzcd} S_*(X \times Y) \arrow[r, "T_*"] \arrow[d, "\alpha"] & S_*(Y \times X) \arrow[d, "\alpha"] \\ S_*(X) \otimes S_*(Y) \arrow[r, "\tau"] & S_*(Y) \otimes S_*(X) \end{tikzcd}$$

commutes and here, $T : X \times Y \to Y \times X$ is the map the obvious homeomorphism.

Combining this fact with dualization, we get a commutative graded R-algebra.

We conclude that if $\alpha \in H^p(X)$ and $\beta \in H^q(X)$, we get $\alpha \smile \beta = (-1)^{pq}\beta \smile \alpha$. $\square$

Remark 26. In some ways, this makes cohomology better than homology. We get to put them all together and have structure!

A natural question one may ask is whether there exists a cochain complex modeling $H^*(X)$ that has a commutative product. The answer is yes when $R = \mathbb{Q}$ and this is due to Sullivan. When R is a field of characteristic $\neq 0$, then it isn't always possible. There are obstructions called Steenrod operations.

Let A, B be graded R-algebras. Let $A \otimes_R B$ be given a natural algebra structure with multiplication

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb'.$$

Next, observe that if A, B are commutative, then $A \otimes_R B$ is commutative.

Proposition 21.4. The cohomology cross product $\times : H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ is an R-algebra homomorphism.

PROOF. This follows by definition. Give the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\Delta_{X\times Y}"] \arrow[d, "\Delta_X \times \Delta_Y"] & (X \times Y) \times (X \times Y) \\ (X \times X) \times (Y \times Y) \arrow[ur, "1\times T\times 1"'] & \end{tikzcd}$$

Let $\alpha_1, \alpha_2 \in H^*(X)$ and $\beta_1, \beta_2 \in H^*(Y)$. Then $\alpha_1, \beta_1, \alpha_2 \times \beta_2 \in H^*(X \times Y)$ is what we want to show. We have

$$\begin{aligned} (\alpha_1 \times \beta_1) \smile (\alpha_2 \times \beta_2) &= \Delta^*_{X\times Y}(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2) \\ &= (\Delta_X \times \Delta_Y)^*(1 \times T \times 1)^*(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2) \\ &= (-1)^{|\beta_1||\alpha_2|}(\Delta_X \times \Delta_Y)^*(\alpha_1 \times \alpha_2 \times \beta_1 \times \beta_2) \\ &= (_1)^{|\beta_1||\alpha_2|}(\alpha_1 \smile \alpha_2) \times (\beta_1 \smile \beta_2). \end{aligned}$$

The last equality follows from naturality of $\smile$ i.e. the following diagram commutes

$$\begin{tikzcd} H^*(X) \otimes H^*(Y) \arrow[r, "\times"] & H^*(X \times Y) \\ H^*(X \times X) \otimes H^*(Y \times Y) \arrow[r, "\times"] \arrow[u, "\Delta_X^* \otimes \Delta_Y^*"] & H^*(X \times X \times Y \times Y) \arrow[u, "(\Delta_X \times \Delta_Y)^*"'] \end{tikzcd}$$

□

Example 21.5. We compute some example.

$H^*(S^n) = \text{Exterior}(x_n)$ is the exterior algebra generated by x_n and $|x_n| = n$ and with $x_n \smile x_n = 0 \in H^{2n}(S^n) = 0$.

By the Künneth theorem, $H^*(S^n \times S^m) = H^*(S^n) \otimes H^*(S^m) = \text{Exterior}(x_n, x_m)$ is the exterior algebra generated by $|x_n| = n, |x_m| = m$.

So we see that

$n+m$	$\mathbb{Z}\{x_n \smile x_m\}$
m	$\mathbb{Z}\{x_m\}$
n	$\mathbb{Z}\{x_n\}$
0	$\mathbb{Z}\{1\}$

where $x_m \smile x_n = (-1)^{mn} x_n \smile x_m, x_n \smile x_n = 0, x_m \smile x_m = 0$.

Corollary 21.5.1. Any continuous map $f : S^{m+n} \to S^m \times S^n$ induces the zero map on H^{m+n}.

PROOF. $f^* : H^*(S^n \times S^m) \to H^*(S^{m+n})$ is a graded algebra map. So that means

$$0 \neq x_n \smile x_m \mapsto f^*(x_n) \smile f^*(x_m) = 0 \smile 0 = 0.$$

because for degree reasons, $f^*(x_n) = f^*(x_m) = 0$. □

Corollary 21.5.2. $S^n \times S^m \not\simeq S^n \vee S^m \vee S^{m+n}$ (the reason one might suspect they were homotopic is because they have one cell in each of dimensions $0, n, m, n+m$).

PROOF. They have the same values for H_*, H^*, but they do not have the same H^* algebra structure. □

22. Lecture 22: February 25th, 2022

ABSTRACT. We state Poincaré duality which asserts that the structure of a manifold forces a symmetry on the cohomology groups. Everything is done in the case of $\mathbb{F}_2$-coefficients but there exists a more general version of the duality theorem for other coefficients. The duality theorem also provides enough structure to compute the cohomology of ring of groups like K and $\mathbb{R}P^n$. We can also construct the intersection form from combining duality with the cup product.

With the duality theorem and the intersection form, we provide an algebraic classification of closed surfaces.

Last time, we had talked about the Eilenberg-Zilber Theorem. From it, there is the **Eilenberg-Zilber map**

$$S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$$

which is a chain homotopy equivalence. A candidate for a map in the other direction was the **Alexander-Whitney map**. Let $\times : S^*(X) \times S^*(Y) \to S^*(X \times Y)$ be the dual of the Alexander-Whitney map. This induces a cross product map which we also denote by $\times$

$$\times : H^*(X) \otimes H^*(Y) \to H^*(S^*(X) \otimes S^*(Y)) \xrightarrow{\times} H^*(X \times Y)$$

If $X = Y$, there is a diagonal map $\Delta : X \to X \times X$ and this induces a map on cohomology. Defined the **cup product** to be the composite

$$\smile: H^p(X) \otimes H^q(Y) \xrightarrow{\times} H^{p+q}(X \times X) \xrightarrow{\Delta^*} H^{p+q}(X).$$

Proposition 22.1. The cohomology ring $H^(X; R)$ for R a commutative rig with unit is actually a commutative graded R-algebra under $\smile$.

We saw some examples last time. In particular,

Example 22.2. $H^*(S^n) = E(x_n)$ was the exterior algebra on x_n i.e. $x_n^2 = 0$.

$H^*(S^n \times S^m) = E(x_n, x_m)$ was the exterior algebra with $x_n \smile x_m = (-1)^{nm} x_m \smile x_n$ and $x_n^2 = x_m^2 = 0$. If we had worked over $\mathbb{F}_2$, the sign would be irrelevant.

For today, we *fix* $R := \mathbb{F}_2$ *as the base ring.* So by abuse of notation, we shall omit the ring in our notations and assume the reader knows to work over $\mathbb{F}_2$ instead of $\mathbb{Z}$.

Fix M a compact connected n-manifold. We shall show over the course of the next few lectures that $H_n(M; \mathbb{F}_2) = \mathbb{F}_2$ and M is orientable over $\mathbb{F}_2$ (made precise later). The fact that $H_n(M; \mathbb{F}_2) = \mathbb{F}_2$ can be assumed for now, however.

Theorem 22.3 (Poincaré Duality). There exists a unique class $[M] \in H_n(M; \mathbb{F}_2) \cong \mathbb{F}_2$ (the nonzero class) called the **fundamental class** of M. It has the property that for all $p \geq 0$,

$$H^p(M) \otimes H^{n-p}(M) \xrightarrow{\smile} H^n(M) \xrightarrow{\langle -, [M] \rangle, \cong} \mathbb{F}_2$$

the composite is a perfect pairing (and the isomorphism for the second map makes sense because $H^n(M) \cong \mathbb{F}_2$ by the universal coefficient theorem). To be a perfect pairing means the adjoint map

$$H^p(M) \to \operatorname{Hom}(H^{n-p}(M), \mathbb{F}_2)$$

is an isomorphism. In particular,

$$H^p(M) \xrightarrow{\cong} \operatorname{Hom}(H^{n-p}(M), \mathbb{F}_2) \xleftarrow{\cong} H_{n-p}(M).$$

The isomorphism $H^p(M) \cong H_{n-p}(M)$ is called the **Poincaré dual**.

In particular,

$$\dim H^p(M) = \dim H_{n-p}(M) = \dim H^{n-p}(M) = \dim H_p(M).$$

Definition 22.4. Using the theorem, we define $\pitchfork$: $H_p(M) \otimes H_q(M) \to H_{p+q-n}(M)$ as the composite

$$\begin{tikzcd} H_p(M) \otimes H_q(M) \arrow[r, "\pitchfork"] \arrow[d, "\cong"] & H_{p+q-n}(M) \arrow[d, "\cong"] \\ H^{n-p}(M) \otimes H^{n-q}(M) \arrow[r, "\smile"] & H^{2n-p-q} \end{tikzcd}$$

making the diagram commute. When $p+q=n$, it is a pairing since $H_{p+q-n}(M) = \mathbb{F}_2$.

Remark 27. The viewpoint here is that each $\alpha \in H_p(M)$ corresponds to a submanifold of dimension p. So when α intersects $\beta \in H_q(M)$ in a transverse manner, $\alpha \pitchfork \beta$ if of dimension $p+q-n$.

Definition 22.5. Let $\alpha.\beta = \langle a \smile b, [M]\rangle$ in the duality with $\alpha, \beta \in H_k(M)$ and $a, b \in H^k(M)$ are their Poincaré dual. This is a nondegenerate symmetric bilinear form on $H^k(M)$ called the **intersection form**.

Example 22.6. In terms of the basis α, β, the intersection form of T^2 has matrix

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

called a **hyperbolic form**.

Recall from basic linear algebra that if there is a nondegenerate bilinear form on a finite dimensional vector space V, and $W \subseteq V$ is a subspace, then the restriction of the form to W is nondegenerate iff $W \cap W^\perp = 0$. Also in this case, there is a splitting $V \cong W \oplus W^\perp$ respecting the forms.

Proposition 22.7. A finite dimensional symmetric bilinear form o er $\mathbb{F}_2$ splits as a direct sum of matrices of forms

$$I := \begin{pmatrix} 1 \end{pmatrix} \qquad \& \qquad H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Also, there is a correspondence,

$$\mathrm{Bil} := \left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of nondegenerate symmetric} \\ \text{bilinear forms over} \\ \mathbb{F}_2 \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{similarity classes} \\ \text{of nonsingular} \\ \text{symmetric matrices} \end{array} \right\}$$

Recall that M is similar to N iff $N = AMA^T$ for some A (this is the definition we take over $\mathbb{F}_2$).

Lemma 22.8. $\begin{pmatrix} & 1 & \\ 1 & & \\ & 1 & \end{pmatrix} \sim \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$

Proof. Take $A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & & 1 \\ & 1 & 1 \end{pmatrix}$. $\square$

So the monoid Bil is generated by I and H with relation $I \oplus H = 3I$. Returning to topology, we see that

$$T^2 \leftrightarrow H \qquad \& \qquad \mathbb{R}P^2 \leftrightarrow I.$$

Recall the connected sum of two surfaces $\Sigma_1 \# \Sigma_2$.

Proposition 22.9. $H_1(\Sigma_1 \# \Sigma_2) = H_1(\Sigma_1) \oplus H_1(\Sigma_2)$.

Proof. Apply the Mayer-Vietoris sequence. $\square$

It follows that the intersection form corresponds to the direct sum of matrices.

Define

$$\mathrm{Surf} := \left\{ \begin{array}{c} \text{homeomorphism} \\ \text{classes of} \\ \text{compact connected} \\ \text{surfaces} \end{array} \right\}$$

which is a monoid with # as its operation. At last, we get the following theorem.

Theorem 22.10 (Classification of Surfaces)**.** Surf $\xrightarrow{\text{intersection form}}$ Bil is an isomorphism of commutative monoids.

For the orientable surfaces $\#_1^n T^2$ corresponds to nH and for nonorientable surfaces, $\#_1^m \mathbb{R}P^2$ corresponds to mI.

Remark 28. Note that $2I \not\sim H$ so that means $\mathbb{R}P^2 \# \mathbb{R}P^2 \ncong T^2$. Indeed, $\mathbb{R}P^2 \# \mathbb{R}P^2$ is a Klein bottle.

23. Lecture 23: February 28th, 2022

Abstract. We can use the structure of a manifold to define orientability using local homology. We explain how the structure of the coefficient ring can affect orientability w.r.t. the ring. All of this is done via the orientable double cover and whether or not it has sections.

Last time that we discuss intersection forms over $\mathbb{F}_2$. We had a form of **Poincaré duality**.

Theorem 23.1. Fix M a connected compact manifold of dimension n. There exists a class $[M] \in H_n(M; \mathbb{F}_2) = \mathbb{F}_2$ that was unique s.t. the bilinear map

$$H^p(M; \mathbb{F}_2) \otimes H^{n-p}(M; \mathbb{F}_2) \overset{\smile}{\to} H^n(M; \mathbb{F}_2) \overset{\langle -, [M] \rangle}{\to} \mathbb{F}_2$$

is a perfect pairing.

Corollary 23.1.1. For M above, $\dim H^p(M; \mathbb{F}_2) = \dim H_p(M; \mathbb{F}_2) = \dim H^{n-p}(M; \mathbb{F}_2) = \dim H^p(M; \mathbb{F}_p)$.

Definition 23.2. If we dualize the cup product, we get the intersection form

$$\pitchfork : H_p(M; \mathbb{F}_2) \otimes H_q(M; \mathbb{F}_2) \to H_{p+q-n}(M; \mathbb{F}_2).$$

Example 23.3. The intersection form for $\mathbb{R}P^2$ is given by $I = (1)$ and for T^2, it is $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = H$.

Theorem 23.4. There is a correspondence that preserves $\oplus$ between Bil, the isomorphism classes of finite dimensional nondegenerate symmetric bilinear forms over $\mathbb{F}_2$, and Sym, the similarity classes of nonsingular symmetric matrices over $\mathbb{F}_2$.

Let Surf be the category of homeomorphism classes of connected closed surfaces. It is a monoid under # connected sums.

Theorem 23.5. Surf $\to$ Bil is an isomorphism of commutative monoids where the map is given by taking the intersection form over $\mathbb{F}_2$.

Corollary 23.5.1. $\#_1^n T^2 = nH$ and $\#_1^m \mathbb{R}P^2 = mI$.

For today, we shall discussion orientations on manifolds. Fix M an n-dimensional manifold (not necessarily compact).

Definition 23.6. The **local homology** of M at a point x is defined as

$$H_*(M, M-x; \mathbb{Z}).$$

Lemma 23.7. $H_k(M, M-x;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=n \\ 0 & \text{if not} \end{cases}$

Proof. Choose an open neighborhood U of x s.t. $U \cong \operatorname{Int} D^n$. By excision,

$$H_k(M, M-x;\mathbb{Z}) \cong H_k(U, U-x;\mathbb{Z}).$$

Now the LES on homology and $U - x \simeq S^{n-1}$ gives

$$H_k(U, U-x;\mathbb{Z}) \xrightarrow{\cong} \widetilde{H_{k-1}}(U-x;\mathbb{Z})$$

and the RHS is what we wanted. The result also holds with a commutative ring with unity R in place of $\mathbb{Z}$. ☐

Definition 23.8. An R-orientation of M at x is an R-module generator of $H_n(M, M-x; R)$.

Example 23.9. In the case of a surface, $H_2(M, M-x;\mathbb{Z}) \cong \mathbb{Z}$ and so there are two possible generators $+1, -1$. This corresponds respectively to counter clockwise orientation and clockwise orientation.

The idea of an **R-orientation on** M should be a collection of R-orientations at each point that are locally compatible.

Definition 23.10. Let $O_M := \coprod_{x\in M} H_n(M, M-x;\mathbb{Z})$. Then $O_M \otimes_{\mathbb{Z}} R = \coprod_{x\in M} H_n(M, M-x; R)$.

There exists a natural map

$$O_M \otimes_{\mathbb{Z}} R \to M \qquad H_n(M, M-x;\mathbb{Z}) \mapsto x$$

and this is an example of a fibre bundle (defined later on).

The topology on O_M is defined as follows.

For any open set $U \subseteq M$ and $x \in U$, consider $j_{U,x} : H_n(X, X-\overline{U}; R) \to H_n(X, X-x)$ which is induced by inclusions. For any $\alpha \in H_n(X, X-\overline{U}; R)$ and $V_{U,\alpha} := \{j_{\overline{U},x}(\alpha)\}_{x\in U} \subseteq O_M$, we declare $V_{U,\alpha}$ to be open.

Lemma 23.11. $\{V_{U,\alpha} : U \text{ open}, \alpha \in H_n(M, M-\overline{U})\}$ forms a basis of a topology on O_M which makes o_M a covering space of U.

Proof. Literally by definitions, the condition is saying that we have made the preimage of open sets over U open and it is clear that the preimage of a disjoint union of spaces homeomorphic to U. ☐

Proof. Let V_{U_i,α_i} be in the claimed base. Let W be the union over all $i \in I$. Then we get

$$W = \bigcup_{i\in I} V_{U_i,\alpha_i} = \bigcup_{i\in I}\{j_{U_i,x}(\alpha_i) : x \in U_i\}.$$

□

Proof. We just prove the second statement. For $U \subseteq M$ and $U \subseteq \operatorname{Int} D^n$,

$$p^{-1}(U) = \bigcup_{x\in U} H_n(M, M-x) = \bigcup_{x\in U} H_n(U, U-x) = \bigcup_{x\in U} \mathbb{Z} = U \times \mathbb{Z}.$$

This bijection is a homeomorphism under the topology defined. □

Definition 23.12. Let $(O_M \otimes R)^\times := \bigcup\{\text{generators of } H_n(M, M-x; R)\} \subseteq O_M \otimes R$.

Consider the $R^\times$-fold covering space $(O_M \otimes R)^\times \to M$.

In the case of $R = \mathbb{Z}$, $(O_M)^\times \to M$ is a 2-fold covering space. This is called the **orientable double cover**.

Definition 23.13. A **section** of the covering space $p : (O_M \otimes_{\mathbb{Z}} R)^\times \to M$ is a continuous map $s : M \to (O_M \otimes R)^\times$ s.t. $p \circ s = \mathrm{id}_M$ aka for all $x \in M$, we can choose a generator $s(x)$ of its local homology (i.e. an R-orientation of x) in a continuous way.

Definition 23.14. Let $\Gamma(M, (O_M \otimes R)^\times) = \{\text{sections of } p : (O_M \otimes R)^\times \to M\}$.

Definition 23.15. An **R-orientation on** M is an element of $\Gamma(M, (O_M \otimes R))$. If the set is nonempty, we say M is R-orientable. We shall say M is orientable if it is $\mathbb{Z}$-orientable.

Proposition 23.16.
(1) All manifolds M are $\mathbb{F}_2$-orientable.
(2) If $\operatorname{char} R = 2$, then all manifolds are R-orientable.
(3) If $\operatorname{char} R \neq 2$, then M is R-orientable iff it is $\mathbb{Z}$-orientable.
(4) $H^1(M; \mathbb{F}_2) = 0$ implies M is orientable.
(5) $\pi_1(M) = 0$ implies M is orientable.

Proof. For (3), see Miller's notes or Hatcher's textbook. The proof of (1) and (2) are trivial and (5) follows from (4).

To prove (4), assume that M is connected. If M is not orientable, $O_M^\times$ is a nontrivial 2-fold cover. By covering space theory, it corresponds to an index 2 subgroup $H \subseteq \pi_1(M)$. So there is a nontrivial map $\pi_1(M) \to \pi_1(M)/H \cong \mathbb{Z}/2$. So, $\operatorname{Hom}(\pi_1 M, \mathbb{Z}/2) \neq 0$. Hence, $\operatorname{Hom}(H_1(M), \mathbb{Z}/2) = 0$ which means $H^1(M; \mathbb{Z}/2) \neq 0$. Contradiction. □

24. Lecture 24: March 2nd, 2022

Abstract. In this lecture, we work toward stating the Orientation Theorem. It essentially says that for finite dimensional compact manifolds, the orientatability of the manifold is determined by the top homology group.

Recall from last time that we introduced local orientations. Given a n-manifold M,

$$H_n(M, M-x; \mathbb{Z}) \cong \mathbb{Z}$$

for any $x \in M$ due to excision. Then set

$$O_M := \bigcup_{x\in M} H_n(M, M-x; \mathbb{Z}) \xrightarrow{\pi} M$$

to get a $\mathbb{Z}$-fold covering space (provided we give it an appropriate topology). This has a subcovering space

$$O_M^\times = \bigcup_{x\in M} \{\text{generators of } H_n(M, M-x;\mathbb{Z})\} = \bigcup_{x\in M} \mathbb{Z}^\times = \bigcup_{x\in M} \{\pm 1\} \xrightarrow{p} M.$$

This is called the orientable double cover. It is always a 2-fold cover.

Example 24.1. The orientable double cover of $\mathbb{R}P^2$ is S^2.

The orientable double cover of K is T^2.

We can consider the situation with R-coefficients

$$O_M \otimes R = \bigcup_{x\in M} H_n(M, M-x; R) = \bigcup_{x\in M} R.$$

Then $(M\otimes R)^\times = \bigcup_{x\in M}\{R\text{-module generators of } H_n(M, M-x; R)\}$.

Definition 24.2. An **R-orientation** on M is a section of the covering map

$$p : (O_M \otimes R)^\times \to M$$

i.e. it is an $s : M \to (O_M\otimes R)^\times$ s.t. $p\circ s = \mathrm{id}_M$.

The essential idea is one should be choosing generators continuously! Such sections may or may not exist.

Remark 29. In general, we say that M is R-orientable if M has an R-orientation. We omit R when referring to $\mathbb{Z}$-orientations. When M is R-orientable, there are always 2 orientations.

Example 24.3. Obviously, $\mathbb{R}^2$ and S^1 are orientable. However, the Möbius band is not.

Remark 30. We shall take as a fact some results. All $\mathbb{F}_2$-manifolds are orientable. The same is true for $\operatorname{char} R = 2$. Also, M is orientable iff $O_M \cong \mathbb{Z}\times M$ iff $O_M^\times \cong \mathbb{Z}/2\times M$.

Recall from covering space theory that if $b \in X$ and X is path connected, then $\pi_1(X, b)$ acts on $p^{-1}(b)$ as follows.

Given $[\gamma] \in \pi_1(X, b)$ and $a \in p^{-1}(b)$. By the unique path lifting property, lift $\widetilde{\gamma}$ to a path that starts at a and ends at possibly some other point a' in the fibre $p^{-1}(b)$. Set $a' = [\gamma]\cdot a$.

Theorem 24.4 (Fact)**.** Recall that

$\{\text{actions of } \pi_1(X, b) \text{ on } S\}/\text{conjugation} \leftrightarrow \{\text{covering spaces over } X \text{ with fibres } S\}/\text{isomorphisms}.$

Now a $\mathbb{Z}$-fold covering space corresponds to $\mathrm{Hom}(\pi_1(X), \mathrm{Aut}(\mathbb{Z}))$ bijectively. It also preserves the group structure so we get an isomorphism between $\mathbb{Z}$-fold covering spaces and $\mathrm{Hom}(\pi_1(X), \mathbb{Z}/2)$.

So we get an action of $\pi_1(M)$ (assuming here M is connected) on $H_n(M, M-x;\mathbb{Z}) \cong \mathbb{Z}$. For each $[\gamma] \in \pi_1(M)$, it acts either as $[\gamma](n) = n$ or $[\gamma](n) = -n$ for all $n \in \mathbb{Z}$.

The former is called **orientation preserving** and the latter is **orientation reversing**. Now M is orientable iff all loops are orientation preserving.

Theorem 24.5. If $\operatorname{char} R \neq 2$, then M is R-orientable iff M is $\mathbb{Z}$-orientable.

Proof. Study the action of $\pi_1 M$ on $R^\times$. $\square$

Theorem 24.6. There is an element $\omega_1 \in H^1(M;\mathbb{F}_2)$ s.t. M is orientable iff $\omega_1 = 0$. We call ω_1 the 1**st Stiefel-Whitney class**.

Now

$$O_M \in \{\mathbb{Z} - \text{fold covering spaces of } M\}.$$

The set of $\mathbb{Z}$-fold covering spaces of M corresponds to the $\pi_1(M)$-actions on $\mathbb{Z}$. Then this corresponds to $\mathrm{Hom}(\pi_1(M), \mathrm{Aut}(\mathbb{Z}) = \mathbb{Z}/2)$ which is equal to $\mathrm{Hom}(H_1(M), \mathbb{Z}/2)$. By UCT, this is equal to $H^1(M; \mathbb{Z}/2)$. The nonzero element of $H^1(M; \mathbb{Z}/2)$ is called the Stiefel-Whitney class and its vanishing implies M is orientable. Then $H^1(M; \mathbb{Z}/2) = 0$ implies M is orientable. So $\pi_1(M) = 0$ implies M is orientable.

We set up the statement of the Orientation Theorem. First, let $\Gamma(M, O_M \otimes R)$ be the sections of the covering map $p : O_M \otimes R \to M$. There is a map $j : H_n(M; R) \to \Gamma(M, O_M \otimes R)$ defined by sending α to the map $x \mapsto j_x(\alpha)$ where

$$j_x : H_n(M; R) \to H_n(M, M - x; R)$$

is the map induced from the inclusion $(M, \emptyset) \to (M, M - x)$.

Theorem 24.7 (**Orientation Theorem**). If M is compact, then there is an isomorphism (for $n = \dim M$)

$$j : H_n(M; R) \to \Gamma(M, O_M \otimes R).$$

Corollary 24.7.1. There is an isomorphism

$$H_n(M, R) \cong \Gamma(M, O_M \otimes R) = \begin{cases} R & M \text{ is orientable} \\ \ker(R \overset{\times 2}{\to} R) & M \text{ is not orientable} \end{cases}.$$

Example 24.8.

$$H_n(M, \mathbb{Z}) = \begin{cases} \mathbb{Z} & M \text{ is orientable} \\ 0 & M \text{ is not orientable} \end{cases}. \qquad \& \qquad H_n(M, \mathbb{Z}/4) = \begin{cases} \mathbb{Z}/4 & M \text{ is orientable} \\ \mathbb{Z}/2 & M \text{ is not orientable} \end{cases}.$$

Sketch of Proof. See Miller's notes for a full proof.

If $A \subseteq M$, then it defines a map

$$j_A : H_n(M, M - A; R) \to \Gamma(A, O_M \otimes R)$$

where elements on the RHS are sections $s : A \to O_M \otimes R$. The map is induced my the inclusion $(M, M - A) \hookrightarrow (M, M - x)$.

A refinement for the orientation is as follows. For general M and compact A (where M is not necessarily compact), the map j_A is an isomorphism and $H - q(M, M - A; R) = 0$ for $q > n = \dim M$. Call these statements (*).

A few lemmas are needed to build up the proof.

Lemma 24.9. If (*) holds for $A, B, A \cap B$ with compact $A, B \subseteq M$, then (*) holds for $A \cup B$.

Proof. Look at the diagram

$$\begin{tikzcd} & (M, M - A) \arrow[dr] & \\ (M, M - (A \cup B)) \arrow[ur] \arrow[dr] & & (M, M - A \cap B) \\ & (M, M - B) \arrow[ur] & \end{tikzcd}$$

Use the Mayer-Vietoris sequence and the 5-lemma to get (*) for $A \cup B$. ☐

Lemma 24.10. If $A_1 \supseteq A_2 \supseteq \dots$ and (*) holds for all A_i, then (*) holds for $\bigcap A_i$.

Proof. There is an isomorphism

$$\varinjlim H_n(M, M - A_i) = H_n(M, M - \cap A_i) = H_n(M, \cup(M - A_i)).$$

□

The goal is to prove (*) for $A \subseteq M$ compact.

The proof follows from five steps.

(i) $M = \mathbb{R}^n$, A convex, then

$$R \cong H_n(M, M - A; R) \to \Gamma(A, O_M \otimes R) \cong R$$

is clearly true.

(ii) $M = \mathbb{R}^n$ and A a finite union of convex sets. Apply the first lemma here.

(iii) $M = \mathbb{R}^n$, A a general compact set. Define $A_i := \bigcup_{x \in B} B_x(\frac{1}{i})$ a union of closed balls. If A is compact, this is a finite union.

(iv) M a general manifold and $A \subseteq \mathbb{R}^n$. Use excision

$$H_n(M, M - A; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - A; R).$$

(v) M a general manifold and A a general compact subset. Then A is a union of finitely many subsets $A_i \subseteq \mathbb{R}^n$.

□

25. Lecture 25: March 4th, 2022

Abstract. Today, we move away from orientations and discuss products in more detail. in particular, we describe the various functorial properties that products like the cup product, cross products, and Kronecker product satisfy. We also construct the cap product and extend it to the relative case. With it, we show that the homology groups $H_*(X)$ form a module over the cohomology ring $H^*(X)$.

Last time, we sketched a proof of the Orientation Theorem. There are modifications for the definition of orientability in the case of M being a smooth manifold. Indeed, the theorem goes wrong when M is not compact (for instance, it would say that $\mathbb{R}^n$ is not orientable for $n > 1$ due to $\mathbb{R}^n$ being contractible).

For today, we shall discuss products in general. Recall the Eilenberg-Zilber Theorem gave a map $\times : S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$ with the Alexander-Whitney map $AW : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ s.t. they induced a chain homotopy equivalence.

From the cross product map, we defined the homological cross product

$$\times : H_*(X) \otimes H_*(Y) \to H_*(S_*(X) \otimes S_*(Y)) \overset{(\times)_*}{\to} H_*(X \times Y)$$

and the cohomological cross product

$$\times : H^*(X) \otimes H^*(Y) \to H^*(S^*(X) \otimes S^*(Y)) \overset{(AW)^*}{\to} H_*(X \times Y).$$

Then we defined the cup product

$$\smile : H^*(X) \otimes H^*(X) \overset{\times}{\to} H^*(X \times X) \overset{\Delta^*}{\to} H^*(X).$$

The cup product made H^* into a commutative ring (in the graded sense). This is considerably more useful than just having cohomology groups.

Next, we want the **Kronecker pairing**. Define $S^v_{-n}(X) := S^n(X)$. Then the evaluation map $S^v_*(X) \otimes S_*(X) \to R$ induces the Kronecker pairing $\langle -, - \rangle : H^n(X) \otimes H_n(X) \to R$.

Clearly, $\langle -, - \rangle$ cannot be natural in the usual sense since it is contravariant in one coordinate and covariant in another.

Theorem 25.1. If $f : X \to Y$, $b \in H^n(Y), x \in H_n(X)$, then $\langle f^*b, x \rangle = \langle b, f_*x \rangle$.

PROOF. This is true on the chain level. So it is true on the homology/cohomology levels.

Say b is represented by $\beta \in S^n(Y)$ and x by $c \in S_n(X)$. Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}(S_n(Y), R) \otimes S_n(X) \arrow[r, "1\otimes f_*"] \arrow[d, "f^*\otimes 1"'] & \mathrm{Hom}(S_n(Y), R) \otimes S_n(Y) \arrow[d, "\langle -,- \rangle"] \\
\mathrm{Hom}(S_n(X), R) \otimes S_n(X) \arrow[r, "\langle -,- \rangle"] & R
\end{tikzcd}$$

and check that it commutes. □

What about naturality of the homological and cohomological cross product? We get it w.r.t. the **Kronecker pairing**.

Lemma 25.2. Let $a \in H^p(X), b \in H^q(Y), x \in H_p(X), y \in H_q(Y)$. Then

$$\langle a \times b, x \times y \rangle = (-1)^{|x||b|} \langle a, x \rangle \langle b, y \rangle.$$

We omit the proof of the lemma since it is easy to do.

Recall that the Künneth Theorem gave a splitting of a SES. Indeed, if R is a PID, then there is a noncanonical splitting of the SES

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \xrightarrow{\times} H_n(X \times Y) \to \bigoplus_{p'+q'=n-1} \mathrm{Tor}_1^R(H_{p'}(X), H_{q'}(Y)) \to 0.$$

In part, if $H_p(X)$ is f.g. and free over R, the **Künneth isomorphism** occurs i.e. $\times : H_*(X) \otimes H_*(Y) \xrightarrow{\cong} H_*(X \times Y)$ is an isomorphism. There is a similar statement for cohomoology.

Theorem 25.3. Let R be a PID. Let $H_p(X)$ be f.g. and free over R. Then

$$\times : H^*(X) \otimes H^*(Y) \xrightarrow{\cong} H^*(X \times Y).$$

What about (X, A) and relative homology? Of courses, there is a chain homotopy equivalence

$$S_*(X, A) \otimes S_*(Y) \leftrightarrow S_*(X \times Y, A \times Y)$$

and this induces the relative cross products

$$\times : H_*(X, A) \otimes H_*(Y) \to H_*(X \times Y, A \times Y) \qquad \& \qquad \times : H^*(X, A) \otimes H^*(Y) \to H^*(X \times Y, A \times Y).$$

There is a relative diagonal map $\Delta(X, A) \to (X \times X, X \times A)$ defined by $(x, a) \to ((x, x), (x, a))$. This induces the relative cup product

$$\smile : H^*(X) \otimes H^*(X, A) \xrightarrow{\times} H^*(X \times X, X \times A) \xrightarrow{\Delta^*} H^*(X, A)$$

and this implies that $H^*(X, A)$ *is a module over* $H^*(X)$.

Let's discuss cup products. The Alexander-Whitney map $AW : S_{p+q}(X \times Y) \to S_p(X) \otimes S_q(Y)$ was defined by $\sigma \mapsto \sigma_1 \circ \alpha_p \otimes \sigma_2 \circ \omega_q$ where $\alpha_p : \Delta^p \to \Delta^{p+q}$ and $\omega_q : \Delta^q \to \Delta^{p+q}$ are the front face and back face maps.

We can define the **cap product** as the composite

$$\cap : S^p(X) \otimes S_n(X) \overset{1\otimes(AW\circ\Delta_*)}{\to} S^p(X) \otimes S_p(X) \otimes S_{n-p}(X) \overset{\langle -,-\rangle\otimes 1}{\to} S_{n-p}(X).$$

The map on elements is

$$\beta \otimes \sigma \to \beta \otimes \sigma \circ \alpha_p \otimes \sigma \circ \omega_q \to \beta(\sigma \circ \alpha_p)(\sigma \circ \omega_q).$$

This is an evaluation map of some sense on the co-chain with parts of the chain. This is actually a chain map so it induces a map in H_*

$$\cap : H^p(X) \otimes H_n(X) \to H_{n-p}(X).$$

Theorem 25.4. The cap product satisfies the following properties:

(1) $(a \smile b) \cap x = a \cap (b \cap x)$;
(2) $1 \cap x = x$;
(3) $H_*(X)$ is a module over $H^*(X)$;
(4) if $f : X \to Y$, $b \in H^p(Y)$, $x \in H_n(X)$, then

$$f_*(f^*(b) \cap x) = b \cap f_*(x).$$

Proof. We will not prove each part of the theorem. The first two parts are easy to see.

Let β be a cocycle representative of b and $\sigma : \Delta^n \to X$. Then

$$\begin{aligned} f_*(f^*(b) \cap x) &= f_*(f^*(b)(\sigma \circ \alpha_p)(\sigma \circ \omega_q)) \\ &= f_*(\beta \circ f \circ \sigma \circ \alpha_p \cdot \sigma \circ \omega_q) \\ &= \beta(f \circ \sigma \circ \alpha_p) f_*(\sigma \circ \omega_p) = \beta \cap f_*(\sigma). \end{aligned}$$

OTOH, if $A \subseteq X$ and $i : A \hookrightarrow X$, we get a diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
S^p(X) \otimes S_n(A) \arrow[r, "i_*"] \arrow[d] & S^p(A) \otimes S_n(A) \arrow[r, "\cap"] & S_{n-p}(A) \arrow[d] \\
S^p(X) \otimes S_n(X) \arrow[rr, "\cap"] \arrow[d] & & S_{n-p}(X) \arrow[d] \\
S^p(X) \otimes S_n(X, A) \arrow[rr, dotted, "\cap"] \arrow[d] & & S_{n-p}(X, A) \arrow[d] \\
0 & & 0
\end{tikzcd}$$

and the existence of the dotted line comes from the exactness of the columns and the commutativity of the diagram. Exactness of the LHS diagram follows from freeness of $S^p(X)$ or the splitting $S_n(X) \cong S_n(A) \oplus S_n(X, A)$ depending on if X satisfies certain finiteness conditions.

In any case, we get a relative cap product

$$\cap : H^p(X) \otimes H_n(X, A) \to H_{n-p}(X, A)$$

which induces an $H^*(X)$-module structure on $H_*(X, A)$. $\square$

26. Lecture 26: March 7th, 2022

ABSTRACT. We state and sketch a proof of the Poincaré duality theorem. This requires construction the Čech cohomology groups. We also provide an example where Čech cohomology does not agree with singular cohomology.

Recall that we discussed the cap product last time. Let $AW : S_{p+q}(X \times Y) \to S_p(X) \otimes S_q(Y)$ be defined by the map

$$\sigma \to (\sigma_1 \circ \alpha_p) \otimes (\sigma_2 \circ \omega_q)$$

where $\sigma_1 : \Delta^{p+q} \to X \times Y \to X$ and $\sigma_2 : \Delta^{p+q} \to X \times Y \to Y$. The maps α_p and ω_q were the affine face maps.

We define a cap product $\cap : S^p(X) \otimes S_n(X) \to S^p(X) \otimes S_p(X) \otimes S_{n-p}(X) \to S_{n-p}(X)$ by the composition

$$\beta \otimes \sigma \to \beta \otimes \sigma_1 \circ \alpha_p \otimes \sigma_2 \circ \omega_{n-p} \to \beta(\sigma \circ \alpha_p) \circ (\sigma_2 \circ \omega_q)$$

which is (in a sense) "evaluation of the cochain on a part of the chain". Checking that this is a chain map, we get an induced map on homology

$$\cap : H^p(X) \otimes H_n(X) \to H_{n-p}(X).$$

Under this cap product, $H_*(X)$ is a module over $H^*(X)$.

Recall that $H^*(X)$ is an algebra under the cup product. Some facts are that $H^*(X, A)$ is a module over $H^*(X)$ via the relative cup product.

Today, we state and sketch a proof of Poincaré duality.

Let $K \subseteq U \subseteq X$ for K a closed subset of X and U an open subset of X. By excision,

$$H_*(U, U-K) \overset{\cong}{\to} H_*(X, X-K)$$

and these two objects are modules over different rings. The LHS is a module over $H^*(U)$ while the RHS is a module over $H^*(X)$. So naturally, we may consider all open sets containing K. That is, $H_*(X, X-K)$ admits an action of $H^*(U)$ and they are compatible.

Lemma 26.1. Let $K \subseteq V \subseteq U \subseteq X$ with the additional assumption that V is open in X. Then there ia commutative diagram

$$\begin{tikzcd}
H^p(U) \otimes H_n(X, X-K) \arrow[r] \arrow[d, "i^* \otimes 1"] & H_{n-p}(X, X-K) \\
H^p(V) \otimes H_n(X, X-K) \arrow[ur] &
\end{tikzcd}$$

We omit the proof of the lemma..

Definition 26.2. The **Čech cohomology** of K is defined as

$$\check{H}^p(K) = \varinjlim_{U \in \mathcal{U}_K} H^p(U)$$

where $\mathcal{U}_K$ is the set of open subsets containing K.

As tensor products commute with direct limits, we get a cap product

$$\cap : \check{H}^p(K) \otimes H_n(X, X-K) \to H_{n-p}(X, K)$$

and so $H_*(X, X-K)$ is a module over $\check{H}^*(K)$. As $K \subseteq U$, we get a map $H^*(U) \to H^*(K)$. This induces a morphism $\phi : \check{H}^*(K) \to H^*(K)$.

When is this map ϕ an isomorphism?

Lemma 26.3. If for all $U \supseteq K$ open neighborhood, there is an open set V s.t. $K \subseteq V \subseteq U$ s.t. $K \hookrightarrow U$ is a homotopy equivalence, then $\check{H}^*(K) \to H^*(K)$ is an isomorphism.

Proof. This is clear from definition. $\square$

Example 26.4. Consider $L := \sin(1/x) \cup (\{0\} \times I) \subseteq \mathbb{R}^2$ which has two path components. Attach a curve from $(1, \sin(1))$ to $(0,0)$ that does not touch K. By the MV-sequence

$$H^*(X) \cong H^*(\text{point}) \qquad \& \qquad \check{H}^*(X) \cong H^*(S^1)$$

which shows that the isomorphism is not true in general.

The goal is to now define Poinecaré duality. Let M be an n-manifold and $K \subseteq M$ a compact subspace. Let $[M]_K \in H_n(M, M-K; R)$ a fundamental class which we call the fundamental class along K (it comes from an orientation along K). Then there is an isomorphism

$$- \cap [M]_K : \check{H}^p(K; R) \xrightarrow{\cong} H_q(M, M-K; R).$$

In particular, take $K = M$ when M is compact to get the statement of Poincaré duality.

Theorem 26.5. If M is a compact orientable n-manifold, there is an isomorphism

$$- \cap [M] : H^p(M; R) \xrightarrow{\cong} H_{n-p}(M; R).$$

The rest of this lecture will discuss the general situation where $K \subseteq M$ is compact. Recall that if $x \in M$ and $x \in U$ is an open subset in M s.t. $U \cong \mathbb{R}^n$, then $H_k(M, M-x) \cong H_k(U, U-x) \xrightarrow{\partial} \widetilde{H_{k-1}}(U-x) \cong \widetilde{H_{k-1}}(S^{n-1})$. Set $O_M := \bigcup_{x\in M} H_n(M, M-x)$ and $p : o_M \to M$ a $\mathbb{Z}$-fold covering. Let R be a commutative ring.

A local R-orientation of x is a choice $[M]_x$ of an R-module generator $H_n(M, M_x; R) \cong R$. An R-orientation of M is an $[M] \in H_n(M; R)$ which is a generator and it restricts to $[M]-x$ for all $x \in M$. That is,m

$$H_n(M; R) \to H_n(M, M-x; R) \qquad [M] \to [M]_x.$$

An R-orientation $[M]_K$ along K, where K is a compact subspace, is an R-module generator of $H_n(M, M-K; R)$ that restricts to $[M]_x$ for all $x \in K$ i.e.

$$H_n(M, M-K; R) \to H_n(M, M-x; R) \qquad [M]_K \to [M]_x.$$

The relative cap product gives a map $\cap : \check{H}^p(K; R) \otimes H_n(M, M-K; R) \to H_{n-p}(M, M-K; R)$. When there exists an R-orientation along K, there exists an $[M]_K$ which gives a map

$$- \cap [M]_K : \check{H}^p(K; R) \to H_{n-p}(M, M-K; R).$$

Theorem 26.6. The map $- \cap [M]_K$ is an isomorphism.

Proof Sketch. See Miller's textbook for a full proof.

For now, let $M = \mathbb{R}^n$ and K be a compact convex set. Then K being convex means that there exists a neighborhood U that is also contractible. So $\check{H}^p(K) \cong H^p(K)$. Now consider the diagram

$$\begin{tikzcd}
H^p(K) \arrow[r, "\cap[\mathbb{R}^n]_K"] & H_{n-p}(\mathbb{R}^n, \mathbb{R}^n - K) \\
H^p(U) \arrow[u, "\cong"] \arrow[r, "\cap[\mathbb{R}^n]_K"] & H_{n-p}(\mathbb{R}^n, \mathbb{R}^n - U) \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
H^p(U) \arrow[u, "\cong"] \arrow[r, "\cap[\mathbb{R}^n]_*"] & H_{n-p}(\mathbb{R}^n, \mathbb{R}^n - \{*\})
\end{tikzcd}$$

and the isomorphism on the bottom horizontal gives the isomorphism on top.

Now consider the case $M = \mathbb{R}^n$ and K is a finite union of convex subsets in $\mathbb{R}^n$. The proof then follows by induction, the MV-sequence, and an application of the 5-lemma.

For the other cases, we obtain the proof via a limit process and excision. □

Next lecture, we shall give some applications of Poincaré duality.

27. Lecture 27: March 9th, 2022

Abstract. Today, we apply the Poincaré duality theorem to prove the Jordan Curve Theorem, the Borsku-Ulam Theorem, and compute the cohomology rings of $\mathbb{R}P^n$ and $\mathbb{C}P^n$. We also state the Alexander Duality Theorem.

Recall that we proved the following version of Poinecaré duality.

Theorem 27.1. Let M be an n-dimensional manifold and $K \subseteq M$ a compact subspace. If there is a fundamental class $[M]_K \in H_n(M, M-K; R)$ along K (i.e. we assume that an R-orientation along K exists), then there is an isomorphism

$$- \cap [M]_K : \check{H}^p(K; R) \to H_{n-p}(M, M-k; R).$$

Corollary 27.1.1. If M is compact R-orientable n-manifold, then

$$\cap[M] : H^p(M; R) \to H_{n-p}(M; R)$$

is an isomorphism.

We discuss the applications of these two results. First, there is a form of duality one can deduce.

Corollary 27.1.2 (Alexander Duality)**.** Set $M = \mathbb{R}$,$K \subseteq M$ a compact subset. Then the composite

$$\check{H}^p(K; R) \overset{\cap[\mathbb{R}^n]_K}{\to} H_{n-p}(\mathbb{R}^n, \mathbb{R}^n - K; R) \to \widetilde{H_{n-p-1}}(\mathbb{R}^n - K; R)$$

is an isomorphism (the boundary map ∂ is an isomorphism from the MV-sequence).

One application is to the Jordan-Curve Theorem.

Theorem 27.2 (Jordan-Curve Theorem)**.** Let K be a Jordan curve i.e. a closed plane curve with no self-intersections. Then $\mathbb{R}^2 - K$ has two path components.

Proof. $\mathbb{Z} \cong H^1(S^1) \cong \widetilde{H_0}(\mathbb{R}^2 - K)$ which means $H_0(\mathbb{R}^2 - K) \cong \mathbb{Z}^2$ so there are two path components. □

Example 27.3. Let $n = 3$ and $K \cong S^1$. Then K is a knot in $\mathbb{R}^3$. Then

$$\widetilde{H}_*(\mathbb{R}^K) \cong \widetilde{H}_*(\mathbb{R}^3 - S^1) \cong \begin{cases} \mathbb{Z} & * = 1, 2 \\ 0 & \text{if not} \end{cases}.$$

So homology cannot distinguish knots. The study of objects like K is called knot theory.

Example 27.4. Let $p := n - 1$ and so $\check{H}^{n-1}(K) \cong \widetilde{H}_0(\mathbb{R}^n - K)$ is an isomorphism if K embeds in $\mathbb{R}^n$. Take $n = 3$ and $K = \mathbb{R}P^2$. Then $H^2(\mathbb{R}P^2) = \mathbb{Z}/2$ implies $\mathbb{R}P^2$ cannot be embedded in $\mathbb{R}^3$ with a regular neighborhood.

Remark 31. Any nonorientable manifold in dimension n cannot be embedded in $\mathbb{R}^{n+1}$ with a regular neighborhood.

$\mathbb{R}P^2$ can be immersed in $\mathbb{R}^3$ (this is called the **Boy surface**).

Theorem 27.5. Let R be a PID and M a compact R-orientable n-manifold. Then

$$\frac{H^p(M;R)}{\text{torsion}} \otimes \frac{H^{n-p}(M;R)}{\text{torsion}} \to R$$

defined by $a \otimes b \to \langle a \cup b, [M] \rangle = \langle a, b \cap [M] \rangle$ is a perfect pairing.

Proof. Use the Universal Coefficient Theorem and Poincaré duality. $\square$

Example 27.6. We know $\mathbb{C}P^2$ has one cell in dimensions $0, 2, 4$. So the 2-skeleton of $\mathbb{C}P^2$ is S^2 and hence, $\pi_1 \mathbb{C}P^2 = \pi_1 S^2 = 0$. Therefore $\mathbb{C}P^2$ is orientable. Now

$$H^*(\mathbb{C}P^2) = \begin{cases} \mathbb{Z} & * = 0, 2, 4 \\ 0 & \text{if not} \end{cases}.$$

We have a perfect pairing

$$H^2(\mathbb{C}P^2) \otimes H^2(\mathbb{C}P^2) \to H^4(\mathbb{C}P^2) \xrightarrow{\cap[\mathbb{C}P^2]} \mathbb{Z}$$

and if a is a generator of H^2, we deduce that a^2 is a generator of $H^4(\mathbb{C}P^2)$. Hence, $H^*(\mathbb{C}P^2) \cong \mathbb{Z}[a]/a^3$ and $|a| = 2$.

Example 27.7. For $\mathbb{C}P^3$, the inclusion

$$\mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3$$

induces a map

$$H^*(\mathbb{C}P^3) \to H^*(\mathbb{C}P^2)$$

which is an isomorphism in $* = 4$. So $H^4(\mathbb{C}P^3)$ is generated by a^2. The perfect pairing says that H^6 is generated by a^3 as well. So

$$H^*(\mathbb{C}P^3) \cong \mathbb{Z}[a]/a^4 \qquad |a| = 2.$$

In general, $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[a]/a^{n+1}$ for $|a| = 2$ and $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[a]$ with $|a| = 2$.

One can also show that

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2[w]/w^{n+1} \qquad |w| = 1.$$

Theorem 27.8 (Borsuk-Ulam Theorem)**.** Let $f : S^n \to \mathbb{R}^n$ be continuous. Then there exists a pair $x \in S^n$ and $-x \in S^n$ s.t. $f(x) = f(-x)$.

PROOF. Suppose not and define $g : S^n \to S^{n-1}$ by $g(x) = \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$ which is well-defined and continuous. Then observe that $g(-x) = -g(x)$. Therefore, $\overline{g} : \mathbb{R}P^n \to \mathbb{R}P^{n-1}$ via

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d, "p"] & S^{n-1} \arrow[d, "p"] \\ \mathbb{R}P^n \arrow[r, "\overline{g}"'] & \mathbb{R}P^{n-1} \end{tikzcd}$$

the diagram (here, the p's are the double cover map). We claim that $\overline{g}$ induces a map

$$\overline{g}_* : H_1(\mathbb{R}P^n; \mathbb{Z}/2) \to H_1(\mathbb{R}P^{n-1}; \mathbb{Z}/2)$$

that is nontrivial.

Let $b \in S^n$ and let $\sigma : [0,1] \to S^n$ be the map $\sigma(0) = b$ and $\sigma(1) = -b$. Then $p\sigma$ is a generator of $H_1(\mathbb{R}P^n)$. Then $pg\sigma : [0,1] \to S^{n-1} \to \mathbb{R}P^{n-1}$ is a generator.

Therefore, $\overline{g}_*$ is nontrivial in dimension 1.

Then $\overline{g}^*$ on $H^1(-;\mathbb{Z}/2)$ is a nontrivial map $H^*(\mathbb{R}P^{n-1};\mathbb{Z}/2) \to H^*(\mathbb{R}P^n;\mathbb{Z}/2)$. But that means it sends w to w. However, $w^n = 0$ while $w^{n+1} \neq 0$ in the target. This is absurd. □

28. Lecture 28: March 11th, 2022

ABSTRACT. This is the last lecture of this course. We provide a rough outline of the materials covered and list out examples that were computed in homework and in class.

Today is a review session and the last lecture with Professor Zhouli Xu teaching the sequence.

- We considered the axioms of singular homology, simplicial homology, and cellular homology;
  - they are functors $h_n : \text{Top} \to \text{Ab}$ for $n \in \mathbb{Z}$,
  - there are natural transformations $\partial : h_n(X, A) \to h_{n-1}(A, \emptyset)$,
  - they are homotopy invariant invariant,
  - excision (which is equivalent to the MV sequence which is equivalent to $(X - U, A - U) \hookrightarrow (X, A)$ for $\overline{U} \subseteq A$ inducing isomorphisms on h_*)
  - long exact sequence exists

    $$\ldots \to h_{q+1}(X) \overset{\partial}{\to} h_q(A) \to h_q(X) \to h_q(X, A) \overset{\partial}{\to} \ldots$$

  - dimension axiom i.e. $h_n(*) = 0$ iff $n = 0$.
- Consider $\mathbb{C}P^n$. Then $\mathbb{C}P^n$ has one $2m$-cell for $2m \in [0, 2n]$. Then

  $$0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \cdots \leftarrow \mathbb{Z} \leftarrow 0$$

  where the $\mathbb{Z}$ are in dimensions $0, 2, 4, \ldots, 2n$. It follows that

  $$H_k(\mathbb{C}P^n) = \begin{cases}\mathbb{Z} & k \text{ is even } k \in [0,n] 0 & \text{if not.}\end{cases}$$

  Now $\mathbb{R}P^n$ has $\text{sk}_k(\mathbb{R}P^n)\mathbb{R}P^k$ for $k \leq n$. We have a chain complex

  $$0 \leftarrow \mathbb{Z} \overset{0}{\leftarrow} \mathbb{Z} \overset{2}{\leftarrow} \mathbb{Z} \overset{0}{\leftarrow} \cdots \leftarrow \mathbb{Z} \leftarrow 0.$$

Then we

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z} & \text{if } k=n,\ n \text{ odd} \\ \mathbb{Z}/2 & \text{if } k \text{ odd},\ 1 \le k \le n-1 \\ 0 & \text{if not} \end{cases}.$$

Meanwhile,

$$H_k(\mathbb{R}P^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 \le k \le n \\ 0 & \text{if not} \end{cases}.$$

- We also discussed free resolutions $F_* \to M \to 0$ and proved the Fundamental Theorem of Homological Algebra. Using this, we defined the Tor and Ext functors. A useful computational fact is

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(m,n)\mathbb{Z} \qquad \& \qquad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(m,n)\mathbb{Z}.$$

- We had the Universal Coefficient Theorems: let R be a PID and M an R-module and then there are nonfunctorial split SESs

$$0 \to H_n(X;R)\otimes M \to H_n(X,M) \to \mathrm{Tor}_1^R(H_{n-1}(X;R),M) \to 0$$
$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R),M) \to H^n(X;M) \to \mathrm{Hom}_k(H_n(X;R),M) \to 0.$$

- The Künneth Theorem gives a split SES

$$0 \to \bigoplus_{p+q=n} H_p(X)\otimes H_q(Y) \to H_n(X\times Y) \to \bigoplus_{p'+q'=n-1} \mathrm{Tor}_1^R(H_{p'}(X),H_{q'}(Y)) \to 0.$$

- We can construct cross products using the EZ map and AW maps. Indeed, EZ: $S_*(X)\otimes S_*(Y) \to S_*(X\times Y)$ while AW goes the other way. This gives cross product

$$\times : H_*(X)\otimes H_*(Y) \to H_*(X\times Y).$$

Then

$$\times : H^*(X)\otimes H^*(X) \to H^*(X\times X) \xrightarrow{\Delta^*} H^*(X).$$

Applying the Künneth Theorem, we see that $H^*(X;R)$ is an R-algebra. This can be used to distinguish $S^n\times S^m$ and $S^n \vee S^m \vee S^{n+m}$. Also $\mathbb{C}P^2 \not\cong S^2\vee S^1$.
- The cap product makes $H_*(X)$ into a module over $H^*(X)$. The relative cup product makes $H_*(X,A)$ into a module over $H^*(X)$. The relative cup product makes $H^*(X,A)$ into a module over $H^*(X)$.
- The Poincaré Duality Theorem states: if M is a compact R-orientable n-manifold, then $-\cap[M] : H^p(M;R) \xrightarrow{\cong} H_{n-p}(M;R)$. This can be used to show

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\omega]/(\omega^{n+1}) \qquad |\omega| = 1$$

and

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[c]/(c^{n+1}) \qquad |c| = 2.$$

CHAPTER 3

Math 290C - UCSD

ABSTRACT. This course continues from Math 290B and was taught at UCSD by Justin Roberts. The lectures appear to be based on Hayne's Miller's notes "Lectures on Algebraic Topology".

1. Lecture 1: March 2nd, 2022

The goal of this course is to cover basic homotopy theory (equivalent of Chapter 4 in Hatcher's textbook). Some major concepts and theorems we shall see are the homotopy groups, Whitehead's Theorem, and Hurewicz's Theorem. The course shall also have problem sets and the final shall be the qualifying exam.

Definition 1.1. $\pi_i S^n$ is the "group" (we show later that this is a group) of maps $(S^i, \text{pt}) \to (S^n, \text{pt})$.

Now what *do* we know about such groups? Some things we know are there is a surjective $\pi_n S^n \to \mathbb{Z}$, $\pi_1 S^n = \mathbb{Z}$, $\pi_1 S^n = 0$ for $n > 1$, there are maps of arbitrary degree $d \in \mathbb{Z}$ of $S^n \to S^n$, that $\pi_3 S^2$ has a nontrivial map $S^3 \to S^2$ called the **Hopf fibration**given by $(X, Y) \to [X : Y]$ if $S^3 \subseteq \mathbb{C}^2$ and $S^2 \cong \mathbb{C}P^1$, and $\pi_i S^n = 1$ for $i < n$ (proven later using the Cellular Approximation Theorem).

Theorem 1.2 (**Cellular Approximation Theorem**)**.** Any map $X \to Y$ of CW-complexes are homotopic to a **cellular map** i.e. a map $f : X \to Y$ s.t. $f(X^{(n)}) \subseteq Y^{(n)}$ (note that $X^{(n)}$ denotes the n-skeleton).

Proposition 1.3. $\pi_3 S^2 \neq 1$.

PROOF. Let $\mathbb{C}P^2 = e^0 \cup e^2 \cup e^4$ be the CW-decomposition. The attaching map for e^4 is given by $\partial S^4 = S^3 \to S^2$ called the Hopf fibration.

Suppose the map were homotopic to the trivial map. Then $\mathbb{C}P^2 \simeq S^2 \vee S^4$. This is absurd by the following argument.

The cohomology ring of $\mathbb{C}P^2$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma$ where $|\beta| = 2$ and $|\gamma| = 4$ and $\beta^2 = \gamma$. Meanwhile, $S^2 \vee S^4$ has $\mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\eta$ where $|\alpha| = 2$ and $|\eta| = 4$ with $\alpha^2 = 0$. These ring structures cannot be isomorphic. Hence, the map could not be homotopic to the trivial map. □

In general, the homotopy groups of spheres are mysterious. There exists the following result.

Theorem 1.4. $\pi_i(S^2) \neq 0$ for $i \geq 2$.

It is know that there is a stabilization i.e. $\pi_{i+k}(S^{n+k})$ becomes constant as $k \to \infty$ and the dependence is only on $n - i$. We denote the limit by π^s_{i-n} which is called the **stable homotopy group of spheres**.

For instance, the stable homotopy groups for $1 \leq i - n \leq 11$ are respectively

$$\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_{24}, 0, 0, \mathbb{Z}_2, \mathbb{Z}_{140}, \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_6, \mathbb{Z}_{504}.$$

The study of these stable homotopy groups have deep connections to number theory (talk to Zhouli if you are curious).

We discuss some categorical aspects of homotopy theory. Given topological spaces X, Y, we may for the product $X \times Y$ and coproduct $X \coprod Y$. In general, if $\mathrm{Map}(X, Y)$ denotes the space of continuous maps,

$$\mathrm{Map}(Z, X \times Y) = \mathrm{Map}(Z, X) \times \mathrm{Map}(Z, Y)$$

via a bijection. We would like to topologize the space of continuous maps. One way to do it is via the **compact-open topology** i.e. we take as a subbase the collection of sets $\mathscr{B}(X, Y) = \{f : K \to U : K \text{ compact } \& \ U \text{ open}\}$. Problems arise because we would like to have the bijection

$$\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \to \mathrm{Map}(X \times Y, Z)$$

to be a homeomorphism. For instance, for homotopy theory, we would want $\mathrm{Map}(I, \mathrm{Map}(Y, Z)) = \mathrm{Map}(I \times Y, Z)$. But this fails! Fortunately, a fix exists...

2. Lecture 2: March 30th, 2022

$\mathrm{Map}(X, Y)$ is the set of continuous $X \to Y$ which we'd like topologize this and get homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

But this doesn't work in with the compact-open topology. However, this works if X Hausdorff, Y locally compact and Hausdorff, and Z arbitrary.

The solution is to redefine the category of topological spaces and work with "compactly generated spaces".

Definition 2.1.
(a) $F \subseteq X$ is **compactly closed** if the preimage $f^{-1}(F)$ is closed for all $f : K \to X$ with K a compact Hausdorff space.
(b) X is **compactly generated** if being closed is equivalent compact closed in X.

Given any space X, there exists a construction of compactly generated space kX given by *redefining* the topology on X i.e. take the closed sets as precisely the compactly closed.

Example 2.2. LCH is compactly generated i.e. $kX = X$.

Fact: kTop can defined a modified mapping space $\mathrm{kMap}(X, Y)$ s.t.

$$\mathrm{kMap}(X \times Y, Z) \cong \mathrm{kMap}(X, \mathrm{kMap}(Y, Z)).$$

The ideas pertaining to these notions were not developed until quite recently. In the 30s-50s there were many theories circulating and each cohomology and homology theory varied in different ways. Eventually the idea became that we should form a sufficiently large category that contains aspects of the cohomology and homology theories we want.

One example of this is with the **fundamental groupoid** πX which is a category whose objects are points in the topological space X and the collection of maps $\mathrm{Map}(x_0, y_0)$ is the collection of homotopy classes of maps.

Some remarks are warranted.

Remark 32. We can always include Top into Top* via the functor $X \to X_+$ where $X_+ := X \coprod \{*\}$ which assigns the base point $*$.

Remark 33. Adjunction is altered in the case of based topological spaces. In this case,

$$\mathrm{Map}(X \wedge Y, Z) = \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

where $X \wedge Y := (X \times Y)/(\{x_0\} \times Y \cup X \times \{y_0\})$.

Remark 34. $X \wedge Y$ not Cartesian product $(X \times Y, \{*\} \times \{**\})$.

Example 2.3. $S^1 \wedge X$ is the **reduced suspension** denoted $\Sigma_{\mathrm{red}}(X, \{*\})$ and

$$\Sigma_{\mathrm{unred}} = \frac{X \times I}{X \times \{0\}, X \times \{1\}}$$

while

$$\Sigma_{\mathrm{red}} X = \frac{\Sigma_{\mathrm{unred}} X}{\{*\} \times I}.$$

Remark 35. There are some conditions in which the **reduced suspension** is the same as the usual **suspension**.

Example 2.4. $(S^0, +1) = X_0$ where $+1 \in \{+1, -1\} = S^0$. Then

$$\Sigma_{\mathrm{red}} X_0 = (S^1, (0, 1)).$$

The n-fold reduced suspension is S^n i.e. $\Sigma^n_{\mathrm{red}} X_0 \cong S^n$ or $S^1 \wedge \cdots \wedge S^1 \cong S^n$ with n-copies of S^1.

Let us begin talking about homotopy groups. The upshot is that we can think about them in various manners. By definition, $[X, Y]$ is the set of based homotopy classes of maps $X \to Y$. Also, we have

$$\pi_0 \mathrm{Map}(X, Y) = [S^0_{+1}, \mathrm{Map}(X, Y)]$$

where S^0_{+1} denotes the based topological space $(S^0, +1)$. In particular,

$$\pi_n(X_{\{**\}}) = [S^n_{\{*\}}, X_{\{**\}} = [\Sigma^n S^0_{\{*\}}, X_{\{*\}}].$$

Definition 2.5. $\Omega X_{\{*\}} = \mathrm{Map}(S^1_{\{*\}}, X)$ which is nice because for all X, Y

$$[\Sigma X, Y] = [X, \Omega Y] = \{f : X \times I \to Y : f(X \times \{0\}) = f(X \times \{1\}) = f(\{*\} \times I) = \{**\}\} \,/\, \mathrm{homotopy}\,.$$

The moral is that $\pi^n(X_{\{*\}}) = [S^0, \Omega^n X_{\{*\}}]$ (using the above results n times). Ultimately, one this of the homotopy group as

$$\pi^n X_{\{*\}} = \pi_0 \mathrm{Map}((I^n, \partial I^n), (X, \{*\})).$$

,

3. Lecture 3: April 1st, 2022

When thinking about $\pi_n(X, x_0)$, there were two mental pictures that people use. The first is viewing its elements as homotopy classes of maps

$$f : (S^n, *) \to (X, x_0)$$

while the other is as homotopy classes of maps

$$f : (I^n, \partial I^n) \to (X, x_0).$$

Additionally, one looked at $[\Sigma^n X, Y] = [X, \Omega^n Y]$. The first method of thinking is nice in some respects since f of the first type is nullhomotopic iff it extends to a map $F : B^{n+1} \to X$ where B^{n+1} is the $(n+1)$-ball.

Product Structures on $\pi_n(X, x_0)$

This is easiest with the picture of $(I^n, \partial I^n)$. For the case $n = 1$, we had the usual concatenation of loops. For $n = 0$, there is really no product structure. The proof for $n \geq 2$ is quite pleasant and in the case of $n \geq 2$, it is clearly abelian. See p. 340 of Hatcher's Algebraic Topology.

In some sense, any other similar attempt to define a product gives the same answer. More conecptually, let $Y = (\Omega X, *)$ be a loop space of any space X. It is called an "H-space" or a "group object in the homotopy category" or a "group up to homotopy" if $Y = \Omega X$ has a product (some sort of composition of loops) $\mu : Y \times Y \to Y$ s.t. there is an identity and it is associative up to homotopy. In this way,

$$\Omega^X_{x_0} = \mathrm{Map}((I, \{0,1\}), (X, x_0)$$

has $\pi_1(X, x_0) = \pi_0(\Omega_{X_0} X)$ defined.

Fact. If Y is an H-group and Z is any space, then $[Z, Y]$ is a group since

$$[Z, Y] \times [Z, Y] \to [Z, Y \times Y] \stackrel{\circ\mu}{\to} [Z, Y]$$

which maps $[f] \times [g] \to [f \times g] \to [\mu \circ (f \times g)]$.

Now $\pi_0(X)$ is mere a set, but $\pi_1(X, x_0) = \pi_0(\omega_{x_0} X) = [(S^0, +1), \Omega_{x_0} X]$ has a group structure with $Z = (S^0, +1)$ and $\Omega_{x_0} X$.

A dual fact also exists. We call $Y = \Sigma X$ a "co-H-group" if it has a map $Y \to Y \vee Y$ where the map is a sort of pinching operation

$$\Sigma Y \stackrel{pinch}{\to} \Sigma X \vee \Sigma X.$$

Essentially, it collapses a copy of X at $X \times \{*\}$ where $*$ is not on the boundary of I in $X \times I$.

Then for any space $Z]$, $[Y, Z]$ gets a group structure defined by precomposing with this pinching operation. That is,

$$[Y, Z] \times [Y, Z] \stackrel{\circ pinch}{\to} [Y \wedge Y, Z] \to [Y, Z].$$

Relative homotopy groups

If $X \supseteq A \supseteq \{x_0\}$, we can define the relative homotopy groups $\pi_n(X, A, \{x_0\})$ for $n \geq 1$ in the following manner. It is the set

$$\pi_n(X, A, \{x_0\}) = [(I^n, I^{n-1}, J) \to (X, A, \{x_0\})]$$

where J is some portion of the boundary e.g. $I^{n-1} \times \{x_0\}$. An alternative picture is a the set

$$[(B^n, S^{n-1}, *), (X, A, x_0)].$$

Remark 36. If $f \in \pi_n(X, A, x_))$ is homotopy to a map valued in A, then it is null-homotopic.

In general, there are two reasons for an element of $\pi_2(X, A)$ to be nontrivial – there is something going on with $\pi_1(A)$ or $\pi_2(X)$. *Justin Roberts drew some topological spaces to demonstrate this, but I will omit them here. The reader can come up with examples themselves.*

This motivates the following theorem.

Theorem 3.1. There is a LES of homotopy groups

$$\begin{aligned}\ldots \to \pi_3(X,A) \to \pi_2(A) \to \pi_2(X)\\ \to \pi_2(X,A) \to \pi_1(A) \to \pi_1(X)\\ \to \pi_1(X,A) \to \pi_0(A) \to \pi_0(X)\\ \to \pi_0(X,A) \to 0.\end{aligned}$$

Some comments are warranted. First, all of the higher groups $\pi_i(A), \pi_i(X), \pi_i(X,A)$ are abelian for $i \geq 3$. Additionally, $\pi_2(A), \pi_2(A)$ are abelian groups. However, $\pi_2(X,A)$ is only a group since it may be nonabelian. Similarly, $\pi_1(A), \pi_1(X)$ are only groups. At the tail-end, $\pi_0(A), \pi_0(X), \pi_0(X,A)$ are only just sets. However, the notation of exactness makes since because we would like $\pi_0(X,A)\text{“}=\text{”}\frac{\pi_0(X)}{\pi_0(A)}$.

We shall explain the construction next time.

4. Lecture 4: April 4rd, 2022

Recall that the relative homotopy groups $\pi_n(X,A,x_0)$ was the group of homotopy classes of maps

$$(I^n, \partial I^n, J) \to (X, A, x_0)$$

where $J = I^{n-1} \times \{x_0\}$. We could also view it as homotopy classes of maps $B^{n+1} \to X$ satisfying certain conditions. The LES was

$$\ldots \to \pi_2 A \to \pi_2 X \to \pi_2(X,A) \to \pi_1(A) \to \ldots \to \pi_0 X \to \pi_0(X,A) \to 0.$$

The proof has six things to check, but we shall omit the verification in these notes. See Hatcher's Algebraic Topology for the full proof.

Change of Basepoints

If $n \geq 2$, then $\pi_1(X,x_0)$ acts on $\pi_n(X,x_0)$ which makes it into a $\mathbb{Z}(\pi_1(X,x_0))$-module. The idea is to use similar change of base-point isomorphisms $\gamma_\#$ to those in for the fundamental group.

Let γ be a path from x_0 to x_1. Then

$$\pi_n(X,x_0) \xrightarrow{\gamma_\#} \pi_n(X,x_1)$$

is given by $[f] \to \gamma_\#[f]$ where $\gamma\#[f]$ just extends the square $f : I^n \to X$ around its edge via γ. This is an isomorphism.

If γ is a loop in $\pi_1(X,x_1)$, we may write

$$[f] \mapsto \gamma_\#[f] = [f]^{[\gamma]}$$

to indicate that this is a right action. That is,

$$([f]^{[a]})^{[b]} = [f]^{[a][b]}.$$

This computation with the abelian group structure of π_n gives us a module.

Example 4.1. Let $X = S^1 \vee S^2$. We claim $\pi_2(X) \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$.

(i) $\pi_2(X) = \pi_2(\widetilde{X})$. This a homework exercise.
(ii) Now $\widetilde{X}$ is a copy of $\mathbb{R}$ with spheres attached at each integer point with $\mathbb{Z}$ acting by translation. By the Hurewics Theorem, $\pi_2(\widetilde{X}) = H_2(\widetilde{X})$. The theorem states that the first nonvanishing homotopy and homology groups agree.

(iii) $\pi_1(X) = \langle t \rangle$ so that $\mathbb{Z}\pi_1(X) = \mathbb{Z}[t^{\pm 1}]$ which acts on $\pi_2(X)$ and hence, on $H_2(\widetilde{X}) = \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$ (this homology computation is an easy application of the Mayer-Vietoris sequence). now t acts on $H_2(\widetilde{X})$ by sending e_i to e_{i+1} where $e_i \in H_2(\widetilde{X})$ is zero in each index except i.

So $H_2(\widetilde{X})$ is a free module over $\pi_1(X)^{e_i}$ generated by $e_0 = x_0$ i.e. the generator given by a single sphere. It is free of rank 1.

In the relative case, $\mathbb{Z}\pi_1(A, x_0)$ acts on $\pi_n(X, A, x_0)$ for $n \geq 3$ making it a module. It does act on $\pi_2(X, A, x_0)$, but $\pi_2(X, a, x_0)$ needs not be an abelian group in general.

Example 4.2. It is a fact that $\pi_3(S^2 \vee S^2) = \mathbb{Z}^3$. This indiates that there is no analogue for the Mayer-Vietoris sequence for homotopy groups. This is one of the main reasons why homotopy groups are so difficult to compute.

Our goal is to work towards Whitehead's Theorem: it says that a map $f : K \to L$ between CW-complexes is a homotopy equivalence iff it induces an isomorphism in all homotopy groups. The method of proof is to examine $f_* : \pi_* K \to \pi_* L$. Of course, $\Rightarrow$ is easy while $\Leftarrow$ is harder.

5. Lecture 5: April 6th, 2022

Theorem 5.1 (Whitehead)**.** A map $f : K \to L$ of CW-complexes is a homotopy equivalence if it induces an isomorphism on all homotopy groups.

Remark 37. Homotopy groups are powerful enough to determine the homotopy classes of CW-complexes according to the theorem. To do this, we need techniques to build homotopies between CW-complexes.

Definition 5.2. A pair of spaces (X, A) has the **homotopy extension property** (or principle) if whenever $f : X \to Y$ is a map and there is a homotopy of $f|_A$ defined on $A \times I$, we can extend this to a homotopy $X \times I \to Y$.

Example 5.3. (B^n, S^n) satisfies the HEP since one can do radial projection.

Corollary 5.3.1. Any CW-pair (L, W) with $L \subseteq W$ a subcomplex has HEP.

Proof. We omit the proof here but this was done in lecture. The idea is to extend the maps inductively. The subtlety that one should be wary of is in the case of an infinite dimensional skeleton. □

Example 5.4. If L is a contractible subcomplex, then $K/L \simeq K$. The method of proof is similar to the one above.

Example 5.5. We prefer to add things than to subtract. For instance, if (K, L) is a CW-pair, then $K \cup C(L)$ (here $C(L)$ is the cone on L) is a good substitute for K/L. That is, one attaches the cone on L to "remove" L since we can collapse the cone.

Theorem 5.6 (Cellular Approximation Theorem)**.** Any map $f : K \to L$ between CW-complexes are cellular i.e. $f(K^{(n)}) \subseteq L^{(n)}$.

Proof. See Hatcher's Algebraic Topology. □

Theorem 5.7. If K has one 0-cell and no i-cells for $1 \leq i \leq n$, then $\pi_i(K) = 0$ for all $i \leq n$. We call K n**-connected** if $\pi_i(K) = 0$ for all $i \leq n$.

Proof. Consider cellular complexes and the cellular approximation theorem. □

Theorem 5.8. There is a partial converse. If K is an n-connected CW-complex, then K is homotopy equivalent to a Cw-complex with one 0-cell and no i-cells for $1 \leq i \leq n$.

6. Lecture 6: April 8th, 2022

Last, the lecture ended with stating the following theorem.

Theorem 6.1. If K has one 0-cell and no i-cell for $1 \leq\leq n$, then $\pi_i K$ is trivial for all $i \leq n$. We call K "n-connected" in this if K has this latter property.

Proof. Follows from the cellular approximation theorem. □

Example 6.2. 1-connected means that K is path-connected and simply-connected.

Corollary 6.2.1. If K is a CW-complex and has $\pi_i K$ trivial for all $i \leq n$, then K is homotopy equivalent to a complex with one 0-cell and no i-cells for $1 \leq i \leq n$.

For instance, if K is 1-connected, then one can collapse the 1-skeleton and focus on the larger dimension cells.

Proof. We can certainly do the $n = 0$ case directly by collapsing a maximal tree L of $K^{(1)}$ by viewing it as a contractible subcomplex. But then $K/L \cong K$ and K/L has a single 0-cell.

Having done this, we have a 1-skeleton that is a bouquet of circles and we glue to this the higher dimensional cells to get the rest of the space. Now we want to get rid of the 1-cells.

The trick is to attach a pair of canceling cells to K.

Glue a 2-cell so that its boundary is a 1-cell we want to kill. Now repeat this for all other 1-cells. Then attach a 3-cell whose boundary is all of the other 2-cells we have added. Now the new complex has a contractible subcomplex which we can collapse. Indeed, we then collapse the 2-skeleton so that the new CW-complex has only dimension 2 or higher cells.

Now that we have destroyed all the 1-cells by "trading" them for 3-cells, in a sense.

Now there is sensible way to do this indefinitely in taking a limit of sorts.

The first attempt of Whitehead's Theorem is to say that

Lemma 6.3. If K is a CW-complex with all $\pi_i K$ trivial, then K is homotopic to a point.

Proof. The idea is to construct a homotopy $K \times I \to K$ between the identity and a constant map.

The method of proof is to use the HEP (homotopy extension property/principle). Fix a vertex $v \in K$. For each vertex $w \in K$ other than v, use $\pi_0 K$ is trivial to find a path from w to v and treat this as a homotopy defined on $\{w\} \times I$. Extend this using the HEP to get a homotopy $K \times I \to K$ that is a homotopy between the identity and the map $f_0 : K^{(0)} \to v$ (which is extended to K).

Now do the same for the higher dimensions with edges instead of vertices. For each 1-cell $e \in K$, observe that $f_0(e)$ is a loop in (K, v) and since $\pi_1 K$ is trivial, it is null-homotopy rel$\{0, 1\}$. Then use this to define a homotopy $K^{(1)} \times I \to K$ between f_0 and a map f_1 that sends $K^{(1)}$ to v. Then repeat inductively.

Assemble all of these homotopies H_i's and use the fact that the topology is defined as the weak topology. Define the value a new homotopy at time 0 to be the constant map and $t = 1$ to be the union of all of these homotopies. It is continuous because H_n does not move anything in $K^{(l)}$ for $l < n$. □

Lemma 6.4 (Relative Version)**.** Suppose that $L \subseteq K$ is a CW complex and $\pi_i(K, L)$ is trivial for i. Then K deformation retracts to L.

Proof. Next time! □

□

Warning. There are pathological examples which exist which show that these results for topological spaces that are not CW-complexes.

7. Lecture 7: April 11th, 2022

Recall that we were proving the following result.

Theorem 7.1. If K is a CW-complex with $\pi_i K = 0$ for all i, then K is contractible.

The method of proof was via induction. There was a relative version that we will now prove.

Theorem 7.2 (Relative Version)**.** If $L \subseteq K$ is a subcomplex and $\pi_i(K, L) = 0$ for all i, then K deformation retracts to L.

Proof. The method of proof is analogous via HEP. Start with v a 0-cell of $K \setminus L$. Since $\pi_0(K, L)$ is trivial, we know v sits in a path-component of L i.e. v can be joined to a point in L via a path γ_v. Do this for all 0-cells and so we get an extension of the identity homotopy to $(L \times I) \cup (K^{(0)} \times I)$ to get a homotopy H_0 between id_K and some $f_0 : K \to K$ which sends $K^{(0)} \cup L \to L$.

To do the 1-cells, the idea is the same. A 1-cell represents an element in $\pi_1(K, L)$ which deformation retracts to a 1-cell in L by $\pi_1(K, L) = 0$. So the HEP gives a $H_1 : (L \times I) \cup (K^0 \times I) \cup (K^1 \times I) \to L \times I$ which is a homotopy between id_K and an $f : K \to K$ s.t. $K^{(1)} \cup L \to L$.

Then use the $\frac{1}{2^n}$ scaling trick to patch all of these homotopies H_i together to a homotopy $\widetilde{H} : K \times I \to K$ so that $\mathrm{id}_K \simeq (f : K \to L)$. □

This was the lemma from the previous lecture. Now we prove the full version which is done via maps between CW-complexes.

Theorem 7.3 (**Whitehead's Theorem**)**.** If $f : L \to K$ is a map of CW-complex inducing isomorphisms on all π_1, then f is a homotopy.

Proof. The first step in the proof is to use the **mapping cylinder** to replace $f : L \to K$ with an inclusion of complexes.

Definition 7.4. Let $f : X \to Y$ be a map. The **mapping cylinder** is $M_f := (X \times I) \cup_f Y = (X \times I) \cup Y / (f : X \times \{1\} \to Y)$.

Example 7.5. If $f : S^1 \to \mathbb{R}^2$ maps to a figure-8, then M_f is the cylinder with one end glued to $\mathbb{R}^2$ via the figure-8 drawing.

Observe that there exists a deformation retract from M_f to Y. There is a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "j"'] & Y \arrow[d, hook, shift right, "i"'] \\ & M_f \arrow[u, shift right, "r"'] \end{tikzcd}$$

which commutes up to homotopy. In fact, $f = r \circ k$ while $i \circ f \simeq j$.

Now we apply this construction to $f : L \to K$ after using the cellular approximation theorem to assume f is cellular. In this case, when we form $M_f : X \times I \cup_f Y$, we actually obtain a CW-complex. Therefore we get

$$\begin{tikzcd} L \arrow[r, dashed, "f"] \arrow[dr, hook, "j"'] & K \arrow[d, "\simeq"] \\ & M_f \end{tikzcd}$$

and we compute the relative homotopy groups

$$\begin{tikzcd} \dots \arrow[r] & \pi_i(L) \arrow[r] \arrow[d, "="] & \pi_i(K) \arrow[r] \arrow[d, "\cong"] & \pi_i(K, L) \arrow[r] \arrow[d, "\cong"] & \pi_{i-1}(L) \arrow[r] \arrow[d, "="] & \dots \\ \dots \arrow[r] & \pi_i(L) \arrow[r] & \pi_i(M_f) \arrow[r] & \pi_i(M_f, L) = 0 \arrow[r] & \pi_{i-1}(L) \arrow[r] & \dots \end{tikzcd}$$

Now apply the 5-lemma to get that $\pi_i(K, L) = 0$.

Apply the relative version of the theorem to see that $j : L \hookrightarrow M_f$ is a homotopy equivalence. □

Example 7.6. The two spaces $S^2 \times \mathbb{R}P^2$ and $\mathbb{R}P^2 \times S^3$ have the same universal cover and hence, the same homotopy groups π_n for $n \geq 2$. It is not hard to see that their fundamental groups are equal. However, they have different cohomology rings because

$$H^*(S^2 \times \mathbb{R}P^2) \cong \mathbb{F}_2[\alpha_{(2)}, \beta_{(3)}]/(\alpha_{(2)}^2, \alpha_{(3)}^4)$$

and

$$H^*(S^3 \times \mathbb{R}P^2) \cong \mathbb{F}_2[\alpha_{(3)}, \beta_{(1)}]/(\alpha_{(3)}^2, \beta_{(1)}^3).$$

One way to compute these cohomology rings is do things via intersections.

The moral of the example is that one *needs* to have a map for Whitehead's Theorem to work.

8. Lecture 8: April 13th, 2022

Example 8.1. From last time, we saw that $S^2 \times \mathbb{R}P^3$ and $S^3 \times \mathbb{R}P^2$ are two spaces with the same homotopy and homology but were not homotopy equivalences. Another example is $L(5, 1)$ and $L(5, 2)$ (the **lens spaces**). For $L(p, q)$ where $(p, q) = 1$, it is obtained by quotienting by the $\mathbb{Z}/p\mathbb{Z}$ action on S^3 given by $(z, w) \mapsto (e^{2\pi i/p}z, e^{2\pi iq/p}w)$.

One should observe that the lens spaces are related to the Hopf fibration and can be used to draw pictures. Vaguely, one can build them from gluing two tori or by using Cw-complexes and gluing one cell of dimension $0, 1, 2, 3$.

How does one distinguish these spaces? One must use the **Bockstein homomorphism** or cohomology. Recall the SES $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ induces a SES of chain complees $0 \to C_*(X, \mathbb{Z}) \to C_*(X, \mathbb{Z}) \to C_*(X, \mathbb{Z}/p\mathbb{Z}) \to 0$. Then we get a LES of homology and the Bockstein homomorphism is

$$\beta : H_i(X, \mathbb{Z}_p) \to H_{i-1}(X, \mathbb{Z}).$$

A few facts. First, if x and y generate $H_2(L, \mathbb{Z}_p)$ and $H_1(L, \mathbb{Z}_p)$ respectively, then $\beta(x) \bmod p = y$. Geometrically, $\beta(x).x = q \bmod p$ where the $\cdot$ indicates the intersection form. If one changes basis $x \to \lambda x$ where $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$, then $\lambda^2\beta(x).x = q \bmod p$ is the change in the intersection form. So if q_1, q_2 do not differ by a square modulo p, then $L(p, q_1) \not\cong L(p, q_2)$.

Now we discuss some Whitehead-like Theorems.

Lemma 8.2 (Compression Lemma)**.** If (K, L) is a CW-pair and $f : (K, L) \to (X, A)$ is a map of topological spaces where $\pi_*(X, A) = 0$, then f is homotopic to a map into A.

Proof. If $v \in K \setminus L$ is a 0-cell, then $f(v)$ is joined to a point of A because $\pi_0(X, A) = 0$. Do this for all vertices. Use the same idea for the 1-cells in $K \setminus L$. Glue all of these homotopies together using the $\frac{1}{2^n}$-trick. ☐

Lemma 8.3. If (K, L) is a Cw-pair and Y a space with $\pi_*(Y) = 0$, then any map $L \to Y$ extends to a map $K \to Y$. That is, we get

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, hook] & Y \\ K \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

Proof. Define $\widetilde{f}$ at random on 0-cells of $K \setminus L$. For each 1-cell of $K \setminus L$, its boundary must go to the same path component of Y because $\pi_0 Y = 0$.

For each 2-cell of $K \setminus L$, use the fact that $\pi_1 Y = 0$ to get a map on the boundaries of 2-cells. Induct and use the $\frac{1}{2^n}$-trick. ☐

Definition 8.4. A map $f : X \to Y$ is a **weak homotopy equivalence** if it is an isomorphism on π_0 and $\pi_*(X, x_1) \xrightarrow{f_*} \pi_*(X, f(x_0))$ is an isomorphism for all choices of base-points x_0.

Remark 38. Using this language, Whitehead's Theorem (which assumed the spaces were path-connected CW-complexes) says

A weak homotopy equivalence of CW-complexes is a homotopy equivalence.

Theorem 8.5. If $X \to Y$ is a weak homotopy equivalence, Z a CW-complex, then f induces isomorphisms $f_* : [Z, X] \to [Z, Y]$.

Remark 39. The theorem says that $[S^n, -]$ determines all the knowledge of $[Z, -]$ for Z a CW-complex.

Proof. Replace Y by the mapping cylinder M_f of f so that we may assume $X \subseteq Y$. Replace Y by M_f and we get $i : X \hookrightarrow Y$ which induces a map $[Z, X] \xrightarrow{i_*} [Z, Y]$.

The map i_* is surjective because $\pi_*(Y, X) = 0$ from the isomorphism $\pi_*(X) \to \pi_*(Y)$ in the LES and then the compression lemma says i_* is surjective.

The map i_* is injective because given a pair of maps $f, g : Z \to X$ which compose with $X \hookrightarrow Y$ to get the same map, there is a homotopy $f \simeq g$ $H : (Z \times I, Z \times \partial I) \to (Y, X)$ which compress to X. ☐

9. Lecture 9: April 15th, 2022

Our goal is to move toward the Hurewicz Theorem.

Theorem 9.1 (Hurewicz Theorem). Given any space (X, x_0), there is a map

$$h_n : \pi_n(X, x_0) \to H_n(X; \mathbb{Z})$$

given by $[f] \to f_*[S^n]$ which is induced by a map $H_n(S^n) \xrightarrow{f_*} H_n(X)$ where $H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$ is generated by the fundamental class $[S^n]$.

First, we have some easy facts.

- The Hurewicz map is a natural transformation $\pi_n(-,-) \to H_n(X)$ i.e.

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{h_n^*} & H_n(X) \\ \big\downarrow g_* & & \big\downarrow g_* \\ \pi_n(Y, y_0) & \xrightarrow{h_n^*} & H_n(Y) \end{array}$$

  is a commutative diagram.
- It is a homomorphism i.e.

$$h_n([f] + [g]) = h_n([f]) + h_n([g])$$

  where $[f] + [g]$ is addition in π_n and the RHS is addition in H_n. Or alternatively, one can view $\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$ and then view the map a map sending

$$[I^n, \partial I^n] \in H_n(I^n, \partial I^n) \cong \mathbb{Z} \to f_*[I^n, \partial I^n] \in H_n(X, \{x_0\}) = H_n(X)$$

  for $n \geq 1$. So the idea is $f + g$ the gluing of two squares and map to (X, x_0).

Theorem 9.2. $\pi_1(X, x_0) \xrightarrow{h_1} H_1(X)$ is the abelianization homomorphism.

Theorem 9.3. Let $n \geq 2$ and suppose X has $\pi_0, \pi_1, \ldots, \pi_n$ trivial. Then

$$h_n : \pi_n(X, x_0) \to H_n(X)$$

is an isomorphism and so $H_1, H_2, \ldots, H_{n-1}$ are all zero.

Corollary 9.3.1. If $\pi_0 X$ and $\pi_1 X$ are trivial with $H_1, H_2, \ldots, H_{n-1} = 0$, then

$$h_n : \pi_n(X, x_0) \to H_n(X)$$

is an isomorphism.

Example 9.4. The universal cover of $S^2 \vee \mathbb{R}P^3$ is $S^2 \vee S^3 \vee S^2$. One knows via the universal cover computation that

$$\pi_2(S^2 \vee \mathbb{R}P^3) = \pi_2(\text{Universal Cover}, S^2 \vee S^3 \vee S^2).$$

By the Hurewicz Theorem, $H_2(S^2 \vee S^3 \vee S^2) \cong \mathbb{Z}^2$.

Remark 40. By the same method, $\pi_2(S^2 \vee \mathbb{R}P^2) \cong \mathbb{Z}^3$ while $\pi_2(S^2 \vee \mathbb{R}P^1)$ is a free module of rank 1 over $\mathbb{Z}[t^{\pm 1}]$.

The idea of the proof of then Hurewicz Theorem is as follows. First, prove it for $X = S^n$. Then, prove it for CW-complexes. Then generalize it to all spaces using a CW-approximation trick. For the first step, one shows that $\deg : \pi_n S^n \to \mathbb{Z}$ is an isomorphism. So we want a theorem to calculate degree geometrically.

Theorem 9.5. Let $f : S^n_L \to S^n_R$ be a map between n-spheres equipped with choices of orientation $gv \in H_n(S^n_L)$ and $h \in H_n(S^n_R)$.

Suppose y S^n_R is a **regular value** of f i.e. $f^{-1}(y) = \{x_1, \ldots, x_k\}$ is a finite set and there is a neighborhood V of $y \in S^n_R$ and U_i neighborhoods of x_i in S^n_L s.t. f carries U_i homeomorphically to V. Then we can define a **local degree**

$$\deg x_i(f) \in \{\pm 1\}$$

via the diagram

$$\begin{tikzcd}
g \in H_n(S^n_L) \arrow[r, "f_*"] \arrow[d, "LES"'] & H_n(S^n_R) \ni h \arrow[d, "LES"] \\
H_n(S^n_L, S^n_L - \{x_i\}) \arrow[d, "Excision"'] & H_n(S^n_R, S^n_R - \{y\}) \arrow[d, "Excision"] \\
g_i \in H_n(U_i, U_i - \{x_i\}) \arrow[r, "(f_*)_i : g_i \mapsto \pm h_y"] & H_n(V, V - \{y\}) \ni h_y
\end{tikzcd}$$

and so

$$\deg f = \sum_i \deg_{x_i}(f).$$

For the proof, see Hatcher's textbook. The idea is to use the MV sequence for pairs to claim that

$$H_n(S^n, S^n - \{x_1, \ldots, x_k\}) \cong \bigoplus_{i=1}^k H_n(S^n, S^n - \{x_i\}).$$

Then there is a commuting diagram

$$\begin{tikzcd}
g \arrow[dddd, mapsto] & H_n S^n \arrow[rr, "f_*"] \arrow[d, "LES"'] & & H_n S^n \arrow[d, "{LES, \cong}"] \\
& H_n(S^n, S^n - \{x_1, \ldots, x_k\}) \arrow[d, "MV"'] & & H_n(S^n, S^n - \{y\}) \arrow[ddd, "Excision"] \\
& \bigoplus_{i=1}^k H_n(S^n, S^n - \{x_i\}) \arrow[d, "Excision"'] & & \\
& & & \\
g_i & \bigoplus_{i=1}^k g_i \in \bigoplus_{i=1}^k H_n(U_i, U_i - \{x_i\}) & & H_n(V, V - \{y\}) \ni h_y
\end{tikzcd}$$

which gives the proof.

10. Lecture 10: April 18th, 2022

Recall that we could compute the degree of a map via local degrees.

Theorem 10.1. If $f : S^n \to S^n$ is a map, then $\deg f = \sum_{i=1}^k \deg_{x_i}(f)$ provided there exists $(V, y) \subseteq S^n_R$ (the subscript R or L means RHS sphere or LHS sphere) s.t. $f^{-1}((V, y)) = \bigcup_{i=1}^k (U_i, x_i)$ s.t. $f : U_i \to V$ is mapped homeomorphically.

Recall that we proved this last lecture via excision and by using the LESs we knew. Now we give an application.

Theorem 10.2. $\deg : \pi_n S^n \to \mathbb{Z}$.

Proof. We already know that this is always surjective so we check injectivity. That is,we show $\deg f = 0$ iff it is null-homotopic. The $\Leftarrow$ direction is clear.

Assume $\deg f = 0$. The base case of $n = 1$ is clear so we start with the base case of $n = 2$. The idea is to pair up local degrees via the "inner-most" method. That is, one pairs up the local degrees $+1$ with -1. and then define a homotopy H via an arc argument which flipts the arcs and map them to the corresponding point.

Define H via this method. Before all of this, one should use simplicial approximation (or similar results) to straighten by homotopy so that the preiamge condition for local degrees can be applied.

Using the pairing argument, one constructs a bunch of embeddings into $B^n \times I$ and the arcs are thickened up arcs pairing the x_i's.

We know the opposite local degrees at ends of each arc lead to maps from $\partial U_1, \partial U_2 \to V$ via a homotopy on $\partial B^2 \times I$ so the base case says that this extends radially to a map from $B^2 \times I \to V$.

Using an analogous method, we get a map $S^n \times \{0\} \cup \bigcup_{i=1}^k B^n \times I \to S^n$ which extends to $S^n \times I \to S^n$ by observing that $\pi_*(S^n - V) = 0$ (apply the extension lemma to fill it out). Finally, we get a null-homotopy where time $= 1$ is a map $S^n \to S^n - V \cong \mathbb{R}^n$ which is nullhomotopic. $\square$

Remark 41. We remark on some of the simplicial approximation theorem-like results.

The **simplicial approximation theorem** says that if $f : |K| \to |L|$ is a continuous map between simplicial complexes in $\mathbb{R}^n$, then there is an r s.t. f is homotopic to a simplicial map from $|K^{(r)}| \to |L|$ where $|K^{(r)}|$ is just $|K|$ barycentrically divided r-times.

Note that simplicial maps are allowed to map vertices and simplices in a noninjective manner.

Remark 42. The proof of this theorem was presented in class with a very geometric argument and a beautiful picture which is omitted from these notes due to technological reasons.

Now we return to the Hurewics Theorem and it's completed proof. This will spill over to the next lecture, but we start with a lemma.

Lemma 10.3. If $n \geq 2$, then the **join** or **wedge sum** $\bigvee_{i\in I} S^n$ of spheres satisfies $\pi_n(\bigvee_{i\in I} S^n) \cong \bigoplus_{i\in I} \mathbb{Z}$.

Proof. Consider $\Pi = \prod_{i\in I} S^n \supseteq \bigvee_{i\in I} S^n = \bigvee$. Observe that $\bigvee$ is the $(n+1)$-skeleton of Π. Indeed all cells in Π have dimension divisible by n so the next ones after dimension 2 are $2n > n + 1$ because $n \geq 2$.

By cellular approximation, we know that $\pi_n(K) = \pi_n(K^{(n+1)})$ by cellular approximation. Hence,

$$\bigoplus_{i\in I} \mathbb{Z} = \pi_n(\Pi) = \pi_n(\Pi^{(n+1)}) = \pi_n(\bigvee)$$

which is what we wanted. $\square$

11. Lecture 11: April 20th, 2022

On our march towards the full Hurewics Theorem, we have shown that $\pi_n S^n \cong \mathbb{Z}$ and $\pi_n(\bigvee_I S^n) \cong \bigoplus_I \mathbb{Z}$ via the cellular approximation theorem.

Theorem 11.1 (**Hurewicz Theorem**)**.** Assume K is an $(n-1)$-connected CW-complex and $\pi_n = \pi_{n-1} = \cdots = \pi_0 = 0$. Then there is an isomorphism

$$h : \pi_n K \to H_n K.$$

The idea of approach is as follows. By $(n-1)$-connectedness, we may replace K by a CW-complex which has one 0-cell and no i-cells for $1 \leq i \leq n-1$.

Also, we can ignore the cells of dimension $\geq n+2$ because these have no effect on $\pi_n K$ by the cellular approximation theorem. That is, we used $\pi_n K = \pi_n K^{(n+1)}$. So WLOG, we are assuming

$$K = e^{(0)} \cup (n\text{-cells}) \cup ((n+1)\text{-cells})$$

and the same can be said and done with $H_*(-)$.

Now for the proof after these reductions.

Proof. Let V be the n-skeleton K i.e. $V = K^{(n)}$. Then $V \cong \bigvee_{i \in I} S^n$. Consider the commutative diagram

$$\begin{tikzcd}
\pi_{n+1}(K,V) \arrow[r] \arrow[d, "h_{rel}"] & \pi_n(V) \arrow[r] \arrow[d, "h_n^V"] & \pi_n(K) \arrow[r] \arrow[d, "h_n^K"] & \pi_n(K,V) \arrow[r] \arrow[d, "h_{rel}"] & 0 \arrow[d] \\
H_{n+1}(K,V) \arrow[r] & H_n(V) \arrow[r] & H_n(K) \arrow[r] & H_n(K,V) \arrow[r] & 0
\end{tikzcd}$$

Now $\pi_n(K,V) = 0$ because $(B^n, S^{n-1}) \to (K,V)$ can be homotoped to land inside V by the cellular approximation theorem. OTOH, $H_n(K,V) = H_n(K/V, \emptyset) = 0$ because K/V has no n-cells nor $n+1$-cells.

Before much can be said about the diagram, the important part here is the relative map h_{rel}. Observe there are elements $\Phi : (B^{n+1}, S^n) \to (K,V)$ corresponding to the characteristic maps Φ_j of the $(n+1)$-cells. Some thought should lead one to deduce that the diagram really is

$$\begin{tikzcd}
\bigoplus \mathbb{Z} \arrow[r] \arrow[d, "h_{rel}"] & \pi_n(V) \arrow[r] \arrow[d, "h_n^V"] & \pi_n(K) \arrow[r] \arrow[d, "h_n^K"] & 0 \arrow[r] \arrow[d, "h_{rel}"] & 0 \arrow[d] \\
\bigoplus \mathbb{Z} \arrow[r] & H_n(V) \arrow[r] & H_n(K) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where the maps on direct sums of $\mathbb{Z}$ is surjective. By the surjection

$$\pi_n K \cong \frac{\pi_n V}{\operatorname{im} \pi_{n+1}(K,V)} \cong \frac{H_n V}{\operatorname{im} H_{n+1}(K,V)}$$

which means we have the desired presentation. That is, $\pi_n K$ is given by the RHS above which is exactly $H_n(K)$.

We are now finished with the proof via a sort of presentation □

We now present some applications. We shall show next lecture that the analogous statement with homology vanishing holds which is called Whitehead's Theorem.

Example 11.2. We now compute the homotopy groups of genus g compact surfaces $\pi_* \Sigma_g$. By Van Kampen's Theorem,

$$\pi_0 \Sigma_g = 0 \qquad \& \qquad \pi_1 \Sigma_g = \langle a_1, \ldots, a_g, b_1, \ldots, b_g : \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle.$$

We can compute $\pi_2\Sigma_g = \pi_2\widetilde{\Sigma_g}$ where $\widetilde{\Sigma_g}$ is the universal cover. Only fact we need is that it's a 2-manifold which is noncompact due to the fact that $\pi_1\widetilde{\Sigma_g}$ is infinite. By the Hurewicz Theorem above, we know

$$\pi_2\widetilde{\Sigma_g} = H_2\widetilde{\Sigma_g} = 0$$

where the RHS is by Poincaré duality theory:

"If M^n is connected n-manifold, then $H_n(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{orientable } + \text{ compact} \\ 0 & \text{if not} \end{cases}$."

But then one can reapply the Hurewicz Theorem to deduce that $H_n\widetilde{\Sigma_g} = \pi_n\widetilde{\Sigma_g}$ for $n \geq 3$ as well. That means $\pi_n\Sigma_g = 0$ for $n \geq 3$ and to summarize:

$$\pi_0\Sigma_g = 0 \qquad \& \qquad \pi_1\Sigma_g = \langle a_1, \dots, a_g, b_1, \dots, b_g : \prod_{i=1}^{g}[a_i, b_i] = 1\rangle. \qquad \& \qquad \pi_n\Sigma_g = 0 \text{ for } n \geq 2.$$

A generalization of these spaces is the following. We write $K(\pi, n)$ for the **Eilenburg-Maclane space** which is definable for any abelian group π for $n \geq 2$ and any group π for $n = 1$.

Fact: All Eilenburg-Maclane spaces are unique up to homtopy equivalence and are "atoms for making up spaces with complicated homotopy groups".

Example 11.3. If M^3 is a 3-manifold with infinite π_1, trivial π_2, then M is $K(\pi, 1)$.

The idea of proof is the same as before. Consider the universal cover $\widetilde{M^3}$. Then $\pi_1\widetilde{M^3} = 0$ and the Hurewicz Theorem says

$$\pi_2\widetilde{M_3} = \pi_2 M^3 = 0.$$

Applying the Hurewicz Theorem,

$$\pi_3 M = \pi_3\widetilde{M^3} = H_3\widetilde{M^3} = 0$$

by the same idea as in the previous example. Continuing inductive

$$\pi_n\widetilde{M^3} = \pi_n(M) \text{ for } n \geq 4.$$

Next, we show Hurewicz Theorem is true for all spaces. The idea of proof is through CW-approximation.

Theorem 11.4 (CW-Approximation). Given X, we can find a CW-complex K s.t. $f : K \to X$ which is a weak homotopy equivalence and also inducing an isomorphism on homology.

Theorem 11.5 (Hurewicz Theorem). A weak homotopy equivalences between spaces always induces an isomorphism.

Proof. Take a CW-approximation $f : K \to X$, then look at the diagram

$$\begin{tikzcd} \pi_i K \arrow[r, "\cong \text{ 1st Theorem}"] \arrow[d, "\text{1st Hurewicz } \cong"'] & \pi_i X \arrow[d, "h^X"] \\ H_i K \arrow[r, "\cong \text{2nd Theorem}"] & H_i X \end{tikzcd}$$

which implies h^X is an isomorphism. Here, the 1st Theorem is the statement that $f : K \to X$ is a weak homotopy equivalence and the 2nd Theorem refers to the the fact that $f : K \to X$ a homotopy equivalence. $\square$

12. Lecture 12: April 22nd, 2022

For today, we shall discuss CW approximation and show that the Hurewicz Theorem is true for all spaces.

Theorem 12.1 (Theorem A)**.** If X is any space, then we can build a CW complex K and a map $f : K \to X$ which is a weak homotopy equivalence (i.e. it induces an isomorphism on π_*).

Theorem 12.2 (Theorem B)**.** Any weak homotopy equivalence $f : X \to Y$ of spaces induces an isomorphism on all homology groups too.

The proof of Hurewicz Theorem is done by

$$\begin{tikzcd} \pi_n K \arrow[r, "{A,\cong}"] \arrow[d, "h_n"'] & \pi_n X \\ H_n K \arrow[r, "{B,\cong}"] & H_n X \end{tikzcd}$$

Proof of A. Assume X is path-connected for convenience.

Define a map f from a point of K^0 which induces an isomorphism on π_0.

Choose generators for $\pi_1(X, x_0)$. One may as well pick all of the elements of $\pi_1(X, x_0)$. Then K^1 is a bouquet of circles and f^1 induces a map on π_1 which is also an isomorphism on π_0.

Choose generators for the kernel of $(f^1)_* : \pi_1 K^1 \to \pi_1 X$. For each one, attach a 2-disk to K^1 with a to a null homotopy of f^1 (i.e. its boundary) in X. Call this K^{1+}. Then $(f^{1+})_*$ is $\cong$ on π_0 and π_1. Now choose generators $\pi_2 X$. Add 2-cells to vertex of K and extend map to get $(f^2)_*$ surjection on π_2 which is an isomorphism on π_1 and π_0. Choose generators for kernel of $(f^2)_*$ namely

$$\begin{tikzcd} K^2 \arrow[r, "f^2"] & X \\ S^2 \arrow[u, "\beta"] \arrow[ur, "\text{null-homotopic}"'] & \end{tikzcd}$$

i.e. $f^2 \circ \beta$ is a boundary of map $\gamma : B^3 \to X$. Attach a 3-cell to K^2 for each such element, with obvious way, to X getting

$$K^{2+} \xrightarrow{f^{2+}} X.$$

By cellular approximation theorem, adding 3-cells deosn't change π_0 and $\pi_1 K$. However,

$$\begin{tikzcd} \pi_2 K^2 \arrow[r, "(f^2)_*"] \arrow[d, "i_*"'] & \pi_2 X \\ \pi_2 K^{2+} \arrow[ur, "(f^{2+})_*"'] & \end{tikzcd}$$

where the map i_* is a surjection by the cellular approximation theorem while $(f^{2+})_*$ is a surjection by the cellular approximation theorem ($K^2 \subseteq K^{2+}$).

So $\pi_2 K^{2+}$ is some "intermediate" quotient, however we have explicitly killed all elements of kernel of $(f^2)_*$, so $(f^{2+})_*$ is an $\cong$. Then repeat. $\square$

Theorem 12.3 (Uniqueness)**.** Any two CW approximation

$$\begin{tikzcd} K_1 \arrow[r, "f_1"] & X \\ K_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

are homotopy equivalent in canonical manner.

Proof. We had a lemma that said when $f_1 : K_1 \to X$ is a weak homotopy equivalence,

$$[K_2, K_1] \overset{(f_1)_*}{\to} [K_2, X]$$

is an isomorphism. Then there exists a $[\emptyset] \in [K_2, K_1]$ s.t. $f_1 \circ \emptyset \simeq f_2$. Clearly $\emptyset$ is a weak homotopy equivalence, so by Whitehead, $\emptyset$ is homotopy equivalence (since K_1 and K_2 are both CW-complexes). $\square$

Proof of Theorem B. Use a standard trick of converting $f : X \to Y$ into an inclusion $X \hookrightarrow Y \simeq M_f$ by replacing Y by M_f. If $X \hookrightarrow Y$, then $\pi_*(X) \simeq \pi_*(Y)$ means $\pi_*(Y, X) = 0$ by a LES.

What we must show is that this implies $H_*(Y, X) = 0$. Then also by the LES we $f_* : H_*X \to H_*Y$ being an isomorphism. Suppose $[Z, -]$ is a cycle in $C_n^{sing}(Y, X)$ and write

$$Z := \mathbb{Z}-\text{linear combination of singular simplexes with } \partial z \in C_{n-1}(X) = \sum_{i \in I} \{\sigma : \Delta^n \to X\}.$$

There is a geometric representation of n-cycles of n-cycles in Y by "n-dimensional pseudomanifolds".

Definition 12.4. An n-dimension pseudomanifold is a space made by gluing a finite set of copies of Δ^n by identifying faces in pairs. Given a cycle in C_nX, we can choose a pairing up of the $+1$ and -1 codimension 1 faces (which must exist because $\partial z = 0$).

If $n = 1$ or $n = 2$, then a pseudomanifold is a manifold (this is not true for $n \geq 3$ can triangulate a torus and then take a cone). Since a pseudomanifold has a fundamental class $[Z = \sum \text{simplexes}] \in H_n(Z; \mathbb{Z})$ and we see

$$H_n(Z) \ni f_*[Z] = [Z] \in H_n(X).$$

$\square$

Consider relative case $Z \in C_n(Y, X)$ with $\partial z \in C_{n-1}X$ can be represented by a pseudomanifold with boundary.

Definition 12.5. If X is a space and $x \in H_n(X; \mathbb{Z})$ is a homology, then x is represented by a map $M^n \to X$ (M^n = closed oriented manifold) if

$$f_*[M] = X \text{ where } f_* : H_nM \to H_nX$$

where $[M] \in_n M$ is the fundamental class.

There will be more next time!

13. Lecture 13: April 25th, 2022

Today we discuss the **relative Hurewicz Theorem**. If (X, A) is a pair of spaces, we can define a **Hurewicz map**

$$h_n : \pi_n(X, A < x_0) \to H_n(X, A)$$

which sends $[f]$ to $f_*([B^n, S^{n-1}]) \in H_n(X, A)$. Indeed, $H_n(B^n, S^{n-1}) \cong \mathbb{Z}$ with generator $[B^n, S^{n-1}]$.

As usual, the Hurewicz map is a homomorphism whenever it makes sense. In this case $\pi_1(X, A)$ is always a set, $\pi_2(X, A)$ is a non-abelian group, and $\pi_*(X, A)$ is always abelian when $* \geq 3$.

Recall that $\pi_1(X, A)$ acts on all higher homotopy groups $\pi_*(X, A)$ for $* \geq 2$.

Theorem 13.1 (Version A)**.** If $\pi_1(A, x_0)$ is zero, then if $\pi_i(X, A) = 0$ for all $i = 0, 1, \ldots, n-1$, then the Hurewiz map

$$h_n : \pi_n(X, A, x_0) \to H_n(X, A)$$

is an isomorphism.

Theorem 13.2 (Version B)**.** If $\pi_1(A, x_0)$ is nonzero but $\pi_i(X, A) = 0$ for all $i = 0, \ldots, m-1$, then

$$h_n : \pi_n(X, A, x_0) \to H_n(X, A)$$

is the coinvariant mapping i.e. the kernel is generated by $[f] - [f]^{[\alpha]}$ where $[f] \in \pi_n(X, A)$ and $[\alpha] \in \pi_1(A)$.

Recall that the coinvariant module for G acting on M is the quotient $M/\{m^g - g : m \in M, g \in G\}$.

We now present some applications. These can be viewed as homological versions of Whitehead's Theorem.

Example 13.3. If $f : K \to L$ is a map between 1-connected CW-complexes which induce an isomorphism on homology, then f is a homotopy equivalence.

PROOF. By taking the mapping cylinder, we may assume $K \subseteq L$. Then because $H_*K \to H_*L$ are isomorphic under f_*, the relative homology groups are trivial i.e. $H_*(L, K) = 0$. Then the relative Hurewixz theorem Version A implies

$$\pi_*(L, K, x_0) = 0.$$

So by the LES of homotopy groups, $f_* : \pi_*K \to \pi_*L$ are all isomorphisms. By Whitehead's Theorem, $f : K \to L$ is a homotopy equivalence. $\square$

Example 13.4. If $f : A \to L$ is a map of CW complexes which are 0-connected s.t.

(a) f_* gives an isomorphism $\pi_1 K \cong \pi_1 L$
(b) which gives an isomorphism on the homology of universal covers,

then f is a homotopy equivalence.

PROOF. We need to explain what (b) says. Let $\widetilde{K} \to K$ and $\widetilde{L} \to L$ be the universal covers. Then we get a composed map $\widetilde{K} \to K \xrightarrow{f} L$. Since $\widetilde{K}$ is simply connected, the lifting criterion for covering spaces says there is a lifting $\widetilde{f} : \widetilde{K} \to \widetilde{L}$.

If $(\widetilde{f})_*$ is an isomorphism on homology, then we can conclude that $\pi_*(\widetilde{L}, \widetilde{K}) = 0$ by the previous example. By the LES, $\pi_*\widetilde{K} \to \pi_*\widetilde{L}$ is an isomorphism for $* \geq 2$. It is also an isomorphism for $* = 1$ since the fundamental groups are zero. But recall that p_* is an isomorphism for $* \geq 2$ where p are the covering maps. Hence,

$$\pi_* K \xrightarrow{\cong} \pi_* K \qquad \& \qquad \pi_* \widetilde{L} \xrightarrow{\cong} \pi_* L.$$

Hence, $\widetilde{f}_*$ being an isomorphism when $* \geq 2$ implies f_* is an isomorphism for $* \geq 2$ and then Whitehead's Theorem lets us win. $\square$

Remark 43. The homology of the universal cover of a space is more or less the same as "taking homology with localcoefficients".

Here is a fact. Givena module M over the group ring $\mathbb{Z}[\pi_1 X]$, we can define $H_*(X; M) = h_*(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} M)$ (in this case, $\mathbb{Z}[\pi_1]$ acting trivially on abelian groups M recovers $H_*(X; M)$). Observe that $C_*(\widetilde{X})$ is a complex of right $\mathbb{Z}[\pi_1 X]$-modules.

Example 13.5. We given an application. If M^3 is a 1-connected 3-manifold, then $M \simeq S^3$.

Proof. If $\pi_1 M = 0$ and $\pi_0 M = 0$, we deduce that $H_1 M = 0$ since it is the abelianization of $\pi_1 M$. Further, $H^1 = \mathrm{Hom}(\pi_1 M, \mathbb{Z}) = 0$.

Using Poincaré duality $H_2 \cong H^1 \cong 0$ and $H_3 \cong H^0 \cong \mathbb{Z}$.

Hence, M is a homology 3-sphere i.e. it has the same homology as S^3.

To prove the result, we construct a map $M \to S^3$ or $S^3 \to M$ that induces an isomorphism on homology and apply Whitehead's Theorem.

To do this, choose a 3-ball B^3 inside some closed char of M. Then look at the quotient (draw a picture if need be)

$$M \to M/(M - \operatorname{Int} B^3) \cong S^3.$$

The claim is that this induces an isomorphism on H_0 and H_3. The fact it does so for H_0 is easy. So the hard part is H_3. We use local degrees for this.

Indeed, since this is the quotient map and we do not touch any point inside $\operatorname{Int} B^3$, we know that the local degree is always ± 1. Since the preimage of any point of S^3 will contain only one point, the local degree is always ± 1. Furthermore, the degree of the map is just ± 1. That means the induces map on H_3 is multiplication by ± 1 which is an isomorphism. $\square$

14. Lecture 14: April 27th, 2022

The whole point of this lecture was to discuss intersection theory. First, we quick review of simplicial homology. Given a simplicial complex K, we can form $C_*^{simp}(K; \mathbb{Z})$. We define an orientation on a finite set S to be a choice of ordering of elements of S considered up to even permutation. Hence, there exists 2 orientations on any set (but for the 1-point set, we require modification). We write $C_n(K)$ for the simplcial chain complex and define

$$C_n(K) := \bigoplus_{\text{oriented } n\text{-simplicies}} \mathbb{Z} \;/\; \{\overline{\sigma} = -\sigma\}$$

where $\overline{\sigma}$ is the reversed orientation for σ. Define

$$\partial : C_n(K) \to C_{n-1}(K)$$

by the formula

$$\partial((a_0, \ldots, a_n)) = \sum_{i=0}^{n} (-1)^i (a_0, \ldots, \widehat{a_i}, \ldots, a_n)$$

and one easily sees that $\partial\partial = 0$. This gives a chain complex and the homology of this complex is denoted by $H_*^{simp}(K)$.

Fact, $H_*^{simp}(K) = H_*^{cell}(K)$ because any simplicial complex can be thought of as a cell-complex with simplex-shaped cells.

Fact. Any smooth manifold can be triangulated. That is, there is a homeomorphism to a simplicial complex. However, for topological manifolds of dimensions $n \geq 4$, there are counterexamples.

For rest of this lecture, we discussed how to view n-cocycles and n-coboundaries as functions on n-cycles and n-boundaries. We then discussed Poincaré duality as an instance of counting intersections. Details of this point of view can be found in Hatcher's Algebraic Topology Chapter 3.

15. Lecture 15: April 29th, 2022

See the next four figures 1, 2, 3, 4.

Lecture 15: Algebraic Topology April 29th, 2022. (1)

Representing a homology class in $H_p(X;\mathbb{Z})$ by a map from a closed manifold (oriented) of dim p.

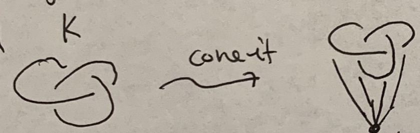
$$\begin{array}{ccc} P^p & \xrightarrow{f} & X \\ [P] & \longrightarrow & f_*[P] \\ \cap & & \cap \\ H_p(P;\mathbb{Z}) & & H_p(X;\mathbb{Z}) \\ \| & & \\ \mathbb{Z} & & \end{array}$$

The best case is if f is an inclusion.

Let $X = M^n$ be a closed oriented manifold. A locally flat p-submf of M is a subspace P^p s.t. P^p a p-mfd and $\forall x \in P$, $\exists\, U \ni x$ local chart homeomorphic to $\mathbb{R}^n$; $\phi : U \xrightarrow{\simeq} \mathbb{R}^n$ s.t. $(U, U \cap P, x) \longrightarrow (\mathbb{R}^n, \mathbb{R}^p \times \{0\}, \{0\})$.

Expl: Non-locally flat mfd is a cone on a trefoil.

Trefoil K cone it



This is not locally flat b/c you can look at the point, b/c you always intersect a copy of the trefoil.

$\cong S^1$ $\quad$ $\cong CS^1 \cong D^2$

Rmk: All smooth manifolds are locally flat. "If we were doing differential top. then it is for free".

An injective immersion of a p-mfd $\hookrightarrow$ n-mfd has a locally flat image.

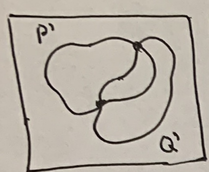
If P^p and Q^q are locally flat submfds of M^n, then we can ask if they're transvers

Def: "$P \pitchfork Q$" transverse if:

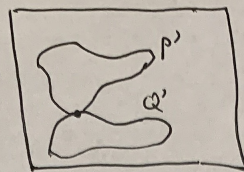
$\boxed{\text{See Below}}$

FIGURE 1. Page 1.

(2)

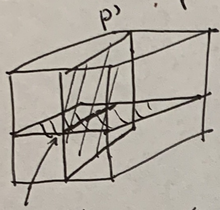


is transverse

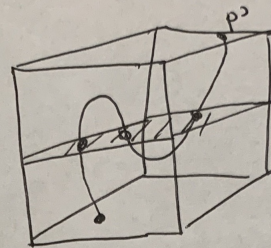


not transverse.

We do not allow cusps not nodes and etc.

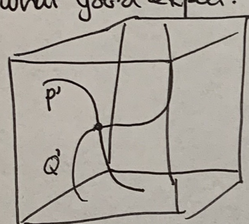


is transverse.



is transverse

Intersection is a "1-dim" mfd. Generally what you'd expect.



is not transverse
"Small perturbation destroys the intersection"

Def: ~~"$P \pitchfork Q$"~~ "$P \pitchfork Q$":

(a) $p+q<n$ then P and Q are disjoint.

(b) $p+q \geq n$ then $\forall x \in P \pitchfork Q$, $\exists$ local chart $(U,x) \xrightarrow{\Phi} (\mathbb{R}^n, 0)$ of M s.t.

$$P \cap U \longrightarrow \mathbb{R}^p \times \{0\}^{n-p}$$
$$Q \cap U \longrightarrow \{0\}^{n-q} \times \mathbb{R}^q$$
$$P \cap Q \cap U \longrightarrow \{0\}^{n-q} \times \mathbb{R}^{p+q-n} \times \{0\}^{n-p}$$

Figure 2. Page 2.

3

Obsv If $P^p \pitchfork Q^q$, then $P \cap Q$ is a closed $(p+q-n)$-mfd if (b) holds.
This is b/c of of our assumptions (use same U as the local chart!).

Orientation: If M, P, and Q are all oriented, then $P \cap Q$ inherits an induced orientation!

Personal description: (By Choose a "frame" for $P \cap Q$ at x i.e. an ordered basis of its tangent space.
Choose "temp" basis for $P \cap Q$ $\{v_1, \dots, v_{n-p-q}\}$. Extend to a basis by choosing $e_1, \dots, e_{n-p}$ for P and $f_1, \dots, f_{n-q}$ and compare the whole basis to the ambient basis of M.
If we disagree, negate of the v_i's.

Thm: If $P^p \pitchfork Q^q$ is in M^n (all closed orientable mfds) then $\underset{H_{p+q-n}(M)}{\underset{\cap}{[P \cap Q]}} =$

$$D(D^{-1}\underset{H_pM}{\underset{\cap}{[P]}} \cup D^{-1}\underset{H_qM}{\underset{\cap}{[Q]}})$$

and D is the Poincaré Duality $D: H^i(M) \to H_{n-i}(M)$.

$D^{-1}[P] \cup D^{-1}[Q] \in H^{2n-p-q}$ b/c of D then D of this lands in H_{p+q-n}.

"intersection is dual to cup product.

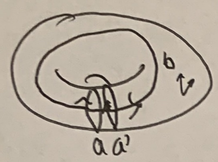
$$\begin{tikzcd}
H_p(M) \otimes H_q(M) \arrow[r, "- \bullet -"] \arrow[d, "D^{-1}", "\wr"'] & H_{p+q-n}(M) \arrow[d, "D^{-1}", "\wr"'] \\
H^{n-p}(M) \otimes H^{n-q}(M) \arrow[r, "\cup"] & H^{2n-p-q}(M)
\end{tikzcd}$$

AKA" If we define $- \cdot -$ then
Thm $[P] \cdot [Q] = [P \cap Q]$ (when $P \pitchfork Q$).

Figure 3. Page 3.

(4)

Ex Compute H^* ring of T^2.



We have $H_1 T \cong \mathbb{Z}^2$ generated by $[a]$ and $[b]$.
View a and b as closed submfd.

Obsv. $[a]\circ[b] = (+1)[\text{point}] \in H_0 T^2$

$[a]\circ[a]$ $\qquad$ $[a]\circ[a'] = \cancel{\emptyset}\, 0$ b/c $a \cap a' = \emptyset$.

not transverse so replace a by a' = parallel pushoff

Also $[b]\circ[a] = -[pt]$

$$\Longrightarrow H^*(T^2) = \Lambda[\alpha^{(1)}, \beta^{(1)}] \text{ and } \alpha = D^{-1}[a] \text{ and } \beta = D^{-1}[B].$$

Rmk product $\circ$ is graded skew-symmetric (??), i.e. $[P]\circ[Q] = (-1)^{p+q}[Q]\circ[P]$.

Next, we will just be discussing qualifying exam questions.

FIGURE 4. Page 4.

16. Lecture 16: May 2nd, 2022

If $D : H^i(M) \to H_{n-i}(M)$ is the duality map for M^n a closed oriented manifold and P^p, Q^q are transversely intersecting submanifiolds of M, then

$$D^{-1}[P] \smile D^{-1}[Q] = D^{-1}[P \cap Q]$$

where $P \cap Q$ is then a $p+q-n$ manifold which is canonically oriented. The idea is that one should think with $\cap$ and prove with $\smile$.

The best case occurs when $p+q=n$. In this case, a transverse intersection counts precisely $\#P \cap Q$. We seen then that for a submanifold P^p, it has a dual in $H^q(M)$ which can be interpreted as a function on $H_q(M)$ in $\mathrm{Hom}(H_q(M), \mathbb{Z})$. That is,

$$D^{-1}[P] \to [Q] \to \#P \cap Q$$

where the second map is "count intersections with P".

Afterwards, we discussed examples in class such as $\mathbb{C}P^n$ and T^3. Some facts were also stated.

Fact 1. A complex manifold always has a canonical orientation.

Fact 2. Bezout's Theorem states that if C_1, C_2 are degree d_1, d_2 curves in $\mathbb{C}P^2$, then if they intersect transversely, then they intersect $d_1 \cdot d_2$ many times.

Definition 16.1. If (M, μ) is an oriented closed n-manifold with μ a generator of $H_n(M;\mathbb{Z})$, then f is **amphirical** if it induces an orientation-reversing self-homotopy equivalence i.e. $f : M \to M$ is.t. $f_*(\mu) = -\mu$.

Remark 44. If no such f exists, we say M is **chiral** and so (M, μ) and $(M, -\mu)$ are nonisomorphic oriented manifolds.

Theorem 16.2. $\mathbb{C}P^{2n}$ is chiral.

Proof. If f were an amphichiral map, we know $f^*(\alpha^{2n}) = -\alpha^{2n}$ for α a generator in degree 2. However, $f^*(\alpha) = \pm\alpha$ implies

$$f^*(\alpha^{2n}) = (\pm 1)^{2n}\alpha^{2n} = \alpha^{2n} \neq -\alpha^{2n}.$$

It follows that f can only induce the identity in top homology since $H^{4n}(\mathbb{C}P^{2n}) \to H^{4n}(\mathbb{C}P^{2n})$ is the identity. $\square$

17. Lecture 17: May 4th, 2022

Today we discussed the Fall 2021 Topology Qualifying Exam at UCSD.

18. Lecture 18: May 6th, 2022

Today we discussed some old qualifying exam problems that were given at UCSD.

19. Lecture 19: May 9th, 2022

Today we discussed some old qualifying exam problems that were given at UCSD.

20. Lecture 20: May 11th, 2022

We discussed some old qualifying exam problems that were given at UCSD.

21. Lecture 21: May 13th, 2022

There is no lecture.

22. Lecture 22: May 16th, 2022

There is no lecture.

23. Lecture 23: May 18th, 2022

There is no lecture.

24. Lecture 24: May 20th, 2022

We discuss **fibre bundles**. The idea is that we have a family of copies of F indexed by some base B.

Definition 24.1. We say $F \hookrightarrow E \xrightarrow{p} B$ is a locally-trivial fibre bundle if B is covered by open sets $\{U_i\}$ s.t.

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "\cong"] \arrow[dr, "p|_{p^{-1}(U_i)}"'] & U_i \times F \arrow[d, "\text{projection}"] \\ & U_i \end{tikzcd}$$

is a commutative diagram. We write ϕ_i for the isomorphisms and we say ϕ_i is a fibre-preserving homeomorphism. We say the fibre bundle is globally trivial if $E \cong B \times F$.

We say two fibre bundles are equivalent it there is a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "\cong"] \arrow[dr, "p_1"] & E_2 \arrow[d, "p_2"] \\ & B \end{tikzcd}$$

Remark 45. A bundle comes with **transition functions** $g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F)$. We call $\mathrm{Aut}(F)$ the structure group.

It is often useful to consider bundles with a restricted structure group to create more refined structures.

Example 24.2. We could look at an $\mathbb{R}^n$-bundle $\mathbb{R}^n \to E \to B$ where $g_{ij}(x) \in \mathrm{GL}(n_j, \mathbb{R}) \leq \mathrm{Homeo}(\mathbb{R}^n)$ which means that the fibres can be equipped with the structure of a real vector space in a canonical way.

Example 24.3. Covering spaces are fibre bundles where the fibre is a discrete space F.

The Möbius strip is an $\mathbb{R}^1$-vector bundle $I \times \mathbb{R}/(0,x) \sim (1,-x) \to I/0 \sim 1 = S^1$

The Hopf fibration is defined by $S^1 \to S^3 \to S^2$. One can write this down via charts and get generalizations like

$$\mathbb{C}^* \to \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n.$$

In a similar vein, $\{\pm 1\} \to S^n \to \mathbb{R}P^n$ and $S^3 \to S^{4n+3} \to \mathbb{H}P^1$ are fibre bundles.

In the case of $n = 1$, we recover $S^3 \to S^7 \to S^4$.

Example 24.4. If G is a Lie group and H is a closed subgroup, then $H \to G \to G/H$ is a fibre bundle.

Example 24.5. There are fibrations $\mathrm{SO}(n) \to \mathrm{SO}(n+1) \to S^n$ and $\mathrm{SU}(n) \to \mathrm{SU}(n+1) \to S^{2n+1}$. Also,

$$\mathrm{Sp}(n) \to \mathrm{Sp}(n+1) \to S^{4n+3}.$$

Example 24.6. Steifel manifolds $V_k(\mathbb{R}^n($ are the space of orthonormal k-frames in $\mathbb{R}^n$ acted on by $O(n)($ transitively with stabilizer $O(n-k)$.

So there is a fibration

$$O(n-k) \to O(n) \to V_k(\mathbb{R}^n).$$

Example 24.7. Grassmanians $G_k(\mathbb{R}^n)$ are the space of k-dimensional subspaces of $\mathbb{R}^n$. There are fibrations

$$O(k) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n) \qquad \& \qquad O(k) \times O(n-k) \to O(n) \to G_k(\mathbb{R}^n).$$

Example 24.8. Letting $n \to \infty$ in the Hopf fibration example,

$$\{\pm 1\} \to S^\infty \to \mathbb{R}P^\infty$$

is a fibration. Other examples are

$$S^1 \to S^\infty \to \mathbb{C}P^\infty \qquad \& \qquad S^3 \to S^\infty \to \mathbb{H}P^\infty.$$

In some sense, fibration are a homotopy version of fibre bundles.

Definition 24.9. Let $p : E \to B$ be called a **fibration** if it satisfies the **homotopy lifting property** (HLP) for all spaces Z. That is, given $H : Z \times I \to B$ and a lift $\widetilde{H_0} : Z \times \{0\} \to E$, then there exists a lift $\widetilde{H} : Z \times I \to E$ extending H_0.

Example 24.10. There are examples of fibrations which are not a fibre bundle. For instance, take a right triangle in the plane's first quadrant and project to the corresponding interval on the real line.

Theorem 24.11. Any fibre bundle has the HLP for CW-complexes. This is technically called a **Serre fibration**.

Proof Sketch. The trivial fibre bundle certainly has the HLP – even in a relative situation where we're given a lift on $A \times I$ for some $A \subseteq Z$ (uses HEP). Then use a cell-by-cell lifting argument. □

Example 24.12. The **path fribation** can be defined as follows. Let (X, x_0) be a based space. Let $P_{x_0}X = \{\text{paths } I \xrightarrow{\gamma} X \text{ starting at } x_0\}$. The projection p maths that path to its initial point.

This has fibres at x_0 given by $\Omega_{x_0}X$. This is a fiirbation because of an explicit lifting formula.

Theorem 24.13. Let $F \to E \to B$ be a fibration. Then there is a LES of homotopy groups

$$\ldots \to \pi_{i+1}B \xrightarrow{\partial} \pi_i F \to \pi_i E \to \pi_i B \xrightarrow{\partial} \pi_{i-1}F \to \ldots.$$

The proof of the theorem can be found in Hatcher.

25. Lecture 25: May 23rd, 2022

Theorem 25.1. If $F \to (E, e_0) \to (B, b_0)$ is a fibration, then there is a LES

$$\ldots \to \pi_i F \to \pi_i E \to \pi_i B \xrightarrow{\partial} \pi_{i-1} F \to \ldots \to \pi_0 F.$$

Proof. See Hatcher's textbook. The proof itself is focused on showing that $\pi_n B$ isomorphic to the relative homotopy group. $\square$

We state some consequences.

(1) $\pi_i E = \pi_i B$ for $i \geq 2$ when $E \to B$ is a covering map.
(2) $S^1 \to S^3 \to S^2$ induces an isomorphism $\pi_i S^3 \cong \pi_i S^2$ for all $i \geq 3$.
(3) $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ induces an isomorphism $\pi_i S^{2n+1} \cong \pi_i \mathbb{C}P^n$. In particular, $\pi_2 S^{2n+1} \cong \pi_2 \mathbb{C}P^n = \mathbb{Z}$. Letting $n \to \infty$, we get the fibre bundle

$$S^1 \to S^\infty \to \mathbb{C}P^\infty$$

and since S^∞ has trivial homotopy from the LES, we deduce that it is contractible by Whitehead's Theorem. Furthermore, we know that $\pi_i \mathbb{C}P^\infty = \mathbb{Z}$ iff $i = 2$ and is zero elsewhere. Therefore, $\mathbb{C}P^\infty$ is an Eilenburg-Maclane space i.e. $K(\mathbb{Z}, 2)$.
(4) $\pi_i O(n)$ stabilizes for large n because $O(n) \to O(n+1) \to S^n$ which means $\pi_i O(n) \cong \pi_i O(n+1)$ for large n.
A natural question is to ask what occurs at $n = \infty$ and if there is a space with such homotopy groups. Additionally, what would happen if we replaced $O(n)$ by $U(n)$. The answer to these questions are encoded in **Bott periodicity** which one can read about in Hatcher's Algebraic Topology.
(5) A fibration with contractible fibres is a weak homotopy equivalence. If E and B CW-complexes, then one gets an actual homotopy equivalence.

Lemma 25.2. The fibre's of a fibration are all homotopy equivalent.

Proof. See Hatcher. The proof itself was also sketched in class. $\square$

The lemma allows us to define an action $\pi_1(B, b_0)$ on F_{b_0} as a group of self-homotopy equivalences. So we get an induced map $\pi_1(B, b_0)$ acting on $H_*(F_{b_0})$. In some sense, "fibrations are a sort of local system". Some may write $H_*(B; H_*(F_{b_0}))$ for the homology with this module structure. In particular, it is defined by $H_*(C_*(\widetilde{B} \otimes_{\mathbb{Z}[\pi_1 B]} H_*(F))$.

Let us discuss functoriality. Fibre bundles on a family of objects parameterized by a base space B are naturally contravariant. Let $f : X \to B$ be a morphism. We can form a pull-back

$$\begin{tikzcd} f^*B \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

via $f^*B = \{(x, e) : f(x) = p(E)\} \subseteq X \times E$. We mention some properties about pull-backs, but do not prove anything.

(1) The pull-back of fibre bundle is also a fibre bundle $f^*E \to X$ and similarly the pull-back of fibrations is also a fibration.
(2) The restriction is also a pull-back i.e. if $i : A \hookrightarrow B$, then $i^*E = E|_A$.

(3) Pull-backs tend to make bundles less "twisted" in a sense. Think of the Möbius strip as a fibre bundle over S^1 and the map $f : S^1 \to S^1$ given by $z \mapsto^2$.
(4) Pull-backs are homotopy invariant. That is, for $f_0 \simeq f_1$, then $f_0^* E \simeq f_1^* E$ is a fibre-wise homotopy equivalence.
(5) As a consequence of (4), any fibration over a contractible space is trivial i.e. $\simeq B \times F$.

For the next lecture, we shall discuss homotopy fibres.

26. Lecture 26: May 25th, 2022

Today we discuss the homoitopy fibre. Recal that the process of taking quotients is not a homotopy invariant concept.

The solution was to use the mapping cylinder. Given $f : X \to Y$, we can replace Y by the mapping cylinder M_f so that $X \hookrightarrow M_f$ and $Y \simeq M_f$. Since $X \hookrightarrow M_f$ as X_0, we can take the quotient M_f/X_0 which call the mapping cone. This is a homotopy invariant object. Indeed, if $f \sim f'$, then $M_f/X_0 \simeq M_{f'}/X_0$. This is somehow the correct notion of a cokernel. The dual concept is called the homotopy fibre.

Definition 26.1. The homotopy fibre replaces the concept of a kernel. In general, fibre's may vary a lot. To define it, take

$$\wp Y = \text{free space } \{\text{paths } \gamma : I \to Y\}$$

which is a fibration over Y in two ways. We can have $\gamma \mapsto \gamma_0$ or $\gamma \mapsto \gamma_1$ for $\wp Y \to Y$.

Let Pf be defined as the pullback where $f : X \to Y$

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] & \wp Y \arrow[d, "\gamma \mapsto \gamma_0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Observe that $P_f \to X$ is a homotopy equivalence. One can define it via the shortening of paths. The map $P_f \to Y$ given by $(x, \gamma) \to \gamma_0$ is a fibration. So there is a homotopy commuting triangle

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ P_f \arrow[u, hook, "\simeq"] \arrow[ur, "\mathrm{ev}_1"'] & \end{tikzcd}$$

and here, ev_1 is the evaluation at 1 map. The evaluation at 1 map is a replacement for $f : X \to Y$ similar to how M_f replaces $f : X \to Y$ which is firbation. The **homotopy fibre** is then then preimage of the basepoint of Y i.e.

$$F_f := \{(x, \gamma : f(x) \to y_0)\}.$$

Example 26.2. If X is a point and $f : X \hookrightarrow Y$ maps X to y_0, then $F_f = \Omega_{y_0} Y$ is the loop space based at y_0.

It is not hard to check the following facts.

(1) If $F \xrightarrow{i} E \xrightarrow{p} (B, b_0$ is a fibration, then the homotopy fibre F_p is homotopic to F.

(2) The homotopy fibre F_i in (1) is homotopic to $\Omega_{b_0}B$. Hence, for a fibration, we get a sequence of spaces

$$\ldots \to \Omega E \to \Omega B \to F_i \to F \to E \to B$$

where $F_i \simeq \Omega_{b_0}B$ and $F \simeq F_p$. On the other hand, each space on the LES is the homotopy fibre of the map on the RHS and this always defines a firbation.

This sequence above is called the **Puppe sequence**.

We can generalize the Puppe sequence as follows. Given any mapping $f : X \to Y$,

$$\ldots \to \Omega F_f \to \Omega X \to \Omega Y \to F_f \to X \xrightarrow{f} Y.$$

This is a sort of exact sequence which is made precise in the next theorem.

Theorem 26.3. If Z is a space, then applying $[Z, -]$ to the sequence gives an exact sequence

$$\ldots \to [Z, \Omega Y] \to [Z, F_f] \to [Z, Y] \to [Z, X] \to [Z, Y]$$

where each set is a pointed set so exactness is defined in the usual sense for sets.

Proof. Omitted. See Hatcher's Algebraic Topology. This was covered in class, but is not included in these notes. □

Remark 46. In the special case of $Z = S^0$, we get the usual LES of fibrations.

Let us discuss the dual notion. First, we define a cofibration and proceed to construct the dual notion of the homotopy fibre.

Definition 26.4. A mapping $A \to X$ is a **cofibration** if the HEP is true i.e.

$$\begin{tikzcd} A \times I \cup_f X \times \{0\} \arrow[r, "H"] \arrow[dr] & X \times I \arrow[d, dotted, "\widetilde{H}"] \\ & Z \end{tikzcd}$$

Any map can be converted into a cofibration via the mapping cylinder. So any map $f : X \to Y$ has a cofibration $X \to Y \to C_f$ given by the mapping cone.

Suppose we iterated this construction. Then we get

$$X \xrightarrow{f} Y \to C_f \to \Sigma X \to \Sigma Y \to \Sigma C_f \to \ldots$$

and this also gives an exact sequence when we apply $[-, Z]$.

Now we discuss a bit of **obstruction theory**. Our goal for now is to explain why one is interested in obstruction theory. Recall that $H^*(X; \pi)$ is an abelian group. It is the natural to study the behavior of these cohomology groups since

$$H^*(X, \pi) = [X, k(\pi, n)].$$

The basic problem is study the extension problem for for CW-pairs (X, A). We are interested in the extension

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \\ X \arrow[ur, dotted, "\exists \widetilde{f}\ ?"'] & \end{tikzcd}$$

When does $\widetilde{f}$ exists? An obvious idea is to try and extend the map A along the cells of $X - A$ one at a time. Obviously, one can get an extension on the 0-skeleton. But once one gets to the 1-skeleton and beyond, there are technical issues.

Let's do this inductively. Suppose we extended f to $X^n \cup A \to Y$. Then to extend over $(n+1)$-cells of $X - A$, we want to associated some object to control when this is permitted. For ease, let's assume $\pi_1(Y)$ acts on $\pi_* Y$ as the trivial action so we can ignore basepoints. We will define an obstruction cochain

$$c(f_n) \in C^{n+1}_{cell}(X, A; \pi_n Y)$$

which we will prove it controls the existence of the extension. We shall see this next time!

27. Lecture 27: May 27th, 2022

Recall that the basic problem we wanted to solve was on how to extend maps for CW-pairs. Let (X, A) be a CW-pair and $f : A \to Y$ be a map. When can we extend to $\widehat{f} : X \to Y$? Assuming f has extended to $f_n : X^n \cup A \to Y$, one can define the **obstruction cochain**

$$c(f_n) \in C^{n+1}_{cell}(X, A; \pi_n Y)$$

which measures our ability to extend over the $(n+1)$-cells of $X - A$.

(1) $c(f_n)$ is always a cocycle,
(2) if f_n and g_n are different extensions $X^n \cup A \to Y$ which agree on $X^{n-1} \cup A \to Y$, then they define a difference cochina $d(f_n, g_n) \in C^n(X, A; \pi_n Y)$ which satisfies $\delta(d(f_n, g_n)) = c(f_n) - c(g_n)$,
(3) the class $[c(f_n)] \in H^{n+1}(X, A; \pi_n Y)$ vanishes iff f_n can be changed on $X^n \cup A$ relative to $X^{n-1} \cup A$ s.t. it extends to $X^{n+1} \cup A$,
(4) for the homotopy problem, if f_0, f_1 are the two functions $X \to Y$ homotopy on $A \subseteq X$, then we can define a sequence of classes in $H^n(X, A; \pi_n Y)$ whose avanishing controls the extension of homotopies on $X^{n-1} \cup A$ to $X^n \cup A$ relative to $X^{n-2} \cup A$.

    Using this, one can show that if M^n is a closed orientable n-manifold, then the homotopy classes are isomorphic to $\mathbb{Z}$ via the degree map i.e. $[M^n, S^n] \cong \mathbb{Z}$.

    To prove this, consider the obstruction to making $M^n \to S^n$ nullhomotopic. The primary obstruction lives in $H^n(M; \pi_n S^n) = H^n(M; \mathbb{Z}) = \mathbb{Z}$. This vanishes iff one can chance the homotopy at the previous stage so that it extends.

We now discuss Eilenburg-Maclane spaces. Let $K(\pi, n)$ be an EM-space which is also a CW-complex. One can construct $K(\pi, n)$ as follows. Take a single 0-cell. Attach n-cells and $(n+1)$-cells so that there is an isomorphism between the nth homology and π. By the Hurewicz Theorem, $\pi_n(X) \cong \pi$ and then we kill all higher homotopy groups by attaching cells of dimension $\geq n+2$.

Lemma 27.1. $K(\pi, n)$ is unique up to homotopy equivalence.

Proof. If $X := K(\pi, n)$ maps over X to $Y := K(\pi, n)$ on the $(n+1)$-skeleton so that we get an isomorphism on H_n and hence, an isomorphism on π_n. Extend the map to the $(n+2)$-skeleton because the obstruction lives in $H^{n+3}(X; \pi_{n+1}(Y)) = H^{n+3}(X; 0) = 0$. Similarly, all higher dimensions vanish and we get a map. This map induces an isomorphism on π_*. By Whitehead's Theorem, we are done. $\square$

Similarly, we can show $[X, K(\pi, n)] \cong H^n(X, \pi)$. So $H^n(-;\pi)$ is a representable functor represented by $\mathrm{Hom}_{Htpy}(-, K(\pi, n))$. One does the proof by consider the obstruction to making $f : X \to K(\pi, n)$ null-homotopic. The first obstruction lives in $H^n(X, \pi_n(K(\pi, n)) = H^n(X, \pi)$. Also, there are no higher obstructions because they sit in $H^{n+i}(X, \pi_{i>n}(K(\pi, n))) = 0$.

Remark 47. In fact, obstruction classes are natural. Given $(Z, B) \xrightarrow{g} (X, A) \xrightarrow{f} Y$, then

$$H^{n+1}(Z, B, \pi_n Y) \ni c(f \circ g) = g^* c(f) \in g^* H^{n+1}(X, A, \pi_n Y).$$

Also, there exists what is called a fundamental class $c \in H^n(K(\pi, n); \pi)$ which is the obstruction to being $\simeq$ to the identity of $K(\pi, n)$. We can view $c(f)$ for $X \to K(\pi, n)$ as $c(f) = f^* c$ for $X \to K(\pi, n)$ because $X \xrightarrow{f} K(\pi, n) \xrightarrow{\mathrm{id}} K(\pi, n)$.

Remark 48. $[K(\pi, n), K(\pi', n)] = \mathrm{Hom}(\pi, \pi')$ illustrates the functoriality of EM-spaces. More generally, the Yoneda lemma says

$$[K(\pi, n), K(\pi', n)] = H^{n+1}(K(\pi, n); \pi') = \mathrm{Nat}(H^n(-, \pi), H^n(-, \pi')).$$

Such things are called **cohomology operations**. Another example of a cohomology operation is the map $\alpha \mapsto \alpha^2$ for $\alpha \in H^n(X) \to H^{2n}(X)$. It corresponds to an element of $H^{2n}(K(\mathbb{Z}, n); \mathbb{Z})$.

From these observations, one sees that $H^*(K(\pi, n))$ are quite important for understanding topological operations and complicated.

28. Lecture 28: June 1st, 2022

Omitted. This lecture and the next lecture are omitted from these notes because they covered spectral sequences and their applications. The lectures are not exactly easy to type notes for. A useful reference is A User's Guide to Spectral Sequences.

29. Lecture 29: June 3rd, 2022

Omitted.

CHAPTER 4

Exercises: Algebraic Topology by Allen Hatcher

Abstract. All exercises in this section are taken from Hatcher's *Algebraic Topology* version as of January 24th, 2022.

1. Chapter 0

Hatcher Chapter 0, Problem 2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.'

Proof. Define $H(x,t) := (1-t)x + t\frac{x}{\|x\|}$ which is a map $(\mathbb{R}^n - \{0\}) \times I \to \mathbb{R}^n - \{0\}$. This map is clearly continuous. Now, we check that this is a deformation retraction (with Zhouli's definition, this is called a strong deformation retraction):

(1) $H(x,0) = (1-0)x + 0\frac{x}{\|x\|} = x$ for all $x \in \mathbb{R}^n - \{0\}$;
(2) $H(x,1) = (1-1)x + 1\frac{x}{\|x\|} = \frac{x}{\|x\|} \in S^1$ for all $x \in \mathbb{R}^n - \{0\}$;
(3) $H(a,1) = \frac{a}{\|a\|} = a$ for all $a \in S^1$;
(4) $H(a,t) = (1-t)a + t\frac{a}{\|a\|} = (1-t)a + ta = a$ for all $a \in S^1$.

□

Hatcher Chapter 0, Problem 3(c).

(a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.
(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.
(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Proof. (c) Let $f : X \to Y$ and $g : Y \to X$ be a homotopy equivalence for X and Y. Let $h \simeq f$ be a map homotopic to the homotopy equivalence. We shall show that h, g is a homotopy equivalence for X and Y as well.

First, show $g \circ h \simeq \mathrm{id}_X$. Let $H(x,t)$ be the homotopy between f and h with $H(x,0) = h(x)$ and $H(x,1) = f(x)$. The homotopy $g \circ h \simeq g \circ f$ is given by

$$G(x,t) := \begin{cases} g \circ f(x) & t = 1 \\ g(H(x,t)) & 0 < t < 1 \\ g \circ h(x) & t = 0 \end{cases}. \tag{4}$$

This map is continuous since $g(H(x,t))$ is a composition of two continuous functions with $g(H(x,0)) = g(h(x))$ and $g(H(x,1)) = g(f(x))$. But homotopy is an equivalence relation for maps so

$$g \circ h \simeq g \circ f \simeq \mathrm{id}_X \implies g \circ h \simeq \mathrm{id}_X. \tag{5}$$

Next, $h \circ g \simeq f \circ g$ since we have a homotopy

$$F(y,t) := \begin{cases} f \circ g(y) & t = 1 \\ H(g(y),t) & 0 < t < 1 \\ h \circ g(y) & t = 0 \end{cases} \tag{6}$$

Again, the map $H(g(y),t)$ is continuous in y since it is a composition of continuous functions, $H(g(y),0) = h \circ g(y)$, and $H(g(y),1) = f \circ g(y)$. Finally,

$$h \circ g \simeq f \circ g \simeq \mathrm{id}_Y. \implies h \circ g \simeq \mathrm{id}_Y. \tag{7}$$

□

Hatcher Chapter 0, Problem 9. Show that a retract of a contractible space is contractible.

Proof. Let $r : X \to A$ be a retract of a contractible space X. A space X is contractible iff $\mathrm{id}_X \simeq C_{x_0}$ where C_{x_0} is a constant map. From Hatcher Exercise 0.12, we can deduce that a contractible space is path-connected. So we may take $x_0 \in A$. Let $G : X \times I \to X$ be a homotopy from id_X to C_{x_0}. We claim that $H(a,t) := G(r(a),t)) : A \times I \to A$ is gives a homotopy $\mathrm{id}_A \simeq C_{x_0}$ and so A is contractible. Indeed,

(1) for all $a \in A$, $H(a,0) = G(r(a),0) = \mathrm{id}_X(r(a)) = \mathrm{id}_A$;
(2) for all $a \in A$, $H(a,1) = G(r(x),1) = C_{x_0}(r(a)) = x_0 = C_{x_0}$;
(3) $H(a,t)$ is continuous because $r(a)$ and $G(a,t)$ are.

□

Remark. This exercise can be used to show that there is no retract of D^2 to S^1 its boundary. Indeed, D^2 is a contractible space while S^1 is not a contractible space. This gives another proof of the second part of Theorem 1.9 in Hatcher's textbook.

Hatcher Chapter 0, Problem 10. Show that a space X is contractible iff every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \to X$ is nullhomotopic.

Proof. We start with a result stated in class that was not proven.

Lemma: A space X is contractible iff id_X is null-homotopic.

Proof. ($\implies$): If $X \simeq \{*\}$, there exist a homotopy equivalence $f : X \to \{*\}$, $g : \{*\} \to X$. So $g \circ f \simeq \mathrm{id}_X$. Observe that $g \circ f$ maps $X \to \{*\} \to x \in X$ where $g(*) = x$ and so $\mathrm{im}(g \circ f) = \{x\}$. Thus, $g \circ f$ is the constant map $C_x : X \to X$ and $g \circ f \simeq \mathrm{id}_X$ implies $\mathrm{id}_X \simeq C_x$.
($\impliedby$): If id_X is null-homotopic, $\mathrm{id}_X \simeq C_x$ for some x and $C_x : X \to X$ a constant map. Let $f : X \to \{x\}$ be the constant map onto $\{x\}$ and $g : \{x\} \to X$ be the inclusion map. These are continuous. Clearly, $f \circ g = \mathrm{id}_{\{x\}}$ so $f \circ g \simeq \mathrm{id}_{\{x\}}$. On the other hand, $g \circ f : X \to X$ is the constant map because for all $y \in X$,

$$g \circ f(y) = g(x) = x = C_x(y).$$

But then $g \circ f = C_x \simeq \mathrm{id}_X$. So f, g is a homotopy equivalence and X is contractible. □

We now proceed to the proof of the exercise.
First Statement: ($\implies$): Let X be contractible. We know $\mathrm{id}_X \simeq C_x$ for an $x \in X$ and suppose it is given by a homotopy $F(s,t)$ where $F(s,0) = \mathrm{id}_X(s)$ and $F(s,1) = C_x(s)$. Let $f : X \to Y$ be a map with Y arbitrary and so,

$$f \simeq f \circ \mathrm{id}_X \simeq f \circ C_x \tag{8}$$

where the last homotopy $H(s,t) : X \times I \to Y$ is given by

$$H(s,t) = f(F(s,t)). \tag{9}$$

Indeed,

$$H(s,0) = f(F(s,0)) = f(\mathrm{id}_X(s))$$
$$H(s,1) = f(F(s,1)) = f(C_x(s)).$$

But we observe that $f \circ C_x : X \to Y$ is a constant map since

$$X \to \{x\} \to Y \qquad \text{is given by} \qquad s \mapsto x \mapsto f(x) \text{ where } s \in X \text{ is arbitrary.} \tag{10}$$

That is, $f \circ C_x = C_{f(x)}$. So, f is nullhomotopic from 8
($\impliedby$): Let $Y := X$ and f be the identity map $\mathrm{id}_X : X \to X$. Then id_X is nullhomotopic and we know that this is equivalent to X being contractible by the lemma.
Second Statement: ($\implies$): If X is contractible, $\mathrm{id}_X \simeq C_x$ for an $x \in X$ and assume the homotopy is given by $F(s,t)$ s.t. $F(s,0) = \mathrm{id}_X$ and $F(s,1) = C_x$. Let $f : Y \to X$ be arbitrary. Then,

$$f \simeq \mathrm{id}_X \circ f \simeq C_x \circ f \tag{11}$$

where the last homotopy $G(s,t) : Y \times I \to Y$ is given by

$$G(s,t) = F(f(s),t). \tag{12}$$

Indeed,

$$G(s,0) = F(f(s),0) = \mathrm{id}_X(f(s))$$
$$G(s,1) = F(f(s),1) = C_x(f(s)).$$

Now, observe that for $L_x : Y \to X$ being the constant map to x,

$$C_x(f(s)) = x = L_x(s)$$

for all $s \in Y$ which means $C_x \circ f = L_x$. So, f is nullhomotopic.
($\impliedby$): Take $Y := X$ and $f : Y \to X$ to be the identity map. Then id_X is nullhomotopic. Our lemma implies X is contractible. $\square$

Hatcher Chapter 0, Problem 12. Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X.

Proof. **Lemma 1.** Every path connected subset of X is contained in a unique path component.
Proof of Lemma 1. Let P be a path connected subset. Suppose there were two path components Q, R containing P. Then $Q \cup R$ is path connected and it contains Q. By maximality, $Q \cup R \subseteq Q$ and this implies $R \subseteq Q$. Interchanging Q and R gives $Q \subseteq R$ and so $Q = R$. For existence of a path component, we define C to be the union over all path-connected subsets of X which contain P. Then C has the desired properties.
Definition. Let $\mathfrak{P}(X)$ denote the set of path components of X. Let $P \in \mathfrak{P}(X)$. Then, there exists a path component Q s.t. $f(P) \subseteq Q \in \mathfrak{P}(Y)$. We define $f_* : \mathfrak{P}(X) \to \mathfrak{P}(Y)$ by $P \mapsto Q$ and this is well defined because if P is path-connected, then so is $f(P)$ and every path-connected set is contained in a unique path component.
Lemma 2. We show $(g \circ f)_* = g_* \circ f_*$.
Proof of Lemma 2. If $P \in \mathfrak{P}(X)$, $Q = f_*(P)$, and $R = g_*(Q)$, then $g_* \circ f_*(P) = R$. Now, $f(P)$ is contained in the path component Q and $g(f(P)) \subseteq g(Q) \subseteq R$ by definition of $g_*(Q) = R$. Since every set is contained in a unique path component, $(g \circ f)_*(P) = R$. This shows $(g \circ f)_* = g_* \circ f_*$.
Lemma 3. We show that homotopic maps induce the same maps on path components.
Proof of Lemma 3. Let $f : X \to Y$ and $h : X \to Y$ be homotopic maps with homotopy $H(s,t)$ s.t. $H(s,0) = f(s)$. Let $P \subseteq X$ be a path-component. Then, $P \times I$ is path-connected and $H(s,t)$ being continuous means $H(P \times I)$ is a path-connected subset of Y. So, it is contained in a path component $Q \in \mathfrak{P}(Y)$. Lemma 1 and the definition implies $f_*(P) = g_*(P)$ because

$$f(P) = H(P \times \{0\}), g(P) = H(P \times \{1\}) \subseteq H(P \times I) \subseteq Q.$$

Proof that homotopy equivalence induces bijection on path components. If $g : Y \to X$ is s.t. f, g is a homotopy equivalence, we have a map $g_* : \mathfrak{P}(Y) \to \mathfrak{P}(X)$. To prove that f induces a bijection between path-components, we need to show that $f_* \circ g_*$ and $g_* \circ f_*$ are the identity. We get

$$g_* \circ f_* \overset{\text{Lemma 2}}{=} (g \circ f)_* \overset{\text{Lemma 3}}{=} (\mathrm{id}_X)_* \qquad \& \qquad f_* \circ g_* \overset{\text{Lemma 2}}{=} (f \circ g)_* \overset{\text{Lemma 3}}{=} (\mathrm{id}_Y)_*$$

The fact that identity maps id_X induce the identity map $(\mathrm{id}_X)_* : \mathfrak{P}(X) \to \mathfrak{P}(X)$ on the set of path components finishes the proof.

Let P be a path component of X and $Q := f_*(P)$. Consider the restricted maps $f|_P : P \to Q$ and $g|_Q : Q \to P$ which are well-defined and continuous (well-definedness follows from our work above). Then,

$$f|_P \circ g|_Q \simeq \mathrm{id}_Q \qquad \& \qquad g|_Q \circ f|_P \simeq \mathrm{id}_P.$$

Proof that homotopy equivalence induces bijection on components. The idea of proof is essentially the same as for the case with path components. We generalize our lemmas. First, it is a standard result of point-set topology that every connected subset of X is contained in a unique connected component. Let $\mathfrak{Q}(X)$ denote the set of components of X. Then define $f^* : \mathfrak{Q}(X) \to \mathfrak{Q}(Y)$ in the same manner as f_* except we replace "path-connected" with "connected" and "path component" with "connected component". The same method of proof shows $(g \circ f)^* = g^* \circ f^*$. The same proof of Lemma 3 holds as well because the product of any family of connected spaces is also connected. Then, $(g^* \circ f^*) =$

$(\mathrm{id}_X)^*$ and $(f^* \circ g^*) = (\mathrm{id}_Y)^*$ implies that there is a bijection $\mathfrak{Q}(X) \to \mathfrak{Q}(Y)$ since id_X induces the identity map $(\mathrm{id}_X)^*$ on $\mathfrak{Q}(X)$.

Last Statement. Suppose path-components of X coincide with components and consider diagram.

$$\begin{tikzcd} \mathfrak{Q}(X) \arrow[r, "\mathrm{id}"] \arrow[d, shift right, "f^*"'] & \mathfrak{P}(X) \arrow[d, shift right, "f_*"'] \\ \mathfrak{Q}(Y) \arrow[u, shift right, "g^*"'] & \mathfrak{P}(Y) \arrow[u, shift right, "g_*"'] \end{tikzcd}$$

The maps $f_* \circ \mathrm{id} \circ g^* : \mathfrak{Q}(Y) \to \mathfrak{P}(Y)$ and $f^* \circ \mathrm{id} \circ g_* : \mathfrak{P}(Y) \to \mathfrak{Q}(Y)$ are bijective. We chase around this map to confirm that a component C is mapped to a path component which is C. Following $f^* \circ \mathrm{id} \circ g_*$,

$$\begin{aligned} C \text{ a component of } Y \mapsto g(C) \subseteq D \text{ a component of } X \mapsto g(C) \subseteq D \text{ a path component of } X \\ \mapsto f(D) \subseteq E \text{ a path component of } Y. \end{aligned}$$

Essentially, the map sends C to the unique path component of Y that contains C since $(f \circ g)^* = (\mathrm{id}_Y)^*$. That path component E must be C because if not, $C \subsetneq E$ and E is connected which contradicts C being a component. Thus, C being component implies it is also a path component. Since $f_* \circ \mathrm{id} \circ g^*$ is a bijection and $f_* \circ \mathrm{id} \circ g^*(C) = C$ for all $C \in \mathfrak{Q}(Y)$, we conclude $\mathfrak{Q}(Y) = \mathfrak{B}(Y)$. So path components and components of Y coincide. □

Hatcher Chapter 0, Problem 14.

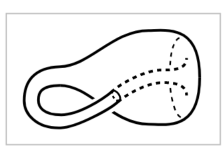
PROOF. Construct the cell structure inductively. Since we know $v - e + f = 2$, it is clear that $v + f = e + 2$ so induct on e. Alternatively, draw any connected graph on the sphere and include the unbounded region once as a face.

[Incomplete] □

Hatcher Chapter 0, Problem 16.

PROOF. □

Hatcher Chapter 0, Problem 20. Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.



PROOF. See figure 1. We give a rough justification as to why each of the steps is a homotopy equivalence. First, we note that the Klein bottle is immersed inside $\mathbb{R}^3$ so the first homotopy is valid. Indeed, we would just shrink the region around the circle of intersection to a point. It is also valid for us to shrink the tube into a line in the second step because of this immersion. The second to last step follows by just moving the end points of the line segment inside the Klein bottle. □

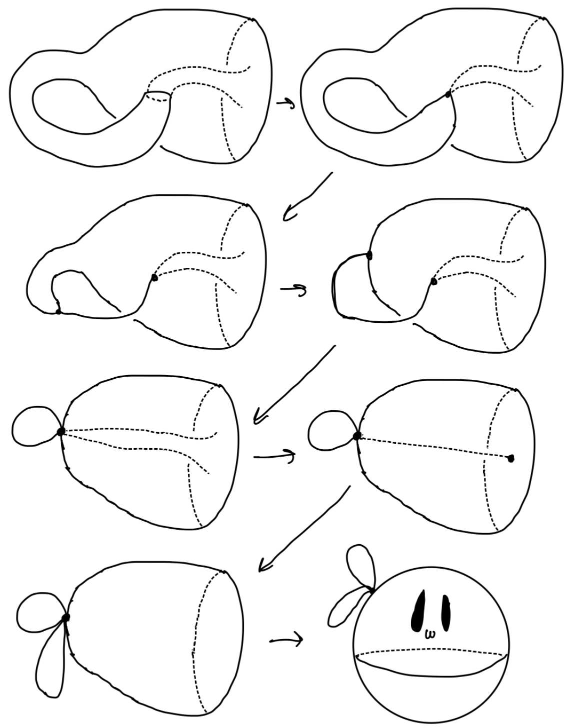


FIGURE 1. A series of homotopy equivalences that give the desired homotopy equivalence $X \simeq S^1 \vee S^1 \vee S^2$. It is happy.

Hatcher Chapter 0, Problem 21. If X is a connected Hausdorff space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that X is homotopy equivalent to a wedge sum of S^1's and S^2's.

PROOF. We shall show X has a natural CW complex structure (see the very end of this proof). We shall also use the result regarding collapsing subcomplexes on p.11 of Hatcher without comment

First, let X_0 be the disjoint union of 0-cells and there is one designated for each sphere. Then, for every pair of spheres that touch at one point, attach a 1-cell connecting their respective 0-cells. Then, attach 2-cells to each point to get the spheres. This gives a space which we call Y. Now, by contracting the 1-cells of Y, we obtain the space X. See figure 2 for an example with three 2-spheres each touching each other at least once.

Now, to obtain the desired homotopy equivalence (also pictured in figure 2), look at the underlying skeleton consisting of only the 1-cells and 0-cells. This is a graph and it is a connected graph because our space X is connected. For every cycle with k points in the graph, we have a homotopy equivalence of the corresponding spheres with 1-cells connecting them to the wedge sum $S^1 \vee \left(\bigvee_{i=1}^k S^2\right)$.

After, we may remove the cycle, add a point in place of it, and any edges not in the cycle but was touching a point removed is attached to this new single point. Then repeat this again with another cycle if there still exists a cycle in the graph.

As there are a finite number of spheres, this process removes all cycles in the graph. This new space is homotopy equivalent to X, but it is also just connecting a finite collection of wedge sums of S^1 and S^2 via 1-cells. Contract all of the 1-cells and we get a new space that is just a wedge sum of S^1's and S^2's. See figure 3 for a visual of what we have done.

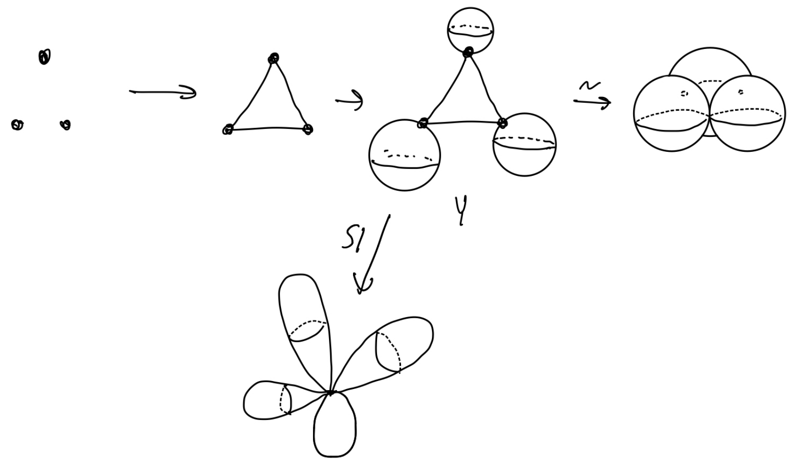


FIGURE 2. The horizontal arrows demonstrate the process by which we construct X when X is the union of three 2-spheres touching a single point. If we follow the proof, then the cycle in the graph second to the left is replaced by a single point.

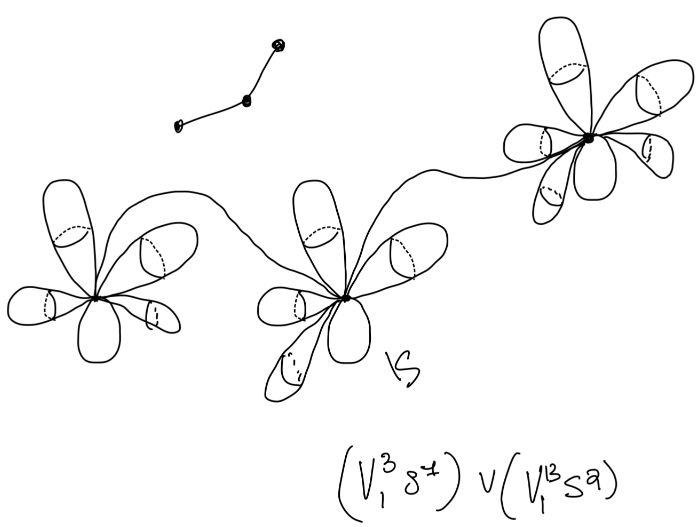


FIGURE 3. A demonstration of the final step of the proof. After removing all of the cycles, we have remaining a bunch of wedge sums of S^1's and S^2's which are connected by a 1-cell. Our graph visual is that we just the graph with three nodes and two edges. Contract the 1-cells represented by the edges to get the desired homotopy equivalence.

Now we actually show that X has a CW complex structure. This requires Proposition A.2 of Hatcher (on p. 522) and this is where we use X Hausdorff. The maps are: $\Phi_\alpha : D^0_\alpha \to X$ which map onto the intersection points, $\Phi_\beta : D^1_\beta \to X$ which map onto bands that connect the intersection points where two spheres meet, $\Phi_\gamma : D^2 \to X$ map onto the 0-cells. In this case, each of these maps restrict to a homeomorphism of the interior of the disk onto the image. Furthermore, each of the cells are disjoint and their union is certainly X. For each cell e^i_η, $\Phi_\eta(\partial D^i_\eta)$ is contained in a union of a finite number of cells (in particular, at most two). Finally, a subset X is closed iff it meets the closure of each cell of X in a closed set because the topology on X is the one induced by the topology on the 2-spheres. □

2. Section 1.1

Hatcher Section 1.1, Problem 1. Show that composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$

Proof. From hypothesis, $f_0 \simeq f_0 \cdot g_0 \cdot \overline{g_0} \simeq f_1 \cdot g_1 \cdot \overline{g_0} \simeq f_1 \cdot g_0 \cdot \overline{g_0} \simeq f_1$. □

Hatcher Section 1.1, Problem 2. Show that the change-of-basepoint homomorphism β_h depends only on the homotopy class of h.

Proof. This is immediate from definition. If $h \simeq h'$, then we find that

$$\beta_h[f] = \beta_{h'}[f] \iff [h \cdot f \cdot \overline{h}] = [h' \cdot f \cdot \overline{h'}] \iff [h \cdot f \cdot \overline{h}] \cdot [h' \cdot \overline{f} \cdot \overline{h'}] = [C_x].$$

But as $h \simeq h'$, we know that $\overline{h}$ and h' cancel. So we have iff $[h \cdot f \cdot \overline{f} \cdot \overline{h'}] = C_x$. This is clearly true. □

Hatcher Section 1.1, Problem 3. For a path-connected space X, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.

Proof. ($\implies$): Assume $\pi_1(X)$ is abelian. We wish to show that $\beta_h = \beta_{h'}$ whenever $x_0 := h(0) = h'(0)$ and $x_1 := h(1) = h'(1)$. Let $[f] \in \pi_1(X, x_0)$ be arbitrary. Then

$$\begin{aligned}\beta_h([f]) = \beta_{h'}([f]) &\iff [\overline{h} \cdot f \cdot h] = [\overline{h'} \cdot f \cdot h'] \in \pi_1(X, x_1) \iff [\overline{h} \cdot f \cdot h][\overline{h'} \cdot \overline{f} \cdot h'] = [L_{x_1}] \\ &\iff [\overline{h} \cdot f \cdot h \cdot \overline{h'} \cdot \overline{f} \cdot h'] = [L_{x_1}]\end{aligned}$$

where $L_{x_1} : I \to X$ is the constant loop at x_1. We will prove this last equality. Since, $[f \cdot h \cdot \overline{h'}]$ and $[\overline{f}]$ are loops based at x_0, and $\pi_1(X, x_0)$ is abelian,

$$[f \cdot h \cdot \overline{h'} \cdot \overline{f}] = [\overline{f} \cdot f \cdot h \cdot \overline{h'}].$$

Therefore,

$$[\overline{h} \cdot f \cdot h \cdot \overline{h'} \cdot \overline{f} \cdot h'] = [\overline{h} \cdot \overline{f} \cdot f \cdot h \cdot \overline{h'} \cdot h'] = [\overline{h} \cdot h \cdot \overline{h'} \cdot h'] = [\overline{h'} \cdot h'] = [L_{x_1}]$$

($\impliedby$): Assume β_h depends only on the endpoints. Let $[f], [g] \in \pi_1(X, x_0) \cong \pi_1(X)$. We need $[f \cdot g] = [g \cdot f]$. Since f is a loop based at x_0, and the base-point change homomorphism $\beta_f : \pi_1(X, x_0) \to \pi_1(X, x_0)$ only depends on the endpoints, $\beta_f = \beta_{L_{x_0}}$ where L_{x_0} is the constant loop at x_0. Then,

$$[g] = \beta_{L_{x_0}}([g]) = \beta_f([g]) = [\overline{f} \cdot g \cdot f].$$

Then, multiplying on the left by $[f]$ gives

$$[f \cdot g] = [g \cdot f].$$

Thus, $\pi_1(X, x_0)$ is abelian and path-connectedness says $\pi_1(X, x_0) \cong \pi_1(X)$. So, $\pi_1(X)$ is abelian. □

Hatcher Section 1.1, Problem 4 A subspace $X \subset \mathbb{R}^n$ is said to be *star-shaped* if there is a point $x_0 \in X$ such that, for each $x \in X$, the line segment from x_0 to x lies in X. Show that if a subspace $X \subset \mathbb{R}^n$ is locally star-shaped, in the sense that every point of X has a star-shaped neighborhood in X, then every path in X is homotopic in X to a piecewise linear path, that is, a path consisting of a finite number of straight line segments traversed at constant speed. Show this applies in particular when X is open or when X is a union of finitely many closed convex sets.

PROOF. *Every path is homotopic to a piecewise linear path.* Let $\gamma : I \to X$ be a path in X. At $\gamma(0)$, choose a star-shaped open set U_0. Then choose the supremum $t_0 \in I$ such that $\gamma(t_0) \in U_0$. Then use a straight line homotopy to deform the part of γ in U_0 to a straight line from $\gamma(0)$ to $\gamma(t_0)$. Now do the same process at t_0. Since the unit interval is compact, $\gamma(I)$ is compact which means this process terminates i.e. are only finitely many straight line segments.

This holds for X open or X a union of finitely many closed convex sets. The result clearly holds for when X is open since every point has an ϵ-ball around it which is necessarily star-shaped. Meanwhile, if X is union of finitely many closed convex sets, every $x \in X$ lays in one of the convex sets C_i which is then star shaped. Then $x \in \operatorname{int}(C_i)$ clearly gives the result so we deal with the case where x is a boundary point. In this case, take $B_\epsilon(x) \cap C_i$ to be the desired neighborhood. $\square$

Hatcher Section 1.1, Problem 5 Show that for a space X, the following three conditions are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.
(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$
(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

PROOF. Throughout this proof, we shall write $e^{2\pi i t}$ to mean the map $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. This is for convenience and there is a homeomorphism between the circle in $\mathbb{C}$ and the circle in $\mathbb{R}^2$ so formally, we have not committed any fallacies.
(a) $\Longrightarrow$ (b): Let $f : S^1 \to X$ be homotopic to a constant path $C_x : S^1 \to X$ with homotopy $H(s,t)$ given by $H(s,1) = f(s)$ and $H(s,0) = C_x$. Elements of D^2 can be viewed in terms of polar coordinates (r,θ) with $|r| \leq 1$ and $0 \leq \theta \leq 2\pi$. We define a map $F : D^2 \to X$ given by

$$F(r,\theta) := \begin{cases} C_x(re^{i\theta}) & r = 0 \\ H(re^{i\theta}, r) & 0 < r < 1 \\ f(re^{i\theta}) & r = 1 \end{cases} = \begin{cases} C_x(0) & r = 0 \\ H(re^{i\theta}, r) & 0 < r < 1 \\ f(e^{i\theta}) & r = 1 \end{cases} \tag{13}$$

We check that this map is well-defined and is the desired map. In particular, $C_x(re^{i\theta})$ is always fixed, $F(r,0) = F(r,2\pi)$ for all r, and $F(r,\theta)|_{S^1} = f$. The first condition is clear since $C_x(re^{i\theta})$ is a constant map. The second condition follows from how $re^{2\pi i} = re^{0\pi i}$ and so, $F(r,0) = F(r,2\pi)$. The third condition is clear by $F(r,\theta)|_{S^1} = F(1,\theta) = f(e^{i\theta})$.
(b) $\Longrightarrow$ (c): Given $x_0 \in X$. Let $\ell : I \to X$ be a loop in X s.t. $\ell(0) = x_0$. We show $\ell \simeq C_{x_0}$. Let $\widetilde{\ell} : S^1 \to X$ be given by $\widetilde{\ell}(e^{2\pi i s}) = \ell(s)$. Then, (b) extends $\widetilde{\ell}$ to an $L : D^2 \to X$ which we

denote by $L(r,\theta)$. Define

$$L(t,s) =: H(e^{2\pi i s}, t) : S^1 \times I \to X. \tag{14}$$

Then, $H(e^{2\pi i s}, 0) = L(0,s)$ which is a constant map $S^1 \to X$. Let x_1 be the image of $L(0,s)$ s.t. $H(e^{2\pi i s}, 0) = C_{x_1}$. Also,

$$H(e^{2\pi i s}, 1) = L(1,s) = \widetilde{\ell}(e^{2\pi i s}).$$

Now, $H : S^1 \times I \to X$ is a continuous map which gives a homotopy $\widetilde{\ell} \simeq C_{x_1}$. Furthermore, the map $L(r,0) : I \to X$ gives a path from x_1 to x_0 so that means $\pi_1(X,x_0) \cong \pi_1(X,x_1)$:

$$L(0,0) = x_1 \qquad \& \qquad L(1,0) = \widetilde{\ell}(e^{2\pi i 0}) = \ell(0) = x_0.$$

Define $L_{x_1} : I \to X$ by $L_{x_1}(s) = x_1$ for all $s \in I$. Then $\ell \simeq L_{x_1}$ by defining $G(s,t) : I \times I \to X$ by $G(s,t) = H(e^{2\pi i s}, t)$ which gives a homotopy. Indeed, H and $s \mapsto e^{2\pi i s}$ continuous means $G(s,t)$ is continuous, and $G(s,0) = L_{x_1}$ and $G(s,1) = \widetilde{\ell}(e^{2\pi i s}) = \ell(s)$.

Therefore, $\ell \simeq L_{x_1} \simeq L_{x_0}$ which means $[\ell] = 0$ inside $\pi_1(X,x_0)$. Since ℓ was arbitrary, $\pi_1(X,x_0) = 0$.

(c) $\implies$ (a): Let $f : S^1 \to X$ be a map. Let $\widetilde{f} : I \to X$ where $\widetilde{f}(t) = f(e^{2\pi i t})$ and this is continuous since it is a composition of two continuous maps. Let $x_0 := f((1,0))$ and since $\pi_1(X,x_0) = 0$, we deduce that $\widetilde{f} \simeq \widetilde{L}_{x_0}$ where $\widetilde{L_{x_0}} : I \to X$ is the constant loop at x_0. Define $L : S^1 \to X$ by $\widetilde{L_{x_0}}(t) = L(e^{2\pi i t})$ and since $\widetilde{L_{x_0}}$ is a constant map, L is constant and continuous. We claim

$$\widetilde{f}(t) \simeq \widetilde{L}_{x_0} \implies f \simeq L_{x_0}.$$

Indeed, if $H(t,u)$ is a homotopy between $\widetilde{f}(t)$ and $\widetilde{L_{x_0}}$ s.t. $H(t,0) = \widetilde{f}(t) = f(e^{2\pi i t})$ and $H(t,1) = \widetilde{L_{x_0}} = L_{x_0}(e^{2\pi i t})$, the map $G(e^{2\pi i t}, u) := H(t,u)$ is a homotopy between f and L. Thus, f is homotopy to a constant map.

We observe that X is simply-connected iff X is path connected and $\pi_1(X) = 0$ iff X is path connected and all maps $S^1 \to X$ is homotopic to a constant map with image a point. We show this last statement is equivalent to the statement that all maps $S^1 \to X$ are homotopic.

($\implies$): Let $f : S^1 \to X$ and $g : S^1 \to X$ be a pair of maps. Then $f \simeq C_x$ and $g \simeq C_{x'}$ for points $x, x' \in X$. Since X is path connected, there is a homotopy between C_x and $C_{x'}$. Indeed, let $\phi : I \to X$ be a path from x to x' and $C_{\phi(t)} : S^1 \to X$ given by mapping of all S^1 to $\phi(t)$. Then, $H(s,t) = C_{\phi(t)}$ is a homotopy between C_x and $C_{x'}$.

($\impliedby$): Assume all maps $S^1 \to X$ are homotopic. Let $x, x' \in X$ and $C_x \simeq C_{x'}$ is given by a homotopy $H(s,t) = f_t(s)$. Then $f_t((1,0)) : I \to X$ is a path in X with $f_0((1,0)) = x$ and $f_1((1,0)) = x'$. Thus, X is path-connected.

Let $f : S^1 \to X$ be an arbitrary map. Let $C_x : S^1 \to \{x\} \subseteq X$ be a constant map. Then $f \simeq C_x$. Thus, $\pi_1(X) = 0$. $\square$

Hatcher Section 1.1, Problem 7. Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles. [Consider what f does to the path $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$.]

PROOF. Take the homotopy $f_t(\theta, s) = (\theta + 2\pi st, s)$. This fixes the circle $S^1 \times \{0\}$. Now there is no homotopy that fixes both circles. Let $\gamma : I \to S^1 \times I$ be the given by $s \mapsto (\theta_0, s)$. Suppose f_t is a homotopy from $f_0 = \mathrm{id}_{S^1 \times I}$ and $f_1 = f$. Then,

$$f_1(\gamma) : I \to S^1 \times I,\ s \mapsto (\theta_0 + 2\pi s, s) \qquad \& \qquad f_0(\gamma) : I \to S^1 \times I,\ s \mapsto (\theta_0, s).$$

So the homotopy must give a path connecting $(\theta_0 + 2\pi s, s)$ to (θ_0, s) for all s. For some $0 < s' < 1$, we would need to connect the points $(\theta_0 + 2\pi s', s')$ to (θ_0, s') via f_t. □

Hatcher Section 1.1, Problem 8. Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?

PROOF. No. Consider the map $f : S^1 \times S^1 \to \mathbb{R}^2$ that maps onto the annulus in the xy-plane. Visually, one can imagine this as flattening the torus. It is clear that there is not anitpodal point that maps to the same point. Indeed, (x, y) and $(-x, y-)$ would lay in a different quadrant of the annulus (here quadrant refers to the usual four quadrants in the xy-plane). □

Hatcher Section 1.1, Problem 9. Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Use the Borsuk-Ulam theorem to show that there is one plane $P \subset \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.

PROOF. See this post for a rough sketch of the proof. □

Hatcher Section 1.1, Problem 10. From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \approx \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

PROOF. The construction is relatively straightforward. □

Hatcher Section 1.1, Problem 11. If X_0 is the path-component of a space X containing the basepoint x_0, show that the inclusion $X_0 \hookrightarrow X$ induces an isomorphism $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

PROOF. The isomorphism can be constructed explicitly. Let $[\gamma] \in \pi_1(X_0, x_0)$ be a loop. Then the inclusion $i : X_0 \hookrightarrow X$ induces $i_*[\gamma] \in \pi_1(X, x_0)$ which is again a loop based at x_0. In fact, $[\gamma] = i_*[\gamma]$ since γ must land entirely inside the path component X_0. Conversely, any loop $[\eta] \in \pi_1(X, x_0)$ must lay entirely in X_0 if it is based at x_0. This gives a bijection and as i_* is clearly a homomorphism, we are done. □

Hatcher Section 1.1, Problem 12. Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

PROOF. A homomorphism $\Phi\pi_1(S^1) \to \pi_1(S^1)$ of cyclic groups is determined by where it maps the generator $[\gamma] \in \pi_1(S^1)$. So $\Phi([\gamma]) = [\gamma]^k$ for some $k \in \mathbb{Z}$. But then take $f_k : S^1 \to S^1$ to be the exponential map $e^{2\pi i\theta} \mapsto e^{2\pi i\theta k}$. Then the induced map $(f_k)_* = \Phi$ is the desired map since it sends $(\cos 2\pi s, \sin 2\pi s)$ to $(\cos 2\pi ns, \sin 2\pi ns)$. □

Hatcher Section 1.1, Problem 13. Given a space X and a path-connected subspace A containing the basepoint x_0, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in X with endpoints in A is homotopic to a path in A.

Proof. Relatively straightforward from the definition, but we sketch out the proof.
($\Rightarrow$): Assume the inclusion is surjective. Let $\gamma : I \to X$ be a path in X with endpoints in A. Draw a path from one endpoint to the other and call it η. Then $\eta \cdot \gamma$ is a path in $\pi_1(X, x_0)$ based at $x_0 =: \gamma(0)$. Then, $\eta \cdot \gamma$ lifts to a loop $\beta \in \pi_1(A, x_0)$. But this map was induced by an inclusion which means $[\eta \cdot \gamma] = [\beta]$. Then $\gamma \simeq \overline{\eta}\beta$ which is a path in A.
($\Leftarrow$): Let $[\gamma] \in \pi_1(X, x_0)$ be a loop. The end point is $x_0 \in A$ so it is homotopic to a loop $[\eta]$ in A. But then $[\eta]$ is the element s.t. $i_*[\eta] = [\gamma]$. $\square$

Hatcher Section 1.1., Problem 14. Show that the isomorphism $\pi_1(X \times Y) \approx \pi_1(X) \times \pi_1(Y)$ in Proposition 1.12 is given by $[f] \mapsto (p_{1*}([f]), p_{2*}([f]))$ where p_1 and p_2 are the projections of $X \times Y$ onto its two factors.

Proof. Relatively straightforward from definitions. $\square$

Hatcher Section 1.1., Problem 15. Given a map $f : X \to Y$ and a path $h : I \to X$ from x_0 to x_1, show that $f_*\beta_h = \beta_{fh}f_*$ in the diagram at the right.

$$\begin{tikzcd}
\pi_1(X, x_1) \arrow[r, "\beta_h"] \arrow[d, "f_*"'] & \pi_1(X, x_0) \arrow[d, "f_*"] \\
\pi_1(Y, f(x_1)) \arrow[r, "\beta_{fh}"] & \pi_1(Y, f(x_0))
\end{tikzcd}$$

Proof. Relatively straightforward diagram chase. $\square$

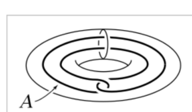
Hatcher Section 1.1, Problem 14. Show that the isomorphism $\pi_1(X \times Y) \approx \pi_1(X) \times \pi_1(Y)$ in Proposition 1.12 is given by $[f] \mapsto (p_{1*}([f]), p_{2*}([f]))$ where p_1 and p_2 are the projections of $X \times Y$ onto its two factors.

Proof. This follows from the correspondence given in the proposition. The homotopy class $[f]$ of loops was equivalent to the homotopy class of loops $[g] \times [h]$ where $f(t) = (g(t), h(t))$ and therefore, the proof had defined the map $\phi : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ by $\phi([f]) = ([g], [h]) = (p_{1*}([f]), p_{2*}([f])$.

Another way to prove the isomorphism is to use the universal property of the direct product of groups and how $\pi_1(-)$ is a functor. $\square$

Hatcher Section 1.1, Problem 16. Show that there are no retractions $r : X \to A$ in the following cases:

(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$
(c) $X = S^1 \times D^2$ and A the circle shown in the figure.



(d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$
(e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.
(f) X the Möbius band and A its boundary circle.

Proof. We fix $i : A \hookrightarrow X$ to be the inclusion map and $i_* : \pi_1(A) \to \pi_1(X)$ the induced map. We will use the result of Proposition 1.17 on p. 36 of Hatcher without comment.
(a) If a retract exists, i_* is injective. However, $\pi_1(X) \cong 0$ and $\pi_1(A) \cong \mathbb{Z}$ which means $i_* : \mathbb{Z} \to 0$ is injective. Contradiction.
(b) By Proposition 1.12 of Hatcher on p. 34,

$$\pi_1(S^1 \times D^2) \cong \pi_1(S^1) \times \pi_1(D^2) \cong \mathbb{Z} \times 0 \cong \mathbb{Z}$$
$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

So if a retraction existed, we have an injective map $i_* : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. Of course, not such map exists. Indeed, $\mathbb{Z}$ is cyclic by $\mathbb{Z} \times \mathbb{Z}$ is not (the direct product of two cyclic groups is cyclic if the orders are coprime).
(c) From above, $\pi_1(X) \cong \mathbb{Z}$ and the induced inclusion map $i_* : \pi_1(A) \to \mathbb{Z}$ should be injective if a retract exists. Let $[\ell]$ be a loop that goes around A once. Then $i_*([\ell]) = [i(\ell)]$ is a loop in X. But this loop is homotopic to the constant map. See figure 4. So the map i_* is actually the zero map. Contradiction since i_* is injective.

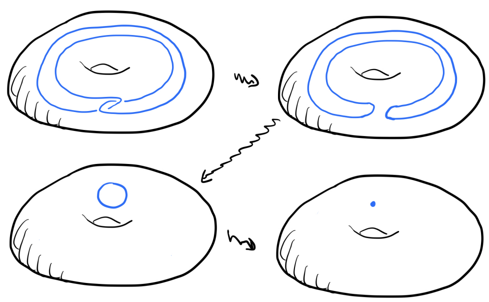


Figure 4. A homotopy that shows ℓ inside $S^1 \times D^2$ is trivial.

(f) Let $r : X \to A$ and $i : A \hookrightarrow X$. In the work of figure 5, we observe that the Möbius strip is homotopy equivalent to the circle and therefore has $\pi_1(X) \cong \mathbb{Z}$. Furthermore, $\pi_1(S^1) = \mathbb{Z}$ so our approach from previous problems will not work here. If a retraction exists, then the inclusion map induces an injective homomorphism $i_* : \pi_1(A, a_0) \hookrightarrow \pi_1(X, a_0)$. Take generators $\pi_1(A, a_0) = \langle [\ell] \rangle$ and $\pi_1(X, a_0) \cong \langle [\gamma] \rangle$. For γ, it is the circle in figure 6. From our work in figure 5, we deduce that $i_*([\ell]) = [\gamma]^2$ since the boundary circle goes around the circle generated by γ twice. Then,

$$(r \circ i)_*([\ell]) = r_*(i_*([\ell])) = r_*([\gamma]^2) = r_*([\gamma])(r_*([\gamma])).$$

However, $r \circ i = \mathrm{id}_A$ which means the induced homomorphism should be the identity. Therefore, $(r \circ i)_*([\ell]) = [\ell]$. However, $[\ell]$ cannot be a square in $\pi_1(A, a_0)$ as it is the generator of a cyclic group isomorphic to $\mathbb{Z}$.

□

Proof. **Note:** There are different versions of the Van Kampen theorem that are made, but they are virtually the same (for instance, Zhouli's statement of the theorem required X to be path-connected but the covering sets need not be). We will follow Hatcher's statement of Van Kampen's theorem when utilizing it.
(d) It follows by Van Kampen's theorem that $\pi_1(D^2 \vee D^2) \cong 0$. Indeed, we can take the open sets X_1, X_2 as indicated by figure 7, apply Van Kampen's theorem to deduce that we have

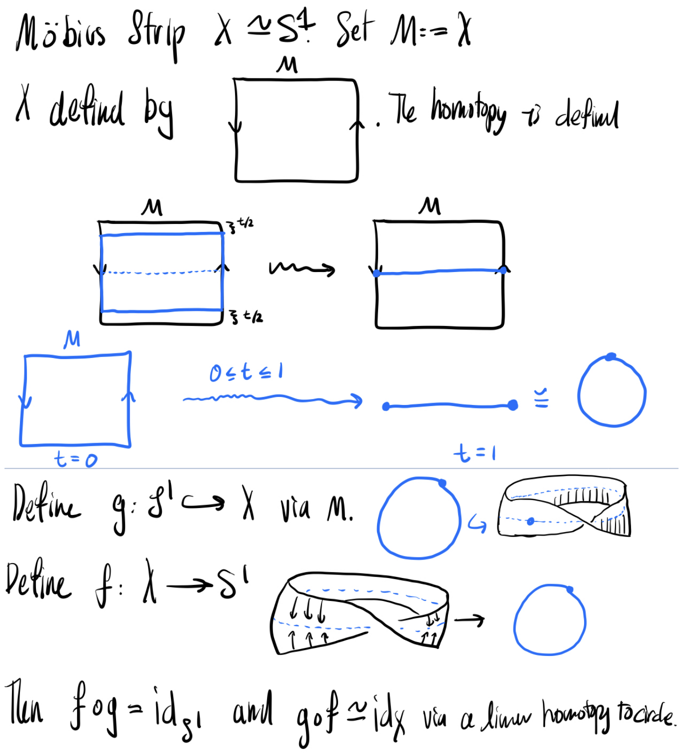


FIGURE 5. Computing the fundamental group of the Möbius strip.

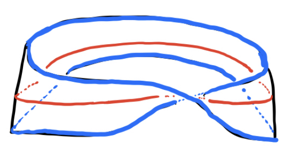


FIGURE 6. The generator for the fundamental group of the Möbius stip. The Möbius strip is path connected so the base point doesn't matter for our proof. It only matters to ensure $\pi_1(A, a_0)$ makes sense in the context of $\pi_1(X, a_0)$. Here, the red circle should really have a point on the boundary circle labeled a_0.

a surjection onto $\pi_1(D^2 \vee D^2)$. However, $\pi_1(X_i) = 0$ for both $i = 1, 2$ so $\pi_1(D^2 \vee D^2) = 0$. Alternatively, one can see that if we wedge $D^2 \vee D^2$ the two disks at their boundary, then we can deformation retract one D^2 to that point, and so $\pi_1(D^2 \vee D^2) \cong \pi_1(D^2) = 0$.

Now if a retraction of $X := D^2 \vee D^2$ onto $A := S^1 \vee S^1$ exists, we would need to have an injection $i_* : \pi_1(A) \hookrightarrow \pi_1(X)$. However, $\pi_1(A) \cong \mathbb{Z} * \mathbb{Z}$ by Van Kampen's theorem which means $\mathbb{Z} * \mathbb{Z} \hookrightarrow 0$. Contradiction.

(e) See figure 8. From that computation we get $\pi_1(X) \cong \mathbb{Z}$. If there was a retraction of X to $S^1 \vee S^1$, then the inclusion $i : S^1 \vee S^1 \hookrightarrow X$ would induce an injective map on fundamental groups. But then $i_* : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$ is injective which is absurd (by a commutativity argument). Therefore, no retraction exists.

$\square$

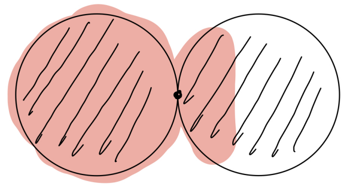


FIGURE 7. The wedge may be based at a point not on the boundary, but we can homotopically deform it so that $D^2 \vee D^2$ has identified points on the boundaries. An open set X_1 is colored. The set X_2 is taken as the same open set except done on the other disk. The intersection $X_1 \cap X_2$ is then path connected. We can take the basepoint to be the identified point in the center.

FIGURE 8. The deformation of X to a circle. Since we identified the boundary points, we get a sort of taco with a touching tip on top. This then deformation retracts onto a circle by retracting along the interior of the disk.

Hatcher Section 1.1, Problem 17. Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$.

PROOF. As before, we will write $e^{2\pi i t}$ to mean the point $(\cos 2\pi t, \sin 2\pi t)$ on S^1. By definition,

$$S^1 \vee S^1 := S^1 \sqcup S^1 / \sim := \frac{\{(e^{2\pi i t}, 0), (e^{2\pi i t}, 1) : t \in [0,1]\}}{\{(e^{2\pi i \cdot 0}, 0) \sim (e^{2\pi i \cdot 0}, 1)\}}$$

where the fraction indicates we have modded out by an identification. We will write $(S^1, 0)$ and $(S^1, 1)$ to indicate the first circle or the second circle. We will want to retract onto the first circle so S^1 in defining our maps is really $(S^1, 0)$. Define a map $r_n : S^1 \vee S^1 \to S^1$ by specifying how it acts on both circles. Set

$$r_n(p) := \begin{cases} (e^{2\pi i t}, 0) & \text{if } p = (e^{2\pi i t}, 0) \\ (e^{2\pi i 0}, 0) & \text{if } p = (e^{2\pi i 0}, 0) = (e^{2\pi i 0}, 1) \\ (e^{2n\pi i t}, 0) & \text{if } p = (e^{2\pi i t}, 1). \end{cases}$$

This is a continuous map since the preimage of any open subset of S^1 is going to be a union of open subsets of $S^1 \vee S^1$. It is a retract since $r_n(S^1 \vee S^1) = S^1$ and $r_n|_{(S^1,0)}(p) = p$.

Furthermore, $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$ by the results on p. 40 of Hatcher. Let a, b be the two generators for $\mathbb{Z} * \mathbb{Z}$. Now,

$$(r_n)_* : \pi_1(S^1 \vee S^1) \to \pi_1(S^1)$$

is given by $a \mapsto a$ and $b \mapsto b^n$ and $(i_n)_* : \pi_1(S^1) \to \pi_1(S^1 \vee S^1)$ is given by $a \mapsto a$. So, $(r_n \circ i_n)_* = (\mathrm{id}_A)_*$.

Now the maps r_n do not induce the same map of fundamental groups. Therefore, they are nonhomotopic i.e. $r_n \not\simeq r_m$ if $n \neq m$. So, $\{r_n\}_{n=1}^{\infty}$ is an infinite family of nonhomotopic maps. $\square$

Hatcher Section 1.1., Problem 18 Using Lemma 1.15, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Apply this to show:

(a) The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.

(b) For a path-connected CW complex X the inclusion map $X^1 \hookrightarrow X$ of its 1 -skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$. [For the case that X has infinitely many cells, see Proposition A.1 in the Appendix.]

Proof. Let $i : A \hookleftarrow X$ be the inclusion. Cover X by the subspace A and the subspace e_2 and the intersection is that path-connected subspace that we glued along. So any loop in X is homotopic to a product of loops in A and in e_n. Since $n \geq 2$, that portion of the loop is homotopic to a constant map. Therefore, every loop in $\pi_1(X)$ lifts to a loop in $\pi_1(A)$.

(a) The wedge sum is obtained via attaching a 2-cell e^2 onto S^1. So $S^1 \vee S^2$ is contained from the path-connected subspace S^1 and the first part says that $S^1 \hookrightarrow S^1 \vee S^2$ induces a surjection $\pi_1(S^1) \twoheadrightarrow \pi_1(S^1 \vee S^2)$. But the kernel is trivial no loop becomes trivial after the addition of e^2. So we get the desired isomorphism.

(b) The space X is obtained by attaching cells e^n for $n \geq 1$ to its one skeleton. So the first result gives the surjection $\pi_1(X^1) \to \pi_1(X)$. $\square$

Hatcher Section 1.1, Problem 20. Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \to X$.]

Proof. Apply the lemma and we know that $f_{0*} = \beta_{f_t(x_0)} f_{1*}$. Then applying to a loop $[\gamma] \in \pi_1(X, x_0)$ gives

$$[\gamma] = f_{0*}([\gamma]) = \beta_{f_t(x_0)} f_{1*}([\gamma]) = \beta_{f_t(x_0)}([\gamma]) = [f_t(x_0) \cdot \gamma \cdot \overline{f_t(x_0)}].$$

But then

$$[\gamma \cdot f_t(x_0)] = [f_t(x_0) \cdot \gamma]$$

which shows that $f_t(x_0) \in \pi_1(X, x_0)$ is a loop. $\square$

3. Section 1.2

Hatcher Section 1.2, Problem 1. Show that the free product $G * H$ of nontrivial groups G and H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.

Proof. If an element $g \in G * H$ commutes with every element of G *and* every element of H, then $xg = gx$ and $yg = gy$ for all $x \in G$ and $y \in H$. So formally, g must lay entirely in G or in H. But the copy of g in G and g in H are not identified unless $g = e$. So, $Z(G * H) = \{e\}$. $\square$

Hatcher Section 1.2, Problem 2. Let $X \subset \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \varnothing$ for all i, j, k. Show that X is simply-connected.

Proof. **Lemma 1:** If U is a convex open set, then U is simply connected.

Proof of Lemma 1. Since U is convex, we know it is path connected. Let γ be a loop in U based at $x_0 \in U$. Then, for every point a on γ, there is a line $(1-t)a + tx_0$ from a to x_0 that lies in U. Take the homotopy $H(x,t) = (1-t)x + tx_0$ which is a map $H : \gamma(I) \times I \to U$. So, $\gamma \simeq C_{x_0}$ and $\pi_1(U) = 0$. □

Lemma 2: If U, V are convex open sets s.t. $U \cap V \neq \emptyset$, then $\pi_1(U \cup V) = 0$. Furthermore, $U \cup V$ is path-connected.

Proof of Lemma 2. We verify the hypothesis of Van Kampen's theorem part (1). Choose a basepoint $x_0 \in U \cap V =: X$.

Given $a, b \in U \cap V$, convexity in U and V shows that the line $(1-t)a + tb$ lies in U and V. So this gives a path from a to b lying entirely in $U \cap V$.

Since $U \cap V$ is path-connected, Van Kampen's theorem says there is a surjection $\pi_1(U) * \pi_1(V) \to \pi_1(X)$. But Lemma 1 showed $\pi_1(U) = \pi_1(V) = 0$ which means $\pi_1(X) = 0$.

For path-connectedness of $U \cup V$, note that the nontrivial case is when $a \in U$ and $b \in V$. In this case, choose $x_0 \in U \cap V$. Draw a line from a to x_0 and from x_0 to b. The desired path from a to b is the composition of these two paths. □

Lemma 3: If U, V, W are convex open sets s.t. $X := U \cap V \cap W \neq \emptyset$, then X is simply connected.

Proof of Lemma 3. We apply Van Kampen's theorem with two sets $U \cup V$ and W. These sets are open. Lemmas 1 and 2 show that they are path-connected.

The intersection $(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$ is path-connected by the next argument. Let a, b be two given points and $x_0 \in U \cap V \cap W \neq \emptyset$. We can draw a path from a to x_0 since a is in the union and a path from x_0 to b for the same reason. The composition is the desired path.

We apply Van Kampen's theorem to get a surjection $\pi_1(U \cup V) * \pi_1(W) \to \pi_1(X)$. Lemmas 1 and 2 imply we have a surjection $0 \to \pi_1(X)$ and so, $\pi_1(X) = 0$. □

Now we prove the main result. It suffices to assume that $n > 3$. If n is even, let $U_1 := X_1 \cup X_2, \dots, U_m := X_{n-1} \cup X_n$. If n is odd, let $U_1 := X_1 \cup X_2 \cup X_3, U_2 := X_4 \cup X_5, \dots, U_m := X_{n-1} \cup X_n$. Let us deal with the case where n is even and note how the argument needs no alteration when n is odd.

First, each of the U_i are open as unions of open sets and path-connected due to Lemmas 2 and 3. Next, the intersection

$$U_i \cap U_j = (X_{2i-1} \cup X_{2i}) \cap (X_{2j-1} \cup X_{2j}) = (X_{2i-1} \cap X_{2j-1}) \cup (X_{2i-1} \cap X_{2j}) \cup (X_{2i} \cap X_{2j-1}) \cup (X_{2i} \cap X_{2j})$$

is path-connected by the our next argument. It follows by our work in the lemmas that the only nontrivial situation is when $a \in X_{2i-1} \cap X_{2j-1}$ and $b \in X_{2i} \cap X_{2j}$. Choose a point $c \in X_{2i-1} \cap X_{2j-1} \cap X_{2i}$ and $d \in X_{2i} \cap X_{2j} \cap X_{2j-1}$. Draw a path from a to c lying in $X_{2i-1} \cap X_{2j-1}$. Draw a path from c to d lying in $X_{2i} \cap X_{2j-1}$. Then draw a path from d to b lying in $X_{2i} \cap X_{2j}$. The composition of these paths gives a path from a to b that lay entirely in $U_i \cap U_j$.

This argument is unaffected in the case when n is odd because $U_1 = X_1 \cap X_2 \cap X_3$ intersect some U_j, say U_2, just gives

$$U_1 \cap U_2 = (X_1 \cap X_2) \cup (X_1 \cap X_5) \cup (X_2 \cap X_4) \cup (X_2 \cap X_5) \cup (X_3 \cap X_4) \cup (X_3 \cap X_5)$$

and the same method of considering triple intersections works to show $U_1 \cap U_2$ is path connected.

By Van Kampen's theorem, we get a surjection $*_{i=1}^m \pi_1(U_i) \to \pi_1(X)$. But $\pi_1(U_i) = 0$ for all i from Lemmas 2 and 3. So $0 \to \pi_1(X)$ is surjective and so, $\pi_1(X) = 0$.

Path-connectedness of X is easy to see. Let $a, b \in X$ be arbitrary points. Assume $a \in X_i$ and $b \in X_j$. Choose a $c \in X_i \cap X_j$. Draw a path from a to c lying in X_i and a path from c to b lying in X_j. Then the composition is a path lying inside X connecting a to b.

We conclude that $\pi_1(X) = 0$ and X is path-connected. So, X is simply connected. □

Hatcher Section 1.2, Problem 3. Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

Proof. First, we show that $\mathbb{R}^n$ stays path connected after removing finitely many points. Let $p^1, \ldots, p^m$ be the points removed, set $X := \mathbb{R}^n - \{p^1, \ldots, p^m\}$, and write $p^i = (p^i_1, \ldots, p^i_n)$ for their coordinates.

For path-connected, a geometric argument is best. Let $a, b \in X$ be two distinct points. Draw a straight line from a to b in $\mathbb{R}^n$. If this line L does not cross any p^i, then we are done. Suppose L crossed some p^i. At worst, it crosses a finite number of points. We alter L by removing a line segment $U \subseteq L$ containing p^i and replace it by a small straight line that makes one of the coordinates distinct from that of p^i and then another straight line that reattaches to the other end of L. By making our line segment U arbitrary small in length, we can ensure that we do not intersect any other point when adding the two line segments. This gives a new curve L_1 lying in $\mathbb{R}^n - \{p^i\}$. If this lies entirely in X, we are done. If not, we can repeat this process again to avoid another point p^j. As we removed only finitely many points to get X, this process terminates at the nth new line L_n. See figure 9.

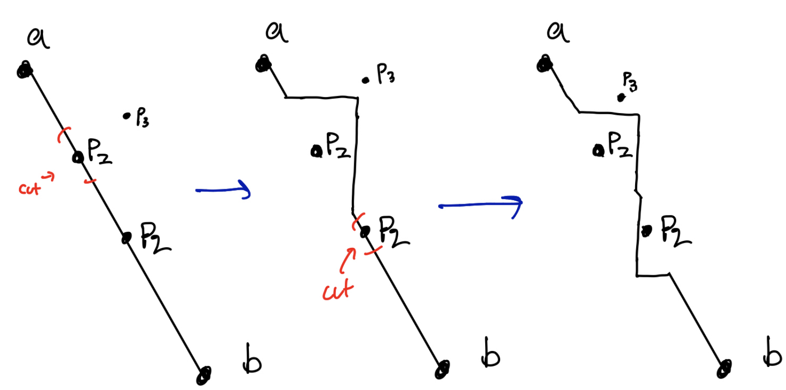


Figure 9. A dummy visual of what we are doing geometrically.

The condition $n \geq 3$ ensures $n - 1 \geq 2$ and so, $\pi_1(S^{n-1}) = 0$. The space X is homotopy equivalent to the wedge sum $\bigvee_{i=1}^m S^{n-1}$. As these spaces are nice, we know deduce that

$$\pi_1(X) \cong \pi_1(\bigvee_{i=1}^m S^{n-1}) \cong *_{i=1}^m \pi_1(S^{n-1}) \cong 0.$$

We conclude that X is simply connected for $n \geq 3$. See figure 10 for a visual of the homotopy used.

Remark: The result is false for $\mathbb{R}^2 - \{0\}$ because it deformation retracts to S^1. □

Hatcher Section 1.2, Problem 4. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

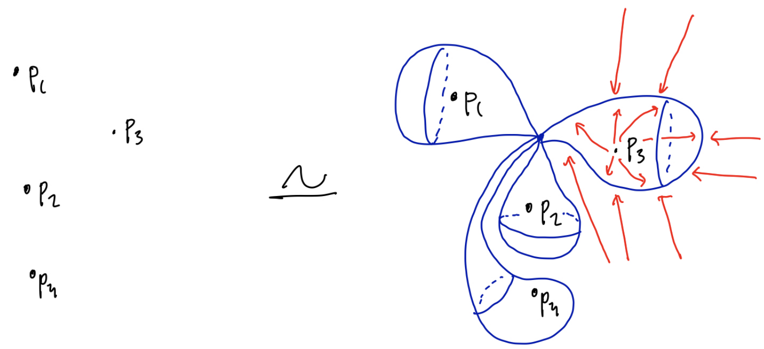


FIGURE 10. The arrows indicate a deformation retract onto the sphere. For each point removed, there is a corresponding sphere around it. The idea is to use the homotopy equivalence $\mathbb{R}^{n+1} - \{0\} \simeq S^n$ for each point.

PROOF. The space $\mathbb{R}^3 - X$ deformation retracts to a sphere with $2n$ points removed. Then this space is homeomorphic to $\mathbb{R}^2$ removing $2n-1$ points via the stereographic projection. But $\mathbb{R}^2$ removing $2n-1$ points deformation retracts to the wedge sum of $2n-1$ copies of S^1. Thus, $\pi_1(\mathbb{R}^3 - X) \cong \mathbb{Z}^{*2n-1}$. □

Hatcher Section 1.2, Problem 6. Use Proposition 1.26 to show that the complement of a closed discrete subspace of $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

PROOF. If the discrete set is empty, then there is nothing to prove. So assume it is nonempty.

Enclose each point of the discrete subspace X by a sufficiently small ball of radius $\epsilon > 0$ fixed. This defines the cell-structure on $\mathbb{R}^n \setminus X$. Indeed, we enclose each removed point with a sphere S^{n-1} of radius $\epsilon > 0$. Then add the rest of the cell structure by connecting each of the spheres together via n-cells. Since $n - 1 \geq 2$, it is clear that everything we did yields a space with trivial fundamental group.

Fix a point of the discrete closed subspace $x_0 \in X$. Now use stereographic projection to $S^{n+1} \setminus \{N\}$ where N is the north pole corresponding to the point x_0. □

Hatcher Section 1.2, Problem 7. Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

PROOF. See figure 11 for two equivalent ways to view X. See figure 12 to see the two "different" ways in which we may build the cell structure (they are actually equivalent).

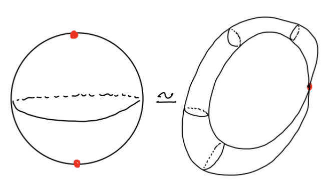


FIGURE 11. Two ways to view the sphere with two boundary points identified.

It follows from Hatcher Proposition 1.26 (or the material in Zhouli's lecture) that

$$\pi_1(X) = \pi_1(\mathrm{sk}_2\, X) = \langle a, b \mid ab, ab \rangle \cong \mathbb{Z}.$$

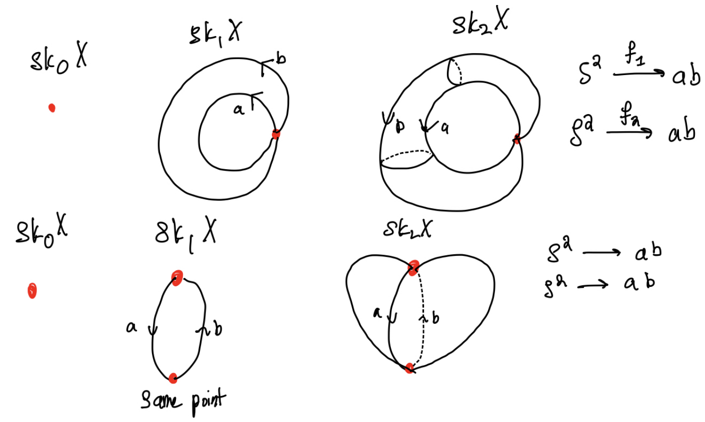


FIGURE 12. The two equivalent cell structures.

The last isomorphism is clear by the following argument. Define a map $\phi : F(a,b) \to \mathbb{Z}$ from the free group on a, b given by $a \mapsto 1$ and $b \mapsto -1$ and extend to a group homomorphism. This map is clearly surjective and $\phi(ab) = 0$ so the kernel contains ab. It is also clear that the kernel precisely contains words w in which the number of powers of a that appear is the same as the number of powers of b. It also follows that ba is in the kernel as well. So the kernel is generated by elements of form $(ab)^m$ for some m. Thereby $F(a,b)/\ker(\phi) \cong \mathbb{Z}$. But the LHS has presentation precisely equal to that of $\pi_1(\operatorname{sk}_2 X)$ and the last isomorphism is valid.

More quickly, we just observe

$$\langle a, b \mid ab, ab \rangle \cong \langle a, b \mid ab \rangle \cong \langle a, a^{-1} \rangle \cong \langle a \rangle \cong \mathbb{Z}.$$

We remark that Example 0.8 of Hatcher demonstrates that this fundamental group is the coorrect one. □

Hatcher Section 1.2, Problem 8. Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

PROOF. Let T_1 and T_2 be the tori pictured in figure 13 with generators of their fundamental group labeled. We will identify them along $S^1 \times \{x_0\}$ to get a new space called X. To compute its fundamental group, we use Van Kampen's theorem.

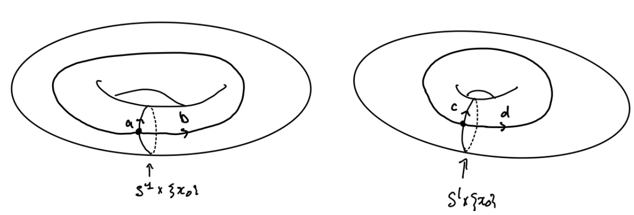


FIGURE 13. Our two given tori.

Let $U_1 := T_1 \cup V$ as pictured in figure 14. In particular, the small open set about $S^1 \times \{x_0\}$ in T_2 is crucial as it makes our sets "nice". Choose another open set U_2 dually. We can retract V onto $S^1 \times \{x_0\}$ so we deduce that $\pi_1(U_1) = \pi_1(T_1) \cong \mathbb{Z}$ and the same for $\pi_1(U_2)$.

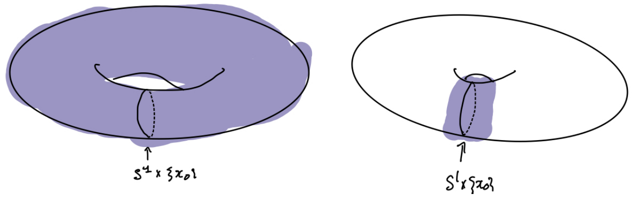


FIGURE 14. An open set U_1 in the open cover $\{U_1, U_2\}$.

The intersection $U_1 \cap U_2$ is path-connected and deformation retracts onto the circle $S^1 \times \{x_0\}$ as pictured in figure 15 (we could have take $U_1 = T_1$ and $U_2 = T_2$ from the start and avoided this).

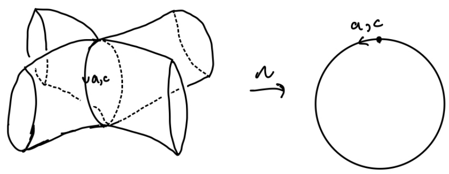


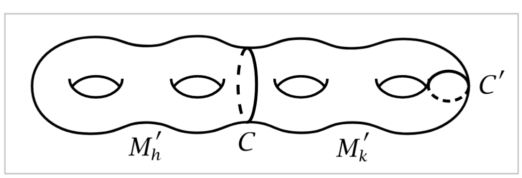
FIGURE 15. The deformation retract of $U_1 \cap U_2$ which demonstrates what the kernel should be.

It is clear from the second part of Van Kampen's theorem that the normal subgroup we quotient by is going to be the one generated by $(a, e)(b^{-1}, e)$ where e is some element of $\mathbb{Z}$. By Van Kampen's theorem, we deduce that

$$\pi_1(X) \cong \frac{\pi_1(U_1) * \pi_1(U_2)}{N} \cong \frac{\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}}{N} \cong \langle a, b, c, d \mid aba^{-1}b^{-1}, cdc^{-1}d^{-1}, ac^{-1} \rangle$$

□

Hatcher Section 1.2, Problem 9. In the surface M_g of genus g, let C be a circle that separates M_g into two compact subsurfaces M_h' and M_k' obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M_h' does not retract onto its boundary circle C, and hence M_g does not retract onto C. [Hint: abelianize π_1.] But show that M_g does retract onto the nonseparating circle C' in the figure.



PROOF. We have a sequence of lemmas which we use to build up the proof. Lemma 5 and 7 were the problems asked of us.

Lemma 1: The torus M_1 removing a disk deformation retracts to a torus removing a disk and then to $S^1 \vee S^1$. So $\pi_1(M_1') = \mathbb{Z} * \mathbb{Z}$.

PROOF. A picture proof suffices so see figure 16.

□

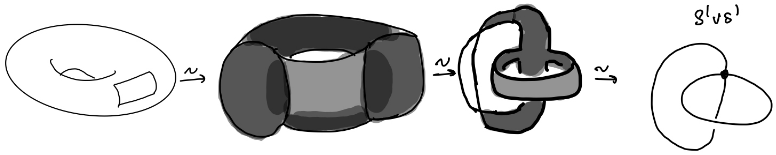


FIGURE 16. A deformation retract onto $S^1 \vee S^1$.

Lemma 2: The torus removing two points/disks deformation retracts to $S^1 \vee S^1 \vee S^1$. So it has fundamental group $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

PROOF. A picture proof suffices so see figure 17.

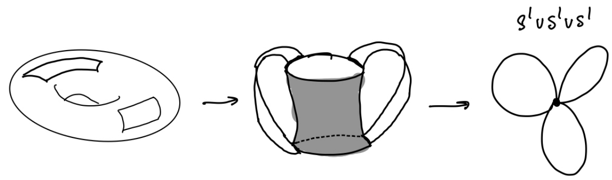


FIGURE 17. A deformation retract onto the wedge sum of three circles.

□

Lemma 3: The surface M_h' has fundamental group $M_h' \simeq \mathbb{Z}^{*2h}$ (our notation for $*_{i=1}^{2h}\mathbb{Z}$).

PROOF. Induct on h. Assume $\pi_1(M_h') \cong \mathbb{Z}^{*2h}$. Let T_2^2 be the torus with two disks removed. Now apply Van Kampen's theorem with Lemma 2 to get

$$\pi_1(M_{h+1}') \cong \frac{\pi_1(M_h') * \pi_1(T_2^2)}{N} \cong \frac{\mathbb{Z}^{*2h} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}}{\mathbb{Z}} \cong \mathbb{Z}^{2h+2}.$$

The open set is given by covering the first h holes and then covering the last "torus" which has a disk removed. This last covering is homotopy equivalent to T_2^2 because we have the boundary from the covering and the disk already removed. □

Lemma 4: There exists no retract of M_h' to C.

PROOF. Assume $[c]$ is the generator of $\pi_1(C)$. We find $i_*(\pi_1(C))$. Draw the $2h$-gon with edges oriented counter-clockwise with $[a_1, b_1] \dots [a_h, b_h]$ and $\pi_1(M_h)$ with presentation given as on p. 51 of Hatcher. Remove a small disk at the center of the $2h$-gon and so, the loop $[c]$ goes around this disk once. It thereby corresponds to the loop given by $[a_1, b_1] \dots [a_h, b_h]$. So $i_*([c])$ is the element $[a_1^{-1}b_1^{-1}a_1b_1 \dots a_g^{-1}b_g^{-1}a_gb_g]$. If a retract existed, we would have a commutative diagram

$$\begin{tikzcd} \mathbb{Z}^{*2h} \arrow[r, "r_*"] & \mathbb{Z} \\ \mathbb{Z} \arrow[u, hook, "i_*"] \arrow[ur, "\mathrm{id}_{\pi_1(C)}"'] & \end{tikzcd}$$

Applying the abelianization functor which we denote by F. Then

$$F(\mathrm{id}_{\pi_1(C)}) = F(r_* \circ i_*) = F(r_*) \circ F(i_*).$$

Then $F(\mathrm{id}_{\pi_1(C)})$ is just the identity map. Meanwhile, $F(r_*) \circ F(i_*)([c])$ is trivial because $F(i_*)$ sends $i_*([c])$ to $[a_1^{-1}b_1^{-1}a_1b_1 \dots a_g^{-1}b_g^{-1}a_gb_g]$ which is trivial due to the abelianization. So,

$$[c] = F(\mathrm{id}_{\pi_1(C)})([c]) \neq 0 = F(r_*)(0) = F(r_*) \circ F(i_*)([c]).$$

Contradiction.

Alternatively, from the universal property for the abelianization of a group, the map r_* factors through

$$\mathbb{Z}^{*2h} \xrightarrow{\phi} \prod_{i=1}^{2h} \mathbb{Z} \xrightarrow{\psi} \mathbb{Z}.$$

so $r_* = \psi \circ \phi$. However, $\phi(i_*([c])) = 0$ which means $r_*i_*([c])$ is trivial. But that means

$$(r \circ i)_*([c]) = 0 \neq [c] = \mathrm{id}_{\pi_1(C)}([c]).$$

Therefore, no such retraction can exist.

□

Lemma 5: There is no retract of M_g onto C.

Proof. If $r : M_g \to C$ is a retract, then we claim $r|_{M_h'} : M_h' \to C$ is a retract of $M_{h'}$ onto C. Indeed, $r|_{M_h'} \circ i|_{M_h'} : C \to M_h' \to C$ is the identity since it would be the restriction of the identity $r \circ i = \mathrm{id}_C$ where i is the inclusion $C \hookrightarrow M_g$. This contradicts the previous lemma. □

Lemma 6: There is a retract of the M_1 onto C'.

Proof. Define $r : M_1 \to C' =: S^1 \times \{x_0\}$ by the map $S^1 \times S^1 \to S^1 \times \{x_0\}$ sending the second component to x_0. Then $r \circ i = \mathrm{id}_{C'}$ is clear. □

Lemma 7: There is a retract of M_g onto C'.

Proof. We induct on the genus and the base case is the previous lemma. Assume the result for M_g and let $s : M_g \to C'$ be the retract. Given M_{g+1}. There is a subspace at the end of M_{g+1} that is similar to a torus. We can retract onto a circle C'' via the the map indicated in figure 17. It clear that $r : M_{g+1} \to M_g$ is a retract onto C''. Now, we claim that the composition $s \circ r$ is a retract of M_{g+1} onto C. Indeed,

$$(s \circ r \circ i)(C') = s(C') = C' = \mathrm{id}_{C'}(C')$$

because r fixes C'.

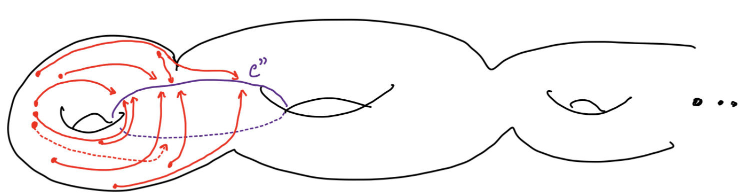


Figure 18. A retract of a subset of M_g onto a circle.

Remark: The proof of the existence of a retract onto C' can be made even more rigorous. If we draw out the gluing pattern of M_g using a $2g$-gon, we could view this as tearing the $2g$-gon at a vertex, choose the pattern given by $b^{-1}a^{-1}ba$. Then retract the other edges until we get to the pattern given by $a^{-1}a$. For ease, one could imagine doing this retract for the gluing pattern of the torus. It is a square (with $ab^{-1}a^{-1}b$ going counter clockwise) and we can tear at the vertex in the upper left corner. Then "fold" the edges along the identification i.e. b^{-1} folds into the edge given by a and b folds into a^{-1}. What remains is the circle C''. The generalization to M_g for $g > 1$ is readily clear. □

□

Hatcher Section 1.2, Problem 14. Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, of order eight.

Proof. There is some ambiguity in what Hatcher means by a one-quarter twist i.e. in which direction. However, it is safe to assume, and this is the only way to get the conclusion, to assume that each face of the cube should have four distinct edges appear. See figure 19.

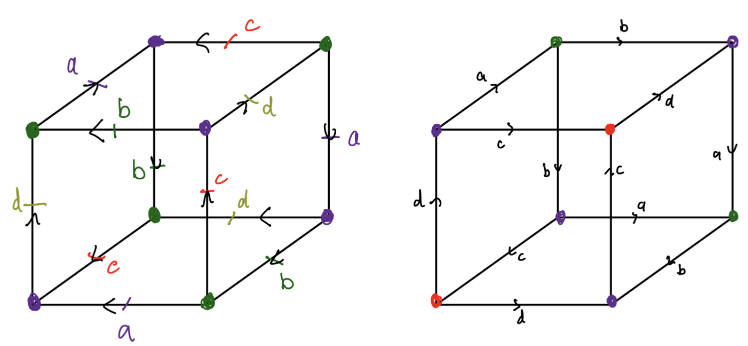


Figure 19. The cube with identifications on the LHS is what we will work with. The cube on the RHS would have three distinct points which is not what Hatcher asked of.

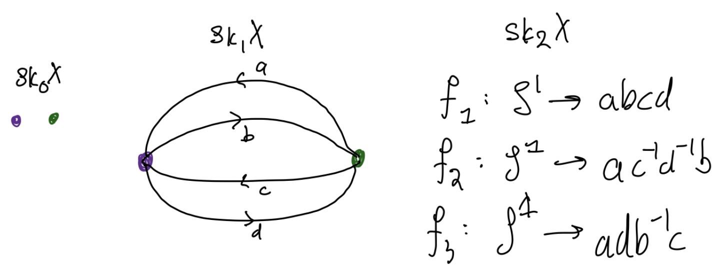


Figure 20. Our choice of cell structure is portrayed. We attached along the faces of the cube.

Now we construct the cell structure as in figure 20. Observe that a maximal tree of $\text{sk}_1 X$ is the subgroup consisting of the two points and the edge a. Therefore, $\pi_1(\text{sk}_1 X) = \langle ab, ac^{-1}, ad \rangle$ is a free group on three generators. It is clear that the loops ab, ac^{-1}, ad are the generators because adding b, c^{-1}, or d respectively gives a loop. The gluing of the 2-cells

follow the three non-opposite faces. By the result in Hatcher (or Zhouli's lecture), we know that
$$\pi_1(X) \cong \pi_1(\mathrm{sk}_2\, X) \cong \langle ab, ac^{-1}, ad \mid abcd, ac^{-1}d^{-1}b, adb^{-1}c\rangle.$$
Observe that the three relations can be rewritten in terms of the generators
$$\begin{aligned} abcd &= abc(a^{-1}a)d = (ab)(ca^{-1})(ad) = (ab)(ac^{-1})^{-1}(ab) \\ ac^{-1}d^{-1}b &= ac^{-1}d^{-1}(a^{-1}a)b = (ac^{-1})(d^{-1}a^{-1})(ab) = (ac^{-1})(ad)^{-1}(ab) \\ adb^{-1}c &= adb^{-1}(a^{-1}a)c = (ad)(b^{-1}a^{-1})(ac) = (ad)(ab)^{-1}(ac). \end{aligned}$$
Therefore, the presentation of $\pi_1(X)$ can be rewritten as
$$\pi_1(X) \cong \langle i, j, k \mid ij^{-1}k, jk^{-1}i, ki^{-1}j\rangle.$$
The three relations imply
$$ki = j \qquad \& \qquad ij = k \qquad \& \qquad jk = i.$$
Thus,
$$\pi_1(X) \cong \langle i, j, k \mid ij = k, jk = i, ki = j\rangle.$$
Now, we have more relations to write out. Take $ij = k$ and multiply on the right by k to get $k^2 = ijk$. Take $jk = i$ and multiply on the left by i to get $i^2 = ijk$. Next, take $ki = j$ and multiply on the right to get $j^2 = kij$. Then, as $ij = k$, we get $j^2 = kij = k^2 = ijk$. Therefore, our presentation can be rewritten as
$$\pi_1(X) \cong \langle i, j, k \mid i^2 = j^2 = k^2 = ijk\rangle.$$
The RHS is known to be a presentation for Q_8. Indeed, to get the usual presentation, we observe that
$$i^4k = i^2i^2k = i^2k^2k = i^2kk^2 = i^2ki^2 = i^2(ki)i = i^2ji = i(ij)i = i(ki) = ij = k.$$
Canceling k implies $i^4 = e$. Thus, we have the usual presentation for the quaternions:
$$\pi_1(X) \cong \langle i, j, k \mid i^4 = e, i^2 = j^2 = k^2 = ijk\rangle \cong Q_8.$$
Or if demanded for it to be clear, our work showed that
$$\pi_1(X) \cong \langle i, j, k \mid i^4 = j^4 = k^4 = e, i^2 = j^2 = k^2 = ijk, ij = k, jk = i, ki = j\rangle$$
which lists all of the usual relations used to define the quaternion group (and thereby this must be the quaternion group). □

Hatcher Section 1.2, Problem 16 Show that the fundamental group of the surface of infinite genus shown below is free on an infinite number of generators. (Image not included).

Proof. Deformation retract into a graph with lines connected circles (think $\cdots - o - o - o - \dots$). This is free on infinitely many generators. An fascinating discussion of a more general fact can be found here. □

Questionable Proof. For each hole that appears, choose an open set U_i that the hole and in particular, a torus around that hole and then require that the open set U_i extends infinitely to the left and right via some rectangular region and does not cross any other holds. This U_i deformation retracts to a torus with two disks removed. The triple intersections $U_i \cap U_j \cap U_k$ will just consist of this rectangular region on the side of the surface. Furthermore, the fundamental group of $\pi(U_i \cap U_j)$ are always either trivial or $\mathbb{Z}$. Quotienting out the normal subgroup by Van Kampen's theorem, we essentially identify two generators so our quotient is still free. □

Hatcher Section 1.2, Problem 19 Show that the subspace of $\mathbb{R}^3$ that is the union of the spheres of radius $1/n$ and center $(1/n, 0, 0)$ for $n = 1, 2, \cdots$ is simply-connected.

Proof. The space X is clearly path-connected. If a, b lay in different spheres, we can draw a path from a to $(0,0,0)$ and then to b. On the other hand, the space is simply-connected because any loop will lay in finitely many spheres by compactness. But then, we can contract each part of the loop to the point $(0,0,0)$ by using the fact that $\pi_1(S^2) = 0$. □

Hatcher Section 1.2, Problem 20 Let X be the subspace of $\mathbb{R}^2$ that is the union of the circles C_n of radius n and center $(n, 0)$ for $n = 1, 2, \ldots$. Show that $\pi_1(X)$ is the free group $*_n \pi_1(C_n)$, the same as for the infinite wedge sum $\vee_\infty S^1$. Show that X and $\vee_\infty S^1$ are in fact homotopy equivalent, but not homeomorphic.

Proof. To observe homotopy equivalence, order the S^1 in $\vee_\infty S^1$ by S^1_n. Then, map each of C_n to S^1_n. Mapping each S^1_n to C_n. This is a homotopy equivalence because the composition is precisely the identity on each space. So, the fundamental groups are the same and so $\pi_1(X) \cong *_{n=1}^\infty \mathbb{Z}$.

To prove the first statement, we can simply apply Van Kampen's theorem. For X, we take an annular region around each circle for our open cover. The pairwise intersections are all simply connected and so are the triple intersections. The open sets also deformation retract onto their respective circle so they are $\pi_1(C_n)$. Thereby $\pi_1(X) \cong *_{n=1}^\infty \pi_1(C_n)$.

To show that the two spaces are *not homeomorphic*, we need to use the differences in the underlying topology. At the set level, they are the same. Topologically, the open subsets of $\vee_\infty S^1$ near the base-point (say) $(0,0)$ are generated by small open arcs that are a part of some circle. Meanwhile, the open subsets of X near the origin $(0,0)$ are generated by the intersection of some ball $B_r(0)$ with X. We observe that X has a countable neighborhood basis at $(0,0)$ while $\vee_\infty S^1$ does not. Therefore, they cannot be homeomorphic. □

4. Section 1.3

Hatcher Section 1.3, Problem 1 For a covering space $p : \tilde{X} \to X$ and a subspace $A \subset X$, let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p : \tilde{A} \to A$ is a covering space.

Proof. Relatively straightforward. □

Hatcher Section 1.3, Problem 2 Show that if $p_1 : \tilde{X}_1 \to X_1$ and $p_2 : \tilde{X}_2 \to X_2$ are covering spaces, so is their product $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be a cover of X_1 s.t. $p_1^{-1}(U_\alpha)$ is a disjoint union of open sets mapping homeomorphically to U_α. Let $\{V_\beta\}_{\beta \in B}$ be a cover of X_2 in an analogous sense. Then the collection $\{U_\alpha \times V_\beta\}_{(\alpha,\beta) \in A \times B}$ covers $X_1 \times X_2$ and $p_1 \times p_2$ maps an open set of $p^{-1}(U_\alpha \times V_\beta)$ homeomorphically onto $U_\alpha \times V_\beta$. This is because the product of two homeomorphisms is a homeomorphism and $p^{-1}(U_\alpha \times V_\beta)$ is a disjoint union of products of pairs of sets in $p^{-1}(U_\alpha)$ and $p^{-1}(V_\beta)$ respectively. □

Hatcher Section 1.3, Problem 3 Let $p : \tilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $\tilde{X}$ is compact Hausdorff iff X is compact Hausdorff.

Proof. Relatively straight forward. For $(\Leftarrow)$, one needs $p^{-1}(x)$ finite for compactness. □

Hatcher Section 1.3, Problem 5 Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1]\times[0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, \cdots$ inside the square. Show that for every covering space $\tilde{X} \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

Proof. For each x in the left edge of X, we obtain an evenly covered neighborhood U_x. By compactness of the left edge, there is a finite subcover of evenly covered neighborhoods which we call $U_1, \ldots, U_k$. Furthermore, we may assume that $p^{-1}(U_1), \ldots, p^{-1}(U_k)$ cover $\widetilde{X}$ by increasing the size of U_i. This is done as follows.

Suppose not and they did not cover. Choose $y \in \widetilde{X}$ not in the sets. Push y down to a point $p(y) \in X$. Choose an evenly covered neighborhood of $V \ni p(y)$. Then $p^{-1}(V)$ will be an open set that contains y. Add this set to the open cover we obtained. The points $p(y)$ will lie in only finitely many of the vertical segments. So compactness of each vertical segment ensures we only need to add finitely many open sets to $\{p^{-1}(U_i)\}_{i=1}^{k}$. There are homeomorphisms from each of these open set to an open subset of X. They behave well w.r.t. intersection so we may glue these homeomorphisms to a homeomorphism from $\widetilde{X}$ to X. The union of the open subsets of X we used forms an open neighborhood of X.

We show that X has no simply-connected covering space. Indeed, if $\widetilde{X}$ is a universal cover, then it is a homeomorphic to some neighborhood of the left edge of X. But any such neighborhood will contain a nontrivial loop (due to there being infinitely many of the vertical line segments in any neighborhood of the left edge, but specifically, the fact that $\{1/n : n \in \mathbb{N}\}$ is has accumulation point 0). $\square$

Hatcher Section 1.3, Problem 8 Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ then $\tilde{X} \simeq \tilde{Y}$. [Exercise 11 in Chapter 0 may be helpful.]

Proof. Assume $X \simeq Y$ and choose $f : X \to Y$ and $g, h : Y \to X$ using Exercise 11. Then, consider the following diagram with lift $\widetilde{f}$ (which exists)

$$\begin{tikzcd}
 & & \widetilde{Y} \arrow[d, "p_2"] \\
\widetilde{X} \arrow[urr, "\widetilde{f}"] \arrow[r, "p_1"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Let $\widetilde{g} : \widetilde{Y} \to \widetilde{X}$ be the analogous lifting. Now, $\widetilde{f}\widetilde{g} : \widetilde{Y} \to \widetilde{X} \to \widetilde{Y}$ and $\widetilde{h}\widetilde{f} : \widetilde{X} \to \widetilde{Y} \to \widetilde{X}$. Now let F^{-1} denote the inverse of in $\widetilde{f}\widetilde{g}$ the group of deck transformations and G^{-1} the inverse of $\widetilde{g}\widetilde{f}$. Then, $\widetilde{g} \circ F^{-1}$ and $G^{-1}\widetilde{g}$ are the desired maps satisfying the condition in Exercise 11 of Chapter 1. $\square$

Hatcher Section 1.3, Problem 9. Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic. [Use the covering space $\mathbb{R} \to S^1$.]

Proof. Given the covering space $p : \mathbb{R} \to S^1$ and $f : X \to S^1$,

$$f_*(\pi_1(X)) \subseteq \pi_1(S^1) \cong \mathbb{Z}.$$

is a subgroup and $|\pi_1(X)| < \infty$ means that $f_*(\pi_1(X))$ is a torsion subgroup. However, $\mathbb{Z}$ is torsion free so $f_*(\pi_1(X)) = 0$. Furthermore, $p_*(\pi_1(\mathbb{R})) = p_*(0) = 0$ which means $f_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R}))$. By Hatcher Proposition 1.33 (we use the hypotheses on X), there is a lift $\widetilde{f}$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dotted, "\widetilde{f}"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

and

$$f = p \circ \widetilde{f} \simeq p \circ C_x \simeq C_{p(x)}$$

since $\mathbb{R}$ is contractible. Indeed, any map into a contractible space is null-homotopic and homotopy behaves well with composition. So, f is null-homotopic. $\square$

Hatcher Section 1.3, Problem 10. Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoints.

Proof. See figures 21 and 22 with their captions for the details.

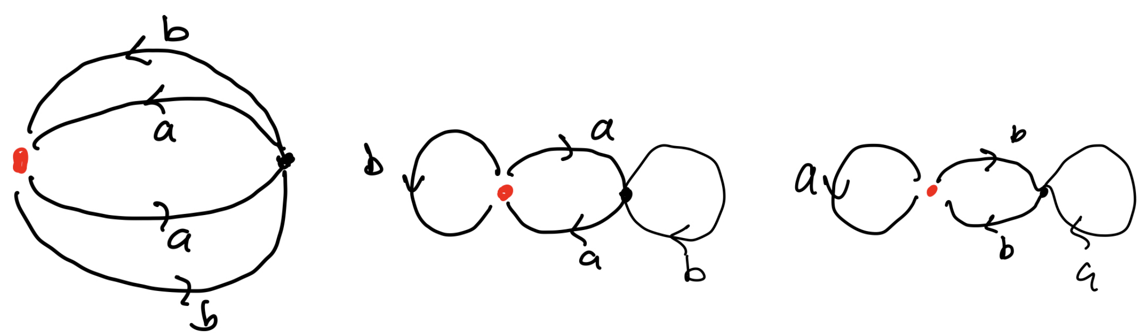


Figure 21. We classify all of the 2-fold connected coverings. This is a complete list due to the following reason. First, there must be two 0-cells. Fix one of them for now. There are three possibilities for the edges labeled a. If the edge is a loop, we must have the RHS covering space. If there are are two edges a, then they must connect to the second point with opposite orientation. Then there are two situations for the edge b. These are the first and second covering spaces pictured. This exhausts the possibilities for a. Furthermore, none of these are isomorphic. They correspond to nonconjugate groups: $\langle a^2, b^2, ab\rangle$, $\langle b, a^2, aba^{-1}\rangle$, $\langle a, b^2, bab^{-1}\rangle$ respectively.
Conjugating the first group, there is no way to obtain b or a which are inside the other two groups respectively.
Conjugating the second group, there is no way to obtain b^2 which is contained in the first group, nor is there any way to obtain a which is in the third group.
By a similar argument, no conjugate of the third group will contain either a^2 or b so it is not conjugate to the first nor second group respectively.

Remark 49 (Notes)**.** See figure 22. Fix a point p (which is the red point pictured). There are two edges of b and a passing through p (and they have one going in and out respectively). Determine ways to get a 3-fold cover.

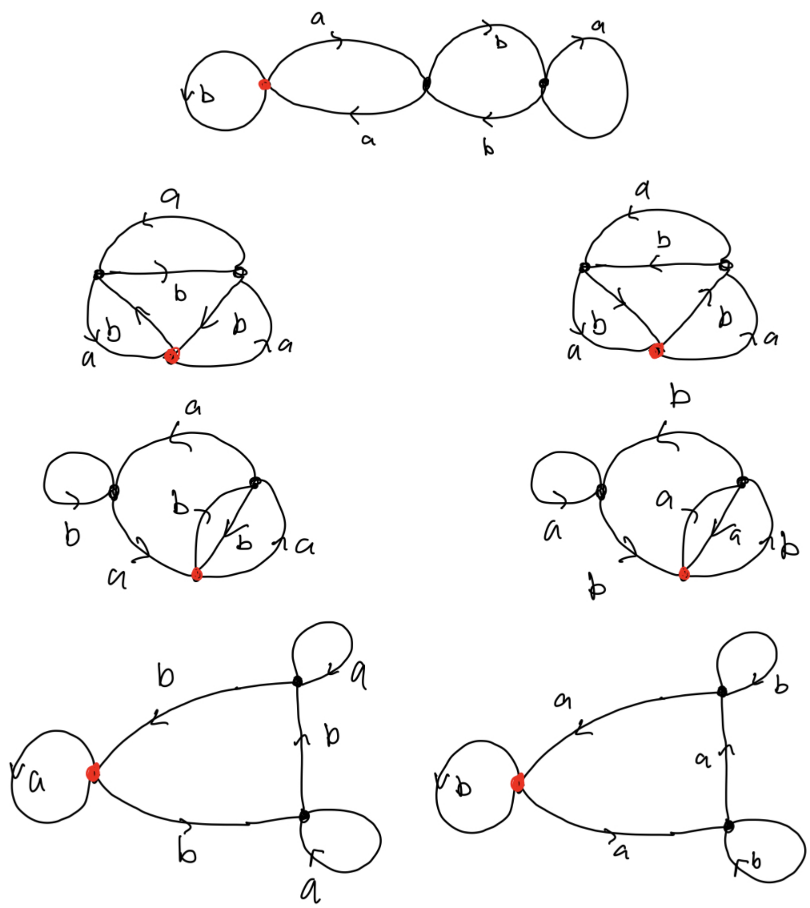


FIGURE 22. The approach is essentially the same. We draw three points and determine all of the possible nonisomorphic covering spaces. We explain why our list is exhaustive. There are cases. See the comments in Remark 49.

- Suppose the edges for b is a loop.
  - Suppose the edge for a is a loop. Then there is no way for us to obtain a connected graph that corresponds to a 3-fold cover. Absurd.
  - Suppose a were not a loop, but both edges were coming and going to the same point. Then we are forced into situation of the first graph in the figure.
  - Suppose the edges for a came from and went to different edges. Then we are forced into the situation in the bottom right graph. It does not matter which edge towards the point p since the covering spaces are homeomorphic (from now on, we do not repeat this and say, "Choice of direction did not matter.").
- Suppose the edges for b were not a loop, but went and came from the same point.
  - Suppose a were a loop. Then we are forced into the situation in the first graph (but our point p would be the point on the right).
  - Suppose a were not a loop, but attached to the same point q.
    - If all four edges for a, b came from and to the same point, we would not get a connected graph that were a 3-fold cover.
    - Suppose they were loops for different points. Then we get the first graph again, but this time, it is the center point.

  - Suppose the two edges for a went to different points. Then we get the fourth graph (we count from left to right and top to bottom for our graphs). Choice of direction did not matter.
- Suppose the edges for b went to different points.
  - Suppose our edges for a was just a loop. Then we get the sixth graph after filling in the rest of the graph. Choice of direction did not matter.
  - Suppose our edges for a were not a loop, but connected to the same point. Then we get the fifth graph once we fill in. Choice of direction did not matter.
  - Suppose our edges for a went to different points. Then we get the second and third graph since we would need to consider what pairs of directions we had for a, b in comparison to each other. Here, the choice matters because our point p will either have both edges a, b going to toward another point, or one going towards the other point and the other going to itself.

□

Hatcher Section 1.3, Problem 11. Construct finite graphs X_1 and X_2 having a common finite-sheeted covering space $\tilde{X}_1 = \tilde{X}_2$, but such that there is no space having both X_1 and X_2 as covering spaces.

Proof. See figure 23. □

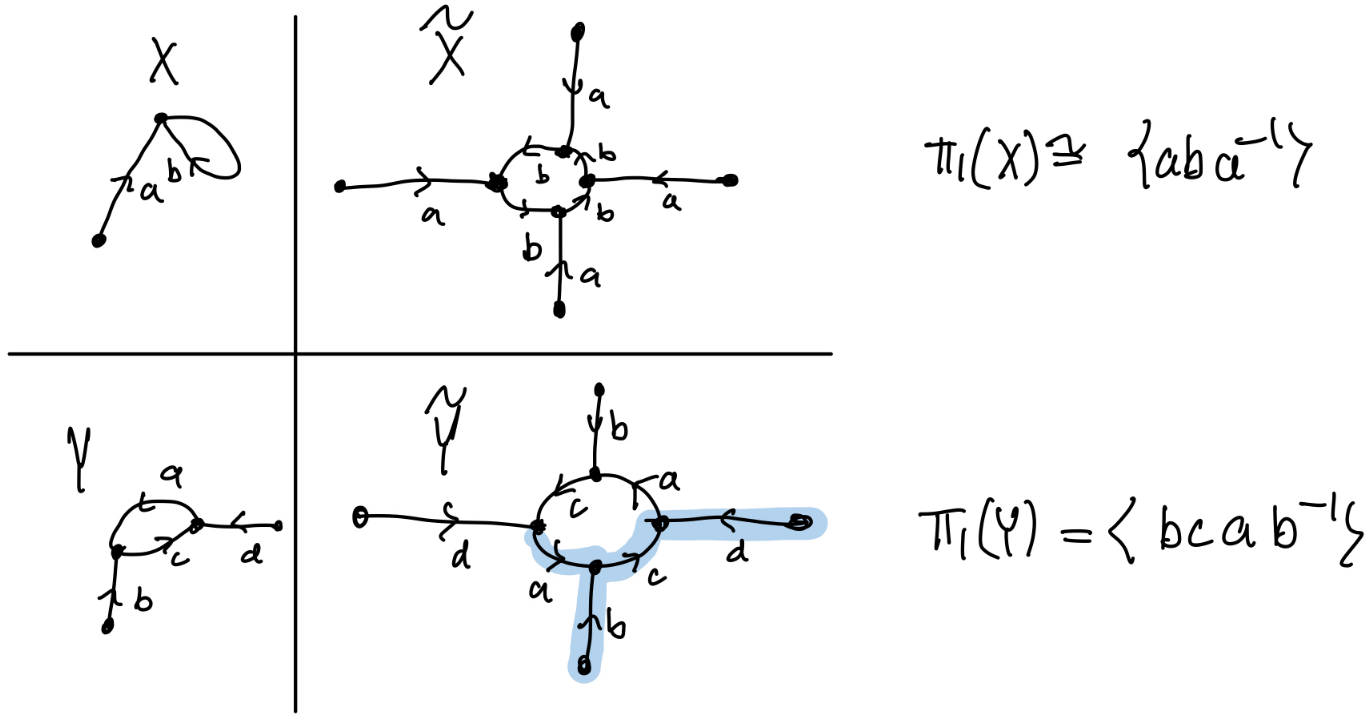


Figure 23. X and Y share the same cover, but do not have any common space they cover.

Hatcher Section 1.3, Problem 12. Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$

corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove that this covering space is indeed the correct one.

PROOF. See figure 24. We show this is correct. Fix a basepoint of $X := S^1 \vee S^1$ where they are wedged and call the loops given by the circles a and b. The covering space has covering map given in the obvious sense (i.e. p sends edged labeled b and a to the loops a, b labeled for $S^1 \vee S^1$ and p maps all of the points to our base-point).

The fundamental group of $\widetilde{X}$, the covering space, is computed as follows. Do not identify the points pictured. Then we have the free group generated by loops (just add back edges starting with the star and go clockwise).

$$\langle (ab)^4, (ab)^3a^2(ab)^{-3}, (ab)^2ab^2a^{-1}(ab)^{-2}, (ab)^2a^2(ab)^{-2}, abab^2a^{-1}(ab)^{-1}, aba^2(ab)^{-1}, ab^2a^{-1}, a^2\rangle$$

Now, taking p identifies the points inside X. Identifying points corresponds to using change-of-basepoints upstairs and this corresponds to conjugating by a loop. For instance, conjugating by a pushes the basepoint at "star" to the point to the right of it. But we could also conjugate by b to identify "star" with the point to the left of it. Since we are can conjugate by a or b without affecting our group, we deduce that the image $p_*(\pi_1(\widetilde{X}, x_0))$ is going to correspond to a normal subgroup. Indeed, any element g we want to conjugate by is a product of a's and b's which then preserve the group.

If we loop at our generators above, we already have $(ab)^4$ and a^2. Taking the loop ab^2a^{-1} and conjugating by a, we get b^2. It is also clear that the remaining generators can be obtained from $a^2, b^2, (ab)^4$ by conjugating by powers of ab or a. So,

$$p_*(\pi_1(\widetilde{X}, x_0)) = \langle (ab)^4, a^2, b^2\rangle.$$

$\square$

Hatcher Section 1.3, Problem 14. Find all the connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

PROOF. First, we draw $\mathbb{R}P^2 \vee \mathbb{R}P^2$ as in figure 26.

Now, we list the possible subgroups of $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2) \cong \pi_1(\mathbb{R}P^2) * \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ (since $\mathbb{R}P^2$ is "nice"). The possible subgroups are listed below:

(1) $\{e\}$,
(2) $\langle (ab)^m\rangle$ for $m \in \mathbb{Z}$,
(3) $\langle (ab)^m a\rangle$ for $m \in \mathbb{Z}$ (note that $(ab)^{-1} = b^{-1}a^{-1} = ba$ inside $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and so, this class also includes the groups of form $\langle (ba)^m b\rangle$ for $m \in \mathbb{Z}$),
(4) $\langle (ab)^m a, (ab)^l\rangle$ for $1 \leq l \leq m$ and $m \in \mathbb{N}$.

We claim this list is exhaustive. Choose a word in a subgroup H of minimal length besides the identity. If there is no other option, H is trivial so suppose not. If this word has length 1, then it is either a or b. If this element does not generate all of H, choose an element (WLOG assuming we had chosen a), $w \in H \setminus \langle a\rangle$ of minimal length. Then w starts and ends with b (for if not, multiply by a). Then $\langle a, w\rangle$ generates all of H. For if not, there is another word $v \in G \setminus \langle a, w\rangle$ of minimal length. Since $v \notin G \setminus \langle a\rangle$ and is of minimal length, assume v starts and ends with b. But then vw is an element of smaller length than that of v and not in $\langle a, w\rangle$. Contradiction.

We deduce that every subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2$ is generated by at most two elements. Listed in 1., 2., 3., are the possible cyclic subgroups (since they contain all possible words), and in 4., we have all possible subgroups generated by two words (since they contain all possible pairs of words).

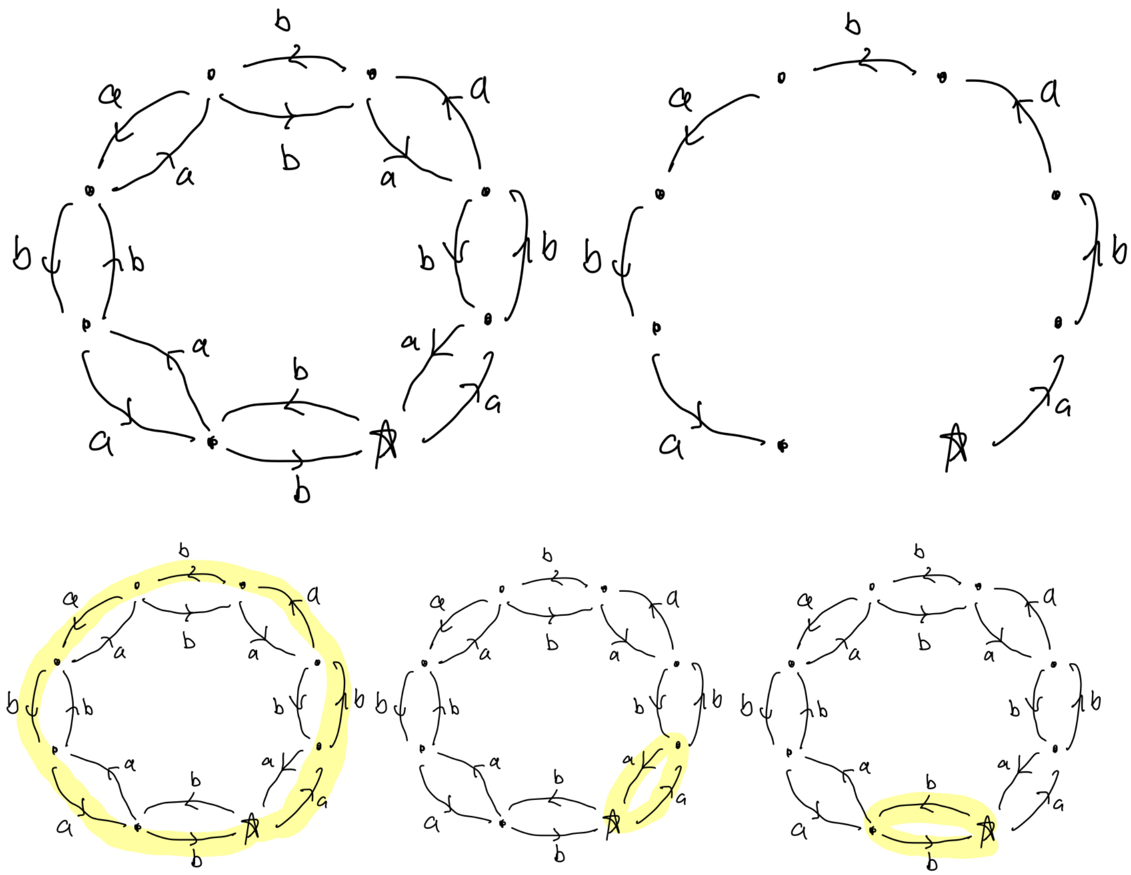


FIGURE 24. Pictured at top is the covering space. On the bottom are the highlighted paths corresponding to $(ab)^4, a^2, b^2$ respectively. Indicated by the star is our selected base-point in the wedge sum $S^1 \vee S^1$.

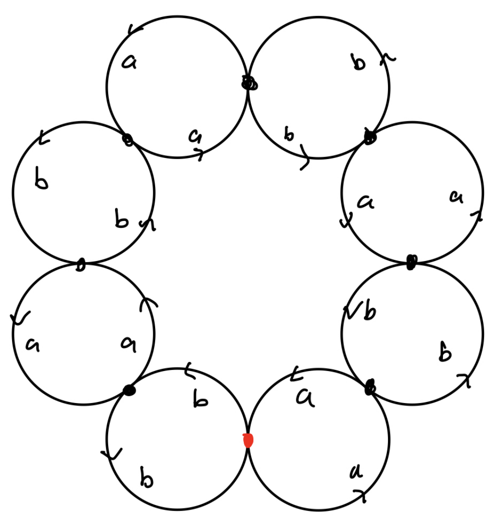


FIGURE 25. A slightly better drawing of the covering space.

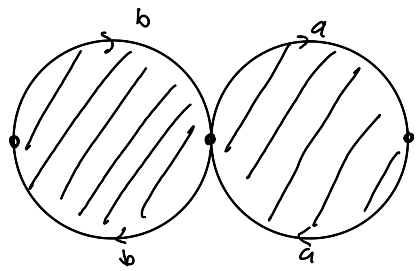


FIGURE 26. Picture of $\mathbb{R}P \vee \mathbb{R}P^2$.

To complete the proof, we simply construct coverings that correspond to each of the families. See figures 27, 28, 29, 30. Using the obvious identifications (for spheres, use an antipodal map), each of the pictured spaces is a covering space. The basepoint is colored in red.

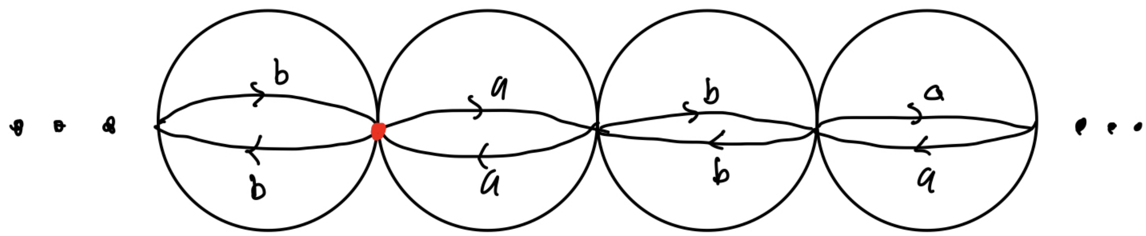


FIGURE 27. The universal covering is a wedge sum of spheres. They map onto X via identifying antipodals. Observe that at each of the points (where the spheres are tangent), there is exactly on edge labeled a going in and one going out. The same for b.

□

Hatcher Section 1.3, Problem 13
Hatcher Section 1.3, Problem 20. Construct nonnormal covering spaces of the Klein bottle by a Klein bottle and by a torus.

PROOF. Draw out the gluing diagram for the torus and Klein bottle and put them together six times so that the rectangle we obtain is a new gluing diagram. Indeed, $\pi_1(K) = \langle a, b \mid baba^{-1} \rangle$ has a finite index nonnormal subgroup $\langle xy^{-1}, y^6 \rangle$. □

Hatcher Section 1.3, Problem 21. Let X be the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus. Compute $\pi_1(X)$, describe the universal cover of X, and describe the action of $\pi_1(X)$ on the universal cover. Do the same for the space Y obtained by attaching a Möbius band to $\mathbb{R}P^2$ via a homeomorphism from its boundary circle to the circle in $\mathbb{RP}^2$ formed by the 1-skeleton of the usual CW structure on $\mathbb{RP}^2$.

PROOF. For computing the fundamental group $\pi_1(X)$, see figure 31. We take the open sets to be U and V as indicated. That is, U is the torus with the open set of the Möbius strip

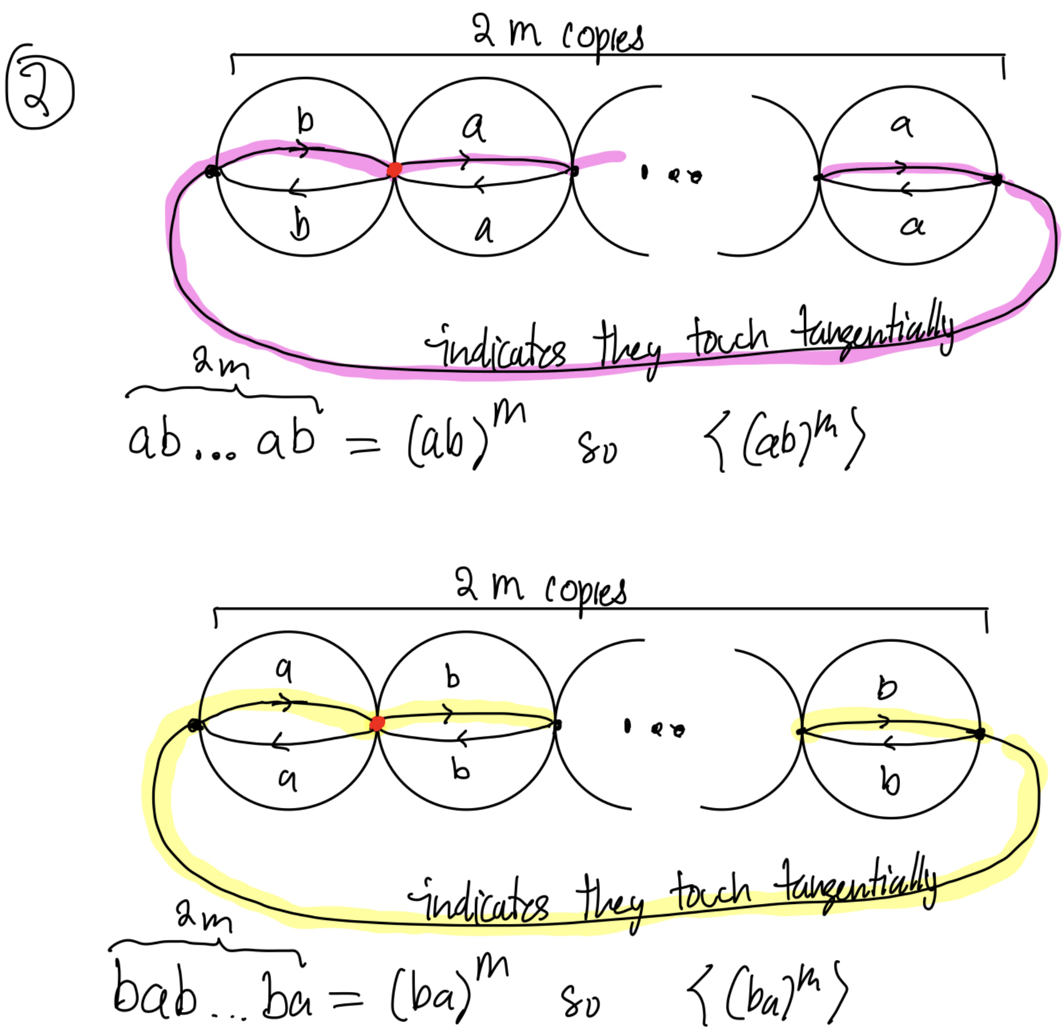


FIGURE 28. Here, we have colored in pink and yellow the paths to take. They go from left to right and circle around. The line we drew just indicates they are touching tangentially i.e. the spheres are placed so that they are going around in a circle in $\mathbb{R}^3$. The second case pictured corresponds to when $\langle (ab)^m \rangle$ has $m < 0$. In which case set $-l = m$, and $(ab)^m = (ab)^l = (b^{-1}a^{-1})^l = (ba)^l$.

along the boundar as that was where it was glued. For V, we take the Möbius strip and a small cylinder open set around the loop a at which we glued. The intersection $U \cap V$ is highlighted in the figure. We observe that the hypotheses of Van Kampen's Theorem is fulfilled. Furthermore, $\pi_1(U) \cong \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(V) \cong \mathbb{Z}$. The loop a in the intersection $U \cap V$ maps to the loop a inside $\pi_1(U)$. On the other hand, if $\langle d \rangle = \pi_1(V)$ is the generating loop for the fundamental group of the Möbius band, then a mapsto d^2. Thus,

$$\pi_1(X) \cong \frac{\pi_1(U) * \pi_1(V)}{\langle a, a^{-1}d^2 \rangle} \cong \langle a, b, d \mid b^{-1}a^{-1}ba, a^{-1}d^2 \rangle \cong \langle a, b, d \mid ab = ba, a = d^2 \rangle.$$

For the universal covering space $\widetilde{X}$, see figure 32 and 33 and the captions respectively. Using our description of universal covering space, we describe the action of $\pi_1(X)$ on $\widetilde{X}$. An action by a simply translates the covering space up vertically along the $\mathbb{R}^1$ axis not pictured

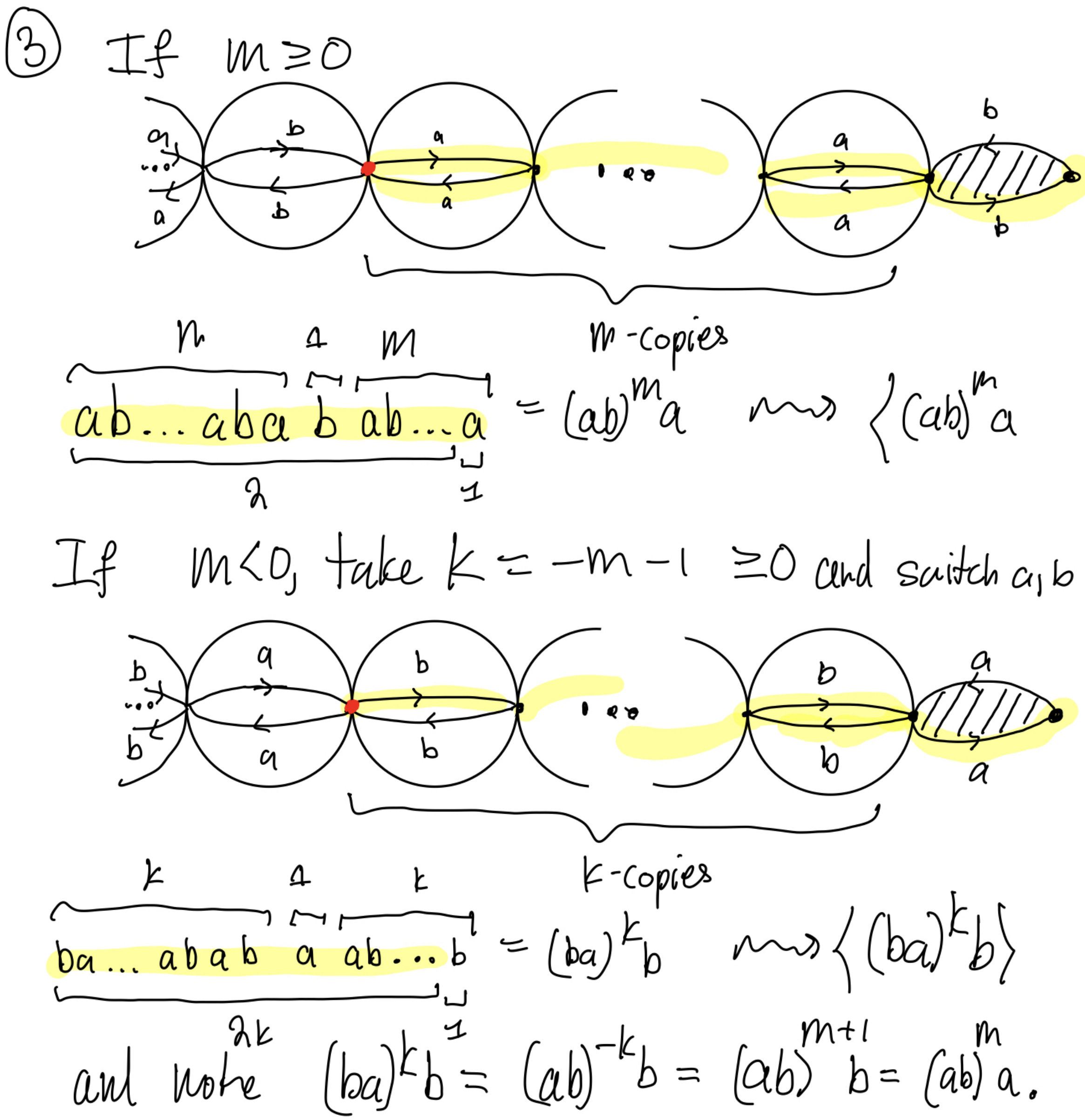


FIGURE 29. We highlighted in yellow the path to take to get the generators. They go from left to right. Observe that at the ends, the two points we see are actually identified because we have a copy of $\mathbb{R}P^2$.

when viewing from above (in figure 33). The shift is done by moving along f and e. An action by b is a shift of $\widetilde{X}$ along the other axis. Clearly, these two actions commute. An action by d is a shift upward along f or e.

Please proceed a few pages over the the second part of the problem.

Our goal now is to do what we did for X with Y. See figure 34 and the caption for the fundamental group computation.

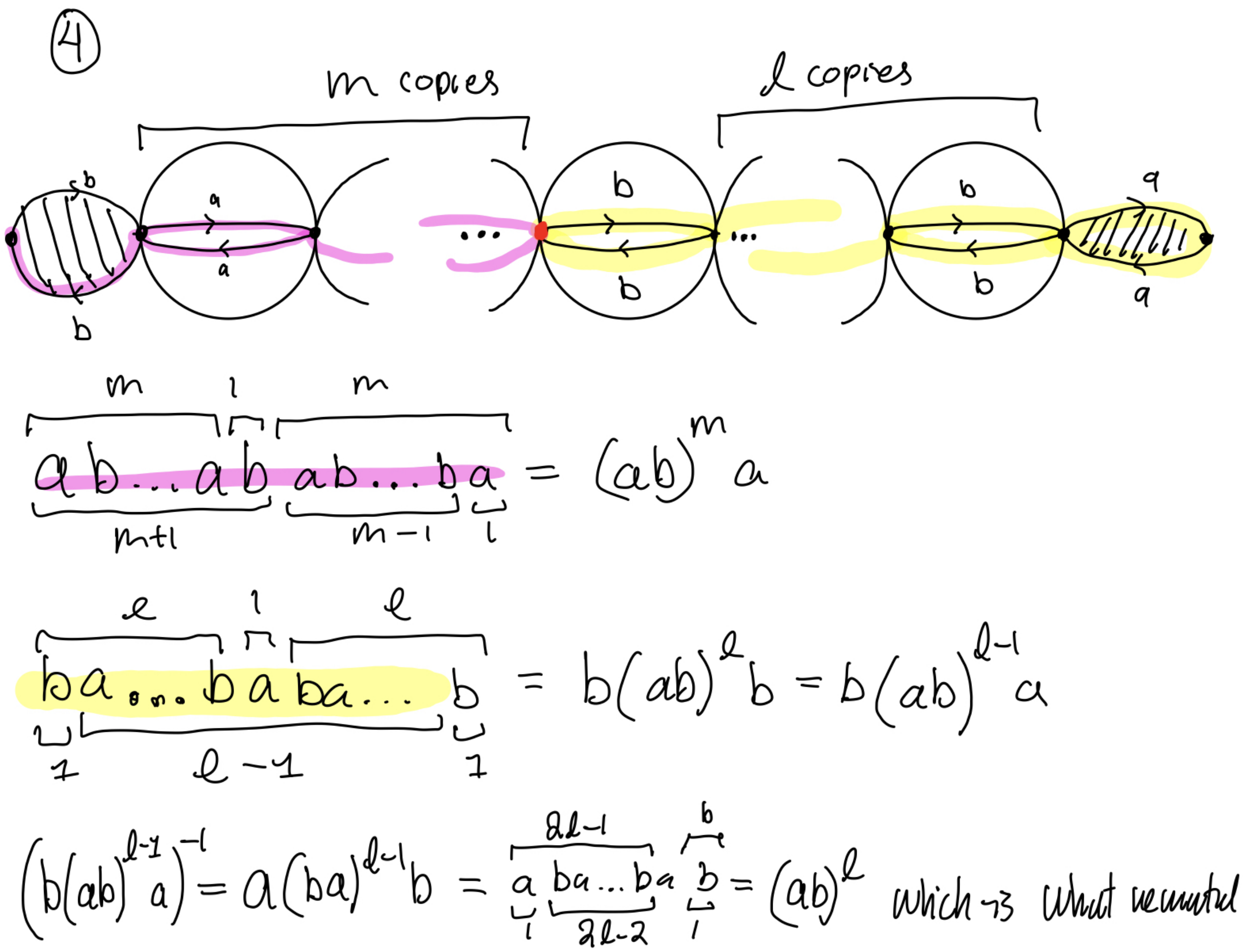


FIGURE 30. Following the path colored in pink (again, note that the ends are copies of $\mathbb{R}P^2$), we have $(ab)^m a$. Meanwhile, the yellow path gives $b(ab)^l bb = b(ab)^{l-1}b$. So we have image being the subgroup $\langle (ab)^m a, b(ab)^l b\rangle$. But the inverse of this element is $a(ba)^{l-1}b$ which equals $(ab)^l$. Thus, $\langle (ab)^m a, b(ab)^l b\rangle = \langle (ab)^m a, (ab)^l\rangle$ is the image (as taking inverse of a generator changes nothing about the group).

Now, we describe our construction of the universal covering of Y and the action of $\pi_1(Y)$ on it. We use the fact that $S^1 \times I$ is a covering space of M (intuitively, this is easy to see - use the meridian circle of M and extend outward for I and add a twist).

For the universal covering space of Y, see figure 35 for the details.

□

5. Section 1.A

6. Section 2.1

Hatcher Section 2.1, Problem 11 Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

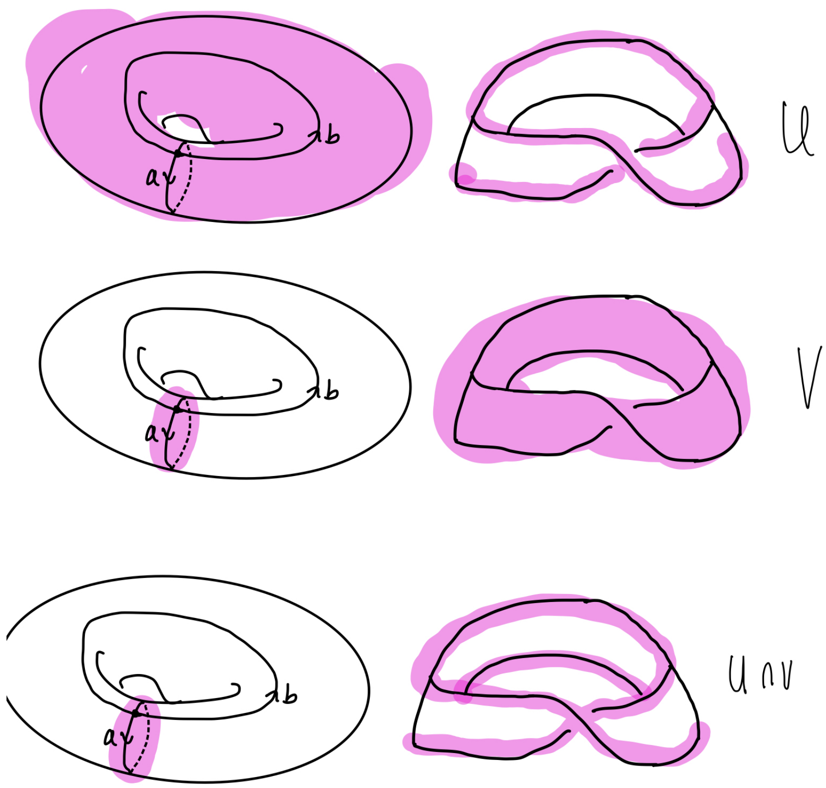


FIGURE 31. Highlighted are the open sets we consider.

PROOF. *Warning*: it is not true that a functor needs to preserve monomorphisms (though counterexamples are hard to find in nature).

Let A be a retract of X with $i : A \hookrightarrow X$ the inclusion. Being retract means there is a map $r : X \to A$ s.t. $r \circ i = \mathrm{id}_A$. These maps induce a map on homology $\mathrm{id}_{H_n(A)} : H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(A)$ which means the map i_* is an injective map. □

Hatcher Section 2.1, Problem 12 (a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.
(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

PROOF. (a) Take the chain complex

$$\dots C_1(X)/C_1(A) \to C_0(X)/C_0(A) \to 0 \to 0 \to \dots.$$

The homology being zero means that $C_1(X)/C_1(X)$ surjects onto $C_0(X)/C_0(A)$. But by chasing elements via this map, we can check that there exists some element of $a \in A$ that meets a chosen path-component of X. Indeed, $[\overline{s}] \in C_1(X)/C_1(A)$ lifts to a $[\overline{t}]$ i.e. $[\partial\overline{t}] = [s] + C_0(A)$. But this essentially says, by definition of the differential, that we can connect a point of X to a point of A via a path. Therefore, A meets each path-component.

Conversely, if A meets each path-component of X, every element of $C_0(X)/C_0(A)$ can be lifted to a 1-chain in $C_1(X)/C_1(A)$. Indeed, consider an element $[s]+C_0(A) \in C_0(X)/C_0(A)$. Now pick a 1-chain $[t] \in C_1(X)$ which connects s to a point in A. Then the differential sends this 1-chain to $[s]+C_0(A)$ precisely because the 0-chain corresponding to the point in A is modded out.

(b) Since $H_1(X,A)=0$, we know $\ker(C_1(X)/C_1(A) \to C_0(X)/C_0(A)) = \operatorname{im}(C_2(X)/C_2(A) \to C_1(X)/C_1(A))$. Now I show that $H_1(A) \to H_1(X)$ is surjective. Let $[s] \in H_1(X)$ be a homology class. Then s is represented by a cycle i.e. $s \in \ker(C_1(X) \to C_0(X))$. But that means $[\overline{s}] \in C_1(X)/C_1(A)$ is a cycle. It then lifts to a $[\overline{t}] \in C_2(X)/C_2(A)$ with $[d\overline{t}] = [\overline{s}]$. But that means $[s]$ differs from the differential of a 2-chain by an element of $C_1(A)$ that is a 1-boundary. So, $[s]$ can be lifted to a 1-cycle $C_1(A)$. This can be proved quite quickly by chasing elements. The proof here is not completely written yet. □

Hatcher Section 2.1, Problem 12 Show that chain homotopy of chain maps is an equivalence relation.

Proof. Relatively straightforward. □

Hatcher Section 2.1, Problem 13 Verify that $f \simeq g$ implies $f_* = g_*$ for induced homomorphisms of reduced homology groups.

Proof. The reduced homology agrees with the usual homology for $n \geq 1$ so nothing needs to be done here. For the case of degree zero, we simply lose a factor of $\oplus\mathbb{Z}$ for both so the induced morphisms stay the same. The proof of this also follows from the five-lemma and the LES. □

Hatcher Section 2.1, Problem 15 For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X,A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X,A) = 0$ for all n.

Proof. Clearly, $C = 0$ implies $A \to B$ surjective and $D \to E$ injective. Now assume the latter. Call the maps ϕ_i for $i = 1,2,3,4$. Now $\phi_1(A) = B$ and $\ker(\phi_4) = 0$. Then $\ker\phi_2 = B$ and $\phi_3(C) = 0$ But now $C/\operatorname{im}\phi_2 \cong C/\ker\phi_3 \cong \phi_3(C) = 0$. So $\operatorname{im}\phi_2 = C$ and so $B/\ker\phi_2 \cong \operatorname{im}\phi_2 \cong C$. Then $\ker\phi_2 = B$ gives $C = 0$.

From the LES,

$$\ldots \to H_n(A) \to H_n(A) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \ldots$$

and since the LHS map and RHS map are isomorphisms, the previous part says $H_n(X,A) = 0$. This proves one direction and the other direction is trivial. □

Hatcher Section 2.1, Problem 16. (a) Show that $H_0(X,A) = 0$ iff A meets each path-component of X.

(b) Show that $H_1(X,A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof. (a) We know $H_0(X,A) = 0$ iff $\to H_1(X,A) \to H_0(X) \to H_0(X) \to 0$ from the LES iff $H_0(A) \to H_0(X)$ is surjective iff A meets each path component of X.

(b) We know $H_1(X,A) = 0$ iff $H_1(A) \to H_1(X) \to 0 \to H_0(X) \to H_0(X) \to H_0(X,A) \to 0$ iff $H_1(A) \to H_1(X)$ is surjective and $H_0(A) \hookleftarrow H_0(X)$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A. □

Hatcher Section 2.1, Problem 17. (a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.
(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown (see figure 38). [What are X/A and X/B ?]

Proof of Part (a). **Case 1. Fix $X = S^2$.** Enumerate $A := \{p_1, \ldots, p_m\}$ a set of points in X. Observe that $H_0(A) \cong \mathbb{Z}^m$ and $H_n(A) = 0$ for $n \geq 1$. Using the LES, we have

$$0 \to H_2(X) \to H_2(X, A) \to H_1(A) \to H_1(X) \to H_1(X, A) \to H_0(A) \to H_0(X) \to H_0(X, A) \to 0.$$

where the LHS zero comes from $H_2(A) = 0$. Of course, we know the singular homology of S^2. Using this, the sequence becomes

$$0 \to \mathbb{Z} \to H_2(X, A) \to 0 \to 0 \to H_1(X, A) \to \mathbb{Z}^m \to \mathbb{Z} \to H_0(X, A) \to 0.$$

It follows that $H_2(X, A) = \mathbb{Z}$. Meanwhile, using the reduced LES, the sequence is

$$0 \to H_1(X, A) \to \mathbb{Z}^{m-1} \to 0 \to H_0(X, A) \to 0.$$

It follows $H_0(X, A) = 0$ and $H_1(X, A) = \mathbb{Z}^{m-1}$. Because $H_n(X) = 0$ for $n \geq 3$ and $H_n(A) = 0$ for $n \geq 3$, continuing the LES from the left implies that $H_n(X, A) = 0$ for all $n \geq 3$ (this requires the use of $H_2(A) = 0$). We conclude that

$$H_0(X, A) = 0, \qquad H_1(X, A) \cong \mathbb{Z}^{m-1}, \qquad H_2(X, A) \cong \mathbb{Z}, \qquad H_n(X, A) = 0 \text{ for } n \geq 3.$$

Case 2. Fix $X := S^1 \times S^1$. Apply the LES of reduced singular homology to get

$$0 = H_2(A) \to H_2(X) \to H_2(X, A) \to H_1(A) \to H_1(X) \to H_1(X, A) \to \mathbb{Z}^{m-1} \to 0 \to H_0(X, A) \to 0.$$

Filling in the information we know about the singular homology of the torus X and the space A, we get

$$0 \to \mathbb{Z} \to H_2(X, A) \to 0 \to \mathbb{Z} \times \mathbb{Z} \to H_1(X, A) \to \mathbb{Z}^{m-1} \to 0 \to H_0(X, A) \to 0.$$

It follows that $H_0(X, A) = 0$ and $H_2(X, A) = 0$. To compute $H_1(X, A)$, we extra a SES

$$0 \to \mathbb{Z}^2 \to H_1(X, A) \to \mathbb{Z}^{m-1} \to 0.$$

Because $\mathbb{Z}^{m-1}$ is a projective abelian group, $H_1(X, A) \cong \mathbb{Z}^{m-1} \oplus \mathbb{Z}^2 \cong \mathbb{Z}^{m+1}$. We conclude, using the same methods as before and the fact $H_n(X) = 0$ for $n \geq 3$ that $H_n(X, A) = 0$ for $n \geq 0$. Hence,

$$H_0(X, A) = 0, \qquad H_1(X, A) \cong \mathbb{Z}^{m+1}, \qquad H_2(X, A) = \mathbb{Z}, \qquad H_n(X, A) \text{ for } n \geq 3.$$

□

Proof of Part (b). **Case of A.** Applying excision, we know that

$$H_n(X, A) \cong H_n(X/A, \{*\}) \cong H_2(T^2 \vee T^2, \{*\}) \cong \widetilde{H_n}(T^2 \vee T^2).$$

The fact that excision can be applied here is we can take a cylindrical open set around A sitting inside X and this deformation retracts to A.

Now we use the fact that $\vee$ behaves w.r.t. singule homology to deduce that for $n \geq 1$,

$$\widetilde{H_n}(T^2 \vee T^2) = \widetilde{H_n}(T^2) \oplus \widetilde{H_n}(T^2) \cong H_n(T^2) \oplus H_n(T^2).$$

Clearly, $T^2 \vee T^2$ has a single path component and hence, $\widetilde{H_0}(T^2 \vee T^2) = 0$. It follows that

$$H_0(X, A) = 0, \qquad H_1(X, A) = \mathbb{Z}^4, \qquad H_2(X, A) = \mathbb{Z}^2, \qquad H_n(X, A) = 0 \text{ for } n \geq 3.$$

Case of B. Let $Y := X/A$. Consider the homotopy equivalence in figure 39

We see that Y is homotopy equivalent to the torus with two points identified. We computed the relative homology of this in Hatcher Section 2.1 Exercise 17(a). In particular, we may use the excision property of singular homology

$$H_n(X, A) \cong H_n(X/A, \{*\}) \cong \widetilde{H}_n(Y).$$

Now using Case 2 in the previous part,

$$H_0(Y) \cong \mathbb{Z}, \qquad H_1(Y) \cong \mathbb{Z}^3, \qquad H_2(Y) \cong \mathbb{Z}, \qquad H_n(Y) = 0 \text{ for } n \geq 3.$$

□

Hatcher Section 2.1 Problem 29 Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Proof. Recall that $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for $n \geq 1$. We already know what the singular homology of the torus is, so we compute $H_*(S^1 \vee S^1 \vee S^2)$ and call this space Y. Clearly $H_0(Y) = \mathbb{Z}$ and $H_1(Y) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_2(Y) \cong \mathbb{Z}$ by the singular homology of S^n. Furthermore, $H_n(Y) = 0$ for all $n \geq 3$. We conclude that $S^1 \times S^1 =: X$ and Y have the isomorphic homology groups in all dimensions.

The universal cover of X is $\mathbb{R}^2$ which is a contractible space. Therefore $H_0(\mathbb{R}^2) = \mathbb{Z}$ and $H_n(\mathbb{R}^2) = 0$ for $n \geq 1$. To compute the universal cover $\widetilde{Y}$ of Y, we take the universal cover of $S^1 \vee S^1$ and attach a sphere at each intersection. See figure 40.

Clearly, there is only one path component so $H_0(\widetilde{Y}) = \mathbb{Z}$. Since it is the universal cover, $H_1(\widetilde{Y}) = 0$. So the only possibility is that $H_2(\widetilde{Y}) \neq 0$ for the homology groups to differ from that of $\widetilde{X} = \mathbb{R}^2$ (we have not show H_1 is the abelianization of π_1, but this observation can be taken as heuristic since we do not rely on $H_1(\widetilde{X}) = H_1(\widetilde{Y}) = 0$).

We use the fact that there is a CW-complex structure for $\widetilde{Y}$. In this case, we can compute $H_2(\widetilde{Y})$ using cellular homology. The cellular chain complex is

$$0 \leftarrow \oplus_{i=1}^{\infty}\mathbb{Z} \leftarrow \oplus_{i=1}^{\infty}\mathbb{Z} \overset{d_2}{\leftarrow} \oplus_{i=1}^{\infty}\mathbb{Z} \overset{d_3}{\leftarrow} 0.$$

It follows that $H_2^C(\widetilde{Y}) = \ker(d_2)/\operatorname{im}(d_3) = \ker(d_2)$. Now we fact $d_2 = j_1 \circ \partial_1$ and using the fact that j_1 is injective, $\ker(d_2) = \ker(\partial_1)$. But ∂_1 is the map $\partial_1 : H_2(\widetilde{Y}_2, \widetilde{Y}_1) \to H_1(\widetilde{Y}_1) = 0$. So $\ker(\partial_1) = H_2(\widetilde{Y}_2, \widetilde{Y}_1) = C_2(\widetilde{Y}) = \oplus_{i=1}^{\infty}\mathbb{Z}$. Hence, $H_2(\widetilde{Y}) = \bigoplus_{i=1}^{\infty}\mathbb{Z}$. The claim follows from $H_2(\widetilde{Y}) = \bigoplus_{i=1}^{\infty}\mathbb{Z} \neq 0 = H_2(\widetilde{X})$. To summarize what we have shown,

$$H_0(\widetilde{Y}) = \mathbb{Z}, \qquad H_1(\widetilde{Y}) = 0, \qquad H_2(\widetilde{Y}) = \bigoplus_{i=1}^{\infty}\mathbb{Z}, \qquad H_n(\widetilde{Y}) = 0 \text{ for } n \geq 3.$$

$$\text{meanwhile} \qquad H_0(\widetilde{X}) = \mathbb{Z}, \qquad H_n(\widetilde{X}) = 0 \text{ for all } n \geq 1.$$

□

7. Section 2.2

Hatcher Section 2.2, Problem 9. Compute the homology groups of the following 2-complexes:

(a) The quotient of S^2 obtained by identifying north and south poles to a point.

(b) $S^1 \times (S^1 \vee S^1)$.

(c) The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.

(d) The quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

Proof 1 of (a). Let X be the space we obtain from quotienting S^2 by the north and south poles. Then the excision principle says that

$$H_n(S^2, \{*, **\}) \cong H_n(X, \{p\}) \cong \widetilde{H_n}(X).$$

We know the LHS from Hatcher Section 2.1 Exercise 17(a). It follows that

$$H_0(X) \cong \mathbb{Z}, \qquad H_1(X) \cong \mathbb{Z}, \qquad H_2(X) \cong \mathbb{Z}, \qquad H_n(X) = 0 \text{ for all } n \geq 3.$$

$\square$

Proof 2 of (a). Let $X := S^2/\{n, s\}$ be the space with north and south pole identified. It is also possible to construct X from the quotient of a copy of S^2 with a 1-cell connecting the north and south pole and quotienting by the 1-cell. See figure 41. Call Y the sphere with the 1-cell and call A the 1-cell connecting north and south pole.

We may apply excision to obtain

$$H_n(Y, A) \cong H_n(Y/A, A/A) \cong H_n(Y/A, \{*\}) \cong \widetilde{H_n}(Y/A) \cong \widetilde{H_n}(X).$$

Now if we study the LES of homology, we extract

$$0 = H_2(A) \to H_2(Y) \to H_2(Y, A) \to H_1(A) \to H_1(Y) \to H_1(Y, A) \to H_0(A) \to H_0(Y) \to H_0(Y, A) \to 0.$$

Of course, using knowledge of cellular homology to compute $H_*(A)$ and taking reduced homology, we get

$$0 \to H_2(Y) \to H_2(Y, A) \to 0 \to \mathbb{Z} \to H_1(Y, A) \to 0 \to 0 \to 0 \to 0.$$

It follows that $H_1(Y, A) \cong \mathbb{Z}$. OTOH, we can extract a SES:

$$0 \to H_2(Y) \to H_2(Y, A) \to 0.$$

We use cellular homology to do this. The cellular chain complex $C_*(Y)$ is

$$0 \xleftarrow{d_0} \mathbb{Z}^2 \xleftarrow{d_1} \mathbb{Z}^3 \xleftarrow{d_2} \mathbb{Z} \xleftarrow{d_3} 0$$

using the cell decomposition in figure 42.

Now, $\ker d_0 = \mathbb{Z}^2$ and since $H_0(Y) = \mathbb{Z}$ (as there is only one path component), $\operatorname{im} d_1 = \mathbb{Z}$. But $\operatorname{im} d_1 = \mathbb{Z}$ implies $\ker(d_1) = \mathbb{Z}^2$. We also know that $H_1(Y) = \mathbb{Z}$ because $H_1(Y)$ is the abelianization of $\pi_1(Y)$. It follows $\operatorname{im} d_2 = \mathbb{Z}$. We know $\operatorname{im} d_2 = \mathbb{Z}$ implies $\ker d_2 = \mathbb{Z}$ and as $\operatorname{im} d_3$, we conclude

$$H_2(C_*(Y)) = \ker d_2 / \operatorname{im} d_3 = \mathbb{Z}/0 \cong \mathbb{Z}.$$

Finally, $H_2(Y, A) \cong H_2(Y) \cong \mathbb{Z}$. We conclude that

$$H_0(X) = \mathbb{Z}, \qquad H_1(X) \cong H_1(Y, A) \cong \mathbb{Z}, \qquad H_2(X) = H_2(Y, A) \cong \mathbb{Z}, \qquad H_n(X) = 0 \text{ for } n \geq 3.$$

$\square$

Proof of (b) via the Mayer-Vietoris Sequence. Pictorially, the space $X := S^1 \times (S^1 \vee S^1)$ looks like that of figure 43.

From our choice of U and V, we know that $U \simeq T^2$, $V \simeq T^2$, and $U \cap V \simeq S^1$. Using the Mayer-Vietoris sequence,

$$H_2(S^1) \to H_2(T^2) \oplus H_2(T^2) \to H_2(X) \to H_1(S^1) \to H_1(T^2) \oplus H_1(T^2) \to H_1(X) \\ \to H_0(S^1) \to H_0(T^2) \oplus H_0(T^2) \to H_0(X) \to 0.$$

Filling in the details with knowledge of the singular homology of T^2 and S^1,

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \stackrel{\phi_1}{\to} H_2(X) \stackrel{\phi_2}{\to} \mathbb{Z} \stackrel{\phi_3}{\to} \mathbb{Z}^2 \oplus \mathbb{Z}^2 \stackrel{\phi_4}{\to} H_1(X) \stackrel{\phi_5}{\to} \mathbb{Z} \stackrel{\phi_6}{\to} \mathbb{Z} \oplus \mathbb{Z} \stackrel{\phi_7}{\to} \mathbb{Z} \stackrel{\phi_8}{\to} 0.$$

Fomr the RHS, we know $\ker \phi_7 = 0$ which means $\operatorname{im} \phi_6 = \mathbb{Z}$. Hence, the tail end gives a SES $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$. So we reduce to studying a sequence

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \stackrel{\phi_1}{\to} H_2(X) \stackrel{\phi_2}{\to} \mathbb{Z} \stackrel{\phi_3}{\to} \mathbb{Z}^2 \oplus \mathbb{Z}^2 \stackrel{\phi_4}{\to} H_1(X) \stackrel{\phi_5}{\to} 0.$$

Now we analyze $\phi_3 : H_1(U \cap V) \to H_1(U) \oplus H_1(V)$. It is clear that the generator of $H - 1(U \cap V)$ gives rise to one of the generators of $H_1(U)$ and of $H_1(V)$. from the may in the Mayer-Vietoris sequence, $H_1(U \cap V)$ has image $\mathbb{Z}$. So $\operatorname{im} \phi_3 = \mathbb{Z}$ which means $\ker \phi_3 = 0$. Thus, there are two sequences

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \stackrel{\phi_1}{\to} H_2(X) \stackrel{\phi_2}{\to} \operatorname{im} \phi_2 = 0 \qquad \& \qquad 0 \to \mathbb{Z} \stackrel{\phi_3}{\to} \mathbb{Z}^4 \stackrel{\phi_4}{\to} H_1(X) \stackrel{\phi_5}{\to} 0.$$

The first sequence gives $H_2(X) \cong \mathbb{Z}^2$ and the second sequence, knowing that the image of ϕ_3 is the copy of $\mathbb{Z}\langle (1,0) \oplus (-1,0) \rangle$ which generates a copy of $\mathbb{Z}$ inside $\mathbb{Z}^4$, gives $H_1(X) \cong \mathbb{Z}^3$. *Warning:* The fact that the image generated by $(1,0) \oplus (-1,0)$ isomorphic to $\mathbb{Z}$ does not imply that we necessarily get a quotient that is free. For instance, the image of $\times 2 : \mathbb{Z} \to \mathbb{Z}$ is $\mathbb{Z}$ but gives rise to $\mathbb{Z}/2\mathbb{Z}$ as the quotient. However, in our case, we can take a change of basis of $\mathbb{Z}^4$ so that $(1,0) \oplus (1,0)$ generates a copy of $\mathbb{Z}$ exactly.

As $\operatorname{im} \phi_3 = \mathbb{Z}$, we know $\ker \phi_3 = 0$. The second sequence gives $H_1(X) \cong \mathbb{Z}^3$. We conclude that

$$H_0(X) \cong \mathbb{Z}, \qquad H_1(X) = \mathbb{Z}^3, \qquad H_2(X) = \mathbb{Z}^2, \qquad H_n(X) = 0 \text{ for } n \geq 3.$$

The last statement follows from X not having any 3-cells in any CW-complex decompositions. □

Proof of (b) via Cellular Homology. Recall that cellular homology coincides with singular homology when the space is a CW-complex. The space has a CW-complex decomposition as drawn in figure 44.

Now the cellular complex is given by

$$0 \leftarrow \mathbb{Z}\langle v \rangle \stackrel{d_1}{\leftarrow} \mathbb{Z}\langle a, b, c \rangle \stackrel{d_2}{\leftarrow} \mathbb{Z}\langle ab^{-1}a^{-1}b, ac^{-1}a^{-1}c \rangle \leftarrow 0.$$

The map d_1 is zero because each edge has just v as its boundary. Set $X := S^1 \times (S^1 \vee S^1)$. It follows that $H_0(X) = \mathbb{Z}\langle v \rangle / \operatorname{im}(d_1) = \mathbb{Z}$.

Now compute $H_1(X)$. Since $\ker(d_1) = \mathbb{Z}\langle a, b, c \rangle = \mathbb{Z}^3$, it suffices to show that the image of d_2 is zero. Recall how the boundary map of the LES of relative homology is computed.

It requires looking at the diagram

$$
\begin{tikzcd}
0 \arrow[r] & S_2(X_1) \arrow[r] \arrow[d, "d_2"'] & S_2(X_2) \arrow[r] \arrow[d, "d_2"] & S_2(X_2)/S_2(X_1) \arrow[r] \arrow[d, "\overline{d_2}"] & 0 \\
0 \arrow[r] & S_1(X_1) \arrow[r] & S_1(X_2) \arrow[r] & S_1(X_2)/S_1(X_1) \arrow[r] & 0
\end{tikzcd}
$$

and chasing an element in $S_2(X_2)/S_2(X_1)$ to $S_1(X_1)$. One pulls back to $S_2(X_2)$, pushes down to $S_1(X_2)$ and then pulls back to $S_1(X_1)$ via the inclusion. Of course, this just means the 2-chains are being computed via the usual differential maps. Of course, the LHS square (call it A, and B the RHS square) gives rise to

$$\partial(A) = a - b - a - b = 0 \qquad \& \qquad \partial(B) = a - c - a - c = 0.$$

Hence, $\operatorname{im}(d_2) = 0$ and

$$H_1(X) = \ker(d_1)/\operatorname{im}(d_2) = \mathbb{Z}^3/0 = \mathbb{Z}^3 \qquad \& \qquad H_2(X) = \ker(d_2)/\operatorname{im}(d_3) = \ker(d_2) = \mathbb{Z}^2.$$

It follows that

$$H_0(X) \cong \mathbb{Z}, \qquad H_1(X) = \mathbb{Z}^3, \qquad H_2(X) = \mathbb{Z}^2, \qquad H_n(X) = 0 \text{ for } n \geq 3.$$

$\square$

Hatcher Section 2.2, Problem 17. Construct a surjective map $S^n \to S^n$ of degree zero, for each $n \geq 1$.

PROOF. We know from p.134 that f being nonsurjective implies $\deg f = 0$. This exercise shows this is not an iff.

Let S^n be a single sphere. We can take a map

$$S^n \to S^n \vee S^n$$

by quotienting a ball around the north and south pole. Then we consider another map

$$S^n \to S^n \vee S^n \to S^n \vee S^n$$

that rotates one of the spheres so that the local degrees $+1$ and -1 for each sphere. Then compose with a map that collapses both spheres to a single sphere

$$S^n \to S^n \vee S^n \to S^n \vee S^n \to S^n.$$

Observe that this map is clearly surjective. $\square$

Hatcher Section 2.2, Exercise 19 Compute $H_i\left(\mathbb{RP}^n/\mathbb{RP}^m\right)$ for $m < n$ by cellular homology, using the standard CW structure on $\mathbb{RP}^n$ with $\mathbb{RP}^m$ as its m-skeleton.

PROOF. Recall that $\mathbb{R}P^n$ is constructed by attaching a single cell in each dimension. To attach the kth cell, we used the attaching map $S^{k-1} \to \mathbb{R}P^{k-1}$ of the 2-sheeted cover.

The space $\mathbb{R}P^n/\mathbb{R}P^m$ has a single cell in dimension 0 and dimensions $k \geq m+1$, but no cells in any other dimension. So the cellular chain complex looks like

$$0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \cdots \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0$$

where the first $\mathbb{Z}$ appears in dimension zero while the next $\mathbb{Z}$ appears at dimension $m+1$ and copies of $\mathbb{Z}$ appear until dimension n. Then computing the homology is easy if we recall that the differential maps are given by $1 + (-1)^i$ when in dimension i. Note that in this case, the differential $d : \mathbb{Z} \to 0$ in dimension m is the zero map. See figure 45 for the table exhibiting the results of the computation.

$\square$

Hatcher Section 2.2, Exercise 21 If a finite CW complex X is the union of subcomplexes A and B, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

Proof. Since $X = A \cup B$, we may apply the Mayer-Vietoris sequence with open covers $\{U, V\}$ where U is an open set containing A and V is an open set containing B s.t. U, V deformation retracts to A, B respectively. Then $U \cap V$ deformation retracts to $A \cap B$. So the Mayer Vietoris sequence looks like

$$\begin{aligned} 0 \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \\ \to \ldots \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0. \end{aligned}$$

Now we use the following lemma.

Lemma 7.1. Let

$$0 \to C_n \to C_{n-1} \to \ldots \to C_1 \to C_0 \to 0$$

be an exact sequence of f.g. abelian groups. Then

$$\sum_{i=0}^{n} (-1)^i \operatorname{rank} C_i = 0.$$

Proof. Applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$ and using the classification of f.g. modules over a PID, we get

$$0 \to \mathbb{Q}^{\operatorname{rank} C_n} \to \mathbb{Q}^{\operatorname{rank} C_{n-1}} \to \ldots \to \mathbb{Q}^{\operatorname{rank} C_0} \to 0$$

because $\mathbb{Q}$ being the localization of $\mathbb{Z}$ means that it is a flat $\mathbb{Z}$-module. Then the sequence above is just an exact sequence of vector spaces. Then the formula

$$\sum_{i=0}^{n} (-1)^i \operatorname{rank} C_i = \sum_{i=0}^{n} (-1)^i \dim(\mathbb{Q}^{\operatorname{rank} C_i}) = 0$$

follows from the rank-nullity theorem of linear algebra. $\square$

Applying the lemma to the M.V. sequence we have above,

$$0 = \sum_{\ell=0}^{n} (-1)^\ell \operatorname{rank} C_\ell = \sum_{i=0}^{n} (-1)^i \operatorname{rank}(H_i(X)) - (-1)^i \operatorname{rank}(H_i(A) \oplus H_i(B)) + (-1)^i \operatorname{rank}(H_i(A \cap B)).$$

Because $\operatorname{rank}(H_i(A) \oplus H_i(B)) = \operatorname{rank} H_i(A) + \operatorname{rank} H_i(B)$, we can rewrite

$$\begin{aligned} \chi(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(X) &= \sum_{i=0}^{n} (-1)^i [\operatorname{rank} H_i(A) + \operatorname{rank} H_i(B)] - \sum_{i=0}^{n} (-1)^i \operatorname{rank}(H_i(A \cap B)) \\ &= \chi(A) + \chi(B) - \chi(A \cap B). \end{aligned}$$

It is important that we note that we sum up to n because $H_i(X), H_i(A), H_i(B), H_i(A \cap B)$ will always have their last nonzero homology at index $\leq n$. This is because every CW-complex here is finite. $\square$

Hatcher Section 2.2, Exercise 22 For X a finite CW complex and $p : \tilde{X} \to X$ an n-sheeted covering space, show that $\chi(\tilde{X}) = n\chi(X)$.

Proof. Assuming we know that the n-sheeted covering space of a finite CW-complex will be a CW-complex with n times many cells in each dimension, the result is obvious. Indeed,

$$\chi(\widetilde{X}) = \sum_{i\geq 0}(-1)^i\#\{i - \text{cells of } \widetilde{X}\} = \sum_{i\geq 0}(-1)^i n\#\{i - \text{cells of } X\} = n\chi(X).$$

It may be wise for us to explain why the first statement is true. Any cell D^k of X has a map $D^k \to X$ in which its boundary is glued to the $(k-1)$-skeleton $\mathrm{sk}_{k-1} X$. The fundamental group of D^k is zero so there is a lift of D^k to $\widetilde{X}$. Since the cover is n-sheeted, there are n corresponding k-cells D^k that map into $\widetilde{X}$ and these n lifts are the corresponding characteristic maps. $\square$

Hatcher Section 2.2, Exercise 24 Suppose we build S^2 from a finite collection of polygons by identifying edges in pairs. Show that in the resulting CW structure on S^2 the 1-skeleton cannot be either of the two graphs shown, with five and six vertices. [This is one step in a proof that neither of these graphs embeds in $\mathbb{R}^2$.]

Proof. The idea of proof is to use knowledge of $H_*(S^2)$ and the Euler characteristic of the graphs to determine that such a cellular chain complex cannot possibly exist.

Suppose $P_1, \ldots, P_m$ are polygons with the obvious CW-complex decompositions. Assume $\frac{(P_1\cup\cdots\cup P_m)}{\sim} \cong S^2$ is obtained by identifying pairs of edges. Then $g(S^2) = 0$ implies

$$2 = \chi(S^2) = m - \#\{\text{1-cells}\} + \#\{\text{0-cells}\} = m + \chi\left(\mathrm{sk}_1\left(\frac{(P_1 \cup \cdots \cup P_m)}{\sim}\right)\right) =: m + k.$$

It follows that $k = 2 - m$ so the Euler characteristic of the one skeleton must equal $2 - m$.

Let G_1 denote the 1-skeletong given by the first graph (resp. G_2 for the second). Clearly, $\chi(G_1) = -5$ and $\chi(G_2) = -3$.

For G_1, we have $m = 7$. The graph itself has 10 edges and 5 vertices. Each edge must arise from identifying a pair of edges from the polygons we are given. Indeed, each edge has two faces that it sits next to. So the number of edges $e_1 + e_2 + \cdots + e_7 = 20$ from each polygon sum to 20. But each polygon necessarily has ≥ 3 edges which means the LHS is at least 21. This is absurd.

For G_2, we have $m = 5$. There are 9 edges. A quick examination of the graph shows that there are no faces that are bounded by 3 edges. So all faces are bounded by ≥ 4 edges. OTOH, we know there must be 5 polygons which means there are at least $\geq 5 \times 4 = 20$ edges. But observe that each edge in the graph can only be the boundary of at most two faces. Some of them are pictured in figure 48. It follows that there are best 9 pairs of edges identified. But that gives at least 11 edges. Our graph has 9. This is absurd.

$\square$

Hatcher Section 2.2, Exercise 24 Suppose we build S^2 from a finite collection of polygons by identifying edges in pairs. Show that in the resulting CW structure on S^2 the 1-skeleton cannot be either of the two graphs shown, with five and six vertices. [This is one step in a proof that neither of these graphs embeds in $\mathbb{R}^2$.] See image 7.

Proof. The idea of proof is to use knowledge of $H_*(S^2)$ and the Euler characteristic of the graphs to determine that such a cellular chain complex cannot possibly exist.

Suppose $P_1, \ldots, P_m$ are polygons with the obvious CW-complex decompositions. Assume $\frac{(P_1 \cup \cdots \cup P_m)}{\sim} \cong S^2$ is obtained by identifying pairs of edges. Then $g(S^2) = 0$ implies

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□

8. Section 2.3

9. Section 2.A

10. Section 2.B

11. Section 2.C

12. Section 3.1

13. Section 3.2

14. Section 3.3

Hatcher 3.3, Problem 5 Show that $M \times N$ is orientable iff M and N are both orientable.

Proof. The result is clear if we assume that M and N are compact since we can just apply the Künneth Theorem. □

15. Section 3.A

16. Section 4.1

17. Section 4.2

18. Section 4.3

19. Section 4.1A

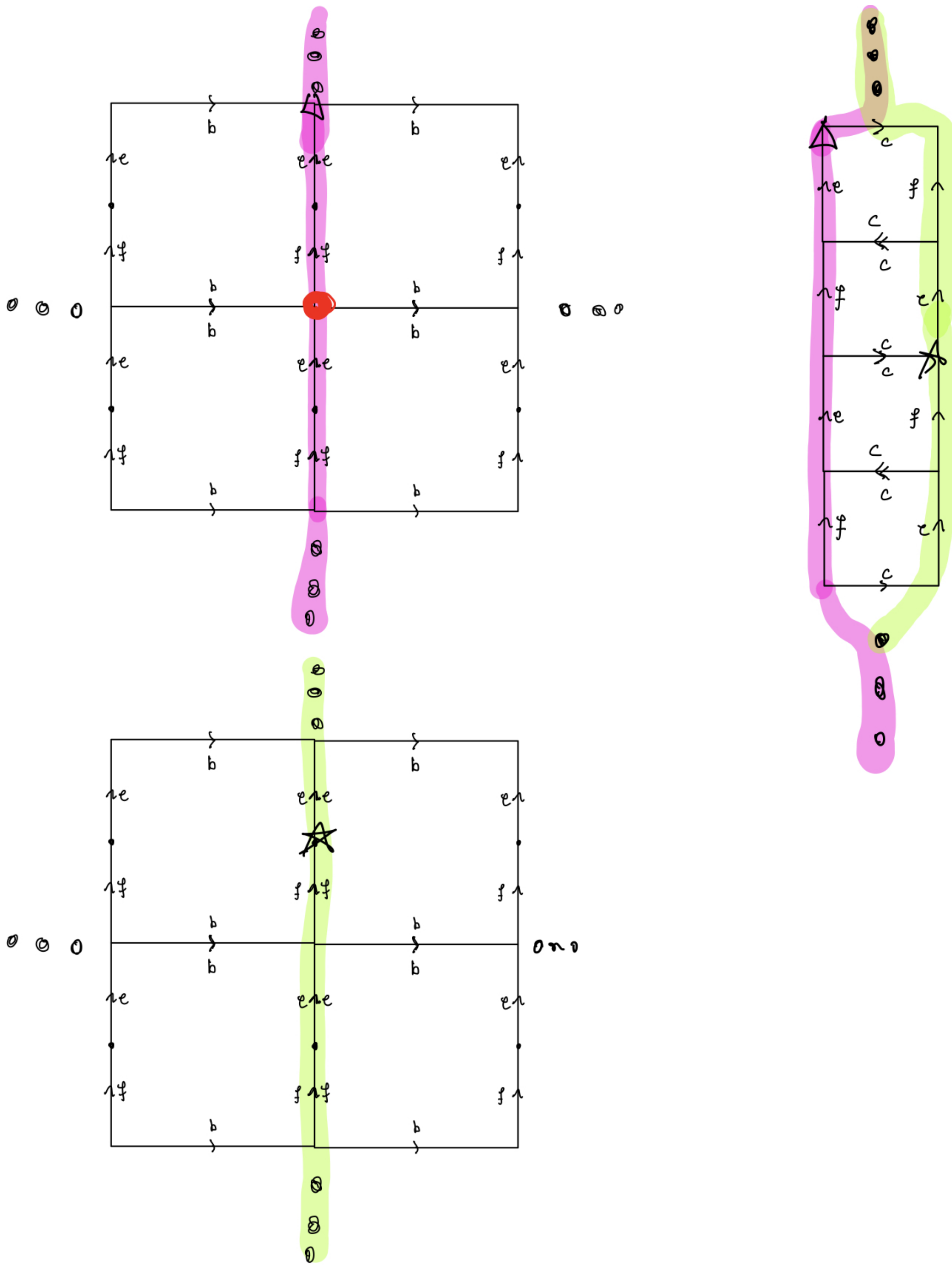

FIGURE 32. The universal covering space of the torus is just what is pictured on the upper left (replace the e, f identification by an edge a). The covering space for the Möbius band is pictured on the right (note that is actually where we ignore the e, f identifications). These universal covering spaces play a role in the universal covering space of X. Here, the edges e, f indicate the gluing of the boundary circle onto the circle of T^2 described to construct X. To construct the covering space, glue $\mathbb{R}^1 \times I$ (on the right) to copies of $\mathbb{R}^2$ as indicated. Use the star and triangle drawn to see where to glue the edges. We must repeat this process inductively to get the desired covering space. See figure 33 for the detailed construction.
To see why would be a covering space, see the caption in figure 33.

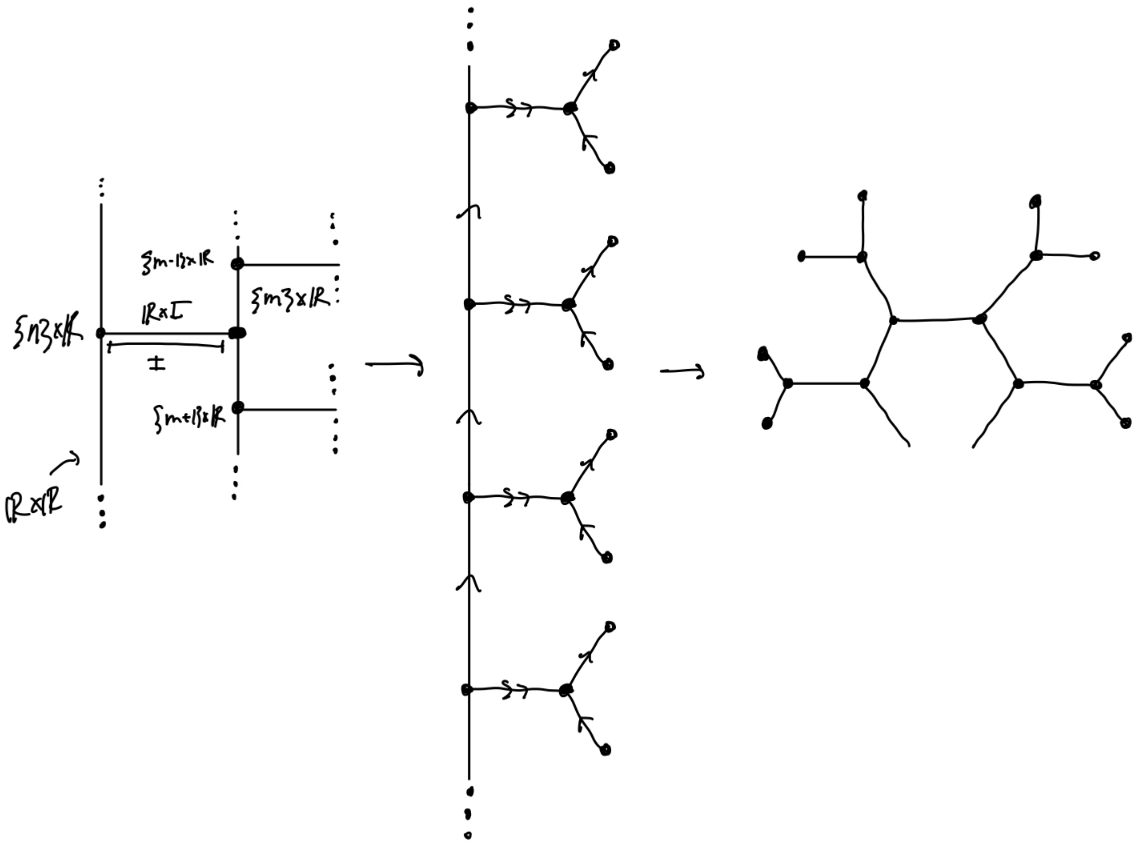

FIGURE 33. On the LHS we start with a line that is simply a copy of $\mathbb{R}^2$ viewed "from above". Choose the line $\{n\} \times \mathbb{R}$ which is a point from our view. Attach a copy of $\mathbb{R} \times I$ (pictured in figure 32). Attach one such copy for each $\{n\} \times \mathbb{R}$ with $n \in \mathbb{Z}$. After that, for each $\mathbb{R} \times I$ we had attached, we can attach another copy of $\mathbb{R}^2$ to each (by the gluing pattern in 32). Repeat this inductively to get the second situation in the figure above (the image is finite, but the construction is inductively continued to the right). If we were to look at this graphically, we get the graph pictured on the RHS (wrap the image in center by folding the line into a "c" and flip). We would continue it symmetrically. Our covering space is then $\widetilde{X} = G \times \mathbb{R}$ where G is the graph obtained.

This is the universal covering because it is clearly path-connected and $\pi_1(\widetilde{X})$ is easy to compute here. Indeed, we can deformation retract the graph to a single point which then gives us just a single copy of $\mathbb{R}$ which has trivial fundamental group. So, $\pi(\widetilde{X}) = 0$.

The action of $\pi_1(X)$ is given as follows. It may be easier to look at the second pictured image. If we act by b that just means shifting along one of the vertical lines (going upwards). If we act by a, we act by shifting along (not pictured) a copy of $\mathbb{R}$. For the action by d, we do exactly half of a in the shift upward the copy of $\mathbb{R}$. For instance, d would send the edge e to the edge f and f to e while a would send the edge fe to fe by a vertical shift. In this way, $d^2 = a$.

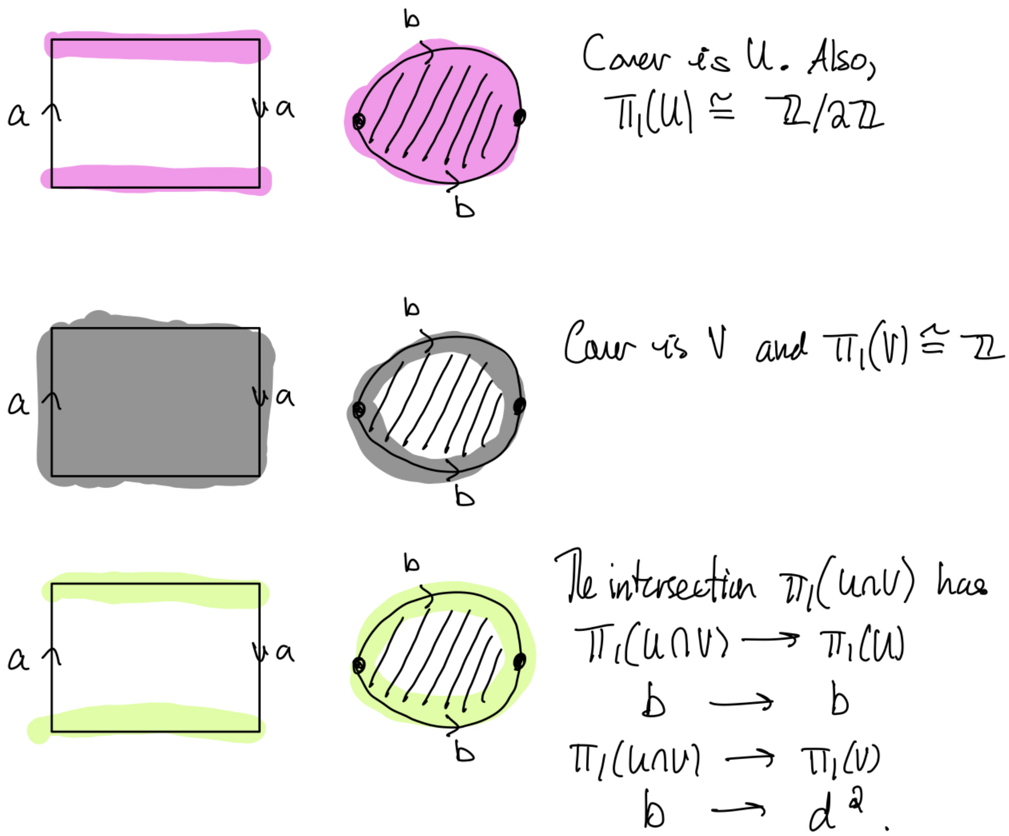

FIGURE 34. Our computation for the fundamental group. Choose the open set to be U and V as indicated in pink and gray. Their intersection is indicated in green. Since these spaces are "nice", we can easily deduce what $\pi_1(U)$ and $\pi_1(V)$ are (as we know the fundamental groups of M and $\mathbb{R}P^2$. In the intersection, the generator is a circle given by the edge b. Inside projective space, it is just the circle b while inside M, it is d^2 where d is the circle generating $\pi_1(M) \cong \langle d \rangle$. By Van Kampen's theorem, $\pi_1(Y) \cong \frac{\pi_1(U) * \pi_1(V)}{N} \cong \frac{\langle b,d \mid b^2 \rangle}{N(bd^2)} \cong \langle b, d \mid b^2 = e, b = d^{-2} \rangle$. This group can be simplified as it is isomorphic to $\langle d \mid (d^{-2})^2 = e \rangle \cong \mathbb{Z}_4$.

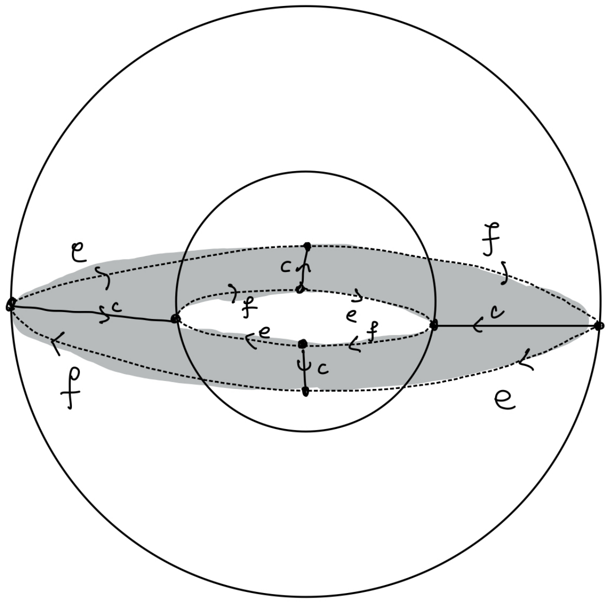

FIGURE 35. See 36 for how we got our space Y. This is the universal cover. We took a square $I \times I$ with the identifications to get the Möbius strip and then the identifications to glue it to $\mathbb{R}P^2$ as stated in the problem. We put together four copies of $I \times I$ with the identifications and form an annulus. The universal covering space of $\mathbb{R}P^2$ is the sphere with its meridian circle as indicated in the larger outer circle. Now, we can fit an annulus inside the larger sphere and then smaller sphere. What we obtain is the surface pictured $\widetilde{Y}$. The surface is clearly path-connected. It has $\pi_1(\widetilde{Y}) = 0$ by the following algebraic argument (and we also give a geometric argument).

Take U to be the open subset consisting of the outer sphere and a part of the outer boundary circle of the annulus so that U is open. Let V be the inner sphere with the annulus except we do not go all the way to the outer boundary circle. So essentially, $U \cap V$ consists of a thin circle lying in the annulus. Both $\pi_1(U)$ and $\pi_1(V)$ are trivial because we can deformation retract the part of the annulus to the spheres, and then use the fact that $\pi_1(S^2) = 0$. Therefore, the surjection $0 = \pi_1(U) * \pi_1(V) \to \pi_1(\widetilde{Y})$ implies $\widetilde{Y}$ is simply connected.

Given a loop in $\widetilde{Y}$. If the loop lies entirely in either sphere, then contracts to point. If a loop lies in the annulus and goes around the inner sphere, we can pull the onto the inner sphere and contract it to the point using $\pi_1(S^2) = 0$. If the loop has any part of its path circling the inner sphere when in the annulus, we can then contract that part to a point by pulling it into the inner sphere. So, any loop we have lies entirely in the outer sphere by retracting the line that remains on the annulus and on the inner sphere. But then we can contract the loop on the outer sphere to a point.

The action of $\pi(Y) \cong \mathbb{Z}_4$ is by rotating the annulus. This is because Hatcher Proposition 1.39 tells us that $\pi_1(\widetilde{Y}) \cong G(\widetilde{Y})$ so we just need to understand the group of deck transformations. If we rotate the annulus and then shrink the outer circle so it becomes the inner circle, we get a deck transformation which has the desired property. See figure 37.

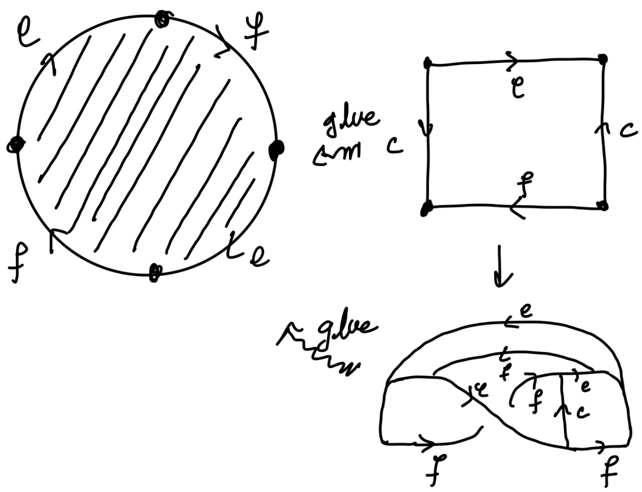

FIGURE 36. How we obtained our surface Y.

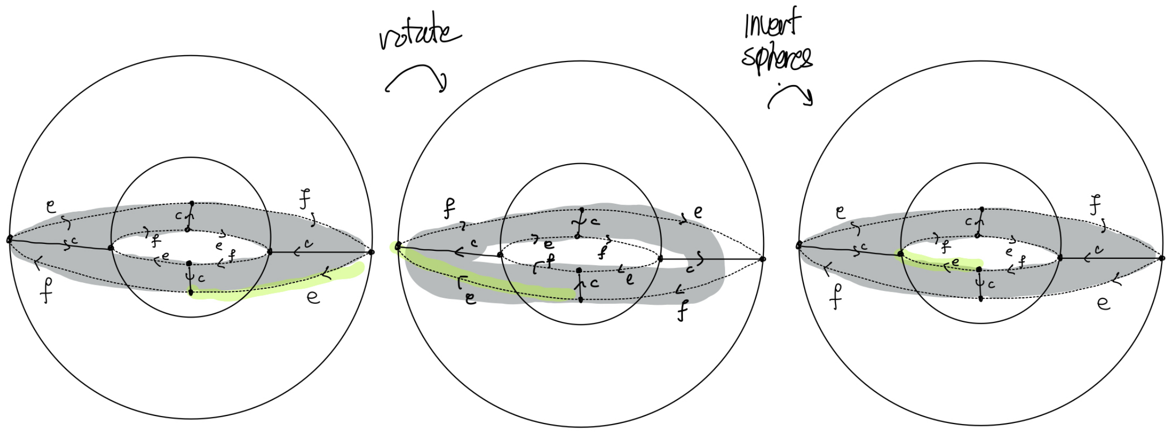

FIGURE 37. The generator for the cyclic group $G(\widetilde{Y})$. Colored in green is an edge for reference to see what happens to an edge due to the transformation. Observe that doing this four times will send the colored edge to its original spot.

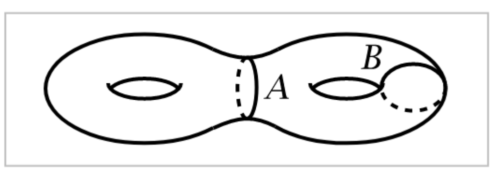

FIGURE 38.

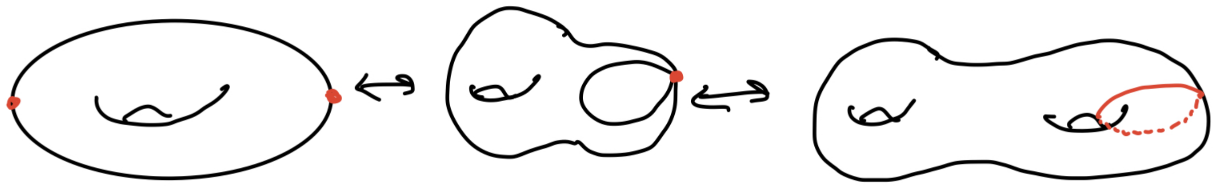

FIGURE 39. The homotopy equivalences between the genus two surface quotiented by b and the torus with two points identified.

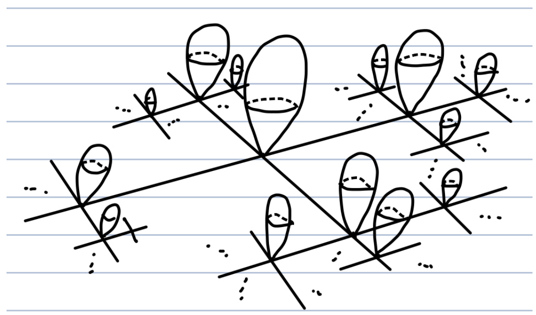

FIGURE 40. The universal cover of $Y = S^1 \vee S^1 \vee S^2$.

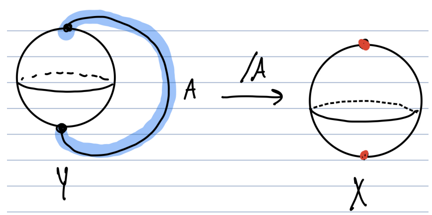

FIGURE 41. An alternative construction of X.

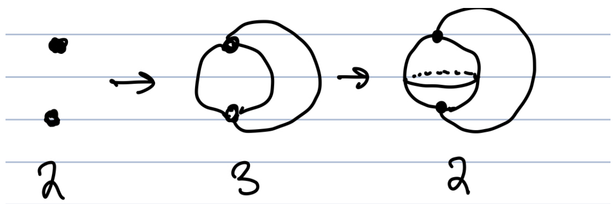

FIGURE 42. The CW-complex decomposition of Y that we take.

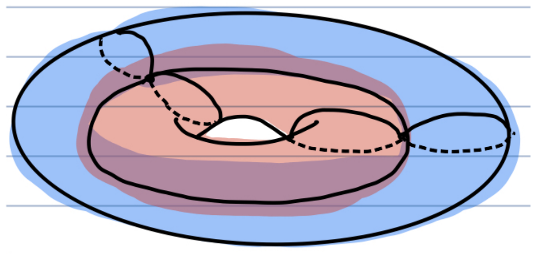

FIGURE 43. The space is X. Pictured as well is the Mayer-Vietoris covering. We take blue to be U and red to be that for V.

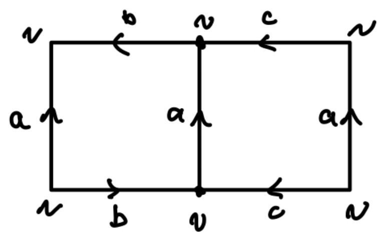

FIGURE 44. The construction of $S^1 \times (S^1 \vee S^1)$ via the cell decomposition. The idea is to wrap the two left and right hand side ends up so get a sort of emboldened 8. Then use the identifications on top and bottom to glue to get the surface in figure 43.

$H_i(\mathbb{R}P^n/\mathbb{R}P^m)$	m odd	m even
n odd	$\begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & m+1 \\ \mathbb{Z}/2\mathbb{Z} & m+j,\ j \text{ even} >0 \\ 0 & m+j\ \ j \text{ odd} \geq 1 \\ \mathbb{Z} & n \end{cases}$	$\begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/2\mathbb{Z} & m+1 \\ 0 & m+j,\ j \text{ even} >0 \\ \mathbb{Z}/2\mathbb{Z} & m+j\ \ j \text{ odd} \geq 1 \\ \mathbb{Z} & n \end{cases}$
n even	$\begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & m+1 \\ \mathbb{Z}/2\mathbb{Z} & m+j,\ j \text{ even} >0 \\ 0 & m+j\ \ j \text{ odd} \geq 1 \\ 0 & n \end{cases}$	$\begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/2\mathbb{Z} & m+1 \\ 0 & m+j,\ j \text{ even} >0 \\ \mathbb{Z}/2\mathbb{Z} & m+j\ \ j \text{ odd} \geq 1 \\ 0 & n \end{cases}$

FIGURE 45. The table exhibiting our results.

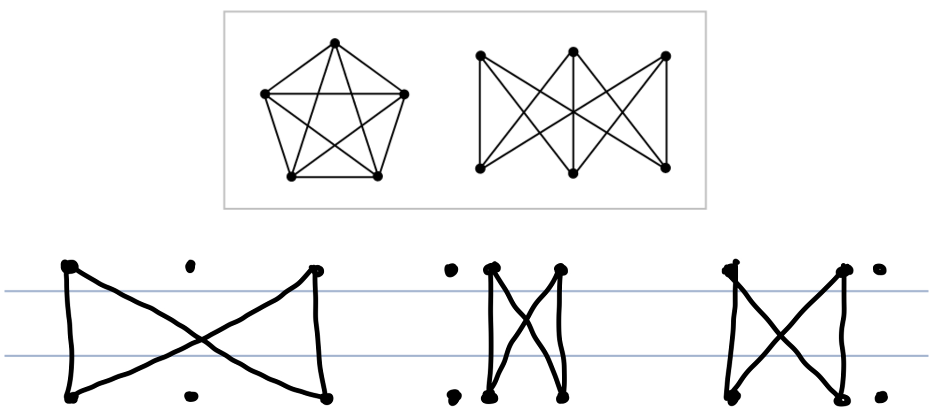

FIGURE 46. Possible faces.

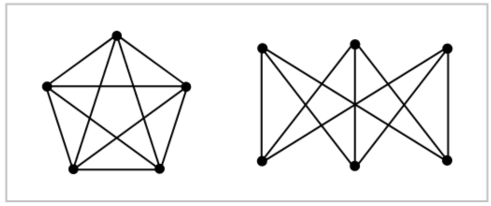

Figure 47. Two graphs.

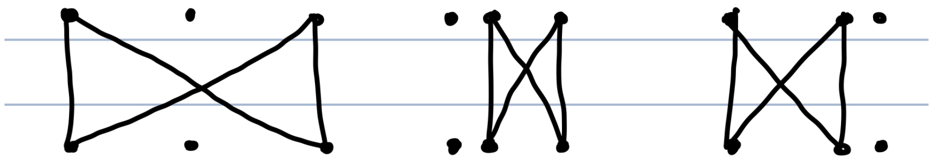

Figure 48. Possible faces.

APPENDIX A

Questions

Question 1. Does there exist a space with trivial fundamental group for which *there does not exist a covering space* (or even there does not exist a universal covering space)?

Answer. This is a trivial question. Every space is its own covering space so the first question is a negative. Meanwhile, the second question depends on the hypothesis on the base space X. If X is path-connected, then it is its own universal covering space. If X is just connected, then the question isn't too worthwhile to consider since we can just look at path-components. If X is allowed to be disconnected, then a disjoint union of spheres will never have a simply connected covering space and all of its fundamental groups are trivial. □

Question 2. For CW-complexes, is homotopy equivalence the same as being homeomorphic?

Answer. Trivially not so. The line segment is homotopy equivalent to a point, but these are not homeomorphic. □

Question 3. Why can the disk not be the universal covering space for $\mathbb{R}P^2$?

Answer. This is false. The disk can be the UCS since all it would entail is identifying antipoldal boundary points of the disk. □

Question 4. What is the group presentation of $\pi_1(\mathbb{R}P^2\#T)$ and $\pi_1(\mathbb{R}P^2 \vee T)$?

Answer. For the first, note that $\mathbb{R}P^2\#T \cong \#_{i=1}^3\mathbb{R}P^2$ by the classification theorem and the formula for the Euler characteristic of connected sums. Also, it is nonorientable since the space contains a Möbius band. So, the fundamental group is $\langle a_1, a_2 \mid a_1^2a_2^2 = e\rangle$.

For the second, we have $\pi_1(\mathbb{R}P^2 \vee T) \cong \langle a_1, a_2, b_1, b_2 \mid a_1^2a_2^2 = b_1b_2b_1^{-1}b_2^{-1} = e\rangle$. □

Question 5. Compute the singular homology of $X := S^2 \vee S^2$ for all degrees.

Answer. The result actually follows from the more general fact that $\widetilde{H}_*(\vee_{i=1}^n X_i) \cong \oplus_{i=1}^n \widetilde{H}_*(X_i)$ when we have good pairs (X_i, x_i). This holds for arbitrary direct sums too. See Corollary 2.2.25 of Hatcher.

Since we do not have computational tool after the first quarter, let me prove this from more direct methods.

Set $A := S^2$ be one copy. Fix $n \geq 3$. Then $H_n(X) \cong H_n(X, A)$ by the LES and the fact that $H_n(A) = 0$ for $n \geq 3$. By the Excision Principle, I know $H_n(X, A) \cong H_n(X/A, \{*\}) \cong H_*(S^2, \{*\})$. To compute this RHS,

$$\to H_n(\{*\}) \to H_n(S^2) \to H_n(S^2, \{*\}) \to H_{n-1}(\{*\}) \to$$

which is $\to 0 \to 0 \to H_n(S^2, \{*\}) \to 0 \to$ so $H_n(S^2, \{*\}) = 0$ for $n \geq 3$. This means $H_n(X, A) \cong 0$ for $n \geq 2$. Therefore,

$$H_n(X) \cong H_n(X, A) \cong H_n(S^2, \{*\}) \cong 0$$

for $n \geq 2$.

We focus on $n = 2$ now. We have a LES

$$\to H_3(X, A) \to H_2(A) \to H_2(X) \to H_2(X, A) \to H_1(A) \to$$

and $H_2(A) \cong \mathbb{Z}$, $H_1(A) \cong 0$, and $H_3(X, A) = 0$ from before. So we have a SES

$$0 \to \mathbb{Z} \to H_2(X) \to H_2(X, A) \to 0.$$

We compute $H_2(X, A)$. We know $H_2(X, A) \cong H_2(X/A, \{*\}) \cong H_2(S^2, \{*\})$ by the excision principle. But that means

$$\to H_2(\{*\}) \to H_2(S^2) \to H_2(S^2, \{*\}) \to H_1(\{*\}) \to$$

gives a SES

$$0 \to \mathbb{Z} \to H_2(S^2, \{*\}) \to 0.$$

So, $H_2(S^2, \{*\}) = \mathbb{Z}$ and so there was a SES

$$0 \to \mathbb{Z} \to H_2(X) \to \mathbb{Z} \to 0 \qquad \Longleftrightarrow \; 0 \to H_2(A) \to H_2(X) \to H_2(X, A) \to 0.$$

I claim that the map $H_2(X) \to H_2(X, A)$ is a split epimorphism. This is relatively easy to see since the relative homology corresponds to those homology elements that lay entirely in the other sphere. Therefore, $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ as it should. □

Question 6. Is singular homology well-behaved with respect to products?

Answer. No. The Mayer-Vietoris theorem is what relates the concepts. Let $X \times Y$ be a given product space. Assume one knows there exists $X \times U_0$ and $V_0 \times Y$ s.t. $X \times U_0 \cap V_0 \times Y$ is contractible. Then for $n \geq 1$, there is an isomorphism $H_n(X \times Y) \cong H_n(X) \times H_n(Y)$. □

Question 7. Neither $\mathbb{R}P^2$ and T^2 can cover the other?

Answer. Yes. The fundamental groups are $\mathbb{Z}/2$ and $\mathbb{Z} \oplus \mathbb{Z}$ which can never be a subgroup of the other by Theorem 1.38 of Hatcher. □

Question 8. The torus is a 2-sheeted cover of the Klein bottle? Can it be 3-sheeted?

Answer. For the 2-sheeted cover, draw out two tori and glue along a common edge. For the 3-sheeted cover, the answer is no. □

Question 9. What is $\mathbb{R}P^n$ removing a point homotopy equivalent to?

Answer. S^{n-1}. □

Question 10. If X is a path-connected space, does every connected covering space of X need be path-connected as well? *Note:* the answer seems to be a likely no given the statement of the Galois correspondence of covering spaces. However, any such example would be extremely pathological.

Question 11. Let X be a simply connected space. Clearly, $H_0(X) = \mathbb{Z}$ and $H_1(X) = 0$. Do all higher singular homology need to vanish as well?

Answer. Clearly no. The space S^2 is simply connected with $H_2(S^2) = \mathbb{Z}$. □

Question 12. Does there exist a space with trivial singular homology at all dimensions except zero which is not contractible?

Answer. Yes, one example can be found here. □

Question 13. Demonstrate how to use local degree to compute the degree of the reflection map.

Proof. The reflection $r_n : S^n \to S^n$ sends $(x_0, \ldots, x_n) \to (-x_0, \ldots, x_n)$. Orienting the spheres, we know r_n sends the north pole to the south pole.

Let $y \in S^n$ be the south pole and observe that $f^{-1}(y)$ consists of the north pole only. Note $f(U - x) \subseteq V - y$ where U, V are open sets (homeomorphic to a ball) around the north pole x and south pole y. Then,

$$f_* : H_1(U, U - x) \to H_1(V, V - y).$$

We compute the degree of this. It is clear that this map is of degree 1 (draw a picture and convince yourself). Indeed, one can pick the standard generator of the homology group it changes direction under the map. □

Question 14. Describe the generator of S^n's nth-homology group explicitly.

Answer. The generator of S^n's nth homology group is the image of the isomorphism

$$H_n(D^n, S^{n-1}) \overset{\cong}{\to} \widetilde{H_n(S^n)}.$$

□

Question 15. Does there exist a Mayer-Vietoris sequence for singular cohomology? If so, why is not as prevalent as the Mayer-Vietoris sequence for singular homology?

Answer. See Hatcher p. 203 for the cohomological version. It is as prevalent and used, it is just because one has not had to use it too often in Math 290B thus far. □

Question 16. Did these notes help you do well on the topology qualifying exams?

Answer. Probably :), but that is for the reader to decide. The writer of these notes scored a 95/100 on the qualifying exam at UCSD for the Spring 2022 qualifying exam. □

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