

MATH 234: CALCULUS – FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT. These were the set of lecture notes used for Math 234: Calculus – Functions of Several Variables taught at UW-Madison for Summer 2025. These notes have not been edited since the end of that summer.

Various resources were used which I will add as needed later.

CONTENTS

1. Setup for Multivariable Calculus	3
1.1. Preliminary Topics	3
1.2. 3d-coordinate systems and vectors	3
1.3. The Dot Product	7
1.4. The Cross Product	10
1.5. Equations of Lines and Planes	14
1.6. Cylinders, Quadric Surfaces	16
1.7. Vector Functions, and Space Curves	21
1.8. Calculus of Vector Functions	22
1.9. Velocity and Acceleration	27
1.10. Arc Length	27
1.11. Curvature, Normal and Binormal Vector	30
1.12. Functions in Several Variables	34
1.13. Limits and Continuity	35
1.14. Review	36
2. Calculus of Multivariable Functions	38
2.1. Partial Derivatives, and Tangent Planes	38
2.2. Multivariable Chain Rule	41
2.3. Directional Derivatives, and Optimization	43
2.4. Extreme Value Theorem, Optimization, and Lagrange Multipliers	50
2.5. More on Lagrange Multipliers, and Double Integrals over Rectangular Regions	55
2.6. *Lagrange Multipliers with Two Constraints	55
2.7. Double Integrals Over Rectangular Regions	56
2.8. Double Integrals over General Regions	59
2.9. Double Integrals in Polar Coordinates	63
2.10. Triple Integrals	66
2.11. Cylindrical Coordinates, and Spherical Coordinates	69
2.12. Change of Variables	73
2.13. Review	77
3. Vector Calculus	83
3.1. Vector Fields and Line Integrals	83

3.2.	Fundamental Theorem of Line Integrals	88
3.3.	Green's Theorem	91
3.4.	Curl and Divergence	96
3.5.	Parametric Surfaces	100
3.6.	Surface Integrals	103
3.7.	Stokes' Theorem	105
3.8.	The Divergence Theorem (aka Gauss's Theorem)	110
3.9.	Review	111

1. SETUP FOR MULTIVARIABLE CALCULUS

1.1. Preliminary Topics. For the first lecture, it is good to go over the syllabus with students.

Here are the key points from the syllabus the students should be aware of:

- the Canvas page has information regarding offices hours, exams, and resources students can use to supplement their studies,
- the student must follow the instructions for the Webassign links to be able to properly access the homework, module quizzes, etc.,
- the first quiz is this week, and the student should be actively checking the course calendar to see material the quizzes will be based on,
- for homework this week, the student needs to complete the online Diagnostic Quiz through Webassign which is meant to help students review their Calculus I and II background.

Students are expected to know the material from Calculus I and II aka single variable calculus. Here are some examples to work out for review.

- (1) Compute the derivative of $F(x) = x^3 \cos(x)$ with respect to x .
Solution: Using the product rule, $F'(x) = 3x^2 \cos(x) - x^3 \sin(x)$.

- (2) Compute the integral $\int \sin^2(x) \cos(x) dx$.
Solution: Let $u := \sin(x)$ so that $du = \cos(x) dx$. Then

$$\int \sin^2(x) \cos(x) dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\sin^3(x)}{3} + C.$$

- (3) Determine if the integral $\int_1^\infty \frac{1}{t^3} dt$ converges or diverges.
Solution: It converges. One can compute this directly

$$\int_1^\infty \frac{1}{t^3} dt = \lim_{t \rightarrow \infty} \int_1^s \frac{1}{t^3} dt = \lim_{s \rightarrow \infty} \left(-\frac{1}{2s^2} + \frac{1}{2} \right) = \frac{1}{2}.$$

- (4) Compute the integral $\int \frac{10x+2}{(3x+1)(x-1)} dx$.

Solution: One should use a partial fraction decomposition of $\frac{10x+2}{(3x+1)(x-1)}$ to compute the integral. Writing

$$\frac{10x+2}{(3x+1)} = \frac{A}{3x+1} + \frac{B}{x-1} = \frac{Ax - A + 3Bx + B}{(3x+1)(x-1)}.$$

One then has that $A + 3B = 10$ and $-A + B = 2$. Then $A = 1, B = 3$ solves this system of linear equations. So,

$$\int \frac{10x+2}{(3x+1)(x-1)} dx = \int \frac{1}{3x+1} + \frac{3}{x-1} dx = \frac{1}{3} \ln |3x+1| + 3 \ln |x-1| + C.$$

1.2. 3d-coordinate systems and vectors. We will work with 3d coordinate systems. Our convention is that the x -axis always points at the viewer aka we follow the “right-hand rule”.

Three dimensional Euclidean space is denoted $\mathbb{R}^3$ and points of this space are specified by coordinates (x, y, z) .

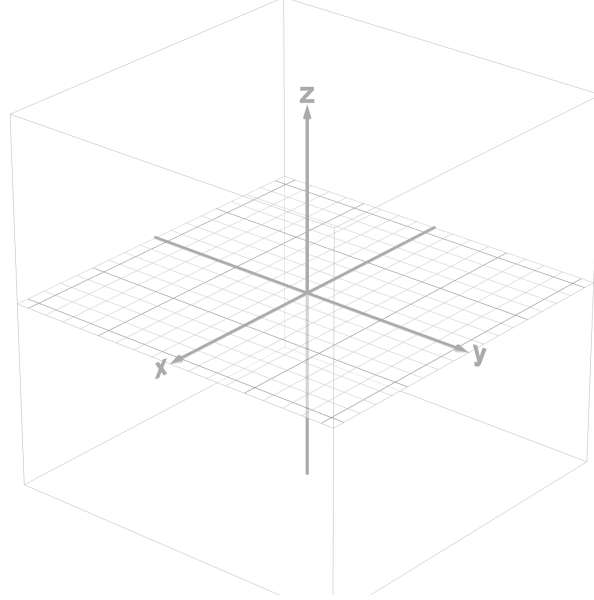


FIGURE 1. The right hand rule. The index finger points in the y -direction, the middle finger points in the x -direction, and z -direction is the direction of the thumb.

Example 1.2.1. For example, $(3, 2, -1)$ is the point in space where one moves in the x -direction by three units, in the y -direction by two, and in the z -direction by -1 .

- (1) Plot the points $(2, 1, -3)$, $(1, 1/2, -2)$, $(-1, 1, 2)$, and $(-1, -1, -1)$.
- (2) The points in $\mathbb{R}^3$ where $z = 2$ is a plane.
- (3) The points in $\mathbb{R}^3$ where $y = 2$ is also a plane.
- (4) The points in $\mathbb{R}^3$ where $x^2 + y^2 = 4$ is a cylinder.
- (5) The points in $\mathbb{R}^3$ where $x^2 + y^2 = 1, z = 2$ is a circle.

In $\mathbb{R}^3$, the distance between a point $P_1 = (x_1, y_1, z_1)$ and another point $P_2 = (x_2, y_2, z_2)$ is given by the **distance formula**:

$$|P_1 P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Example 1.2.2. For example, the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are all distance 1 away from the origin $(0, 0, 0)$.

- (1) Compute the distance between $(2, 3, 0)$ and $(3, 1, 1)$.
- (2) Compute the distance between $(1, 3, 8)$, and $(1, 3, 5)$.

Solution: (1) The distance between the two points equals

$$\sqrt{(3-2)^2 + (1-3)^2 + (1-0)^2} = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{1+4+1} = \sqrt{6}.$$

(2) The distance between the two points equals

$$\sqrt{(1-1)^2 + (3-3)^2 + (8-5)^2} = \sqrt{0^2 + 0^2 + 3^2} = \sqrt{9} = 3.$$

□

Using the distance formula, the equation that cuts out a sphere whose center is $P_0 = (x_0, y_0, z_0)$ and has radius r is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

Example 1.2.3. For example, the sphere given by $x^2 + y^2 + z^2 = 1$ is the unit sphere centered at the origin.

- (1) Draw the sphere given by the equation $x^2 + y^2 + z^2 = 9$.
- (2) Draw the sphere given by $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$. (Hint: recall the method of completing the square).
- (3) Draw the sphere given by $(x - 1)^2 + y^2 + 2y + (z - 3)^2 = 2$.

Solution: These problems are left as an exercise for the student. □

Definition 1.2.4. A **vector** is a “directed arrow” with an initial point and a terminal point. One writes $\vec{AB}$ for the vector whose initial point is A and terminal point is $\vec{B}$. Addition and scalar multiplication of vectors can easily be defined. Since we work in $\mathbb{R}^3$, we use vectors whose initial points are at the origin $(0, 0, 0)$. Then, a vector can be described as a tuple $\langle v_1, v_2, v_3 \rangle$. We use this notation to emphasize that we are viewing these as *vectors* and not as *points*.

Example 1.2.5. To draw a vector like $\langle 1, 1, 1 \rangle$, one draws an arrow from $(0, 0, 0)$ to the point $(1, 1, 1)$.

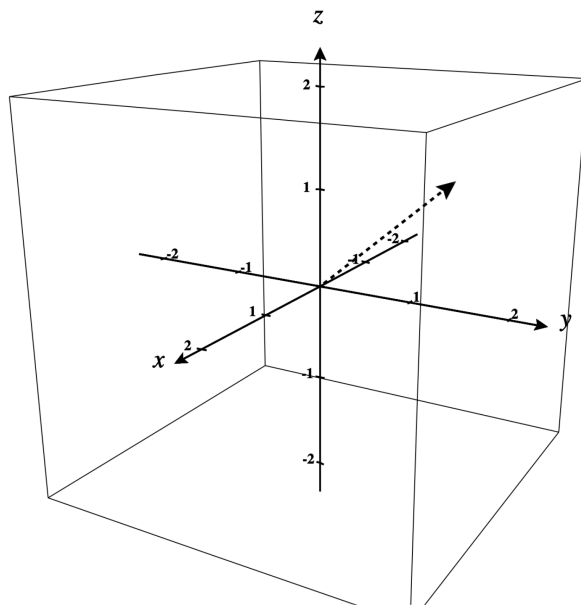


FIGURE 2. The vector $\langle 1, 1, 1 \rangle$.

As practice for the student, try the following exercises.

- (1) Draw the vector $\langle 1, 2, -1 \rangle$.
- (2) Draw the vector $\langle 2, 1, 1 \rangle$.

Definition 1.2.6. The **length/magnitude** of a vector in $\mathbb{R}^3$ is defined by the formula

$$|v| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Example 1.2.7. (1) Compute the length of $\langle 1, 2, -1 \rangle$.

- (2) Compute the length of $\langle 1, 1, 1 \rangle$.
- (3) Compute the length of $\langle 0, 1, 0 \rangle$.

Some people might use the notations $\vec{i}, \vec{j}, \vec{k}$ for the **standard basis vectors** $\langle 1, 0, 0 \rangle$, $\langle 0, 1, 0 \rangle$, $\langle 0, 0, 1 \rangle$. Any vector v can then be written in the form

$$v = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

Due to the potential for confusion, we will limit our usage of this notation.

Definition 1.2.8. If v is a nonzero vector (i.e. $v \neq \langle 0, 0, 0 \rangle$), the **unit vector in v 's direction** is the vector

$$\frac{v}{|v|} = \left\langle \frac{v_1}{\sqrt{v_1^2 + v_2^2 + v_3^2}}, \frac{v_2}{\sqrt{v_1^2 + v_2^2 + v_3^2}}, \frac{v_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \right\rangle.$$

Example 1.2.9. Given the following vectors, (a) compute their magnitude, (b) compute the unit vector in its direction, (c) and write the vector using standard basis vectors.

- (1) $v = \langle 1, 2, 3 \rangle$,
- (2) $v = \langle \frac{1}{2}, 2, 3 \rangle$.

Let $u = \langle 1, 2, -1 \rangle$ and $v = \langle 3, 2, 5 \rangle$. Compute the following

- (a) $u + 2v$,
- (b) $3u + v$,
- (c) $|u + v|$,
- (d) $\frac{1}{3}(u + 2v)$.

Solution: (1) $|v| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, the unit vector in v 's direction is $\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle$, and $v = \vec{i} + 2\vec{j} + 3\vec{k}$.

(2) Left for the student.

(a) $u + 2v = \langle 1, 2, -1 \rangle + \langle 6, 4, 10 \rangle = \langle 7, 6, 9 \rangle$.

(b) $3u + v = \langle 3, 6, -3 \rangle + \langle 3, 2, 5 \rangle = \langle 6, 8, 2 \rangle$.

(c) Left for the student.

(d) Left for the student. □

1.2.1. *Area and Volume of Intersecting Spheres.* The material in this subsection is not covered in class or on the exams. It is solely entertainment for the curious student.

Example 1.2.10 (Challenge Problem). Find the area of region inside of the the intersection of the two circles $x^2 + y^2 = 4$ and $(x - 3)^2 + y^2 = 4$.

Solution: The intersection is symmetrical along the line $x = 1.5$. The situation is pictured below.

Anyway, the region to the left of the line can be computed as follows. First, we write $x^2 + y^2 = 4$ and rewrite this in terms of y . So we are writing $y = \sqrt{4 - x^2}$ if we want the part above the x -axis. Then to compute what is in the RHS of the bounded region, we can setup the integral

$$2 \int_{1.5}^2 \sqrt{4 - x^2} dx$$

The factor of 2 accounts for the symmetry across the x -axis. Anyway, a trig substitution (the computation is omitted) gives us

$$4 \left(\frac{\pi}{2} - \arcsin \left(\frac{3}{4} \right) - \frac{3\sqrt{7}}{16} \right).$$

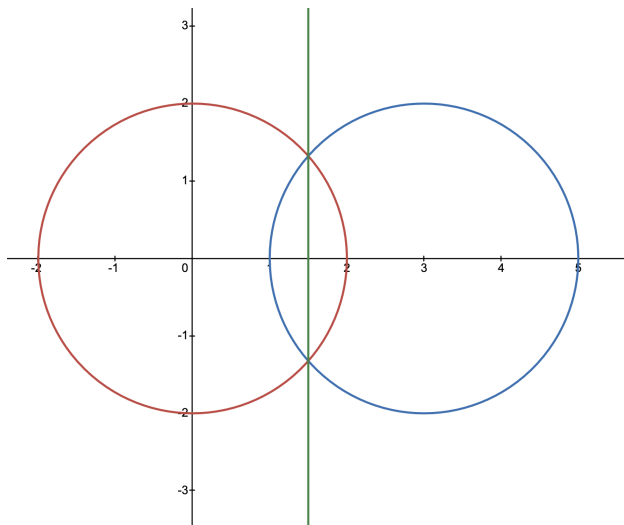


FIGURE 3. The intersection of two circles and the region bounded by it.

Then, multiplying this by 2 due to the symmetry across the $x = 1.5$ line gives the area

$$8 \left(\frac{\pi}{2} - \arcsin \left(\frac{3}{4} \right) - \frac{3\sqrt{7}}{16} \right).$$

□

Remark 1.2.11. For two circles of equal radius r with centers of distance d apart, one can show that the area of the intersection is exactly

$$A := 2r^2 \cos^{-1} \left(\frac{d}{2r} \right) - \frac{d}{2} \sqrt{4r^2 - d^2}.$$

This agrees with the example above. Showing this formula is theoretically within student's capabilities (it only uses tools from Calculus I and II) but perhaps is far more difficult for most students. Give it a try, but it won't be on the exam!

Here's a harder variant that was originally on the homework but taken off: compute the volume of the solid enclosed by two spheres of radius 2 but with centers distance $d = 3$ apart. Similar ideas as above work!

1.3. The Dot Product.

Definition 1.3.1. Let $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ be two vectors (again, we are in $\mathbb{R}^3$). Then, the **dot product** is

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Note that the input for the dot product is two vectors and the output is a scalar.

Example 1.3.2. The dot product of any vector u with the zero vector $\langle 0, 0, 0 \rangle$ is zero.

- (1) Compute $\langle 1, 2, 3 \rangle \cdot \langle 3, 5, 8 \rangle$.
- (2) Compute $\langle 1, -1, 2 \rangle \cdot \langle 1, 5, 2 \rangle$.

Solution: (1) The dot product is $3 + 10 + 24 = 37$.

(2) The dot product is $1 + (-5) + 4 = 0$. □

Here are some **properties of the dot product** you should be aware of:

- $a \cdot a = |a|^2$,
- the dot product behaves like multiplication of numbers in the sense that it is commutative, distributive, and the dot product with the zero vector is zero,
- the dot product is bilinear in the sense that if l is a number, then $(\vec{a} + l\vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + l(\vec{b} \cdot \vec{c})$.

Theorem 1.3.3 (Main Theorem about Dot Products). If a, b are vectors and θ is the angle between them, then

$$a \cdot b = |a||b| \cos \theta.$$

Proof. Apply the Law of Cosines to the triangle determined by $a, b, a - b$ to deduce that

$$|a - b|^2 = |a|^2 + |b|^2 - 2|a||b| \cos \theta.$$

Rewrite the left hand side

$$|a - b|^2 = (a - b) \cdot (a - b) = |a|^2 - 2(a \cdot b) + |b|^2.$$

Subtract $|a|^2 + |b|^2$ from both sides to get

$$-2(a \cdot b) = -2|a||b| \cos \theta.$$

To conclude, divide by -2 . □

Example 1.3.4. The main applications of the theorem above is to determine the angle between two vectors.

- (1) Find the angle between the vectors $\langle 2, 1, 0 \rangle$ and $\langle -1, 3, 5 \rangle$.
- (2) Find the angle between the vectors $\langle 1, 1, 0 \rangle$ and $\langle 0, 0, 5 \rangle$.
- (3) Find the angle between the vectors $\langle -1, 2, 3 \rangle$ and $\langle 1, -2, 3 \rangle$.

Solution: We will work out (1) and leave (2) and (3) for the student. First, apply the theorem to deduce that

$$\cos(\theta) = \frac{\langle 2, 1, 0 \rangle \cdot \langle -1, 3, 5 \rangle}{|\langle 2, 1, 0 \rangle| |\langle -1, 3, 5 \rangle|}.$$

The dot product equals $-2 + 3 + 0 = 1$. The magnitudes are $|\langle 2, 1, 0 \rangle| = \sqrt{5}$ and $|\langle -1, 3, 5 \rangle| = \sqrt{1 + 9 + 25} = \sqrt{35}$. So,

$$\theta = \cos^{-1} \left(\frac{\langle 2, 1, 0 \rangle \cdot \langle -1, 3, 5 \rangle}{|\langle 2, 1, 0 \rangle| |\langle -1, 3, 5 \rangle|} \right) = \cos^{-1} \left(\frac{1}{\sqrt{5} \cdot \sqrt{35}} \right) = \cos^{-1} \left(\frac{1}{5\sqrt{7}} \right).$$

□

Corollary 1.3.5 (Orthogonality). If u and v are **perpendicular/orthogonal** (i.e. the angle between them is 90°), then $u \cdot v = 0$.

Proof. Use the theorem and set $\theta = \frac{\pi}{2}$. □

Remark 1.3.6. The sign of the dot product determines when the angle between two vectors is acute/right/obtuse:

- (1) $a \cdot b < 0$, then $\theta > \frac{\pi}{2}$
- (2) $a \cdot b = 0$, then $\theta = \frac{\pi}{2}$

(3) $a \cdot b > 0$, then $\theta < \frac{\pi}{2}$.

Example 1.3.7. Determine which of the following pairs of vectors are orthogonal:

- (1) $\langle 2, 1, 3 \rangle, \langle -3, -2, -1 \rangle$,
- (2) $\langle 1, 1, 1 \rangle, \langle -1, -1, 2 \rangle$,
- (3) $2\vec{i} + 3\vec{j} - \vec{k}, 5\vec{i} - 4\vec{j} + 2\vec{k}$,
- (4) $\vec{i}, \vec{j}$.

Solution: (1) The dot product is -6 so the vectors cannot be orthogonal.

(2) The dot product is 0 so the two vectors are orthogonal.

(3) and (4) are left to the student. □

Definition 1.3.8. Let a, b be vectors. Then we define the **scalar projection of b onto a** as

$$\text{comp}_a b = \frac{a \cdot b}{|a|}$$

and the **vector projection of b onto a** as

$$\text{proj}_a b = \frac{a \cdot b}{|a|^2} a = \frac{a \cdot b}{|a|} \frac{a}{|a|}.$$

Remark 1.3.9. The better way to memorize how these formulas work is to memorize the formula for $\text{comp}_a b$ and then realize that $\text{proj}_a b = \text{comp}_a b \frac{a}{|a|}$ i.e. one scales the unit vector using the scalar projection.

Example 1.3.10. Compute both the scalar projection and the vector projection of b onto a in the following examples.

- (1) $a = \langle -2, 3, 1 \rangle$ and $b = \langle 1, 1, 2 \rangle$,
- (2) $a = \langle 1, 0, 0 \rangle$ and $b = \langle 5, 1, 2 \rangle$,
- (3) $a = \vec{i} + \vec{j} + \vec{k}$ and $b = 5\vec{i} - \vec{k}$.

Solution: The biggest piece of advice is that the vector projection is taking the component projection and using it to scalar the unit vector in the direction of the vector you are projecting on.

We shall do (1) and leave (2) and (3) to the student. For (1), we first compute the scalar projection

$$\text{comp}_a b = \frac{a \cdot b}{|b|} = \frac{3}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{3}{\sqrt{6}}.$$

The unit tangent vector in the direction of a is then $\langle \frac{-2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}} \rangle$. So the vector projection is precisely

$$\text{proj}_a b = \left\langle \frac{-6}{6}, \frac{9}{6}, \frac{3}{6} \right\rangle = \left\langle -\frac{2}{3}, \frac{3}{2}, \frac{1}{2} \right\rangle.$$

□

Example 1.3.11. Find the angle of the triangle Δ at the point $(1, 2, -1)$ where Δ has corner points $(0, 0, 0)$, $(3, 4, 5)$, $(1, 2, -1)$.

Solution: First, we compute the vectors which are the sides of the triangle with initial point $(1, 2, -1)$. In this case, we are looking at

$$\begin{aligned}\langle 3, 4, 5 \rangle - \langle 1, 2, -1 \rangle &= \langle 2, 2, 4 \rangle \\ \langle 0, 0, 0 \rangle - \langle 1, 2, -1 \rangle &= \langle -1, -2, 1 \rangle.\end{aligned}$$

The dot product between these two vectors is -2 . Their magnitudes are equal to $|\langle 2, 2, 4 \rangle| = \sqrt{4 + 4 + 16}$ and $|\langle -1, -2, 1 \rangle| = \sqrt{1 + 4 + 1}$. So the angle equals

$$\theta = \arccos \left(\frac{-2}{\sqrt{4 + 4 + 16}\sqrt{1 + 4 + 1}} \right) = \arccos \left(\frac{-2}{\sqrt{24}\sqrt{6}} \right) = \arccos \left(\frac{-1}{12} \right).$$

□

Example 1.3.12. (1) Find the angle between the lines $y = 4 - 3x$ and $y = 3x + 2$.
 (2) Find the angle between the lines $y = 5x - 8$ and $x + 3y = 15$.
 (3) Find the x values so that the angle between $\langle 2, 1, -1 \rangle$ and $\langle 1, x, 0 \rangle$ is $\frac{\pi}{4}$.

Solution: (1) The vector that describes the two lines are $\langle 1, -3 \rangle$ and $\langle 1, 3 \rangle$. The angle between these vectors is given by $\cos^{-1} \left(\frac{-8}{10} \right)$. We leave (2) and (3). □

1.4. The Cross Product.

Definition 1.4.1. The **cross product** between two vectors a and b is defined by the formula

$$a \times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Here, the notation means that one takes the determinant of the (3×3) -matrix inside the bars. Notice that the output is a vector.

Remark 1.4.2. The determinant of a (3×3) -matrix follows as simple formula. First, recall that the determinant of a (2×2) -matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Then, the determinant of the (3×3) -matrix is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}.$$

Warning: notice that there is a negative sign for the second term.

Example 1.4.3. Compute the following cross products.

- (1) $\langle 1, 3, 4 \rangle \times \langle 2, 7, 5 \rangle$,
- (2) $\langle 2, 1, 3 \rangle \times \langle -1, 2, -3 \rangle$.

Solution: (1) The cross product is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & 5 \end{vmatrix} = (15 - 28)\vec{i} - (5 - 8)\vec{j} + (7 - 6)\vec{k} = -13\vec{i} + 3\vec{j} + \vec{k}.$$

(2) The cross product is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ -1 & 2 & -3 \end{vmatrix} = (-3 - 6)\vec{i} - (-6 + 3)\vec{j} + (4 + 1)\vec{k} = -9\vec{i} + 3\vec{j} + 5\vec{k}.$$

□

Now we list out some properties regarding cross products. The key thing the student should realize are the following facts

$$a \cdot (a \times b) = 0$$

$$b \cdot (a \times b) = 0.$$

We check the first of these and leave the details of the second one to the student. First off, $a \cdot (a \times b)$ expands into

$$\langle a_1, a_2, a_3 \rangle \cdot \langle a_2b_3 - a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_1 \rangle.$$

Now expanding out the dot product, we get

$$\begin{aligned} & (a_1a_2b_3 - a_1a_3b_2) \\ & + (-a_1a_2b_3 + a_2a_3b_1) \\ & + (a_1a_3b_2 - a_2a_3b_1). \end{aligned}$$

One can check quite easily that all of the terms cancel out.

Theorem 1.4.4 (Properties of the Cross Product). Let a, b be two nonzero vectors in $\mathbb{R}^3$. Then

- (1) $a \times b$ is orthogonal to a and b ,
- (2) $a \times b$ is linear in the first coordinate i.e. if $r \in \mathbb{R}$ and c is a third vector, then

$$(ra + c) \times b = r(a \times b) + (c \times b)$$

- (3) the cross product is alternating i.e. $a \times b = -(b \times a)$
- (4) the cross product interacts well with the dot product i.e.

$$a \cdot (b \times c) = (a \times b) \cdot c.$$

Remark 1.4.5. As a warning, the cross product is **not associative**. Indeed, one can check that $\vec{i} \times (\vec{j} \times \vec{j}) = \mathbf{0}$ but $(\vec{i} \times \vec{j}) \times \vec{j} = -\vec{i} \times \vec{j} = -\vec{k}$.

Remark 1.4.6. The student that is curious on **why** the cross product is defined in the way it is can try the following exercise.

Let $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ be two vectors. We seek a third vector $\langle x, y, z \rangle$ which is perpendicular to the two vectors just give. One can show that this is the same as requiring

$$a_1x + a_2y + a_3z = 0 \quad \& \quad b_1x + b_2y + b_3z = 0.$$

Multiply the first equation by b_1 and the second by a_1 and subtract the first by the second. This gives,

$$(a_2b_1 - b_2a_1)y + (a_3b_1 - a_1b_3)z = 0.$$

This implies that

$$y = \frac{(a_1b_3 - a_3b_1)}{(a_2b_1 - b_2a_1)}z = \frac{(a_3b_1 - a_1b_3)}{(b_2a_1 - a_2b_1)}z$$

Next, multiply the first equation by b_2 and the second equation by a_2 and then subtract the first by the second. This gives

$$(a_1b_2 - a_2b_1)x + (a_3b_2 - a_2b_3)z = 0.$$

This implies

$$x = \frac{(a_2b_3 - a_3b_2)}{(a_1b_2 - a_2b_1)}z.$$

Now in the case where the denominators are all nonzero, it makes sense to say that our desired vector equals

$$\left\langle \frac{(a_2b_3 - a_3b_2)}{(a_1b_2 - a_2b_1)}z, \frac{(a_3b_1 - a_1b_3)}{(b_2a_1 - a_2b_1)}z, z \right\rangle.$$

Now set $z = (a_1b_2 - a_2b_1)$ and this gives the formula for the cross product

$$a \times b = \langle a_2b_3 - a_3b_2, a_1b_3 - a_3b_1, a_2b_1 - b_2a_1 \rangle.$$

Theorem 1.4.7. If a, b are two nonzero vectors, θ is the angle between them, then

$$|a \times b| = |a||b| \sin \theta.$$

Proof. The proof of this formula is not particularly exciting. If one expands the definition, we get

$$|a \times b|^2 = |a|^2|b|^2 - (a \cdot b)^2.$$

Using the formula $a \cdot b = |a||b| \cos \theta$, this equals

$$|a|^2|b|^2 - |a|^2|b|^2 \cos^2 \theta = |a|^2|b|^2 \sin^2 \theta.$$

Altogether, we get

$$|a \times b|^2 = |a|^2|b|^2 \sin^2 \theta.$$

Now take the square root (which is permitted since the angle between a and b is always $0 \leq \theta \leq \pi$ which forces $\sin \theta$ to be nonnegative). $\square$

Corollary 1.4.8. If $|a \times b| = 0$, then a and b are parallel in the sense that $a = lb$ for some number l .

Proof. For $|a \times b| = 0$, then $\theta = 0$ and so the angle between a and b equals zero. So the two vectors are parallel. $\square$

Remark 1.4.9. The theorem itself is not used as often in this class. The corollary is “more” useful and is probably what you should be taking away from this part of the class.

Remark 1.4.10. It is intriguing to observe $a \times b$ has direction that is independent of varying a, b in the plane that the span (excluding the case you allow $\theta = 0$). The vector $a \times b$ always stays on a fixed line and the length of the vector is maximized precisely when $\theta = \frac{\pi}{2}$.

Example 1.4.11. Determine if the following pairs of vectors are parallel.

$$(1) \vec{i}, \vec{j},$$

- (2) $\vec{k}, \vec{i}$,
- (3) $\langle 1, 1, 2 \rangle, \langle 2, 2, 4 \rangle$,
- (4) $\langle 1, 0, 2 \rangle, \langle 1, 0, 1 \rangle$.

Solution:

- (1) They are not since $\vec{i} \cdot \vec{j} = 0$
- (2) They are not for the same reason as (1).
- (3) Yes, $\langle 2, 2, 4 \rangle = 2\langle 1, 1, 2 \rangle$.
- (4) No, for $\langle 1, 0, 2 \rangle = c\langle 1, 0, 1 \rangle$ to hold, we would need the first coordinates to match up and that means $c = 1$ and clearly $\langle 1, 0, 2 \rangle \neq \langle 1, 0, 1 \rangle$.

□

Example 1.4.12. Answer Yes/No to the statement “the expression makes sense”. Here, a, b, c, d are all vectors in $\mathbb{R}^3$.

- | | |
|---------------------------------------|--|
| (1) $a \cdot (b \times c)$ | (6) $ a \times b $ |
| (2) $a \times (b \cdot c)$ | (7) $(\vec{i} + \vec{j}) \times (\vec{k} - \vec{j})$ |
| (3) $a \cdot (b \cdot c)$ | (8) $(\vec{i} \times \vec{j}) - \vec{i} \cdot \vec{j}$ |
| (4) $(a \cdot b) \times (c \cdot d)$ | (9) $\vec{i} \cdot \vec{j}$ |
| (5) $(a \times b) \cdot (c \times d)$ | (10) $(\vec{i} \cdot \vec{j}) \times \vec{k}$. |

Solution: We provide the answers along with a justification.

- | | |
|--|---|
| (1) Yes, $b \times c$ is a vector and so the dot product makes sense. | (6) Yes, you can certainly take the magnitude of a vector and $a \times b$ produces a vector. |
| (2) No, $b \cdot c$ is a scalar while the cross product requires one to use two vectors. | (7) Yes, for obvious reasons. |
| (3) No, $b \cdot c$ is a scalar but the dot product takes two vectors as input. | (8) No, you cannot subtract a vector by a scalar. |
| (4) No, cd is undefined as vector multiplication is not well-defined. | (9) Yes, $\vec{i}$ and $\vec{j}$ are vectors and so you can certainly take their dot product. |
| (5) Yes, the cross product produces vectors and so the dot product here makes sense. | (10) No, $\vec{i} \cdot \vec{j}$ produces a scalar which you cannot have as an input for a cross product. |

□

1.4.1. *The Scalar Triple Product.* The material in this subsection will not be covered in class nor in the exam. They are meant solely for entertainment for the curious student.

Theorem 1.4.13 (Scalar Triple Product and Volume of a Parallelepiped). Let a, b, c be three vectors. Then the **scalar triple product** is

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and it is the volume of the **parallelepiped** determined by a, b , and c .

Example 1.4.14. Compute the volume of the parallelepiped determined by $\langle 1, 4, -7 \rangle$, $\langle 2, -1, 4 \rangle$, and $\langle 0, -9, -19 \rangle$.

Solution: The scalar product is the determinant of the (3×3) -matrix

$$\begin{pmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & -19 \end{pmatrix}.$$

In this case, the scalar triple product equals

$$1(19 + 36) - 4(-38 - 0) + (-7)(-18 - 0) = 55 - 4(-38) + (-7)(-18) = 55 + 142 + 126 = 323.$$

□

1.5. Equations of Lines and Planes.

Definition 1.5.1. The **vector equation** of a line containing the point given by the vector $r_0 = \langle x_0, y_0, z_0 \rangle$ and pointing in the direction of $v = \langle a, b, c \rangle$ is

$$r = r_0 + tv \text{ for } t \in \mathbb{R}.$$

If one expands out what this means, we get a system of **parametric equations for the line**

$$x = x_0 + at \quad \& \quad y = y_0 + bt \quad \& \quad z = z_0 + ct \text{ for } t \in \mathbb{R}.$$

Example 1.5.2. Find the vector equation and parametric equation for a line going through the point $(4, 3, 2)$ and which is parallel to the line given by $x = 1 + 4t$, $y = 2 + 5t$, $z = 1 - t$.

Solution: The direction of the given line is $\langle 4, 5, -1 \rangle$. Our line needs to go through $(4, 3, 2)$. Therefore, we have a point and a direction for our desired line. So, we have vector equation given by

$$\langle x, y, z \rangle = \langle 4 + 4t, 3 + 5t, 2 - t \rangle.$$

The parametric equations are

$$x = 4 + 4t \quad \& \quad y = 3 + 5t \quad \& \quad z = 2 - t$$

and one lets t vary in $\mathbb{R}$. □

Now we talk about the equation of planes. First, observe that a plane is determined by the data of a **normal vector** $\vec{n} = \langle a, b, c \rangle$ and a point $r_0 = \langle x_0, y_0, z_0 \rangle$ on the plane.

Definition 1.5.3. An equation of a plane with normal vector $\vec{n}$ and containing the point r_0 is given by

$$(r - r_0) \cdot \vec{n} = 0$$

where $r = \langle x, y, z \rangle$. Writing this out, the equation is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

The **standard form** of the equation is

$$Ax + By + Cz + D = 0.$$

Example 1.5.4. Find the equation of a plane containing $(1, 2, -1)$ with normal vector $\vec{n} = 2\vec{i} + 3\vec{j} - 4\vec{k}$.

Solution: The normal vector determines the scalars in front of x, y, z . So, it is not hard to see that an equation is

$$2(x - 1) + 3(y - 2) - 4(z + 1) = 0.$$

□

Example 1.5.5. Find the equation of a plane containing the point $(4, 2, 1)$ and which is orthogonal to the line $x = 1 + 5t, y = -1 - 4t, z = 2 + 3t$.

Solution: This is left as an exercise to the student.

□

Remark 1.5.6. Given an equation, for example $2x + 3y + 4z = 12$, one might ask how to draw the plane. The easiest method is to mark the points that are both on the plane and on one of the x, y, z -axis.

Example 1.5.7. Write the equation of a plane perpendicular to the plane $3x + 2y + 6z + 10 = 0$ containing the point $(0, 0, -2)$.

Solution: The first step is to find the normal vector of the desired plane. The normal vector of our given plane is $\langle 3, 2, 6 \rangle$. To a plane that is perpendicular to the given plane, we simply pick a normal vector which is orthogonal to $\langle 3, 2, 6 \rangle$. Let's make it easy and pick $\langle 2, 0, -1 \rangle$. We are already given a point. So the equation is

$$2(x - 0) + 0(y - 0) - (z - (-2)) = 0 \Rightarrow 2x - z + 2 = 0.$$

One can plot their equations ($z = (1/6)(-3x - 2y - 10)$, and $z = 2x + 2$) into Desmos or your other favorite graphing program to see that they are indeed perpendicular planes. □

Example 1.5.8. Write in standard form the equation of a plane containing the three points $P = (0, 3, 1)$, $Q = (2, 1, 0)$ and $R = (1, 0, 1)$.

Solution: This is a tedious exercise, but the idea is straightforward. First, we find two vectors that sit in the plane defined by the three points.

We shall take

$$PQ = \langle -2, 2, 1 \rangle \quad \& \quad RQ = \langle -1, -1, 1 \rangle.$$

Then, we compute the cross product of these vectors

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 1 \\ -1 & -1 & 1 \end{vmatrix} = 3\vec{i} + \vec{j} + 4\vec{k}.$$

Now we can use the point $Q = (2, 1, 0)$ and the vector above to write the equation of the plane. In this case, we get

$$3(x - 2) + (y - 1) + 4(z - 0) = 0.$$

□

Remark 1.5.9. On an exam, it does not hurt to be cautious and check that the plane really does contain all three points given. It clearly contains Q in the solution above. If we plug in P , the LHS is

$$3(0 - 2) + (3 - 1) + 4(1 - 0) = -6 + 2 + 4 = 0$$

so all is good. If we plug in R , the LHS is

$$3(1 - 2) + (0 - 1) + 4(1 - 0) = -3 - 1 + 4 = 0$$

and all is good.

Example 1.5.10. Write down the parametric equation of the line formed by intersecting the planes given by the equations

$$3x + y - z = -10 \quad \& \quad x + 3y + 2z = -6.$$

You know that $(-26/7, 0, -8/7)$ is a point in the intersection of the two planes.

Solution: There are a two ways to do this. We will do it one way and only describe the second method.

First off, the two planes are determined by the normal vectors $\langle 3, 1, -1 \rangle$ and $\langle 1, 3, 2 \rangle$ respectively. The cross product equals $\langle 5, -7, 8 \rangle$. This tells us a direction that will determine the line. We are already given a point in the intersection, so

$$x = -26/7 + 5t \quad \& \quad y = -7t \quad \& \quad z = (-8/7) + t.$$

Note that if we were not given a point to begin with, we could still solve for the point. One can multiply the first equation by 2 and add it to the second equation to get $7x + 5y + 26 = 0$. Then setting $y = 0$ gives $7x + 26 = 0$. Therefore $x = -\frac{26}{7}$. Then, plugging this into the second equation, we get $-\frac{26}{7} + 2z = -6$ and solving for z gives $z = \frac{1}{2}(\frac{26}{7} - 6) = \frac{-16}{14} = \frac{-8}{7}$. $\square$

1.6. Cylinders, Quadric Surfaces. Let's just begin with graphing in $\mathbb{R}^3$. The strategy to graph an equation is to fix one of the variables and see what the graph in the associated plane looks like. See Figure 4 for an example.

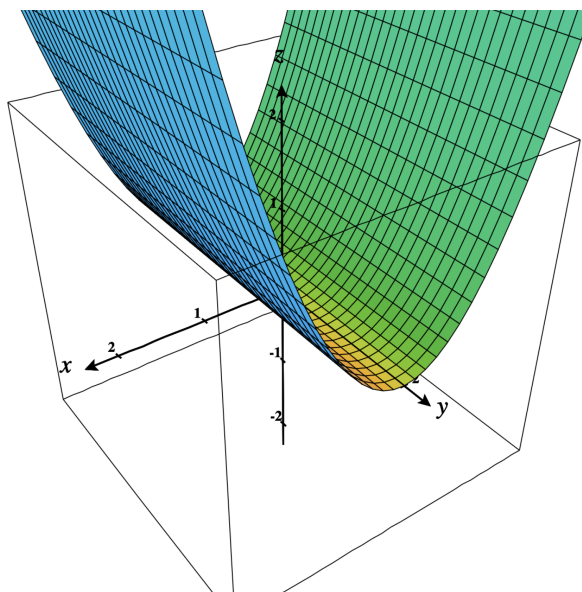


FIGURE 4. Here we have graphed $z = x^2$. To see what is happening, the idea is to fix different values of y e.g. if $y = 0$, we are looking at the usual xy -plane and there, we have a parabola $z = x^2$. In general, the vertical traces look like parabolas. The horizontal traces are just a pair of lines.

Definition 1.6.1. The **horizontal trace** of a surface in $\mathbb{R}^3$ is the intersection of the surface with a plane given by $z = k$.

The **vertical trace** of a surface is the intersection of the surface with a plane orthogonal to the vector $\langle 0, 0, 1 \rangle$.

Definition 1.6.2. A **conic** is the graph of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

in the xy -plane. For example, parabolas, circles, ellipses, and hyperbolas are all the possible conics.

One has a similar classification in $\mathbb{R}^3$. Instead, one considers degree 2 equations

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

and the surfaces one gets from the (x, y, z) satisfying equation are called **quadric surfaces**. The table (5) describes all of the possibilities. You are expected to be able to determine when a given degree 2 equation gives which of these surfaces.

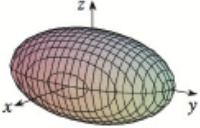
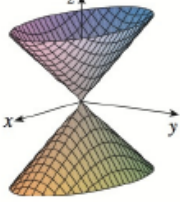
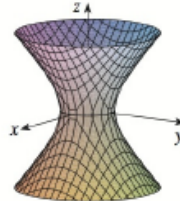
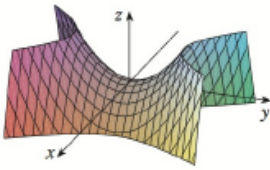
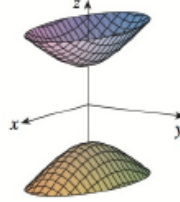
Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

FIGURE 5

Warning: Do not attempt to memorize the table. it is a waste of time. A better approach is to graph traces of the various equations. **The emphasis is on your ability to determine the graph of the function – not your ability to name which quadric it is.**

Example 1.6.3. To determine which quadric surface the equation

$$4x^2 + 9y^2 + z^2 = 36$$

corresponds to, we pick different values of $z = a$ and attempt to determine the conic one gets in the plane $z = a$. For example, if $a = 0$ we get the equation

$$4x^2 + 9y^2 = 36 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$

which is an ellipse. In fact, no matter which a value one picks, one gets an ellipse. This means it can either be an *elliptic paraboloid*, *cone*, *hyperboloid of one sheet*, or an *ellipsoid*. One can rule out it being a hyperboloid of two sheets since there's no "gap".

Now set $y = a$ and observe that we also always get an ellipse e.g. $y = 0$ gives $4x^2 + z^2 = 36$. So both vertical and horizontal **traces** are ellipses. So, it must be an ellipsoid.

Example 1.6.4. Sketch out the surface implicitly defined by

$$-\frac{x^2}{4} - y^2 + \frac{z^2}{4} = 1.$$

Solution: See Figure 6 for a sketch of the surface.

One can see that this is a hyperboloid of two sheets because when $z = 0$, the equation becomes

$$-\frac{x^2}{4} - y^2 = 1$$

which has no solutions. In fact, for $-2 < k < 2$, then the intersection the plane $z = k$ is zero. Meanwhile, we can consider the situation where $y = k$ is a fixed constant. Then the equation becomes

$$-\frac{x^2}{4} + \frac{z^2}{4} = 1 + k^2$$

and this gives hyperbola. The same occurs if one had fixed $x = k$.

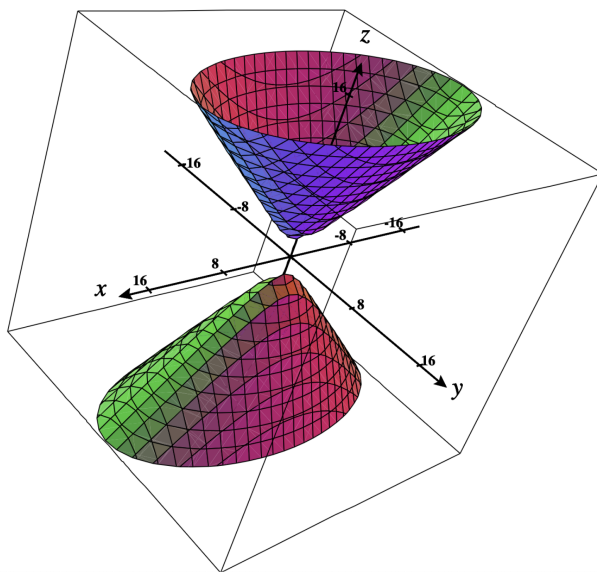


FIGURE 6. The implicit surface given by $-\frac{x^2}{4} - y^2 + \frac{z^2}{4} = 1$.

□

Example 1.6.5. Determine which surface corresponds to the equation $-x^2 + y^2 + z^2 = 1$. Sketch the surface given by the equation.

Solution: It is a hyperboloid of one sheet but it is rotated. A graph of it is provided in Figure (7). One can see this by sketching what happens if one fixes different x -values.

If $x = 0$, then we get $y^2 + z^2 = 1$. If $x = 8$, then we get $x^2 + y^2 = 9$.

On the other hand, if we fix the z -value e.g. $z = 2$, then we get $-x^2 + y^2 = -3$ which is going to produce a hyperbola in the plane $z = 2$.

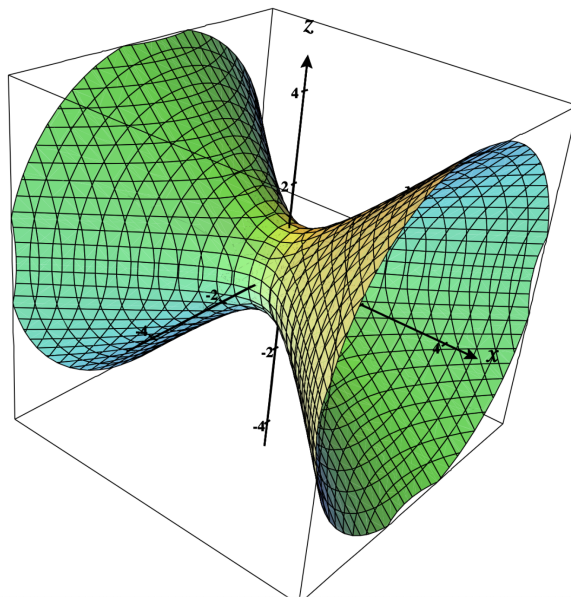


FIGURE 7. A graph of $-x^2 + y^2 + z^2 = 1$.

□

Example 1.6.6. Draw the traces $z = 0, z = 1, z = -1, z = 2$ and $x = 0, x = -9, x = 9$ for the surface

$$z = \frac{x^2}{81} - \frac{y^2}{144}.$$

Solution: See Figure 8 for a graph of the surface. The traces $z = 0, z = 1, z = -1, z = 2$ in this case are all hyperbolas. The $x = 0$, we get the curve $z = -\frac{y^2}{144}$ which is a parabola. For $x = -9, x = 9$ we both get $z = 1 - \frac{y^2}{144}$.

□

The two most important surfaces for this class one should be familiar with are the cylinder and the cone. The equation describing them, respectively, are

$$x^2 + y^2 = r^2$$

and

$$ax^2 + by^2 = cz^2$$

where a, b, c are nonzero integers (some care is needed here since there are choices of a, b, c which “degenerate” in nature). See Figure 9 and Figure 10 for some visuals.

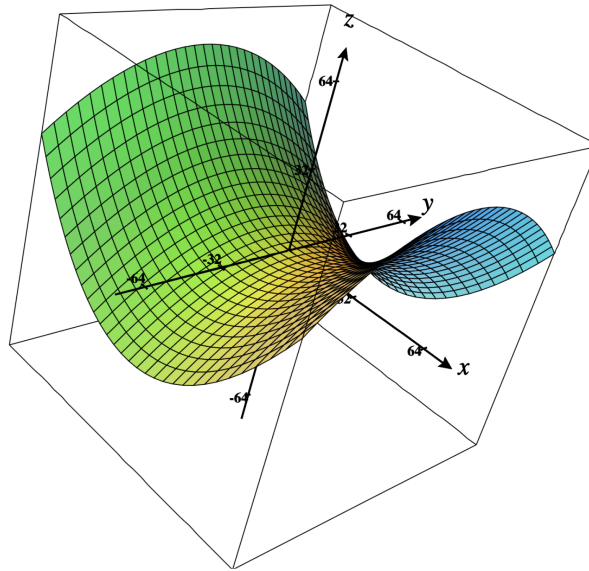


FIGURE 8. The graph of $z = \frac{x^2}{81} - \frac{y^2}{144}$.

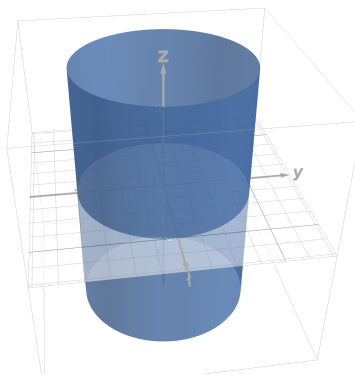


FIGURE 9. Pictured is a cylinder given by the equation $x^2 + y^2 = 4$

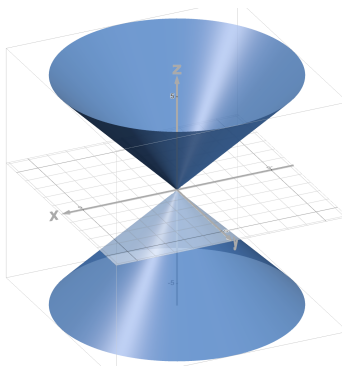


FIGURE 10. Pictured is the cone given by the equation $x^2 + y^2 = z^2$.

Remark 1.6.7 (What should the student expect to be examinable?). Questions which are fair game are of the following nature: (a) “Sketch the trace $z = 2$ of (insert degree 2 equation)” and (b) “Match the graph pictured to the appropriate degree 2 equation”. We

will not expect the student to do any 3d-sketches on paper nor do any actual “naming” of surfaces.

1.7. Vector Functions, and Space Curves.

Definition 1.7.1. A **vector-valued function** (or simply **vector function**) for us is a function $\vec{r}(t)$ that takes some input and returns a vector. In the case where the input is a number $t \in \mathbb{R}$ and the output is a vector in $\mathbb{R}^3$, we can write

$$\vec{r}(t) := \langle f(t), g(t), h(t) \rangle$$

for functions $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$. In this case, we are considering $\vec{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^3$ where $\mathbb{R}^3$ is viewed as a vector space.

The functions f, g, h are called the **component functions**. Obviously, all of this can be done more generally. All of our vector-valued functions will have range in $\mathbb{R}^3$ and domain in $\mathbb{R}$.

Definition 1.7.2. Assume $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Then the set of points C in $\mathbb{R}^3$ given by

$$x = f(t) \quad \& \quad y = g(t) \quad \& \quad z = h(t)$$

for t ranging over some interval I is called a **space curve**.

Example 1.7.3. Find the domain and range of

$$\vec{r}(t) = (2t + 2)\vec{i} + \sqrt{t}\vec{j} - \ln(t + 3)\vec{k}.$$

To compute limits with vector-valued functions, one can work component-wise. Indeed, we have that

$$\lim_{t \rightarrow a} \vec{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle$$

and the limit exists if and only if the component-wise limits exist.

Example 1.7.4. Compute the limit

$$\lim_{t \rightarrow 2\pi} \langle 2 \cos t, \sin \frac{t}{2}, 3 \sin \frac{t}{4} \rangle.$$

Example 1.7.5. Compute the limit

$$\lim_{t \rightarrow \infty} \langle \frac{1}{t}, \frac{1}{t^2 + 1}, \frac{t}{t^2 + 1} \rangle.$$

Definition 1.7.6. A vector valued function $\vec{r}(t)$ is continuous if $\lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a)$ for all a in the domain of $\vec{r}(t)$.

Using vector-valued functions, one can imagine that $\vec{r}(t)$ draws a curve in 3d-space as t varies. We already saw this previously with vector equations of lines.

Example 1.7.7. Sketch the following curves as t ranges over all of $\mathbb{R}$.

- (1) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$,
- (2) $\vec{r}(t) = \langle 1 + t, 2 - t \rangle$,
- (3) $\vec{r}(t) = \langle t, t^2, t^3 \rangle$.

Solution: (1) This curve is called a helix: see this [Desmos3d graph](#) for an image.
 (2) This is just the straight line in the direction $\langle 1, -1 \rangle$ containing the point $(1, 2)$.
 (3) This famous curve is known as the twisted cubic. See this [Desmos3d graph](#) for an image.
 $\square$

Example 1.7.8. Let's find a vector-valued function for the curve obtained by intersecting $x^2 + y^2 = 1$ and $x + y + z = 2$.

Solution: First off, the equation $x^2 + y^2 = 1$ should make one think to use $x = \cos(t)$ and $y = \sin(t)$ for the parameterization. If one does this, then $z = 2 - \cos(t) - \sin(t)$. So the vector-valued function that describes the curve is

$$\langle \cos(t), \sin(t), 2 - \cos(t) - \sin(t) \rangle.$$

$\square$

Example 1.7.9. Write down a vector function for the curve given by intersecting $x^2 + z^2 = 4$ with the hyperplane $4x + y + z = 0$.

Solution: First off, we can set $x = 2 \cos(t)$ and $z = 2 \sin(t)$ and let t range over all of $\mathbb{R}$. The equation of the hyperplane forces $y = -4x - z = -8 \cos(t) - 2 \sin(t)$. So, we can parameterize the intersection by

$$\langle 2 \cos(t), -8 \cos(t) - 2 \sin(t), 2 \sin(t) \rangle$$

where we let t range over all of $\mathbb{R}$. Strictly speaking though, we really only need $0 \leq t \leq 2\pi$. See this [Desmos3d graph](#) for an image. $\square$

Example 1.7.10. Parameterize and sketch the follow space curves:

- (1) $\vec{r}(t) := \langle t^2, t^4, t^6 \rangle$,
- (2) $\vec{r}(t) = \cos(t)\vec{i} - \cos(t)\vec{j} + \sin(t)\vec{k}$,
- (3) $\vec{r}(t) = \langle 3, t, 2 - t^2 \rangle$,
- (4) the curve obtained by intersecting $3x^2 - y^2 - z^2 = 1$ with the plane $x - y = 0$,
- (5) the intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$.

Solution: These are left mostly as an exercise for the student. We work out (5) for the reader.

Let $x(t) := t$ with t our parameter. Then we need to rewrite y, z in terms of t . For this purpose, the first equation becomes $z^2 = x^2 + y^2$ and the second equation, when squared, becomes $z^2 = 1 + 2y + y^2$. Subtract $z^2 = x^2 + y^2$ by $z^2 = 1 + 2y + y^2$ to get $0 = x^2 - 1 - 2y$ and so, $y = \frac{t^2 - 1}{2}$. Also, $z = 1 + y$, we get $z = \frac{t^2 + 1}{2}$. Altogether,

$$\vec{r}(t) = t\vec{i} + \frac{t^2 - 1}{2}\vec{j} + \frac{t^2 + 1}{2}\vec{k}.$$

Warning: One might be tempted to try and use $x := \cos(t)$ and $y = \sin(t)$ for the parameterization. This will not work because the absolute value of x (and y) is not fixed. This is in contrast to Example 1.7.8. $\square$

1.8. Calculus of Vector Functions. Before proceeding onto a new topic, let's recall what we know about vector functions via a sequence of exercises.

Example 1.8.1. Compute the limit

$$\lim_{t \rightarrow 2} \langle 2t^2, t, e^t \rangle.$$

Solution: Each component is continuous so the limit is $\langle 8, 2, e^2 \rangle$. $\square$

Example 1.8.2. Write down a vector function that describes the intersection of the following implicitly defined surfaces:

$$x^2 + y^2 = 8 \quad \& \quad x^3 - z = y.$$

Solution: Again, the trick for these types of problems is when you see the equation of a circle, it is best to use trigonometric functions.

In this case, we set $x := \sqrt{8} \cos(t)$ and $y := \sqrt{8} \sin(t)$ and this obviously satisfies the first equation. The second equation we must have says that $z = x^3 - y$. So plugging in our choices above, we get $z = 8\sqrt{8} \cos^3(t) - \sqrt{8} \sin(t)$. So a vector function that describes the curve obtained by intersecting the two surfaces is

$$\mathbf{r}(t) = \langle \sqrt{8} \cos(t), \sqrt{8} \sin(t), 8\sqrt{8} \cos^3(t) - \sqrt{8} \sin(t) \rangle.$$

$\square$

Definition 1.8.3. If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and the component functions are all differentiable, then the derivative of $\mathbf{r}(t)$ is defined to be

$$\mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle.$$

Example 1.8.4. Compute $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ of the following vector function

$$\mathbf{r}(t) = e^{2t} \mathbf{i} + \ln(1+t) \mathbf{j} + (\cos t) \mathbf{k}.$$

Solution: One has

$$\mathbf{r}'(t) = \langle 2e^{2t}, \frac{1}{1+t}, -\sin(t) \rangle$$

and

$$\mathbf{r}''(t) = \langle 4e^{2t}, -\frac{1}{(1+t)^2}, -\cos(t) \rangle.$$

$\square$

In physics, one can regard $\mathbf{r}(t)$ as a position vector for a particle moving in space. Then the first and second derivatives are sometimes denoted $\mathbf{v}(t)$ and $\mathbf{a}(t)$ for the velocity and acceleration. In this interpretation of the derivatives, the value $|\mathbf{v}(t)|$ is the **speed** of the particle moving in space.

Definition 1.8.5. The vector $\mathbf{r}'(t)$ describes how $\mathbf{r}(t)$ changes with respect to t . If one cares only about the “direction” in which $\mathbf{r}(t)$ will go, then one is asking about the **unit tangent vector**. By definition, the **unit tangent vector** of $\mathbf{r}(t)$ depends on t and is given by

$$\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Example 1.8.6. Find the unit tangent vector to $\mathbf{r}(t) = \langle 2 \cos t, 1, 2 \sin t \rangle$ at $t = 0$. Then find the tangent line at $t = 0$. Hint: Use $\mathbf{r}'(t)$ and recall what one does in single variable calculus.

Solution: To compute the unit tangent vector, we evaluate the derivative: $\mathbf{r}'(t) = \langle -2 \sin(t), 0, 2 \cos(t) \rangle$. Then,

$$\mathbf{T}(t) := \frac{1}{\sqrt{4 \sin^2(t) + 4 \cos^2(t)}} \langle -2 \sin(t), 0, 2 \cos(t) \rangle = \left\langle -\frac{\sin(t)}{2}, 0, \frac{\cos(t)}{2} \right\rangle.$$

At $t = 0$, the unit tangent vector equals $\mathbf{T}(0) = \langle 0, 0, \frac{1}{2} \rangle$.

The tangent line at $t = 0$ is the line given by the vector $\langle 0, 0, \frac{1}{2} \rangle$ and the point $\mathbf{r}(0) = \langle 2, 1, 0 \rangle$. So the tangent line is

$$\ell(t) := \langle 2 + 0 \cdot t, 1 + 0 \cdot t, 0 + \frac{1}{2} \cdot t \rangle = \langle 2, 1, t \rangle$$

□

Definition 1.8.7. Let $\mathbf{r}(t)$ be a (nice) space curve. The tangent line to $\mathbf{r}(t)$ at $t = t_0$ can be given by the vector equation

$$\langle x, y, z \rangle = \mathbf{r}(t_0) + t\mathbf{T}(t_0).$$

Example 1.8.8. Find the unit tangent vector to $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$ and use it to find the tangent line at $t = 1$.

Solution: The unit tangent vector at $t = 1$ is $\mathbf{T}(1) = \frac{\langle 1, 2, 3 \rangle}{\sqrt{14}}$. So, the tangent line is given by

$$\mathbf{r}(t) = \langle 1 + t, 1 + 2t, 1 + 3t \rangle \text{ for } t \in \mathbb{R}.$$

□

Solution: Find the unit tangent vector to $\mathbf{r}(t) = \langle \sin(2t), \cos(2t), t \rangle$ at time $t = \pi$ and use it to find the tangent line to the space curve.

□

Proposition 1.8.9 (Rules of Differentiation). Differentiation of vector-valued functions, whenever it is defined, has the following properties

- (1) it is linear,
- (2) $\frac{d}{dt} f(t)\mathbf{u}(t) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ for any single variable function $f : \mathbb{R} \rightarrow \mathbb{R}$,
- (3) $\frac{d}{dt} \mathbf{u}(t) \cdot \mathbf{v}(t) = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$
- (4) $\frac{d}{dt} \mathbf{u}(t) \times \mathbf{v}(t) = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$,
- (5) the chain rule is true i.e. $\frac{d}{dt} \mathbf{u}(f(t)) = f'(t)\mathbf{u}'(f(t))$.

Proof of (2). We only explain why (2) is true and leave the other parts as an exercise for the precocious student. First off, let $\mathbf{u}(t) = \langle g(t), h(t), i(t) \rangle$. Then,

$$\frac{d}{dt} f(t)\mathbf{u}(t) = \left\langle \frac{d}{dt} f(t)g(t), \frac{d}{dt} f(t)h(t), \frac{d}{dt} f(t)i(t) \right\rangle.$$

Then, apply the product rule to each component.

□

Proposition 1.8.10. If $|\mathbf{r}(t)| = c$ is constant, then $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$ i.e. the tangent vector is orthogonal to the position vector.

Proof. Since $c^2 = |\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$, we can differentiate both sides by t to get

$$0 = 2\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

Dividing by 2 implies $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

□

Remark 1.8.11. The converse statement is also true i.e. if $\mathbf{r}(t)$ is a (differentiable) space curve such that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ for all t , then $\mathbf{r}(t)$ lies on a sphere centered at the origin.

Example 1.8.12. Find $\frac{d}{dt}u(t) \cdot v(t)$ and $\frac{d}{dt}(u(t) \times v(t))$ when

$$u(t) = \langle \cos t, \sin t, t \rangle \quad \& \quad v(t) = \langle t, \ln t, 1 \rangle.$$

Solution: Since it is relevant, we compute that

$$\begin{aligned} u'(t) &= \langle -\sin(t), \cos(t), 1 \rangle \\ v'(t) &= \langle 1, \frac{1}{t}, 0 \rangle. \end{aligned}$$

The we use the properties described in the proposition. In that case, we get

$$\frac{d}{dt}(u(t) \cdot v(t)) = (\cos(t) + \frac{1}{t} \sin(t) + 0) + (-t \sin(t) + \ln(t) \cos(t) + 1).$$

For the cross product computation, things get a little messier. This is left as an exercise the ardent student. $\square$

Now we move onto integration. Just like how differentiation can be done componentwise, integration is done componentwise. More precisely, if $\mathbf{r}(t) := \langle f(t), g(t), h(t) \rangle$ is a vector-valued function, then for indefinite integrals, we set

$$\int \mathbf{r}(t) dt := \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle$$

and for definite integrals, we set

$$\int_a^b \mathbf{r}(t) dt := \langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \rangle.$$

Just the properties of differentiation carry over nicely to differentiating vector functions, the properties of integrals carry over as well. The reason for this is precisely because we can consider the components of each vector function and use theorems from single variable calculus. For example, we have the following version of the Fundamental Theorem of Calculus.

Theorem 1.8.13 (Fundamental Theorem of Calculus for Vector Function). Assume $\mathbf{r}(t)$ is a vector function and $\mathbf{R}(t)$ is a vector function with well-defined derivative satisfying $\mathbf{R}'(t) = \mathbf{r}(t)$. Then,

$$\int_a^b \mathbf{r}(t) dt = \mathbf{R}(b) - \mathbf{R}(a).$$

Example 1.8.14. The following are essentially exercises in integration.

- (1) Let $\mathbf{r}(t) = \langle 4t^{1/3}, 6t + 1, \frac{2}{t+1} \rangle$. Compute $\int \mathbf{r}(t) dt$.
- (2) Let $\mathbf{r}(t) = \langle 1 + 3\sqrt{t}, 3t^2 + 1, 2t - 1 \rangle$. Compute $\int_0^2 \mathbf{r}(t) dt$.
- (3) Let $\mathbf{r}(t) = \langle \cos(t), \sec^2(t), \sin(t) \rangle$. Compute $\int \mathbf{r}(t) dt$.

Solution: We shall do (1) in these notes and leave the remainder as an exercise. To compute $\int \mathbf{r}(t) dt$ entails integrating each component. In that case,

$$\int \mathbf{r}(t) dt = \langle 3t^{4/3} + A, 3t^2 + t + B, 2 \ln |t + 1| + C \rangle.$$

$\square$

Example 1.8.15. Suppose you know a particle has acceleration $\mathbf{a}(t) = \langle t, 1, 0 \rangle$ and that $\mathbf{v}(0) = \langle 0, 0, 0 \rangle$, and $\mathbf{r}(0) = \langle 2, -1, -1 \rangle$. Compute $\mathbf{r}(t)$ and the speed of the particle at time $t = 1$.

Solution: First off, we can integrate $\mathbf{a}(t)$ to get

$$\mathbf{v}(t) = \left\langle \frac{t^2}{2} + A, t + B, C \right\rangle.$$

Setting $t = 0$ and using $\mathbf{v}(0) = \mathbf{0}$ implies that $A = 0$, $B = 0$, and $C = 0$. So,

$$\mathbf{v}(t) = \left\langle \frac{t^2}{2}, t, 0 \right\rangle.$$

Now integrating again, we get

$$\mathbf{r}(t) = \left\langle \frac{t^3}{6} + A, \frac{t^2}{2} + B, C \right\rangle.$$

Since $\mathbf{r}(0) = \langle 2, -1, -1 \rangle$, we see that

$$\mathbf{r}(t) = \left\langle \frac{t^3}{6} + 2, \frac{t^2}{2} - 1, -1 \right\rangle.$$

The speed at time $t = 1$ is $|\mathbf{v}(1)|$ and that equals

$$|\mathbf{v}(1)| = \sqrt{(1/2)^2 + 1^2 + 0^2}.$$

□

Example 1.8.16. Suppose an object has acceleration $\mathbf{a}(t) = \langle e^{2t} + 1, 3t^2 + 1, 4 \rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ (which are its velocity function and position function) subject to the initial conditions $\mathbf{v}(0) = \langle 0, 1, 0 \rangle$ and $\mathbf{r}(0) = \langle 2, 0, 1 \rangle$. Afterwards, compute its **speed** at time t i.e. $|\mathbf{v}(t)|$.

Solution: First off, I know that $\mathbf{v}'(t) = \mathbf{a}(t)$, and $\mathbf{r}'(t) = \mathbf{v}(t)$. Therefore, it makes sense to integrate to try and solve for $\mathbf{v}(t)$ and $\mathbf{r}(t)$.

First,

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \left\langle \frac{1}{2}e^{2t} + t + C_1, t^3 + t + C_2, 4t + C_3 \right\rangle.$$

To solve for the constants, we must solve for

$$\left\langle \frac{1}{2} + C_1, C_2, C_3 \right\rangle = \mathbf{v}(0) = \langle 0, 1, 0 \rangle.$$

So, $C_1 = -\frac{1}{2}$, $C_2 = 1$, and $C_3 = 0$. So,

$$\mathbf{v}(t) = \left\langle \frac{1}{2}e^{2t} + t - \frac{1}{2}, t^3 + t + 1, 4t \right\rangle.$$

We do the same thing to solve for $\mathbf{r}(t)$. First, we integrate

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \left\langle \frac{1}{4}e^{2t} + \frac{1}{2}t^2 - \frac{1}{2}t + C_1, \frac{1}{4}t^3 + \frac{1}{2}t^2 + t + C_2, 2t^2 + C_3 \right\rangle$$

Now I plug in $t = 0$ to get

$$\mathbf{r}(0) = \left\langle \frac{1}{4} + C_1, C_2, C_3 \right\rangle = \langle 2, 0, 1 \rangle.$$

So I need $C_1 = \frac{7}{4}$, $C_2 = 0$, $C_3 = 1$. Altogether, I have

$$\mathbf{r}(t) = \left\langle \frac{1}{4}e^{2t} + \frac{1}{2}t^2 - \frac{1}{2}t + \frac{7}{4}, \frac{1}{4}t^3 + \frac{1}{2}t^2 + t, 2t^2 + 1 \right\rangle$$

as the position vector. To find the speed at time t , I need to compute the magnitude of $\mathbf{v}(t)$. This gives me

$$|\mathbf{v}(t)| = \sqrt{\left(\frac{1}{2}e^{2t} + t - \frac{1}{2}\right)^2 + (t^3 + t + 1)^2 + (4t)^2}.$$

□

Example 1.8.17. Suppose you know Sarah is driving their car in $\mathbb{R}^3$ and you know that Sarah's acceleration vector is given by $\mathbf{a}(t) = \langle 2t, 25t^2, \cos(t) \rangle$. Assume their initial conditions are $\mathbf{v}(0) = \langle 0, 1, 1 \rangle$ and $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$. Find the position vector that describes her position at time t .

Solution: This is left as an exercise for the student.

□

1.9. Velocity and Acceleration. We had already mentioned velocity and acceleration vectors before, but we isolate the details into a short subsection.

Definition 1.9.1. If $\mathbf{r}(t)$ is the position of a particle in space at time t , then its **velocity**, **speed**, and **acceleration** can all be defined

- (1) velocity is $\mathbf{v}(t) := \mathbf{r}'(t)$,
- (2) speed is $|\mathbf{v}(t)| = |\mathbf{r}'(t)|$
- (3) acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example 1.9.2. Compute the velocity, speed, and acceleration of the particle in space whose position is given by $\mathbf{r}(t) = \langle \sin t, t^2 + 1, e^{3t} \rangle$.

Solution: The velocity vector is $\mathbf{v}(t) = \langle 1, 2t, 3e^{3t} \rangle$.

The speed is $|\mathbf{v}(t)| = \sqrt{1 + 4t^2 + 9e^{6t}}$.

The acceleration vector is $\mathbf{a}(t) = \langle 0, 2, 9e^{3t} \rangle$.

□

Example 1.9.3. Find the velocity, speed, and position if you know that a particle's acceleration is $\mathbf{a}(t) = -9.8\mathbf{k}$, its velocity at time $t = 0$ is $\mathbf{v}(0) = \mathbf{i}$, and its position at time $t = 0$ is $\mathbf{r}(0) = 5\mathbf{k}$.

If one draws a picture of what is happening, one sees that this is determining the position of an object thrown horizontally off a cliff of height 5 with gravity acting on the object.

1.10. Arc Length.

Definition 1.10.1. The **arc length** of a curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ between $a \leq t \leq b$ is given by the formula

$$L := \int_a^b |\mathbf{r}'(t)| dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt.$$

This is the correct formula for the length of the curve one can view it as approximating the curve by infinitely many line segments and the length of those line segments are given by the distance formula. For a neat visual, see [“Interactive Visualization of the Arc Length Approximation”](#) by Larry Green.

Remark 1.10.2. For a more suggestive notation, I prefer to let C be the curve traced out by $\mathbf{r}(t)$ over the interval $t \in [a, b]$ and then write

$$L := \int_C d\mathbf{r}.$$

Example 1.10.3. Compute the arc length from $t = 0$ to $t = 1$ of $\mathbf{r}(t) = \langle e^t, e^t, e^t \rangle$.

Solution: The arc length equals

$$L := \int_0^1 \sqrt{3e^{2t}} dt = \sqrt{3} \int_0^1 e^t dt = \sqrt{3}(e^1 - e^0).$$

□

Example 1.10.4. Compute the arc length from $t = 0$ to $t = 1$ of the curve $\mathbf{r}(t) = \langle t, \frac{\sqrt{2}}{2}t^2, \frac{1}{3}t^3 \rangle$.

Solution: The arc length equals

$$L = \int_0^1 \sqrt{1 + 2t^2 + t^4} dt = \int_0^1 \sqrt{(1 + t^2)^2} dt = \int_0^1 (1 + t^2) dt = 1 + \frac{1}{3}.$$

□

Example 1.10.5. Use the arc length formula to compute the circumference of a circle.

Suppose we were focused on computing the arc length of a curve in the xy -plane given by $f(x) = y$. This is a very natural question to ask and one could've asked it in Single Variable Calculus. The language of vector functions makes this not so difficult to answer. First off, the curve can be parameterized by $\langle x, f(x), 0 \rangle$ (basically x is our “ t ” here). Then, we get

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

by the formula above.

Example 1.10.6. Use the arc length formula to compute the length of the curve given by $f(x) = \frac{2}{3}x^{3/2}$ over the interval $0 \leq x \leq 3$.

Solution: In this case, the arc length formula gives

$$L = \int_0^3 \sqrt{1 + x} dx = \frac{2}{3}(x + 1)^{3/2} \Big|_0^3 = \frac{2}{3}4^{3/2} - \frac{2}{3} = \frac{14}{3}.$$

□

Definition 1.10.7. The function that outputs the arc length of a curve from $u = a$ to $u = t$ is given by the function

$$S(t) = \int_a^t |\mathbf{r}'(u)| du.$$

Example 1.10.8. Find the arc length function for the curve $\mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2} \rangle$ starting at $t = 0$.

Definition 1.10.9. A natural question to ask is given a curve $\mathbf{r}(t)$, can we reparameterize it so that its derivative $\mathbf{r}'(t)$ satisfies $|\mathbf{r}'(t)| = 1$. This is akin to making the curve “move” at speed 1. Here is how to do it. This leads to the notion of **parameterization of a curve using arc length**.

Algorithm to find a parameterization of a space curve via arc length.

- (1) Compute $S(t)$, the arc length function.
- (2) Solve for t in terms of S .
- (3) Then use $\mathbf{r}(t(S))$.

(4) Observe that $|\mathbf{r}'(t(S))| = 1$.

Example 1.10.10. Parameterize the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ by arc length for $t \geq 0$.

Solution: First, we compute the arc length function:

$$S(t) = \int_0^t \sqrt{(-\sin^2(s)) + \cos^2(s) + 1} ds = \int_0^t \sqrt{2} ds = \sqrt{2}t.$$

So, $S = \sqrt{2}t$. Then to solve for t in terms of S means $t = \frac{S}{\sqrt{2}}$. So setting $t := \frac{S}{\sqrt{2}}$, we have the new parameterization

$$\mathbf{r}(S) = \langle \cos(S/\sqrt{2}), \sin(S/\sqrt{2}), S/\sqrt{2} \rangle \quad \text{where } S \geq 0.$$

Notice though that

$$\mathbf{r}'(S) = \langle -(1/\sqrt{2}) \sin(S/2), (1/\sqrt{2}) \cos(S/2), 1/\sqrt{2} \rangle$$

has magnitude 1 for all S . □

Example 1.10.11. Parameterize the curve $\mathbf{r}(t) = \langle t, t, t \rangle$ by arc length for $t \geq 0$.

Solution: In this case,

$$S(t) = \int_0^t \sqrt{3} ds = \sqrt{3}t.$$

So, $t = \frac{S}{\sqrt{3}}$ and so the reparameterized curve is $\mathbf{r}(S) = \langle \frac{S}{\sqrt{3}}, \frac{S}{\sqrt{3}}, \frac{S}{\sqrt{3}} \rangle$. □

Remark 1.10.12. A curious student might ask: why is it the case that $|\mathbf{r}'(t(S))| = 1$? Here, we provide a rough answer (not rigorous) answer.

First off, we consider $\mathbf{r}(t(S))$ as a function in terms of S . The first thing one should observe is that

$$\frac{d}{dt}S(t) = |\mathbf{r}'(t)|$$

by the Fundamental Theorem of Calculus. Here is where we are waving out hands – we say that this implies

$$\frac{dt}{dS} = \frac{d}{dS}t(S) = \frac{1}{|\mathbf{r}'(t)|}.$$

Next, if we take $\mathbf{r}(t(S))$ and differentiate w.r.t S , we get

$$\mathbf{r}'(t(S)) \frac{dt}{dS} = \mathbf{r}'(t) \frac{dt}{dS}.$$

So,

$$\left| \frac{d}{dS} \mathbf{r}(t(S)) \right| = \left| \mathbf{r}'(t) \frac{dt}{dS} \right| = \left| \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right| = 1.$$

1.11. Curvature, Normal and Binormal Vector.

Definition 1.11.1. A curve C parameterized by $\mathbf{r}(t)$ is **smooth** if (1) $\mathbf{r}'(t)$ is continuous and (2) $\mathbf{r}'(t)$ is nonzero for all t . We say a curve C is **smoothly parameterized** if there exists a parameterization $\mathbf{r}(t)$ like above.

Remark 1.11.2. We've seen a lot of examples of **smooth space curves**, but we have not seen too many examples that are not. Some obvious examples arise if we pick $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ with a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is not differentiable at some point. For example, $f(t) = |t|$ is not differentiable at $t = 0$.

Furthermore, condition (2) rules out uninteresting examples like points e.g. $\mathbf{r}(t) = \langle 2, 3, 1 \rangle$ has $\mathbf{r}'(t) = \mathbf{0}$.

To motivate the definition of curvature, one needs to ask the question: how does one measure the “curviness of a curve”? One guesses that this has to do with how the tangent vector changes over time. If the tangent vector changes direction a lot, then it is the case that the curve is “curvier”. So we want to think about $\mathbf{T}'(t)$ over time. Unfortunately, this runs into the issue that this depends on the parameterization, but there is a parameterization associated to each curve – the parameterization by arc length. So, we ganders that we should be thinking about $d\mathbf{T}$ (change of $\mathbf{T}$) with respect to dS (the change in arc length).

Definition 1.11.3. Assume $\mathbf{r}(t)$ smoothly parameterizes the curve C . Then the **curvature** of C is a function in t defined by

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

where $\mathbf{T}(t)$ is the unit tangent vector. One can rewrite this into many forms

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|S'(t)|} = \frac{\left| \frac{d\mathbf{T}}{dt} \right|}{\frac{dS}{dt}} = \left| \frac{d\mathbf{T}}{dS} \right|.$$

Note that by the fundamental theorem of calculus, $dS = |\mathbf{r}'(t)| dt$.

Remark 1.11.4. Here is some explanation on why this is the definition of “curvature”. This is something the student needs to grapple with at some point – the issue isn’t about giving an answer to the question “what is curvature”, but rather an answer to the question “what is curvature measuring”. Well, curvature measures how much a curve...curves through space. Okay, so how do you measure that? Well, we know that the unit tangent vector tells us in which direction a curve is “tending towards”. Certainly, if the unit tangent vector is constant then the curvature is constant e.g. a circle has constant curvature. Now if we imagined a case where the unit tangent vector changed wildly, then one can imagine that the curviness must be quite large. So, we come to the realization that curvature should be about how much the unit tangent vector changes i.e. $\mathbf{T}'(t)$. This still is not an entirely happy answer – in part because this answer depends only on t and therefore depends on the parameterization chosen. For this reason, one would rather study $\mathbf{T}(t)$ ’s change in terms of the arc length. This leads one to consider $\kappa(t) = \left| \frac{d\mathbf{T}}{dS} \right|$.

Warning: It is strongly advised that one does not memorize all of these formulas for the class. There are only “two good formulas” which are

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad \& \quad \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Why is this formula true? First off, $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$. Then we recall that $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. So,

$$\mathbf{r}'(t) = |\mathbf{r}'(t)|\mathbf{T}(t) = \frac{dS}{dt}\mathbf{T}(t).$$

Not differentiate one more time and use the product rule to get

$$\mathbf{r}''(t) = \frac{d^2S}{dt^2}\mathbf{T}(t) + \frac{dS}{dt}\mathbf{T}'(t).$$

Since $\mathbf{T}(t) \times \mathbf{T}(t) = 0$, we then know

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{dS}{dt}\right)^2 (\mathbf{T}(t) \times \mathbf{T}'(t)).$$

Now we compute the magnitude of both sides. Since $\mathbf{T}(t)$ is orthogonal to $\mathbf{T}'(t)$, the magnitude of their cross product is the product of their magnitudes. So,

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \left|\frac{dS}{dt}\right|^2 |\mathbf{T}(t)| |\mathbf{T}'(t)|.$$

Since $\mathbf{T}(t)$ is the *unit* tangent vector, $|\mathbf{T}(t)| = 1$. Therefore,

$$|\mathbf{T}'(t)| = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|(dS/dt)^2|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2}.$$

□

For a plane curve $y = f(x)$, the curvature becomes

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}.$$

Example 1.11.5. Consider the plane curve $y = \sin(x)$. Compute the curvature using the above formula.

Solution: In this case, $f''(x) = -\sin(x)$ and $f'(x) = \cos(x)$. Altogether, the curvature is

$$\kappa(x) = \frac{|\sin(x)|}{(1 + \cos^2(x))^{3/2}}.$$

□

Example 1.11.6. Compute the curvature of the following space curves:

- (1) when $\mathbf{r}(t)$ parameterizes the line $\langle t, t, t \rangle$.
- (2) when $\mathbf{r}(t)$ parameterizes the plane circle of radius 2 i.e. $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 0 \rangle$.

Solution: (1) First off, $\mathbf{r}'(t) = \langle 1, 1, 1 \rangle$ and $\mathbf{r}''(t) = \langle 0, 0, 0 \rangle$ and so $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 0$. So, $\kappa(t) = 0$ in this case.

(2) The curvature, if one computed using the formula correctly, is $\frac{1}{2}$.

□

Remark 1.11.7. An inquisitive person might ask: can we determine all space curves which have constant curvature? The answer is rather tricky in $\mathbb{R}^3$ since there are many nontrivial ones (see the following [webpage](#) for examples to play around with). On the other hand, the classification of **planar curves** or with constant curvature is not too bad – they are lines or circles.

Example 1.11.8. Compute the curvature of the helix $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $t \in \mathbb{R}$.

Solution: We have $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$ and $\mathbf{r}''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$. The magnitude of $\mathbf{r}'(t)$ is $|\mathbf{r}'(t)| = \sqrt{2}$. The cross product equals

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(t) & \cos(t) & 1 \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} = \langle \sin(t), -\cos(t), \sin^2(t) + \cos^2(t) \rangle = \langle \sin(t), -\cos(t), 1 \rangle.$$

The magnitude of this cross product equals $\sqrt{2}$. Altogether,

$$\kappa(t) = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}.$$

□

Example 1.11.9. Find the curvature of the curve $\langle 4t, -t^2, 2t^3 \rangle$ by choosing one of the formulas

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \quad \& \quad \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Which formula would be easier to use? Why?

Example 1.11.10. Compute the curvature of the space curve $\langle e^t, 2t^3, t^2 \rangle$.

Solution: This is left as a tedious exercise for the student.

□

Definition 1.11.11. If $\mathbf{r}(t)$ smoothly parameterizes a curve C , then there are some interesting vectors to consider

- the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
- the unit normal vector $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$
- and the binormal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

It is worth noting that the three vectors are all **orthogonal** to one another. The importance of these vectors will not be seen here – they will play a bigger role (in particular $\mathbf{N}(t)$), later in our study of vector calculus.

Example 1.11.12. Compute the unit tangent vector, unit normal vector, and binormal vector of $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution: The unit tangent vector is

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle.$$

The unit normal vector is

$$\mathbf{N}(t) = \langle -\cos(t), -\sin(t), 0 \rangle.$$

The binormal vector is

$$\mathbf{B}(t) = \left\langle \frac{\sin(t)}{\sqrt{2}}, -\frac{\cos(t)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

□

Example 1.11.13. Consider the curve $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), \sqrt{12}t \rangle$. Compute its unit tangent vector, unit normal vector, and unit binormal vector.

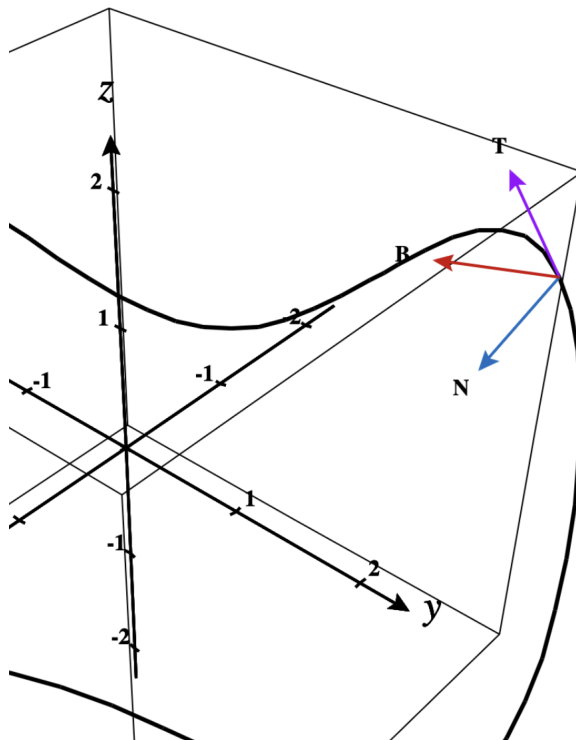


FIGURE 11. Pictured is a space curve. The unit tangent $\mathbf{T}$ points in the direction the position vector is moving towards at time t . The normal vector $\mathbf{N}$ points to the “left” of the direction of the tangent vector. Recalling our discussion regarding cross products, $\mathbf{B}$ points upwards from the plane determined by $\mathbf{T}$ and $\mathbf{N}$.

Solution: First off, $\mathbf{r}'(t) = \langle -2 \sin(2t), 2 \cos(2t), \sqrt{12} \rangle$ and $|\mathbf{r}'(t)| = \sqrt{4 \sin^2(2t) + 4 \cos^2(2t) + 12} = \sqrt{16} = 4$. So,

$$\mathbf{T}(t) = \frac{\langle -2 \sin(2t), 2 \cos(2t), \sqrt{12} \rangle}{4} = \left\langle -\frac{1}{2} \sin(2t), \frac{1}{2} \cos(2t), \frac{\sqrt{3}}{2} \right\rangle.$$

To compute the unit normal vector, we compute

$$\mathbf{T}'(t) = \langle -\cos(2t), -\sin(2t), 0 \rangle.$$

The magnitude of $\mathbf{T}'(t)$ is always 1. So,

$$\mathbf{N}(t) = \langle -\cos(2t), -\sin(2t), 0 \rangle.$$

Next, we need to compute the binormal vector.

$$\begin{aligned} \mathbf{B}(t) &= \left\langle \frac{\sqrt{3}}{2} \sin(2t), -\frac{\sqrt{3}}{2} \cos(2t), \frac{1}{2} \sin^2(2t) + \frac{1}{2} \cos^2(2t) \right\rangle \\ &= \left\langle \frac{\sqrt{3}}{2} \sin(2t), -\frac{\sqrt{3}}{2} \cos(2t), \frac{1}{2} \right\rangle. \end{aligned}$$

□

Example 1.11.14. Find the unit tangent, unit normal, and binormal vectors of for the curve $\mathbf{r}(t) = \langle t, \sqrt{2} \ln(t), \frac{1}{t} \rangle$ at $(1, 0, 1)$. Then, determine its curvature at $(1, 0, 1)$.

Solution: First off, $\mathbf{r}'(t) = \langle 1, \frac{\sqrt{2}}{t}, -\frac{1}{t^2} \rangle$. Its magnitude is $\sqrt{1 + \frac{2}{t} + \frac{1}{t^4}} = (1 + \frac{1}{t^2})$. $\square$

1.11.1. *Normal and Osculating Plane, and the Osculating Circle.* This material will not be tested on. Perhaps I will say some words in these notes about them.

1.12. **Functions in Several Variables.** In this class, we will consider functions with more than one number as input. For example, we shall consider two variable and three variable functions

$$\begin{aligned} f(x, y) &: \mathbb{R}^2 \rightarrow \mathbb{R} \\ f(x, y, z) &: \mathbb{R}^3 \rightarrow \mathbb{R}. \end{aligned}$$

It is good to get acquainted with the domain and range of such functions and the following examples are meant to do this.

Example 1.12.1. Sketch the domain of $f(x, y) = \ln(y^2 - 4x)$ in the xy -plane. What is its range?

Example 1.12.2. Sketch the domain of each function

- (1) $f(x, y) = \frac{\sqrt{x}}{\sqrt{y}}$,
- (2) $f(x, y) = e^x \sin(y)$,
- (3) $f(x, y) = \ln(y - x^2) + \sqrt{x}$,
- (4) $f(x, y) = \frac{1}{x+1} - \frac{1}{y-1}$.

One can also graph a function in two variables $f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ by considering the points $(x, y, f(x, y))$.

Example 1.12.3. Graph the following functions. Their graphs are in $\mathbb{R}^3$.

- (1) $f(x, y) = 1 - x - y$
- (2) $f(x, y) = 2\sqrt{x^2 + y^2}$
- (3) $f(x, y) = x^2 + y^2$.

The **level curves** of a function $f(x, y)$ are the curves in $\mathbb{R}^2$ described by $f(x, y) = k$ for some fixed $k \in \mathbb{R}$. These level curves help to describe the topography of the graph of $f(x, y)$ in $\mathbb{R}^3$.

Example 1.12.4. Draw the level curves of the following functions

- (1) $f(x, y) = 1 - x - y$
- (2) $f(x, y) = 2\sqrt{x^2 + y^2}$
- (3) $f(x, y) = x^2 + y^2$
- (4) $-\sqrt{x^2 + y^2}$
- (5) $x^2 - y^2$.

1.13. Limits and Continuity.

Definition 1.13.1. We shall write

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$$

for the limit of $f(x,y)$ as (x,y) approaches (a,b) exists and is equal to L . We will not go into rigorous definitions. What the student should understand is that $(x,y) \rightarrow (a,b)$ is a rather ambiguous notation – how should one “tend to” (a,b) ?

For the most part, there is no issue if the function is **continuous** and in these cases,

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b).$$

Recall that in single variable calculus, the **Squeeze Theorem** states that if $g(x) \leq f(x) \leq h(x)$ and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$, then $\lim_{x \rightarrow c} f(x)$ exists and equals L . For multivariable functions, this also holds true.

Example 1.13.2. Show that the limit $\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \sin(x+y)$ exists and equals zero.

Solution: First off,

$$-(x^2 + y^2) \leq (x^2 + y^2) \sin(x+y) \leq x^2 + y^2.$$

The functions on the left and right are both continuous so their limit as $(x,y) \rightarrow (0,0)$ is zero. That means the limit in the middle exists and is equal to zero (this is a version of the Squeeze Theorem). $\square$

Example 1.13.3. Show that $\lim_{(x,y) \rightarrow (0,0)} x^2 y^2 \cos(x^2 - y^2) = 0$.

Solution: Since $-1 \leq \cos(x^2 - y^2) \leq 1$, one has

$$-x^2 y^2 \leq x^2 y^2 \cos(x^2 - y^2) \leq x^2 y^2.$$

Applying $\lim_{(x,y) \rightarrow (0,0)}$ to the LHS and RHS gives zero. So the Squeeze Theorem applies to the middle term as well: $\lim_{(x,y) \rightarrow (0,0)} x^2 y^2 \cos(x^2 - y^2) = 0$ $\square$

Example 1.13.4. Show that $\lim_{(x,y) \rightarrow (0,5)} (y-5)^2 \sin(x^2 + y^2 - 25) = 0$.

Solution: This is left as an exercise for the student. $\square$

Example 1.13.5. Show that the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$ does not exist by considering $(x,y) \rightarrow (0,0)$ along the line $y = kx$ for different values of k .

Remark 1.13.6. Note that it is not true that if one checks the limits along all lines through the origin and that they all agree, then the limit $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ exists. One must consider also other possible paths e.g. $y = x^2$.

Example 1.13.7.

- (1) $\lim_{(x,y) \rightarrow (0,0)} \frac{e^{x^2 + y^2}}{e^{3x - y + 1} + x},$
- (2) $\lim_{(x,y) \rightarrow (1,2)} \frac{xy}{x^3 + y^3}$
- (3) $\lim_{(x,y) \rightarrow (-1,1)} \frac{x+1}{x^2 + y^2 + xy}.$

Example 1.13.8. The student should review the “Squeeze Theorem” from Calculus II which provides a useful way to determine when a limit might exist. Compute the following limits

- (1) $\lim_{(x,y) \rightarrow (3,2)} x^2 + xy,$
- (2) $\lim_{(x,y) \rightarrow (0,0)} x^2 + y^2 - xy$

- (3) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y}{xy-1}$
- (4) $\lim_{(x,y) \rightarrow (0,0)} \ln(xy)$
- (5) $\lim_{(x,y) \rightarrow (0,0)} x^3 + x^2y + 5x - y$
- (6) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2-y^2}{x-y}$
- (7) $\lim_{(x,y) \rightarrow (0,0)} \sin(x^2 - y^2)$
- (8) $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x+y)}{x^2+y^2}$

Solution:

- (1) The limit is 15.
- (2) The limit is 0.
- (3) The limit is zero and it exists. **Warning:** This is not completely obvious!
- (4) The limit does not exist since $\ln(xy) \rightarrow \infty$ as $xy \rightarrow 0$.
- (5) The limit is zero.
- (6) The limit exists and it equals zero. The reason is because $\frac{x^2-y^2}{x-y} = x + y$.
- (7) The limit is zero since $\sin(x^2 - y^2)$ is continuous and $\sin(0^2 - 0^2) = \sin(0) = 0$.
- (8) The limit does not exist.

□

Example 1.13.9. Show that the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y^2}{x^2+y^2}$ exists and equals zero.

Solution: Provided $(x, y) \neq (0, 0)$, the following is true $0 \leq \frac{x^2y^2}{x^2+y^2} \leq \frac{x^2y^2}{\max(x^2, y^2)}$. Letting $(x, y) \rightarrow (0, 0)$, the RHS goes to zero and so the limit exists and equals zero by the Squeeze Theorem. One can justify $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y^2}{\max(x^2, y^2)}$ by saying noting that $\frac{x^2y^2}{\max(x^2, y^2)} = \min(x^2, y^2)$ and since $\min(x^2, y^2)$ is continuous and can be evaluated at 0. □

1.14. **Review.** We discuss some problems which the student can use for review. The solutions are provided in detail so students know what an accurate solution looks like.

Example 1.14.1. Let $\vec{u} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ and $\vec{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Compute the following quantities:

- (a) The dot product $\vec{u} \cdot \vec{v}$.
- (b) The unit vector in the direction of $\vec{v}$.
- (c) The scalar projection $\text{comp}_{\vec{v}} \vec{u}$.
- (d) The vector projection $\text{proj}_{\vec{v}} \vec{u}$.
- (e) The cross product $\vec{u} \times \vec{v}$.

Solution.

- (a) The dot product equals $1 + 2(-2) + (5)(1) = 2$.
- (b) First, compute the length of $\mathbf{v}$ which is $\sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$. Then the unit vector in the direction of $\mathbf{v}$ is $\langle \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \rangle$.
- (c) The scalar projection of $\mathbf{u}$ onto $\mathbf{v}$ is equal to $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{6}}$.
- (d) The vector projection is the scalar projection times the unit vector in the direction of $\mathbf{v}$. So the answer is $\langle \frac{2}{6}, -\frac{4}{6}, \frac{2}{6} \rangle$.
- (e) The cross product is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 5 \\ 1 & -2 & 1 \end{vmatrix} = 12\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}.$$

Example 1.14.2. Consider the lines given by $\langle x, y, z \rangle = \langle 1 - t, 1 + t, 2t \rangle$ and $\langle x, y, z \rangle = \langle 3t, t, -t \rangle$. Find the equation of the plane that contains both lines.

Solution. The lines point in the direction of $\langle -1, 1, 2 \rangle$ and $\langle 3, 1, -1 \rangle$ respectively. A plane that contains both lines must have normal vector equal to the cross product of these two vectors. So

$$\mathbf{n} = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = -3\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}.$$

Next, we just pick a point that must be on this plane. For ease, we set $t = 0$ for the first line and we get the point $(1, 1, 0)$. So an equation for our plane is

$$-3(x - 1) + 4(y - 1) - 4z = 0.$$

In standard form, this becomes

$$-3x + 4y - 4z - 1 = 0.$$

Example 1.14.3. Determine if the following limit exists and if it does, compute it: $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin^2(x+y)}{x^2+y^2}$.

Solution. Since $0 \leq \sin^2(x+y) \leq 1$, we have the bound

$$0 \leq \frac{\sin^2(x+y)}{x^2+y^2} \leq \frac{1}{x^2+y^2}.$$

Taking the limit as $(x, y) \rightarrow (0, 0)$ gives me

$$0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{\sin^2(x+y)}{x^2+y^2} \leq 0$$

and so the limit exists and it equals zero.

Example 1.14.4. Consider the intersection of the two surfaces $x^2 + y + z^2 = 4$ and $x^2 + z^2 = 5$. Write down a parameterization for that curve.

Solution. We can easily parameterize $x^2 + z^2 = 5$ by $x = \sqrt{5} \cos(t)$ and $z = \sqrt{5} \sin(t)$ for $0 \leq t \leq 2\pi$. The first equation says that $y = 4 - x^2 - z^2$ and so $y = 4 - \sqrt{5}(\cos(t) + \sin(t))$. So the desired parameterization is

$$\langle \sqrt{5} \cos(t), 4 - \sqrt{5}(\cos(t) + \sin(t)), \sqrt{5} \sin(t) \rangle \text{ where } 0 \leq t \leq 2\pi.$$

Example 1.14.5. Consider the curve parameterized by $\langle t, 1, t^2 \rangle$ for $0 \leq t \leq 1$. Compute the arc length of the curve.

Solution. Use the arc length formula

$$L := \int_0^1 \sqrt{1^2 + 0^2 + (2t)^2} dt = \int_0^1 \sqrt{1 + 4t^2} dt = \frac{1}{2} \left(\sqrt{5} + \frac{1}{2} \ln(2 + \sqrt{5}) \right).$$

One can compute this integral using trig-substitution.

2. CALCULUS OF MULTIVARIABLE FUNCTIONS

2.1. Partial Derivatives, and Tangent Planes.

Definition 2.1.1. If $f(x, y)$ is a function in 2 variables, then its **partial derivative** w.r.t. x at (a, b) is

$$\frac{\partial f}{\partial x} = D_x f(a, b) = f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

By making the appropriate modification, one can define the partial derivative w.r.t. y at (a, b) . Above, we have given various notations that one might use for the partial derivative.

The *rules* for computing the derivative of $f(x, y)$ w.r.t. x is straightforward – differentiate as you would by viewing $f(x, y)$ as a function in the variable x and treat any instances of y as constant. Due to this perspective, the usual differentiation rules generalize quite easily.

Example 2.1.2. Find $f_x(x, y)$ and $f_y(x, y)$ of the following two variable functions:

- (1) $f(x, y) = x^3y + 2x$,
- (2) $f(x, y) = x \sin(y) + y \sin(x)$,
- (3) $f(x, y) = x^2e^{x+y}$,
- (4) $f(x, y) = \frac{x}{y+x}$.

Solution:

- (1) $f_x(x, y) = 3x^2y + 2$, $f_y(x, y) = x^3$,
- (2) $f_x(x, y) = \sin(y) + y \cos(x)$, $f_y(x, y) = x \cos(y) + \sin(x)$,
- (3) $f_x(x, y) = x^2e^{x+y} + 2xe^{x+y}$, $f_y(x, y) = x^2e^{x+y}$,
- (4) this one is left for the student to do.

□

One can view the partial derivative as the slope of the tangent line pointing in some direction: $f_x(a, b)$ is the slope of the tangent line at (a, b) pointing in the x direction. *Students should draw a picture for themselves.*

Example 2.1.3. Compute $f_x(2, \pi)$ and $f_y(1, 6)$ for the following functions:

- (1) $f(x, y) = 16 - x^2 + y^2$,
- (2) $f(x, y) = 2 + \pi x^3 + 6xy$,
- (3) $f(x, y) = x^3y^2$.

Solution: The details are left to the student. The answers are

- (1) $f_x(2, \pi) = -4$ and $f_y(1, 6) = 12$.
- (2) $f_x(2, \pi) = 2 + 12\pi + 6\pi$ and $f_y(1, 6) = 6$.
- (3) $f_x(2, \pi) = 12\pi^2$ and $f_y(1, 6) = 12$.

□

One can also generalize this process to more than two variables and even implicitly differentiate.

Example 2.1.4 (Implicit Differentiation Example). Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ from

$$z^2 + yz + y \ln(x) = 0$$

where one assumes y is independent of x .

Solution: We leave finding $\frac{\partial z}{\partial y}$ to the student. In this case, implicitly differentiation gives

$$2z \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial x} + \frac{y}{x} = 0.$$

Therefore,

$$\frac{\partial z}{\partial x} = \frac{-(y/x)}{2z + y} = \frac{-y}{2xz + xy}$$

assuming $2xz + xy \neq 0$. □

You can also compute higher order derivatives and here are some notations for it

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial}{\partial x} \right) \frac{\partial f}{\partial x}$$

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} = \left(\frac{\partial}{\partial y} \right) \frac{\partial f}{\partial x}.$$

Remark 2.1.5. As a **warning**, it is not generally true that $f_{xy} = f_{yx}$. Care must be taken, but for this class we shall always be given functions $f(x, y)$ where $f_{xy} = f_{yx}$. An example is to define $f(x, y) = \frac{xy(y^2 - x^2)}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) := 0$. Then f_{xy} and f_{yx} differ at $(0, 0)$.

Remark 2.1.6. Due to the previous remark, students should explicitly compute f_{yx} , and f_{xy} in order. If a problem asks to compute both, then students should show the work for both.

Remark 2.1.7 (Properties of $\partial/\partial x$). Almost all properties of the usual single variable derivative transfers over to the partial derivatives e.g. product rule, quotient rule, power rule, and linearity. Some care must be taken with compositions of functions which is discussed later.

Example 2.1.8. Find f_{xx} , f_{xy} , f_{yx} , and f_{yy} for the following function

$$f(x, y) = xy^2 + y^3 + y^2 + \sin(x).$$

Now do the same for

$$f(x, y) = \ln(x^4 + y^2).$$

Solution: We leave $f(x, y) = \ln(x^4 + y^2)$ to the student. For the first function, we get

$$f_{xx} = -\sin(x)$$

$$f_{xy} = \frac{\partial}{\partial y}(y^2 + \cos(x)) = 2y$$

$$f_{yx} = \frac{\partial}{\partial x}(2xy + 3y^2 + 2y) = 2y$$

$$f_{yy} = 2x + 6y + 2.$$

□

In single variable calculus, you used tangent lines to approximate a function. Recall that if $f(x)$ is a differentiable function, then the linear approximation at $x = a$ is the line

$$L(x) = f(a) + f'(a)(x - a).$$

This is the **tangent line** at $x = a$.

For $f(x, y)$ a differentiable function in two variables, its **linear approximation at** (a, b) is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Example 2.1.9. Find the linear approximation of $f(x, y) = e^x \cos(y)$ at the point $(0, \frac{\pi}{2})$.

Solution: In this case, $f_x(0, \pi/2) = e^0 \cos(\pi/2) = 0$ and $f_y(0, \pi/2) = e^0(-\sin(\pi/2)) = -1$. So,

$$L(x, y) = e^0 \cos(\pi/2) + 0(x - 0) + (-1)(y - \pi/2) = -y + \pi/2$$

is the linear approximation at $(0, \pi/2)$. $\square$

Example 2.1.10. Use linear approximation at the point $(1, 1)$ to approximate $f(x, y) = \sqrt{x^2 + y^3}$'s value at $(2, 2)$.

Solution: This is left as an exercise for the student. $\square$

Now we explain how to find the tangent plane at a point. To be honest, this is what we computed when we were looking at the linear approximation.

Recall that a tangent plane is given by a choice of normal vector and a point/vector. So for us, we need to figure out the normal vector pointing away from the surface $f(x, y) = z$ and then consider a point on this surface.

A normal vector at the point¹ $(a, b, f(a, b))$ is

$$\langle f_x(a, b), f_y(a, b), -1 \rangle.$$

Then the equation of the **tangent plane** is

$$\langle f_x(a, b), f_y(a, b), -1 \rangle \cdot \langle x - a, y - b, z - f(a, b) \rangle = 0.$$

If we expand the dot product, the equation is then

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0.$$

For a visual/neat tool to play around with tangent planes, consider checking out this [Desmos3d graph](#).

Example 2.1.11. Let $f(x, y) = x^2 - 2y$. Then the tangent plane at $(0, 0, 0)$ can be computed as follows. First, $f_x(0, 0) = 0$, $f_y(0, 0) = -2$. So, the equation says

$$0 \cdot (x - 0) + -2(y - 0) - (z - f(0, 0)) \Rightarrow -2y = z.$$

Example 2.1.12. Compute the tangent planes at the point $(x, y) = (0, 0)$ for the following:

- (1) $f(x, y) = x^3 + 2xy$,
- (2) $f(x, y) = (x - 1)^3 + (y - 2)^3$,
- (3) $f(x, y) = x + xy + y^2$,
- (4) $f(x, y) = 4x + 5y + 6y^2$.

Solution: We leave (2), (3), and (4) to the student to do. For (1), we get

$$f_x(x, y) = 3x^2 + 2y \quad \& \quad f_y(x, y) = 2x.$$

Plugging in $(0, 0)$ to both gives zero. So, the tangent plane in this case is given by

$$-z + f(0, 0) = 0 \Rightarrow z = 0.$$

$\square$

¹We will see later why this is a valid choice of normal vector to the surface. This normal vector is often referred to as the *gradient vector*.

2.2. Multivariable Chain Rule. In one variable, the chain rule says that

$$\frac{d}{dt}f(g(t)) = f'(g(t)) \cdot g'(t).$$

In the multivariable situation, the chain rule has more “layers”.

Case 1. If $z = f(x, y)$ and the variables $x = x(t)$ and $y = y(t)$ are functions in t , then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

Example 2.2.1. Remember that it is best practice to write the final answer in the variable that you are differentiating w.r.t to.

(1) Let $z = x^2y - xy^2 + y$ and $x = \sin(t)$, $y = e^t$. Compute $\frac{dz}{dt}$.

(2) Do the same when $z = x^2 + y^3$ and $x = \sin(t)$ and $y = \cos(4t)$.

Solution: (1) In this case, the answer is

$$\frac{dz}{dt} = (2xy - y^2)(\cos(t)) + (x^2 - 2xy)(e^t) = (2\sin(t)e^t - e^{2t})(\cos(t)) + (\sin^2(t) - 2\sin(t)e^t)(e^t).$$

The student should take care to make sure their final answer is written w.r.t the variable one is differentiating w.r.t.

(2) In this case,

$$\frac{dz}{dt} = (2x)(\cos(t)) + (3y^2)(-4\sin(4t)) = 2\sin(t)\cos(t) + 3\cos^2(4t)(-4\sin(4t)).$$

□

Remark 2.2.2. Consider (2) from the previous problem. We could rewrite

$$z(t) = \sin^2(t) + \cos^3(4t).$$

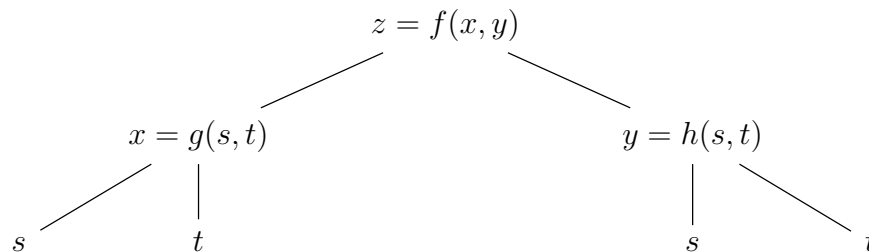
Then, differentiating w.r.t t gives

$$2\sin(t) + 3\cos^2(4t)(-4\sin(4t)).$$

Case 2. Let $z = f(x, y)$ and assume instead $x = f(s, t)$, $y = g(s, t)$ are functions in two variables. Then the chain rule looks like

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}. \end{aligned}$$

If one cannot remember these two rules, the easiest method is to just draw the “tree of dependencies”.



Example 2.2.3. Find $\partial z/\partial t$ and $\partial z/\partial s$ where $z := (x + y)^2$, $x := e^{2s+t}$ and $y := \ln(t/s)$.

Solution: One has

$$\frac{\partial z}{\partial s} = 2(x+y)(2e^{2s+t}) + 2(x+y)\left(-\frac{1}{1/s}\right) = 2(e^{2s+t} + \ln(t/s))(2e^{2s+t}) + 2(e^{2s+t} + \ln(t/s))\left(-\frac{1}{1/s}\right)$$

$$\frac{\partial z}{\partial t} = 2(x+y)(e^{2s+t}) + 2(x+y)\left(\frac{1}{t}\right) = 2(e^{2s+t} + \ln(t/s))(e^{2s+t}) + 2(e^{2s+t} + \ln(t/s))\left(\frac{1}{t}\right).$$

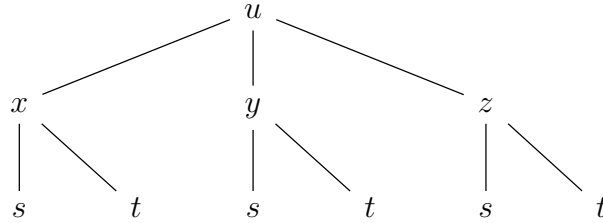
□

Example 2.2.4. Repeat the previous example except $z := xe^y$, $x := s^2 + t^2$, and $y := \ln(st)$.

Solution: Left for the student.

□

The previous two examples are special cases in a more general story. For example, let $f(x, y, z) := u$ be a function in variables x, y, z and assume x, y, z are functions in the variables s, t . Then, one can draw the “tree of dependencies” once more



Using this, it is easier to see that

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}.$$

As a general rule, one has the following fact.

Theorem 2.2.5. If $u(x_1, x_2, \dots, x_m)$ is a function in m variables and each x_i is a function in the n variables $t_1, \dots, t_n$, then

$$\frac{\partial u}{\partial t_i} = \sum_{j=1}^m \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial t_i}$$

whenever the partial derivatives are defined.

Example 2.2.6. Find $\frac{\partial u}{\partial s}$ if $u(x, y, z) := x^3y + yz + z^2$ and $x := \sin(t) + s$, $y := \cos(t)$, $z := t^2s$.

Solution: Left for the student.

□

Example 2.2.7. Find $\frac{\partial f}{\partial s}$ where $f(x, y := \pi x^2 + y + xy$ and $x := \sqrt{\pi} \sin(st)$ and $y := -\pi \cos(t)$.

Solution: The derivative equals

$$\frac{\partial f}{\partial s} = (2\pi x + y)(t\sqrt{\pi} \cos(st)) + (1 + x)(0) = (2\pi x + y)(t\sqrt{\pi} \cos(st)).$$

□

Recall from Calculus I that the chain rule was used to differentiate implicitly defined functions. For example, when one writes $x^2 + y^2 = 1$, we know that y is an implicitly defined

function since y depends on the variable x . Then in Calculus I, differentiating with respect to x gives

$$2x + 2y \frac{dy}{dx} = 0.$$

The student should know that the multivariable chain rule allows them to do the same thing for functions with multiple variables. We shall not spend time on this for the course.

2.3. Directional Derivatives, and Optimization. Recall the formulae

$$f_x(x_0, y_0) := \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) := \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

One interprets the value of these limits as the amount of change in the x -direction (resp. y -direction). These are the directions given by the unit vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$. In principle, one can rewrite these formulas as

$$f_x(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle 1, 0 \rangle$$

and

$$f_y(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle 0, 1 \rangle.$$

This might feel circular, but it provides the answer for differentiating in a general direction.

Definition 2.3.1. If $f(x, y)$ is a differentiable function, then the **directional derivative at (x_0, y_0) in the direction of the unit vector $\mathbf{u} := \langle a, b \rangle$** is defined as

$$D_{\mathbf{u}}f(x_0, y_0) := \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle a, b \rangle = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

Remark 2.3.2. A natural question to ask is: why do we need to take a unit vector $\mathbf{u}$? The reason² is in the name – the *directional* derivative should only care about the direction involved. Vectors contain more data than just “direction” – it has length as well. Imposing the condition that $\mathbf{u}$ is a unit vector ensures that the data of length is irrelevant.

This is especially important if we want things to match up when dealing with a single direction. For example, we ideally would like the directional derivative of $f(x, y)$ in the direction of $\langle 1, 0 \rangle$ to just be $f_x(x, y)$.

Example 2.3.3. Find $D_{\mathbf{u}}f(x, y)$ where $\mathbf{u} := \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ and $f(x, y) := x^2y + y^2$ at $(1, 2)$.

Solution: Clearly, $f_x(x, y) = 2xy$, $f_y(x, y) = x^2 + 2y$. At $(1, 2)$, we get the values 4 and 5 respectively. So,

$$D_{\mathbf{u}}f(1, 2) = 4\frac{1}{\sqrt{2}} + 5\frac{1}{\sqrt{2}} = \frac{9}{\sqrt{2}}.$$

□

Example 2.3.4. Find $D_{\mathbf{u}}f(x, y)$ in the direction of $\mathbf{u} := \langle -1, \sqrt{3} \rangle$ for $f(x, y) = 3xy - y^2$ at the point $(-1, 3)$.

²Though, personally, it is not a very compelling reason since we could also think about directional derivatives with respect to any vector. The answer one gets in this case is then a scalar multiple of the directional derivative we get here in these notes.

Solution: Left for the student. □

Definition 2.3.5. The **gradient of** $f(x, y)$ is the vector function / vector field

$$\text{grad}(f) = \nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle.$$

This is a process to take a function $f(x, y)$ and produce a vector field which quantifies the “growth rate” of $f(x, y)$ at all points. So,

$$D_{\mathbf{u}}f(x, y) := (\nabla f) \cdot \mathbf{u}.$$

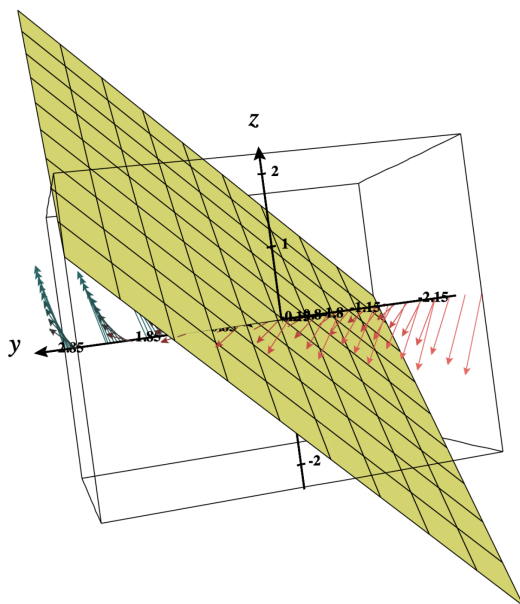


FIGURE 12. The gradient vector field of $f(x, y) = x + y$ is pictured. Notice where vectors point on the direction where $f(x, y)$ is changing and the length quantifies how much it is changing.

Example 2.3.6. Find the gradient of $f(x, y) = x^2y^3$ at the point $(2, 2)$. Use it to find the directional derivative in the direction from $(1, 1)$ to $(2, 3)$ at the point $(2, 2)$.

Solution: The gradient is $\langle 2xy^3, 3x^2y^2 \rangle$ and at $(2, 2)$ we get the vector $\langle 2^5, 3 \cdot 2^4 \rangle = \langle 32, 48 \rangle$.

The direction from $(1, 1)$ to $(2, 3)$ is given by the unit vector $\mathbf{u} = \frac{1}{\sqrt{5}}\langle 1, 2 \rangle$. So, the directional derivative in the direction $\frac{1}{\sqrt{5}}\langle 1, 2 \rangle$ is

$$D_{\mathbf{u}}(f, 2, 2) = \frac{32}{\sqrt{5}} + \frac{96}{\sqrt{5}}.$$

□

Example 2.3.7. Find the gradient of $f(x, y) = x^3e^y$ at $(1, 3)$. Use the gradient to find the directional derivative of f at $(2, 0)$ in the direction $\langle 1, 0 \rangle$.

Solution: First off, $\nabla f = \langle 3x^2e^y, x^3e^y \rangle$ and so $\nabla f(1, 3) = \langle 3e^3, e^3 \rangle$.

So the directional derivative equals

$$D_{\mathbf{u}}f(2, 0) = \langle 12, 8 \rangle \cdot \langle 1, 0 \rangle = 12.$$

□

Here is a good place to summarize what we have so far:

- $D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b = \nabla f(x, y) \cdot \mathbf{u}$, and
- $\text{grad}(f) = \nabla(f) = \langle f_x, f_y \rangle$.

Writing this out, it becomes clear that there is no need to restrict to two variables. Everything we wrote above works with 3 or more variables as well. For three variables where $\mathbf{u} = \langle a, b, c \rangle$, these formulas turn into

- $D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c = \nabla f(x, y, z) \cdot \mathbf{u}$, and
- $\text{grad}(f) = \nabla(f) = \langle f_x, f_y, f_z \rangle$.

Example 2.3.8. Let $f(x, y, z) := x^2y - xyz^2$. Find the directional derivative of f where $\mathbf{u}$ is the unit vector centered at $(0, 1, 2)$ pointing in the direction of $(1, 4, 3)$.

Solution: Left to the student. □

Here's a very natural question to ask: if the directional derivative is supposed to quantify the growth of a function in various directions, then how does one determine the direction of **maximal** growth?

Theorem 2.3.9. For a reasonable function $f(\mathbf{x})$, the direction of maximal growth i.e. the unit vector $\mathbf{u}$ where $|D_{\mathbf{u}}f|$ is largest at (x_0, y_0) is when $\mathbf{u}$ points in the same direction as $\nabla f(\mathbf{x}_0)$. The maximum value of the directional derivative is then $|\nabla f(\mathbf{x}_0)|$.

By considering $-f(\mathbf{x})$ in place of $f(\mathbf{x})$, one can deduce a similar statement about the direction of least growth.

Proof. By definition,

$$|D_{\mathbf{u}}f(\mathbf{x})| = |\nabla f(\mathbf{x}) \cdot \mathbf{u}| = |\nabla f(\mathbf{x})||\mathbf{u}| \cos(\theta) = |\nabla f(\mathbf{x})| \cos(\theta).$$

This is maximized when $\cos(\theta)$ is the largest it can be. In particular, when $\cos(\theta) = 1$. This occurs only when $\theta = 0$ (since we restrict $0 \leq \theta < \pi$). So, $\mathbf{u}$ must be pointing in the same direction as $\nabla f(\mathbf{x})$. □

Example 2.3.10. What direction maximizes $D_{\mathbf{u}}f(x, y)$ at the point $(1, -2)$ and where $f(x, y) := x^2 + y^2$.

Solution: The direction is given by $\nabla f(1, -2)$. Since $\nabla f(x, y) = \langle 2x, 2y \rangle$, the direction that maximizes the directional derivative is the unit vector pointing in the direction of $\langle 2, -4 \rangle$. In other words, we take $\mathbf{u} := \langle \frac{2}{\sqrt{20}}, \frac{-4}{\sqrt{20}} \rangle$. □

Remark 2.3.11. There are instances where the gradient is the zero vector $\mathbf{0}$ at a point. In this case, this means the function grows at the same rate in all direction. For example, the constant function $f(\mathbf{x}) = 1$ directional derivative equal to zero in all directions at the origin.

Theorem 2.3.12. The gradient $\nabla f(x_0, y_0)$ is always orthogonal to the level curve of $f(x, y)$ through the point (x_0, y_0) . For a visual, see this [Desmos graph](#).

Proof. Assume $f(x_0, y_0) = c_0$ is the level curve C which one draws in the xy -plane. Assume C is parameterized by $\langle g(t), h(t) \rangle$. Consider the composition

$$F(t) := f(g(t), h(t)).$$

Now, $F(t)$ also describes the level curve C . By the multivariable chain rule,

$$F'(t) = \frac{\partial f}{\partial y} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial h}{\partial t}.$$

But this means $F'(t) = \nabla f(x, y) \cdot \langle g'(t), h'(t) \rangle$. So, $F'(t_0) = \nabla f(x_0, y_0) \cdot \langle g'(t_0), h'(t_0) \rangle$. Now, $\langle g'(t_0), h'(t_0) \rangle$ is tangent to the level curve and since $F(t) = c_0$, we know $F'(t) = 0$. Altogether, we get

$$0 = F'(t_0) = \nabla f(x_0, y_0) \cdot \langle g'(t_0), h'(t_0) \rangle.$$

The vector $\langle g'(t_0), h'(t_0) \rangle$ is tangent to the curve and $\nabla f(x_0, y_0)$ is orthogonal to it which implies the gradient orthogonal to the level curve. $\square$

The discussion in the preceding theorem applied to functions in two variables, but the argument generalizes to any number of variables. Because of this property, one can compute the tangent plane and normal line to a surface. Recall that the equation $f(x, y, z) = K$ graphs a surface in $\mathbb{R}^3$. Let S denote this surface.

- (1) The **tangent plane** at (x_0, y_0, z_0) is the plane with normal vector $\nabla f(x_0, y_0, z_0)$ passing through the point. In other words, the tangent plane has equation given by

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

- (2) The **normal line** to S at (x_0, y_0, z_0) is the line going through this point with the direction $\nabla f(x_0, y_0, z_0)$.

Remark 2.3.13. If the reader is confused on why the gradient is playing such an important role here, consider the sphere $f(x, y, z) = x^2 + y^2 + z^2 = K$ of radius $\sqrt{K}$ where K is a positive number. The gradient is $\langle 2x, 2y, 2z \rangle$ in this case. It is always pointing out of the sphere. The reason is because if we increase x, y , or z , then $f(x, y, z)$ is increasing. For a visual, see this [Desmos graph](#).

Example 2.3.14. Find the tangent plane and normal line to $x^2 + y^2 - z^2 = 24$ at $(3, 4, -1)$.

Solution: In this case, the gradient vector is $\nabla f(3, 4, -1) = \langle 6, 8, 2 \rangle$. So, the tangent line is given by

$$6(x - 3) + 8(y - 4) + 2(z + 1) = 0.$$

The normal line can be described by

$$\mathbf{r}(t) = \langle 3, 4, -1 \rangle + t\langle 6, 8, 2 \rangle.$$

$\square$

Example 2.3.15. Find the tangent plane and normal line to the surface $z - x^2 + y^2 = 1$ at the point $(1, 1, 1)$.

Solution: The gradient is $\nabla f(x, y, z) = \langle -2x, 2y, 1 \rangle$ where $f(x, y, z) = z - x^2 + y^2$. So, $\nabla f(1, 1, 1) = \langle -2, 2, 1 \rangle$ and the tangent plane is given by the equation

$$-2(x - 1) + 2(y - 1) + (z - 1) = 0.$$

Meanwhile, the equation for the normal line is

$$\mathbf{r}(t) = \langle 1, 1, 1 \rangle + t\langle -2, 2, 1 \rangle$$

□

Before the next lecture, it might help to think about the following thought experiment. Suppose I give you a function $f(x, y, z)$ and to you, a starting point (x_0, y_0, z_0) . I give you the command to compute the vector $\nabla f(x_0, y_0, z_0)$ and then, using this vector, move in the direction this vector points to. Now, suppose you were to follow this command “infinitesimally” so i.e. you do it at basically every small movement – you move a tiny bit, compute the gradient, and then move a little bit in the direction it points to. If you were to listen to these commands, then (a) is there ever a point in time where you stop moving i.e. the gradient you get is zero? and (b) if you do stop moving, what sort of point are you at?

To provide answers to these two questions, we begin with some definitions.

Definition 2.3.16. Let $f(x, y)$ be a function. A point (a, b) is a

- **local maximum** if $f(a, b) \geq f(x, y)$ for all (x, y) near (a, b) ,
- **local minimum** if $f(a, b) \leq f(x, y)$ for all (x, y) near (a, b) ,
- **global maximum** if $f(a, b) \geq f(x, y)$ for all (x, y) in the domain of f ,
- **global minimum** if $f(a, b) \leq f(x, y)$ for all (x, y) in the domain of f .

Remark 2.3.17. Note from the definition that a global maximum/minimum is also a local maximum/minimum.

Definition 2.3.18. If $f(x, y)$ is a function, then a point (a, b) is called a **critical point** if $f_x(a, b) = f_y(a, b) = 0$ or one of the partial derivatives does not exist.

Theorem 2.3.19. If (a, b) is a local maximum or local minimum, then (a, b) is a critical point for $f(x, y)$.

Remark 2.3.20. Why is this true? More or less for the same reason as in the single variable case.

Example 2.3.21. Find the critical points for the following functions:

- (1) $f(x, y) = x^3 + xy + 3x$,
- (2) $f(x, y) = y^2 - x^2$,
- (3) $f(x, y) = x^2y^2 + 2xy$,
- (4) $f(x, y) = x^2 + y^3$.

Solution: (3) and (4) are left as exercises to the student. We discuss (1) and (2).

(1) $f_x(x, y) = 3x^2 + y + 3$ and $f_y(x, y) = x$. If these are equal to zero, then $x = 0$ and the first equation becomes $y + 3 = 0$. So, $y = -3$ and the sole critical point is $(0, -3)$.

(2) $f_x(x, y) = -2x$ and $f_y(x, y) = 2y$ and for these to both be zero, $x = 0$ and $y = 0$. So the sole critical point is $(0, 0)$. □

In the case of 2-variables, some new phenomena can occur and $f(x, y) = x^2 + y^3$ is an example of this. This function has a **saddle point** at $(0, 0)$.

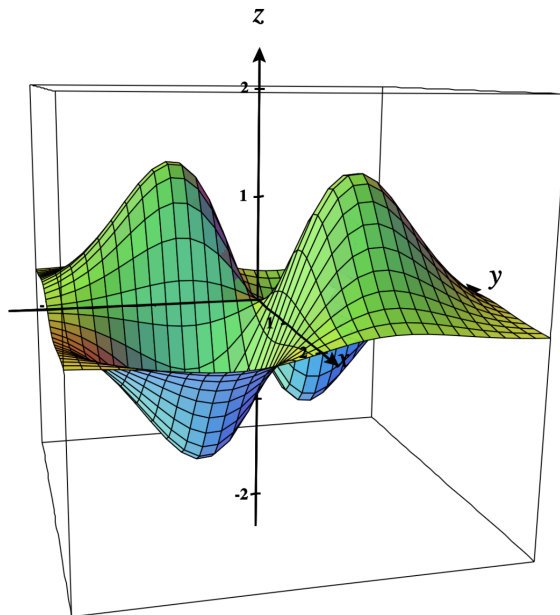


FIGURE 13. The graph of some function $f(x, y)$. Notice that the local maximum are the peaks of mountains and local minimums are the deepest points of the valleys.

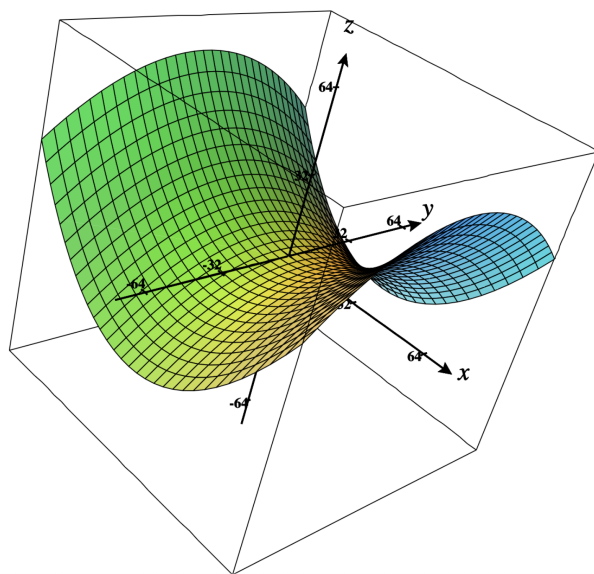


FIGURE 14. The graph of a hyperbolic paraboloid which has a saddle point.

Theorem 2.3.22 (Second Derivative Test). Let $f(x, y)$ be a sufficiently nice function. Let

$$D(x, y) := f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)^2 = \det \begin{pmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{pmatrix}.$$

Let (a, b) be a critical point of $f(x, y)$. Then

- if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local minimum,

- if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local maximum,
- if $D(a, b) < 0$, then (a, b) is a saddle point,
- if $D(a, b) = 0$, then the second derivative test is inconclusive.

Remark 2.3.23. In situations where the second derivative test is inconclusive, what can be done? Usually, a more direct analysis of the function is needed. Take for example $f(x, y) = -x^4 - y^4$. The second derivative test is inconclusive at $(0, 0)$. However, one should recognize that $x^4 + y^4 \geq 0$ and so $-x^4 - y^4 \leq 0$. At the origin, $f(0, 0) = 0 \leq 0$ and so it is certainly a local maximum. It is not hard to then check that it is a global maximum.

Example 2.3.24. Find all local maxima, local minima, and saddle points of $f(x, y) = x^2 + xy + y^2 - 6x + 6$. Extra: Use [CalcPlot3D](#) to check that the answer is correct.

Solution: First, we need to find the critical points so we solve

$$f_x(x, y) = 2x + y - 6 = 0 \quad \& \quad f_y(x, y) = x + 2y = 0.$$

The first equation says that $y = 6 - 2x$. Plugging this into the second equation says that $x + 2(6 - 2x) = 0$ and so $-3x + 12 = 0$. Therefore, $x = 4$. It follows that $y = -2$ since $x + 2y = 0$ needs to be true.

So there is a single critical point $(4, -2)$. Next, we compute

$$D(x, y) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3.$$

So $D(4, -2) = 3 > 0$. Next, we know that $f_{xx}(4, -2) = 2 > 0$. Therefore, we have a local minimum. In fact, it is a global minimum. $\square$

Example 2.3.25. Find all local maxima, local minima, and saddle points of $f(x, y) = x^3 + y^3 + 2xy - 4x - 3y + 5$.

Solution: This problem is left as an exercise for the student. $\square$

Example 2.3.26. Find all local maxima, local minima, and saddle points of $f(x, y) = 4x^2 + 3y^2 - xy + 2y$.

Solution: This problem is left as an exercise for the student. $\square$

Algorithm to find Local Minimums and Local Maximums Let $f(x, y)$ be a sufficiently nice function.

- (1) First, solve for all points (x, y) such that $f_x(x, y) = f_y(x, y) = 0$. These are called **critical points**.
- (2) Second, take a critical point (a, b) and plug it into the function

$$D(x, y) = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

- (3) Third, if you need to, take (a, b) and plug it into $f_{xx}(a, b)$.
- (4) Fourth, compare what you have against the table below:

$D(a, b) > 0$	$f_{xx}(a, b) > 0$	(a, b) is a local minimum
$D(a, b) > 0$	$f_{xx}(a, b) < 0$	(a, b) is a local maximum
$D(a, b) < 0$		(a, b) is a saddle point
$D(a, b) = 0$		no information can be obtained

2.4. Extreme Value Theorem, Optimization, and Lagrange Multipliers. To optimize means to “make as perfect or as effective as possible”. In mathematics, the most common variant of this task is to find inputs to a function f so that the output is the biggest it can be.

However, this is often impossible to do if one does not impose constraints. For example, if I were to ask you to find the largest value of $f(x) = x$ where $x \in \mathbb{R}$, then there is no good answer to my question. You can pick x arbitrarily large, afterall.

The typical situation where one has hope for a good answer is when one imposes constraints on the domain of the function. This comes in the following form in single variable calculus.

Theorem 2.4.1 (Extreme Value Theorem). If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function, then there is a $a \leq c \leq b$ such that $f(x) \leq f(c)$ for all $a \leq x \leq b$.

The correct formulation for functions in two variables requires some preparations.

First, we say a subset $D \subseteq \mathbb{R}^2$ is **closed** if it contains all of its boundary points. A boundary point p of D is a point where all open neighborhoods intersect D nontrivial. This is best understood in examples.

- $\{(x, y) \mid x^2 + y^2 \leq 1\}$ is closed.
- $\{(x, y) \mid x \geq 1\}$ is closed.
- $\{(x, y) \mid x^2 + y^2 = 1\}$ is closed.
- $\{(x, y) \mid x^2 + y^2 < 1\}$ is not closed.
- $\{(x, y) \mid 1 \leq x \leq 2, y > 1\}$ is not closed.
- $\{(x, y) \mid x^2 + y^2 = 1\} - \{(0, 1)\}$ is not closed.

Theorem 2.4.2 (Extreme Value Theorem). If $f : D \rightarrow \mathbb{R}$ is a continuous function where $D \subseteq \mathbb{R}^2$ is a closed and bounded region, then f attains a maximum on D .

Remark 2.4.3. Why is the Extreme Value Theorem true? A rigorous precise justification requires some explaining so we focus on an intuitive explanation. The conditions closed and bounded essentially say roughly “any sequence of points in D that looks like it has a limit does have a limit and that limit is in D ” and “any sequence of points . Now pick a sequence of points (x_n, y_n) subject to the condition that $f(x_n, y_n) > f(x_{n-1}, y_{n-1})$. If f did not attain a maximum, one could make this sequence infinite. One can reason that this means (x_n, y_n) has to either (a) approach the boundary of D or (b) accumulate somewhere in D and both of these lead to a contradiction.

Here are two motivating examples to keep in mind.

Example 2.4.4. The function $f(x, y) = x + y$ going from $\mathbb{R}^2 \rightarrow \mathbb{R}$ never attains a maximum. We could make x and y arbitrarily large. The region $\mathbb{R}^2$ is closed but not bounded.

Example 2.4.5. The function $f(x, y) = \frac{1}{x^2 + y^2}$ on the region $D := \{(x, y) \mid x^2 + y^2 \leq 1\} \setminus \{(0, 0)\}$ never attains a maximum since we can make (x, y) arbitrarily close to $(0, 0)$. On the other hand, the region is bounded but not closed.

Combining the Extreme Value Theorem with the fact that local maximums occur at critical points, we have the following algorithm for computing absolute maximums (and also minimums by looking at $-f$ instead of f).

Algorithm to Find Absolute Maximums and Minimums: Assume $f : D \rightarrow \mathbb{R}$ is continuous and that D is closed and bounded.

- (1) Find all critical points interior to D .
- (2) Find the absolute maximum and minimum on the boundary of D .
- (3) The largest/smallest values for $f(x, y)$ from among (1) and (2) is the absolute maximum / absolute minimum of f on D .

Example 2.4.6. Let $f(x, y) := 2x - 2xy + y^2$ whose domain is defined by $D := \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$. Find the points where the global maximum / global minimum occur.

Solution: To find the critical points, we need to solve the equations

$$\begin{aligned} f_x(x, y) &= x - 2y = 0 \\ f_y(x, y) &= -2x + 2y = 0 \end{aligned}$$

Adding the two equations together implies $-x = 0$. So, $x = 0$. If $x = 0$, then $-2y = 0$ and $2y = 0$ which means $y = 0$. Therefore, the function has a single critical point at $(x, y) = (0, 0)$. This covers step (1).

To handle step (2), we notice that the boundary is the box determined by the lines $x = 0, x = 4, y = 0, y = 3$.

If $x = 0$, then $f(0, y) = y^2$ is the largest when $y = 3$ and smallest when $y = 0$.

If $x = 4$, then $f(4, y) = 8 - 8y + y^2 = y^2 - 8y + 8$. This is the smallest when $y = \frac{8}{2} = 4$. So in the range $0 \leq y \leq 3$, the largest $f(4, y)$ is occurs when $y = 0$ where $f(4, 0) = 8$ and the smallest $f(4, y)$ is occurs when $y = 3$ where $f(4, 3) = 9 - 24 + 8 = -7$.

If $y = 0$, then $f(x, 0) = 2x$. Assume $0 \leq x \leq 4$. It is greatest when $x = 4$ where $f(4, 0) = 8$ and smallest when $x = 0$ where $f(0, 0) = 0$.

If $y = 3$, then $f(x, 3) = 2x - 6x + 9 = 9 - 4x$. This is greatest when $0 \leq x \leq 4$ has $x = 0$ where $f(0, 3) = 9$ and it is smallest when $x = 4$ where $f(4, 3) = 9 - 16 = -7$.

Altogether, the critical point is $(0, 0)$, the maximum along the boundary is where $f(0, 3) = 9$, and the minimum along the boundary is where $f(4, 3) = -7$. Since $f(0, 0) = 0$, we know that the

- the global maximum is at $(0, 3)$ where $f(0, 3) = 9$,
- the global minimum is at $(4, 3)$ where $f(4, 3) = -7$.

□

Example 2.4.7. Let $f(x, y) := x^2 + y^2 - x - y$ whose domain is defined by $D := \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2\}$. Find the points where the global maximum / global minimum occur.

The previous examples are nice, but run into a serious issue – the boundary of the domains D are always “nice”. In particular, D was always some sort of rectangular region. It is quite difficult to deal with general compact regions D .

Another situation one is often interested is the following: find the extreme values of a function $f(x, y, z)$ subject to the constraint that $g(x, y, z) = K$ (with the technical assumption that $\nabla g(x, y, z) \neq 0$). We formulate this in three variables but everything said here works with two variables.

The Method of Lagrange Multipliers. Assume $\nabla g(x, y, z) \neq 0$. To find the extreme values of $f(x, y, z)$ subject to $g(x, y, z) = K$, one runs the following algorithm.

- (1) Solve for the values x, y, z, λ satisfying the system of equations

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$g(x, y, z) = K.$$

- (2) Plug the values found in the previous part into $f(x, y, z)$. The largest output is the absolute maximum value. The smallest output is the absolute minimum value.

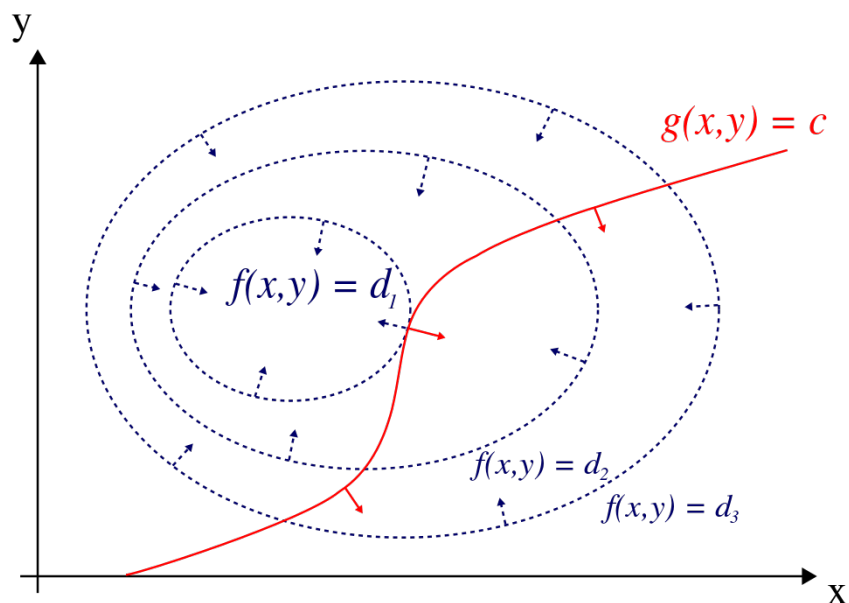


FIGURE 15. Image from [Wikipedia](#)'s article on Lagrange Multipliers. The idea of Lagrange multipliers can be phrased roughly as determining the points where the gradient of $f(x, y, z)$ and its constraint are parallel.

Remark 2.4.8. The equations in part (1) of the algorithm translate into:

$$f_x(x, y, z) = \lambda g_x(x, y, z) \ \& \ f_y(x, y, z) = \lambda g_y(x, y, z) \ \& \ f_z(x, y, z) = \lambda g_z(x, y, z) \ \& \ g(x, y, z) = K.$$

Remark 2.4.9. There is nothing special about formulating the method of Lagrange multipliers with three variables versus two.

Remark 2.4.10. We attempt to explain why the method of Lagrange multipliers works to the student. Since it is easier to visualize things in two dimensions, we drop the z variable in our explanation.

First off, we recall that $\nabla g(x, y)$ is always orthogonal to the level curve $g(x, y) = K$. Meanwhile, $\nabla f(x, y)$ always points in the direction of greatest growth. Now suppose one finds a point (x_0, y_0) where $f(x_0, y_0)$ is an absolute maximum on the level curve $g(x, y) = K$. Then, $\nabla f(x, y)$ should be pointing away from the curve because $f(x, y)$ will not grow larger than $f(x_0, y_0)$ if $f(x, y)$ stays along the curve. The condition that $\nabla f(x, y)$ has to point away from the curve is precisely the condition that $\nabla f(x, y)$ is parallel to $\nabla g(x, y)$. So, one is solving for $\nabla f(x, y) = \lambda \nabla g(x, y)$.

Now, the discussion above focused on maximum values, but it also applies equally well to minimum values by considering $-f(x, y)$ instead of $f(x, y)$. So, the solutions to $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = K$ are the local minimum and local maximum values along the level curve. The largest among these is the absolute maximum and the smallest among these is the absolute minimum.

Example 2.4.11. Find the extremal values of $f(x, y) = 3x + y$ subject to the condition $x^2 + y^2 = 10$.

Solution: If we write out the equations we need to solve, then we get the system

$$\begin{aligned} 3 &= \lambda 2x \\ 1 &= \lambda 2y \\ x^2 + y^2 &= 10. \end{aligned}$$

The first two equations imply that $x = \frac{3}{2\lambda}$ and $y = \frac{1}{2\lambda}$ provided $\lambda \neq 0$ (which of course cannot happen because that would contradict the first two equations). Plugging this into the equation $x^2 + y^2 = 10$, we get

$$\frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 10 \Rightarrow 10 = 40\lambda^2 \Rightarrow \lambda = \pm \frac{1}{2}.$$

If we plug these values of λ back into the equations $x = \frac{3}{2\lambda}$, $y = \frac{1}{2\lambda}$. So the possible points are $(\pm 3, \pm 1)$. The possible values for $f(x, y)$ in this case are (corresponding to $(3, 1)$, $(3, -1)$, $(-3, 1)$, and $(-3, -1)$ resp.)

$$\{10, 8, -8, -10\}$$

So the absolute maximum is 10 and absolute minimum is -10 . These are the extremal values. $\square$

Example 2.4.12. Find the absolute maximum of $f(x, y) = x^2 + y^3$ subject to the condition that $x + y = 20$.

Solution: The system of equations we need to solve is

$$\begin{aligned} 2x &= \lambda \\ 3y^2 &= \lambda \\ x + y &= 20. \end{aligned}$$

The first two equations imply $2x = 3y^2$ and so $x = \frac{3}{2}y^2$. Plugging this into the third equation gives you

$$\frac{3}{2}y^2 + y - 20 = 0.$$

Now you need to solve for y . It helps to multiply by $\frac{2}{3}$ to instead solve the quadratic

$$y^2 + \frac{2}{3}y - \frac{40}{3} = 0 \Rightarrow (y - \frac{10}{3})(y + 4) = 0$$

So, $y = \frac{10}{3}, -4$ and that means $x = \frac{50}{3}, 24$ respectively. So the possible points are $(\frac{50}{3}, \frac{10}{3})$ and $(-4, 24)$. Plugging these in

$$\begin{aligned} f(\frac{50}{3}, \frac{10}{3}) &= \frac{2500}{9} + \frac{1000}{27} = \frac{8500}{27} \approx 314. \\ f(24, -4) &= (24)^2 + (-4)^3 = 512. \end{aligned}$$

Altogether, 512 is the absolute maximum. $\square$

Example 2.4.13. Find the absolute maximum of $f(x, y) = x^4 + y^4$ subject to the condition that $x^2 + y^2 = 25$.

Solution: This is left as a challenging exercise for the student. $\square$

Example 2.4.14. Find the extremal values of $f(x, y, z) = x^3 + 2y^2 + 3z^2$ subject to $x^2 + y^2 + z^2 = 1$.

Solution: The system for the method Lagrange multipliers in this case is

$$3x^2 = 2\lambda x \quad \& \quad 4y = 2\lambda y \quad \& \quad 6z = 2\lambda z \quad \& \quad x^2 + y^2 + z^2 = 1.$$

Solving this system can be tricky. First off, $\lambda = 0$ cannot be possible since at least one of x, y , or z must be nonzero. So, assume $\lambda \neq 0$.

If $z \neq 0$, then equation three says that $\lambda = 3$. Plugging this into equation two, this means $y = 0$. Finally, $3x^2 = 2\lambda x$ becomes $3x^2 = 6x$ which means $x^2 - 3x = 0$ so $x = 0$ or $x = \sqrt{3}$. If $x = 0$, then $(0, 0, 1)$ is the point we get. If $x = \sqrt{3}$, then $x^2 = 3$ and $x^2 + y^2 + z^2 = 1$ is false. So, there are only two points where the system is true i.e. $(0, 0, \pm 1)$.

If instead $z = 0$, then one of x or y must be nonzero. Assume for now that $y \neq 0$. Then $\lambda = 2$ by the second equation. This means $3x^2 = 4x$ so either $x = 0$ or $x = \frac{4}{3}$. We can rule out $x = \frac{4}{3}$ since $x^2 = \frac{16}{9} > 1$ leading to a contradiction of the fourth equation $x^2 + y^2 + z^2 = 1$. Altogether, if $z = 0$ and $y \neq 0$, then $x = 0$ which means the points we get are $(0, \pm 1, 0)$.

By the same reasoning, if $z = 0$ and $x \neq 0$ were true instead, then $3x^2 = 2\lambda x$ forces $3x = 2\lambda$. Then, the second equation says $4y = 3xy$. There are two cases from here. If $y = 0$, then we get $(\pm 1, 0, 0)$. If $y \neq 0$, then $x = \frac{4}{3}$ which is a contradiction due to the fourth equation.

Given everything, we see that the points we must consider are $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$. One can check that values we get are $\{1, -1, 2, 2, 3, 3\}$ where the value -1 corresponds to $(-1, 0, 0)$ and the value 3 correspond to the two points $(0, 0, 1)$ and $(0, 0, -1)$. $\square$

Example 2.4.15. Find the point of the plane $x - 2y + 3z = 6$ closes to the point $(0, 2, 4)$.

Solution: Left as a very challenging exercise the student. $\square$

Example 2.4.16. Maximize the area of a rectangle that is inscribed in the ellipse $\frac{x^2}{4} + y^2 = 1$.

Solution: If (x, y) marks a the corner point of the rectangle in the 1st quadrant, then the area of the rectangle is $4xy$. So we are maximizing $4xy$ w.r.t. the constraint $\frac{x^2}{4} + y^2 = 1$.

Using Lagrange multipliers, one finds that $x = \sqrt{2}$ and $y = \frac{1}{\sqrt{2}}$ and so the maximal area is $4\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 4$.

Note that there are many ways to do solve the system arising from Lagrange multipliers. One should realize though that since we are maximizing the area of a rectangle, we may assume $x \neq 0$ and $y \neq 0$. $\square$

2.5. More on Lagrange Multipliers, and Double Integrals over Rectangular Regions. Before we proceed to the next topic, let's do one or two more Lagrange multipliers problems. Given that it is algorithmic in nature, students are strongly encouraged to do lots of practice problems so they have the algorithm ingrained in their mind for the exams.

Example 2.5.1. Find the extremal values of $f(x, y) = xy$ subject to $2x^2 + y^2 = 1$.

Solution: First, we need to solve the relevant system of equations

$$\begin{aligned}y &= \lambda 2x \\x &= \lambda 2y \\2x^2 + y^2 &= 1.\end{aligned}$$

The first equation implies $\lambda = \frac{y}{2x}$ and the second implies $\lambda = \frac{x}{2y}$ provided $x \neq 0$ and $y \neq 0$. So, $\frac{y}{2x} = \frac{x}{2y}$ and therefore $2y^2 = 2x^2$. This means $y^2 = x^2$. Plugging this into $2x^2 + y^2 = 1$, we get that $3x^2 = 1$ and so $x = \pm \frac{1}{\sqrt{3}}$. This means $y = \pm \frac{1}{\sqrt{3}}$ as well.

What if $x = 0$ or $y = 0$? We note that this is impossible. If $x = 0$, then $y = \lambda 2x = 0$ and so $2x^2 + y^2 = 1$ is false. Similarly, if $y = 0$, then $x = \lambda 2y = 0$ and so $2x^2 + y^2 = 1$.

Altogether, we have four relevant points to check $(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}})$. The function we are studying is $f(x, y) = xy$ and it is not hard to check that the maximum value and minium value are

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \frac{1}{3} \quad \& \quad f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -\frac{1}{3}.$$

$\square$

Example 2.5.2. Find the extremal values of $f(x, y) = xy + 14$ subject to the condition $x^2 + y^2 = 18$.

Solution: Left to the student. $\square$

2.6. *Lagrange Multipliers with Two Constraints. We will not use this in this course and you can safely not have to worry about this next material being on the exam: Lagrange multipliers with more than one constraint.

Algorithm for Lagrange Multipliers with Two Constraints. The following algorithm finds the extremal values of $f(x, y, z)$ subject to $g(x, y, z) = K$ and $h(x, y, z) = C$.

(1) First, solve for all x, y, z, λ, μ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

$$g(x, y, z) = K$$

$$h(x, y, z) = C.$$

(2) Plug these points into $f(x, y, z)$. The largest value is the maximum value and the smallest is the minimum value.

Example 2.6.1. Find the extremal values of $f(x, y, z) = z$ subject to the conditions $x - y + z = 1$ and $x^2 + 4y^2 = \frac{5}{4}$.

Solution: This is left as a challenging exercise for the curious student. $\square$

2.7. Double Integrals Over Rectangular Regions. By this point in time, we have enough to discuss integration in multiple variables. The student is advised to focus on the syntax of multiple integrals as opposed to why it is true. Recall that in single variable calculus, the perspective was that

$$\int_a^b f(x) dx$$

expressed the “area” under the curve $y = f(x)$. Under this perspective, one defined the integral as a limit of Riemann sums

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x_i.$$

Let us generalize this into two variables. First, $z = f(x, y)$ can be regarded as a “surface” in $\mathbb{R}^3$. Then, $\iint_R f(x, y) dA$ should really be giving us the volume of what is underneath the surface $f(x, y) = z$ over the region R . In this case, one can define

$$\iint_R f(x, y) dA = \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j) \Delta A_{ij}.$$

These definitions are not particularly useful for computations. Fubini’s Theorem is a statement that allows us to actually get things done. The advice for the student here is to just use Fubini’s Theorem freely whenever dealing with rectangular regions.

Theorem 2.7.1 (Fubini’s Theorem). Assume $f(x, y)$ is continuous on the rectangle

$$R := \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}.$$

Then,

$$\iint_R f(x, y) dA = \int_c^d \left(\int_a^b f(x, y) dx \right) dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

i.e. one can integrate w.r.t. x and then integrate w.r.t. y (or vice-versa). Note that the second equality is special to rectangular regions. One calls the expressions on the right the associated **iterated integrals** for $\iint_R f(x, y) dA$.

Example 2.7.2. Find the value of $\iint_R 2x + y^2 dA$ for $R = [1/2, 1] \times [7/6, 2]$.

Solution: By Fubini's Theorem, we can first compute the integral w.r.t. x :

$$\int_{1/2}^1 2x + y^2 dx = [x^2 + xy^2]_{1/2}^1 = (1 + y^2) - (1/4 + y^2/2).$$

Then we put this into an integral in terms of y

$$\int_{7/6}^2 (1 + y^2) - (1/4 + y^2/2) dy = \left(\frac{3}{4}\right) \left(2 - \frac{7}{6}\right) + \int_2^{7/6} \frac{1}{2} y^2 dy.$$

Since $\frac{1}{2}y^2 = \frac{1}{6}y^3$, the integral above gives me $\frac{1}{6} \left(\frac{7}{6}\right)^3 - \frac{1}{6} (2)^3$. So the final answer is

$$\left(\frac{3}{4}\right) \left(2 - \frac{7}{6}\right) + \frac{1}{6} \left(\frac{7}{6}\right)^3 - \frac{1}{6} (2)^3$$

□

Remark 2.7.3. A useful for technique for evaluating iterated integrals is the following formula

$$\int_a^b \int_c^d f(x)g(y) dx dy = \int_a^b f(x) dx \int_c^d g(y) dy.$$

Example 2.7.4. Evaluate $\iint_R 2xe^{3y} dA$ for

$$R := \{(x, y) \mid 1 \leq x \leq 2, 0 \leq y \leq \ln(7)\}.$$

Solution: The double integral can be written as

$$\int_1^2 \int_0^{\ln(7)} 2xe^{3y} dy dx.$$

Since x, y are independent variables, we can write this integral as

$$\int_1^2 2x dx \cdot \int_0^{\ln(7)} e^{3y} dy = (2^2 - 1^2) \cdot \left(\frac{1}{3}e^{3\ln(7)} - \frac{1}{3}e^{3 \cdot 0}\right) = 3 \cdot \left(\frac{1}{3}(7^3 - 1)\right) = 7^3 - 1.$$

□

Example 2.7.5. Evaluate $\iint_R 1 dA$ where

$$R := \{(x, y) \mid 1 \leq x \leq 3, 2 \leq y \leq 7\}.$$

Solution: The double integral computes the area of the rectangle R . The rectangle has length 2 and width 5 and so the area is 10. □

Example 2.7.6. Find the volume under the cone $x^2 + y^2 = z$ over the rectangle

$$R := \{(x, y) \mid 1 \leq x \leq 3, 2 \leq y \leq 5\}.$$

Solution: In this case, we are evaluating

$$\int_2^5 \int_1^3 x^2 + y^2 dx dy.$$

First off,

$$\int_1^3 x^2 + y^2 dx = \left.\frac{1}{3}x^3 + xy^2\right|_1^3 = (9 + 3y^2) - \left(\frac{1}{3} + y^2\right) = \frac{26}{3} + 2y^2.$$

So,

$$\int_2^5 \int_1^3 x^2 + y^2 \, dx \, dy = \int_2^5 \frac{26}{3} + 2y^2 \, dy = 26 + \int_2^5 2y^2 \, dy = 26 + \left(\frac{2}{3}5^3 - \frac{2}{3}2^3\right).$$

□

Example 2.7.7. Find the volume under the paraboloid $x^2 + y^2 + z = 4$ over the rectangle

$$R := \{(x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Solution: The volume under the paraboloid is

$$\int_0^1 \int_{-1}^1 4 - x^2 - y^2 \, dx \, dy = \frac{20}{3}.$$

□

Example 2.7.8. Compute the following integrals

- (1) $\int_1^3 \int_0^{\pi/2} x + x \sin(y) \, dy \, dx$
- (2) $\int_2^3 \int_1^5 xye^y \, dy \, dx$
- (3) $\int_1^3 \int_{\pi}^{\pi^3} \cos(x) + y^2 \, dx \, dy.$

Hint: $\int_R g(x)h(y) \, dA = \int_a^b g(x) \, dx \int_c^d h(y) \, dy.$

Solution: These problems are left to the student for practice.

□

Definition 2.7.9. The **average value** of $f(x, y)$ over a bounded region R is defined to be

$$\frac{1}{\text{area}(R)} \iint_R f(x, y) \, dA.$$

Example 2.7.10. What is the average value of $f(x, y) = 2x + y$ over $R = \{(x, y) \mid 1 \leq x \leq 5, 0 \leq y \leq 4\}$?

Example 2.7.11. What is the average value of $f(x, y) = 4x^2y$ over $R := \{(x, y) \mid 0 \leq x \leq 4, 2 \leq y \leq 3\}$?

Solution: The area of the rectangle in this case is 16. So,

$$\begin{aligned} \text{avg}_R(f) &= \frac{1}{16} \iint_R 2x + y \, dx \, dy = \frac{1}{16} \int_0^4 \int_1^5 2x + y \, dx \, dy = \frac{1}{16} \int_0^4 24 + 4y \, dy \\ &= \frac{1}{16} \left(24y + 2y^2 \Big|_0^4 \right) = \frac{1}{16} (96 + 38) = \frac{134}{16}. \end{aligned}$$

□

Example 2.7.12. Compute the average value of $f(x, y) = xy + e + xy^2$ over the rectangle

$$R := \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Solution: We did this one in class (and there was a typo on the board for those that were paying attention). First off, the area of R is 1 since it is the unit square. So we only need to compute the double integral. In that case,

$$\iint_R e + xy + xy^2 dA = \int_0^1 \int_0^1 e + x(y + y^2) dx dy.$$

Integration is linear so distributes over addition. So,

$$\int_0^1 \int_0^1 e + x(y + y^2) dx dy = \int_0^1 \int_0^1 e dx dy + \int_0^1 \int_0^1 x(y + y^2) dx dy.$$

The first double integral equals e since its the volume under the plane $z = e$ over the unit square. The second equals

$$\int_0^1 x dx \int_0^1 y + y^2 dy = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} \right).$$

So, the final answer is

$$e + \frac{1}{4} + \frac{1}{6} = e + \frac{5}{12}.$$

□

Solution: The integral can be set up as

$$\int_0^1 \int_{\sqrt{x}}^{x^2} xy dy dx = -\frac{1}{12}.$$

□

2.8. Double Integrals over General Regions.

Example 2.8.1 (Motivating Example). Find $\iint_D xy dA$ where D is the region bounded by $y = x^2$ and $y = \sqrt{x}$.

Solution: Draw a picture of the region D . We claim that the double integral can be written as

$$\int_0^1 \int_{x^2}^{\sqrt{x}} xy dy dx = \int_0^1 x \left(\frac{x}{2} - \frac{x^4}{2} \right) dx = \int_0^1 \frac{x^2}{2} - \frac{x^5}{2} dx = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}.$$

□

Question: How does one set up integrals over general regions?

Answer: Often in more than one way and in the most general setting, it is almost impossible to find a good way to write it down. We shall stick to “nice” situations in this class, however.

Type I Regions. Suppose we have a region

$$D := \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

Then the integral can set up as

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

Type II Regions. Suppose we have a region

$$D := \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}.$$

Then the integral can be set up as

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

Example 2.8.2. Evaluate $\iint_D 3y^2 x dA$ where D is the region bounded by $x = \sqrt{y}$, $x = -y$, $y = 0$, and $y = 1$.

Solution: The student is encouraged to draw a picture. One can describe the region D as

$$D = \{(x, y) \mid -y \leq x \leq \sqrt{y}, 0 \leq y \leq 1\}.$$

This is a type II region and so we can set this up as

$$\int_0^1 \int_{-y}^{\sqrt{y}} 3y^2 x dx dy.$$

One can evaluate this double integral via an iterated integral. First, the inner integral is

$$\int_{-y}^{\sqrt{y}} 3y^2 x dx = \frac{3}{2} y^2 x^2 \Big|_{-y}^{\sqrt{y}} = \frac{3}{2} (y^3 - y^4).$$

Then, the double integral equals

□

Example 2.8.3. Consider the triangular region T enclosed by the points $(0, 0)$, $(2, 1)$ and $(2, 0)$. Compute the integral $\iint_T x + y dA$.

Solution: The right triangle has hypotenuse given by $y = x - 1$. So we can set up the integral as

$$\int_0^2 \int_0^{x-1} x + y dy dx.$$

This evaluates to

$$\begin{aligned} \int_0^2 [xy + \frac{y^2}{2}]_0^{x-1} dx &= \int_0^2 x(x-1) + \frac{(x-1)^2}{2} dx = \int_0^2 x^2 - x + \frac{x^2}{2} - x + \frac{1}{2} dx \\ &= \frac{2^3}{3} - \frac{2^2}{2} + \frac{2^3}{6} - \frac{2^2}{2} + \frac{2}{2} = 1. \end{aligned}$$

□

Example 2.8.4. Evaluate $\iint_D e^{y^2} dA$ where D is the region bounded by $y = 2x$, $y = 2$, and $x = 0$ by writing it as either a type I integral or a type II integral. Which one is actually doable?

Solution: The integral that is actually doable is the one given by

$$\int_0^2 \int_0^{y/2} e^{y^2} dx dy = \int_0^2 \frac{y}{2} e^{y^2} dy \stackrel{u:=y^2}{=} \int_0^4 \frac{1}{4} e^u du = \frac{1}{4}(e^4 - e^0)$$

since $du = 2y dy$. Notice that setting up the problem the other way i.e.

$$\int_0^1 \int_{2x}^2 e^{y^2} dy dx$$

is not a feasible way to compute the integral. The issue is that one cannot write down $\int e^{y^2} dy$ using only elementary functions. $\square$

Example 2.8.5. Compute $\iint_D 4 dA$ where D is the disc of radius 1 centered at the origin in three different ways (a) using the fact you know the area of D , and (b) by writing it as a type I region.

Solution: (a) The area of D is πr^2 and $r = 1$. So, the double integral equals 4π .
(b) As a type I region, we can write D as

$$D := \{(x, y) \mid -1 \leq x \leq 1 \mid -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}.$$

So, we have that the double integral equals

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 4 dy dx = \int_{-1}^1 8\sqrt{1-x^2} dx.$$

This is a problem that you can really only solve (so far) by setting $x := \sin \theta$. Then $dx = \cos \theta d\theta$ and therefore, this integral equals

$$8 \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2 \theta} \cos \theta d\theta.$$

Then, $1 - \sin^2 \theta = \cos^2 \theta$ and the square root equals $\cos \theta$. So, the integral becomes

$$8 \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta.$$

We can rewrite $\cos^2 \theta = \frac{\cos(2\theta)+1}{2}$. So, we get

$$8 \int_{-\pi/2}^{\pi/2} \frac{\cos(2\theta)+1}{2} d\theta = 8 \left(\frac{\sin(2\theta)}{4} + \frac{1}{2}\theta \right) \Big|_{-\pi/2}^{\pi/2} = 4\pi.$$

$\square$

We've already been using the properties of single variable integrals. It would be of value to make it clear what properties transfer over to double integrals. More or less, everything transfers over:

- (1) $\iint_D (-) dA$ is linear
- (2) $\iint_D (-) dA$ preserves inequalities,
- (3) if $D = D_1 \sqcup D_2$, then $\iint_D (-) dA = \iint_{D_1} (-) dA + \iint_{D_2} (-) dA$
- (4) $\iint_D dA = \text{area}(D)$,
- (5) if $m \leq f(x, y) \leq M$, on D , then $\text{area}(D)m \leq \iint_D f(x, y) dA \leq \text{area}(D)M$.

A useful tool of for a lot of problems is to change the order of integration. Often times, trying to solve an integral like $\iint_D f(x, y) dx dy$ won't work nicely so rewriting it as $\iint_D f(x, y) dy dx$ makes it easier.

Example 2.8.6. Evaluate $\int_0^1 \int_x^1 e^{y^2} dy dx$.

Solution: If we change the order of integration, we get

$$\int_0^1 \int_0^y e^{y^2} dx dy = \int_0^1 y e^{y^2} dy = \left(\frac{1}{2} e^{y^2} \right)_0^1 = \frac{1}{2} (e - 1).$$

□

Example 2.8.7. Set up what happens if you change the order of integration $\int_0^1 \int_y^{3y} e^{x^2} dx dy$.

Solution: If one changes the order of integration, one needs to have two integrals involved. Then the answer is

$$\int_0^1 \int_{(1/3)x}^x e^{x^2} dy dx + \int_1^3 \int_{(1/3)x}^2 e^{x^2} dy dx$$

This is not an integral that one can readily compute because there is no method that would allow you to compute the second integral. However, the first integral above is computable – so you know a lower bound for $\int_0^1 \int_y^{3y} e^{x^2} dx dy$. □

Example 2.8.8. Set up the integral $\iint_T f(x, y) dA$ for the region T which is the triangle bounded by the points $(0, 0)$, $(2, 0)$, and $(1, 1)$.

Solution: It is easier to let y be the variable of integration for the outer integral. Then,

$$\int_0^1 \int_y^{y-2} f(x, y) dx dy$$

□

Example 2.8.9. Find the volume of the solid under the surface $z = xy$ and above the triangle with corners $(1, 1)$, $(4, 1)$, and $(1, 2)$.

Solution: This is left as an activity for the student. □

Example 2.8.10. Change the order of integration of

$$\int_1^2 \int_0^{\ln(x)} f(x, y) dy dx.$$

Solution: This is left as an activity for the student. □

Example 2.8.11. Setup the double integral $\iint_T f(x, y) dx dy$ for the trapezoid bounded by the lines $y = 1, y = 2, y = x + 5, y = -x + 10$ so that one only has a single double integral to compute.

Solution: It is easier to let y be the variable of integration for the outer integral. In that case,

$$\int_1^2 \int_{y-5}^{10-y} f(x, y) dx dy$$

is the correct set-up. □

As a short summary, here are the main points:

- (1) Depending on the region, letting y be the variable of integration for the outer integral may be more advantageous than letting x be the variable of integration.
- (2) It helps to draw arrows $\rightarrow$ for dy and dx to help one orient which curves serve as the upper bound or lower bound for the bounds of integration.
- (3) The regions in which it is “easier” to set-up are almost always one of the following: triangles, regions bounded between two curves and with easy cutoffs, trapezoids with its parallel sides parallel to one of the axes.

A region we have not attempted to set-up an integral for is the disk/filled in circle. We will do this next time using polar coordinates.

2.9. Double Integrals in Polar Coordinates. Next we discuss how to integrate using polar coordinates. First we start with a refresher on polar coordinates.

Definition 2.9.1. The **polar coordinate system** is defined by the formulas. The point (x, y) in the xy -plane are assigned to a point $P(r, \theta)$ given by $r \geq 0$ and $\theta \in \mathbb{R}$ by the formulas

$$x = r \cos \theta \quad \& \quad y = r \sin \theta.$$

It is important to note that if $r = 0$, then the point $P(0, \theta)$ is the origin.

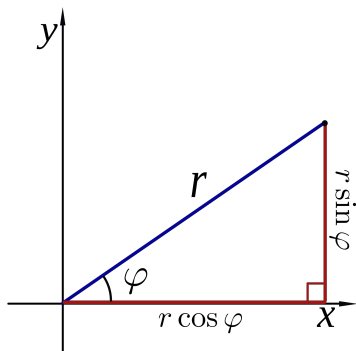


FIGURE 16. Image taken from [Wikipedia](#). The image describes how to convert from Cartesian coordinates (x, y) to polar coordinates (r, φ) .

Example 2.9.2. Plot the points with the specified polar coordinates.

- (1) $(3, \frac{5\pi}{3})$,
- (2) $(2, -\frac{\pi}{4})$,
- (3) $(3, 0)$.

Example 2.9.3. Convert into polar coordinates / rectangle coordinates from the point in the specified coordinate system.

- (1) Polar $(3, \pi/4)$
- (2) Rectangular $(2, 0)$
- (3) Polar $(-2, \frac{3\pi}{4})$
- (4) Rectangular $(-1, -\sqrt{3})$

Graphing using polar coordinates can get somewhat trickier, so we consider some examples.

Example 2.9.4. Graph the following using polar coordinates. A description of the graph is next to them.

- (1) $\theta = \frac{\pi}{3}$ is a line through the origin with angle $\frac{\pi}{3}$ to the x -axis.
- (2) $r \sin \theta = 1$ is the horizontal line where $y = 1$.
- (3) $r = \sin \theta$ is a circle when one converts it into rectangular coordinates.
- (4) $r = (\cos \theta)(\sin \theta)$ looks like a flower with four petals.

Example 2.9.5. Graph the following functions in polar coordinates.

- (1) $r = 3$,
- (2) $r \sin \theta = 3$,
- (3) $r = \cos(2\theta)$.

Recall from above that we can convert Cartesian coordinates into polar coordinates via $x = r \cos \theta$ and $y = r \sin \theta$. The natural thing to do now is to ask how to integrate with respect to these new coordinates.

Last time, we say that

$$\iint_D 4 \, dA = 4\pi$$

where $D := \{(x, y) \mid x^2 + y^2 \leq 1\}$. We can also redo the computation using polar coordinates.

So,

$$\iint_D 4 \, dA = \int_0^{2\pi} \int_0^1 4r \, dr \, d\theta = \int_0^{2\pi} 2 \, d\theta = 4\pi.$$

More generally, if

$$D := \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

then

$$\iint_D f(x, y) \, dA = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.$$

Example 2.9.6. Evaluate

$$\int_0^2 \int_0^{\sqrt{4-x^2}} x^2 + y^2 \, dy \, dx.$$

Solution: It is best to swap to polar coordinates. In this case, we can rewrite it as

$$\int_0^{\pi/2} \int_0^2 r^2 \cdot r \, dr \, d\theta = \int_0^{\pi/2} 2 \, d\theta = \pi.$$

□

Example 2.9.7. Find the volume of the solid under the paraboloid $z = 12 - 3y^2 - 3x^2$ and above the xy -plane.

Solution: One can set this up as follows. When $z = 0$, we are looking at the equation $3x^2 + 3y^2 = 12$ and so the intersection of the paraboloid with the xy -plane is the circle $x^2 + y^2 = 4$. So, we can set-up the integral producing the volume as

$$\int_0^{2\pi} \int_0^2 (12 - 3r^2) r \, dr \, d\theta.$$

□

Example 2.9.8. Consider the annulus $D := \{(x, y) \mid 2 \leq x^2 + y^2 \leq 5\}$. Evaluate

$$\iint_D \exp(x^2 + y^2) dA.$$

We can also setup double integrals for regions of the form

$$D := \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}.$$

The integral becomes

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

Remark 2.9.9. It is important that the student always draws a picture. Also, remember that it is $r dr d\theta$ and not just $dr d\theta$. We will provide more general explanation for why this r factor appears will be discussed later.

Example 2.9.10. Find the area of the cardioid bounded by $r = 2(1 - \cos \theta)$. Hint: Recall that $\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$.

Example 2.9.11. Find the area of the region in the first quadrant inside $r = 3 + 2 \sin \theta$ and outside $r = 4 \sin \theta$.

Example 2.9.12. Evaluate

$$\iint_D \sin(3x^2 + 3y^2) dA$$

where D is the region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 7$.

Solution: We leave evaluating the integral in polar coordinates to the reader. If one sets this up in polar coordinates, one can write

$$\int_0^{2\pi} \int_1^{\sqrt{7}} \sin(3r^2) r dr d\theta.$$

□

Example 2.9.13. Use a double integral to find the volume of the solid inside both the cylinder $x^2 + y^2 = 9$ and the sphere $x^2 + y^2 + z^2 = 16$ and above the $z = 0$ plane.

Solution: We consider $z = \sqrt{16 - x^2 - y^2}$. One can set-up the double integral as

$$\int_0^{2\pi} \int_0^3 \sqrt{16 - r^2} r dr d\theta.$$

□

Example 2.9.14 (Fun Fact). Using polar coordinates, one can show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. The idea is to square the integral which gives you

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2 - y^2} dA.$$

You can rewrite this into polar coordinates as

$$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi.$$

So, $\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \pi$ and taking the square root gives you the claim.

Example 2.9.15 (Challenge). All the way back in Example 1.2.10 we were given the problem of computing the area of the region formed by intersecting two circles whose centers sit on a common line. Using polar coordinates, this is a lot easier to do now and we illustrate this in an example.

Let C_1 denote the unit circle centered at the origin.

Let C_2 denote the unit circle centered at the point $(1, 0)$.

What is the area of the bounded region given by these two circles?

Solution: This is left as a challenging exercise for the student. $\square$

2.9.1. *Applications of Double Integrals.* We illustrate one applications of double integrals. These will not be covered in class nor tested on.

Let $\rho(x, y)$ be a density function and D a region. Then $\rho(x, y)$ can be viewed as associating the density at a point of D . Anyway, one can then interprety the mass of D as the double integral

$$\text{mass}(D) = \iint_D \rho(x, y) dA.$$

For a physical interpretation, one can imagine D is some plate made out of an allow where the density varies throughout different points of the plate.

One can then attempt to determine the center of mass. In this case, there is a pretty precise answer: the center of mass is at the point $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{M} \iint_D x\rho(x, y) dA \quad \& \quad \bar{y} = \frac{1}{M} \iint_D y\rho(x, y) dA.$$

Example 2.9.16. Let D be the triangle given by $(0, 0)$, $(1, 0)$, and $(1, 1)$. If D has density $\rho(x, y) = 1 + 6x + 6y$, then find the center of mass of D .

Solution: $\bar{x} \approx 0.738$, $\bar{y} \approx 0.405$. $\square$

2.10. **Triple Integrals.** Let's now discuss triple integrals. We already discussed double integrals and what occurs for triple integrals is very much analogous.

First, we should ask about how to describe

$$\iiint_E f(x, y, z) dV.$$

Everything we did for double integrals carries over: Fubini's Theorem, changing order of integration, linearity of the integral, etc.. One could make a picture akin to how we imagine double integrals as the volume under the function $f(x, y)$. However, this is not as helpful since we cannot really imagine "hypervolumes"³ very well as 3-dimensional beings

Example 2.10.1. Evaluate $\iiint_B 4xyz dV$ where B is the box

$$B = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 3\} = [0, 2] \times [0, 1] \times [0, 3].$$

³I am also not convinced hypervolume is the term used. If anyone knows the correct word that fits next into the pattern "length, area, volume,..." with a proper justification on why that's the correct word, then please let me know.

Solution: We set up the triple integral

$$\int_0^3 \int_0^1 \int_0^2 4xyz \, dx \, dy \, dz = 4 \left(\frac{2^2}{2} \right) \left(\frac{1^2}{2} \right) \left(\frac{3^2}{2} \right).$$

□

Example 2.10.2. Set-up and evaluate the triple integral

$$\iiint_B x^2 y^2 z^2 \, dV$$

where B is the box given by $[0, 1] \times [0, 3] \times [0, 5]$.

Solution: Setting it up is easy – $\int_0^5 \int_0^3 \int_0^1 x^2 y^2 z^2 \, dV$. Evaluating it is just annoying – the answer is

$$\frac{5^3}{3} \frac{3^3}{3} \frac{1^3}{3}.$$

□

Example 2.10.3. Evaluate

$$\int_0^{\pi/2} \int_1^{2\pi} \int_{\ln 2}^{\ln 3} e^x \cdot \sin y \cdot \cos(z) \, dx \, dy \, dz.$$

Solution: This is left as an exercise for the student. Observe that this is a product of functions in terms of the variables x, y, z respectively. □

In the previous two examples, we focused on rectangular regions. For more general regions, things get vastly more complicated. Here is a strategy that one can try to use.

- (1) Draw E and project E onto the xy -plane. Call the region D .
- (2) Parameterize D .
- (3) Find the upper and lower functions which bound the z -values of E which we call $u_2(x, y)$ and $u_1(x, y)$.
- (4) In this case, one has that

$$\iiint_E f(x, y, z) \, dV = \iint_D \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \, dA.$$

Notice that there is nothing special with the xy -plane. Any of the two other coordinate planes (i.e. given by $x = 0$ and $y = 0$) can be used for this purpose.

Example 2.10.4. Evaluate the integral

$$\int_0^1 \int_0^{1-x} \int_0^{1-x+y^2} dz \, dy \, dx.$$

Can you imagine what is the solid E we are integrating over here is?

Solution: We are finding the volume of the solid under $1 - x + y^2$ over the triangle with corners $(0, 0)$, $(1, 0)$, $(0, 1)$ in the xy -plane. □

Example 2.10.5. Let E be the tetrahedron formed by the coordinate planes and the plane $x + y + z = 1$. Then evaluate $\iiint_E dV$.

Solution: Following the outlined strategy, we get

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx = \frac{1}{6}.$$

□

Example 2.10.6. Let E be the tetrahedron formed by the coordinate planes and the plane $2x + 3y + 4z = 12$. Then evaluate the integral $\iiint_E 3 dV$.

Solution: When $z = 0$, we have the line $2x + 3y = 12$. We can also solve for z to get $z = \frac{1}{4}(12 - 2x - 3y)$. What we get then is the triple integral

$$\int_0^6 \int_0^{(1/3)(12-2x)} \int_0^{(1/4)(12-2x-3y)} 3 dz dy dx$$

□

Example 2.10.7 (General Strategy for Tetrahedrons bounded by coordinate planes and another plane). Suppose I give you the equation of a plane $4x + 2y + z = 1$ and ask you to compute the solid of the tetrahedron bounded by the coordinate planes and this plane as before.

Every plane is determined by 3 points. What are some of them? Well, this plane contains the points $(1/4, 0, 0)$, $(0, 1/2, 0)$, and $(0, 0, 1)$. So this helps me set-up the integral as follows. First, we write $z = 1 - 4x - 2y$. Then,

$$\int_0^{1/4} \int_0^{(1/2)(1-4x)} \int_0^{1-4x-2y} dz dy dx.$$

Note that this works a lot better when the numbers are nice.

Example 2.10.8. Setup a triple integral that determines the volume of the region below $z = 4 - xy$ and sits above the region $0 \leq x \leq 2$ and $0 \leq y \leq 1$.

Solution: $V = \int_0^2 \int_0^1 \int_0^{4-xy} dz dy dx$.

□

Example 2.10.9. Evaluate $\iiint_E yz dV$ where E is the region bounded by $x = 2y^2 + 2z^2 - 5$ and the plane $x = 1$.

Solution: The way to do it is $\iint_D \int_{2y^2+2z^2-5}^1 dx dA$. Here, D is the intersection of $x = 1$ and $x = 2y^2 + 2z^2 - 5$ i.e. it is a circle in the plane $x = 1$ given by $2y^2 + 2z^2 = 6$. We can rewrite this as $y^2 + z^2 = 3$ and realize that it is easier to use polar coordinates to parameterize this. Indeed,

$$\int_0^{2\pi} \int_0^3 \int_{2r^2-5}^1 r dx dr d\theta.$$

We leave the rest of the computation of this triple integral to the reader.

□

Example 2.10.10. Set up a triple integral that gives the volume of E the solid in the first octant ($x \geq 0, y \geq 0, z \geq 0$) enclosed by $z = 4 - 2x^2 - y^2$ and the xy -plane. Here, D is the circle of radius $\sqrt{3}$ so one sets up D in polar coordinates to get

$$\int_0^{2\pi} \int_0^{\sqrt{3}} [6 - 2r^2](r \sin \theta r \cos \theta) r dr d\theta$$

by setting $y = r \sin \theta$ and $z = r \cos \theta$. One can quickly compute this integral and get 0 as the answer.

Example 2.10.11. Set-up but do not evaluate the volume of the region below $z = 8 - x^2 - y^2$ and above $z = -\sqrt{4x^2 + 4y^2}$ and inside $x^2 + y^2 = 4$. Make sure to try setting it up using polar coordinates.

Solution: Again, we shall use polar coordinates to do this. In this case, we get

$$\int_0^{2\pi} \int_0^2 \int_{-\sqrt{4r^2}}^{8-r^2} r dz dr d\theta$$

□

Notice that there is nothing special about using the z -values for setting up the integral. Here's another example.

Example 2.10.12. Find the volume of the solid E enclosed by $y^2 + z^2 = 4$, $x = 0$, $z = 0$, and $x + z = 5$.

Solution: In Cartesian coordinates, this can be set-up as

$$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_0^{5-z} dx dz dy.$$

Using polar coordinates, $y = r \cos \theta$, $z = r \sin \theta$, one can set this up as

$$\int_0^\pi \int_0^2 \int_0^{5-r \cos \theta} r dz dr d\theta$$

□

2.11. Cylindrical Coordinates, and Spherical Coordinates. To convert x, y, z coordinates into cylindrical / polar coordinates, one sets $x = r \cos \theta$, $y = r \sin \theta$ and leaves z alone. We were already doing this last lecture.

For example, in polar coordinates, $r = 3$ cuts out an infinitely long cylinder with radius 3 centered at the origin. One can rewrite many surfaces using cylindrical coordinates.

Surface	Rectangular	Cylindrical
Plane	$ax + by + cz + d = 0$	
Cylinder	$x^2 + y^2 = r^2$	$r = \text{radius}$
Sphere	$x^2 + y^2 + z^2 = R^2$	$r^2 + z^2 = R^2$
Circular Cone	$z^2 = x^2 + y^2$	$z^2 = r^2$

The trick for rewriting almost all of these equations into cylindrical coordinates is to keep in mind that $x^2 + y^2$ becomes r^2 due to the Pythagorean identity.

One should also note that when changing coordinates, $dx dy dz$ transforms into $r dr d\theta dz$.

Example 2.11.1. Set up a the triple integral that is the volume of the solid E bounded by

$$x^2 + y^2 + z^2 = 4$$

for $z \geq 0$ and the cylinder $x^2 + y^2 = 4$.

Solution: We have a sphere and a cylinder. We can rewrite the equation of the sphere into two equations $z = \pm\sqrt{4 - x^2 - y^2}$ and this will give the bounds for z . We can set-up the bounds for a circle as usual and this would give the bounds

$$\int_0^2 \int_0^{2\pi} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

□

Example 2.11.2. Setup integrals representing volumes of the following solids

- 1) Solid enclosed by planes $z = 1, z = 4$, and cylinders $x^2 + y^2 = 1$, and $x^2 + y^2 = 9$.
- 2) Solid enclosed by intersecting the sphere $x^2 + y^2 + z^2 = 9$ and the cylinder $x^2 + y^2 = 2$.
- 3) Solid above the plane $z = 0$, below the cone $z = \sqrt{4x^2 + 4y^2}$ and on the inside of the cone $x^2 + y^2 = 9$.

Solution: All three are left as an exercise for the student.

□

Example 2.11.3. Use cylindrical coordinates to compute the volume of a cylinder with height h and radius R .

Solution: In cylindrical coordinates, this can be set-up as

$$\int_0^{2\pi} \int_0^R \int_0^h r \, dz \, dr \, d\theta = h \frac{R^2}{2} 2\pi = \pi R^2 h.$$

□

Example 2.11.4. Use cylindrical/polar coordinates to compute the integral

$$\iint_D e^{x^2+y^2} \, dA$$

where D is the solid disk

$$D := \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

Solution: If we swap to polar coordinates, then $x^2 + y^2 = r^2$ and dA becomes $r \, dr \, d\theta$. So the double integral equals

$$\int_0^1 \int_0^{2\pi} e^{r^2} r \, d\theta \, dr = 2\pi \int_0^1 e^{r^2} r \, dr \stackrel{u=r^2}{=} \pi \int_0^1 e^u \, du = \pi(e^1 - e^0) = \pi(e - 1).$$

□

There is a third coordinate system that becomes very useful in 3 dimensions. It is called **spherical coordinates**. In this situation, one sets

$$x := \rho \sin(\phi) \cos(\theta) \quad \& \quad y := \rho \sin(\phi) \sin(\theta) \quad \& \quad z := \rho \cos(\phi)$$

and the spherical coordinates take in ρ, θ, ϕ . Refer to Figure 17 for a visual explanation of why this is how x, y, z transform.

Example 2.11.5. Draw the following objects in $\mathbb{R}^3$ using spherical coordinates.

- (1) $\rho = 1$,
- (2) $\theta = 1$,
- (3) $\phi = c$ where $0 < c < \frac{\pi}{2}$,
- (4) $\phi = c$ where $\frac{\pi}{2} < c < \pi$.

There are three key points about spherical coordinates the student should understand:

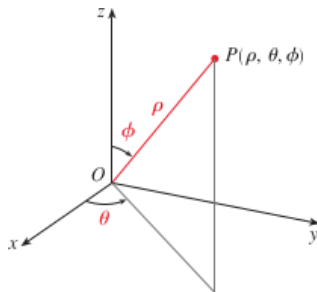


FIGURE 17. ρ is for “radius”, θ measures the angle from the x -axis, and ϕ measures the angle from the z -axis. In the literature, some people refer to ϕ as the angle of depression.

- $x^2 + y^2 + z^2 = \rho^2$,
- $0 \leq \phi < \pi$,
- the Jacobian of the transformation equals

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = \rho^2 \sin \phi.$$

Example 2.11.6. Check that the Jacobian is as claimed from before.

In this case, we can rewrite triple integrals where

$$E := \{(\rho, \theta, \phi) \mid \alpha \leq \theta \leq \beta, e \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\}$$

as

$$\iint_E f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta, \phi)}^{g_2(\theta, \phi)} \rho^2 \sin(\phi) d\rho d\theta d\phi.$$

Example 2.11.7. Use spherical coordinates to compute the volume of a solid sphere with radius 1. Note: The ancient Greeks knew the answer to be $\frac{4}{3}\pi$ in this case.

Solution: This is straightforward – the volume equals the value of the following integral

$$\int_0^\pi \int_0^{2\pi} \int_0^1 \rho^2 \sin(\phi) d\rho d\theta d\phi = \frac{4}{3}\pi.$$

□

Example 2.11.8. Set up the integral for the volume of the solid E bounded by the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$ and sitting above the xy -plane.

Solution: $\int_0^{2\pi} \int_0^{\pi/2} \int_1^{\sqrt{2}} \rho^2 \sin \phi d\rho d\phi d\theta$.

□

Example 2.11.9. Convert $\iiint_E z dV$ into spherical coors where E is the solid defined by inequalities $0 \leq \sqrt{x^2 + y^2} \leq z$ and $0 \leq x^2 + y^2 + z^2 \leq 1$.

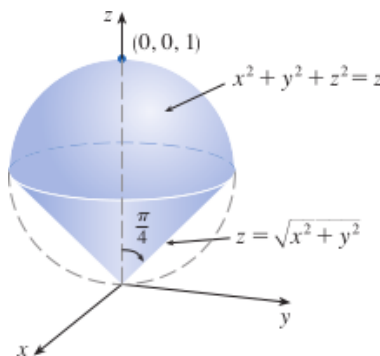
Example 2.11.10. Use an integral to compute the volume of the solid cone enclosed by $\sqrt{x^2 + y^2} = z$, and $z = 3$, and $z = 0$.

Solution: $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\frac{3}{\cos \phi}} \rho^2 \sin \phi d\rho d\phi d\theta$

The answer can be determined by considering the yz -plane and how it cuts through the cone. It is not hard to see that the radius has $y = z$ and is given by $\frac{3}{\cos(\phi)}$.

□

Example 2.11.11. Compute the volume of the solid drawn below:



Solution: The answer is $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta = \frac{\pi}{8}$. One can describe the solid E as

$$E = \{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4, 0 \leq \rho \leq \cos \phi\}.$$

The student might be curious on how to figure this out, so we provide some explanation. First off, there is no question how we figured out the bounds for θ and ϕ , so one must recognize that the difficulty comes from determining the bound for ρ . The equation of the sphere $x^2 + y^2 + z^2 = z$ translates into the equation $\rho^2 = \rho \cos \phi$. In particular, $\rho = \cos \phi$. $\square$

Example 2.11.12. Write down an integral computing the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and intersects the solid

$$1 \leq x^2 + y^2 + z^2 \leq 9.$$

Solution: In spherical coordinates, the solid displayed is given by $1 \leq \rho \leq 3$. Meanwhile, the equation of the cone becomes

$$\rho \cos(\phi) = \sqrt{\rho^2 \sin^2(\phi) \cos^2(\theta) + \rho^2 \sin^2(\phi) \sin^2(\theta)} = \rho \sin(\phi).$$

In that case, we would need $\cos(\phi) = \sin(\phi)$ which occurs iff $\phi = \pi/4$.

Now we can set-up the triple integral as

$$\int_0^{2\pi} \int_0^{\pi/4} \int_1^3 \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$\square$

Example 2.11.13. Set up an integral using spherical coordinates computing the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 2$.

As an aside, set up an integral using cylindrical coordinates.

Solution: Again, the cone corresponds to the equation $\rho = \frac{\pi}{4}$. Now we need to consider the equation of the plane $z = 2$ which becomes $\rho \cos(\phi) = 2$. This can be rewritten as the condition that $\rho = \frac{2}{\cos(\phi)}$ provided $\cos(\phi) \neq 0$ (which is not an issue because the condition $0 \leq \phi \leq \pi/2$ implies $\phi = \pi/2$ is the only time this occurs). So altogether, we get

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^{2/\cos(\phi)} \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$\square$

Example 2.11.14. Set up an integral that computes the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 25$, above the $z = 0$ plane, and below the cone $z = \sqrt{x^2 + y^2}$.

Solution: One can set up the integral as $\int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^5 \rho^2 \sin(\phi) d\rho d\theta d\phi$. Here, the bounds for ϕ are given the way they are because I need to be below the cone and above the $z = 0$ -plane. $\square$

2.12. Change of Variables. In Calculus I, students used u -substitution to change the variable being integrated. Here's a quick review of what that process looked like in an example. Suppose you wanted to rewrite $\int_a^b f(x) dx$ using the new variable $u = 3x$. Then, one has $x = \frac{u}{3}$ and $du = 3 dx$ and must rewrite this definite integral as

$$\int_{3a}^{3b} f(u/3)(u/3) du.$$

In that this case, the factor of “1/3” that appears is telling us that a small change in x leads to one-third the change in u .

One can do this for multiple variables as well e.g. we already saw that three different change of variables:

$$\begin{aligned} dA &\rightarrow r dr d\theta \\ dV &\rightarrow r dr d\theta dz \\ dV &\rightarrow \rho^2 \sin \phi d\rho d\theta d\phi. \end{aligned}$$

Question: Why did the differentials transform like this?

Answer: The answer, due to Jacobi, is that the differentials transform and are scaled by the Jacobian matrix's determinant.

To clarify this answer, we start with $2D$ -transformations.

Let $x = x(u, v)$ and $y = y(u, v)$. Then consider

$$\vec{r}(u, v) = \langle x(u, v), y(u, v) \rangle.$$

Then we can imagine that the map $T : uv\text{-plane} \rightarrow xy\text{-plane}$ given by $T(u, v) = (x(u, v), y(u, v))$ is our change of coordinate from (u, v) to (x, y) .

Now, a rectangle in the uv -plane has area equal to $\Delta u \times \Delta v$. When transformed, the sides of the new region is given by $\Delta u \vec{r}_u$ and $\Delta v \vec{r}_v$. So the area of this new region is approximately

$$|\Delta u \vec{r}_u \times \Delta v \vec{r}_v| = \Delta u \Delta v |\vec{r}_u \times \vec{r}_v|.$$

Recall that the absolute value of the cross product always computes the area of the parallelogram determined by the two vectors.

Anyway, we have

$$|\vec{r}_u \times \vec{r}_v| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \left| \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right) \vec{k} \right| = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right|.$$

Now, if we let the change in u, v and x, y become arbitrarily small, we get the relationship

$$dxdy = \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right) dudv.$$

Definition 2.12.1. The Jacobian of a transformation $T(u, v) = \langle x(u, v), y(u, v) \rangle$ is

$$J(u, v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left| \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right) \right|$$

The standard notation in this case is to write

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right|$$

for $J(u, v)$.

Remark 2.12.2 (Warning!). One should remember that the Jacobian matrix always has its absolute value taken i.e. one takes the positive part only. This was never an issue for cylindrical coordinates nor spherical coordinates (the student should check this) but it can be in general!

Example 2.12.3. If $x = r \cos \theta$ and $y = r \sin \theta$, then compute $J(r, \theta)$.

Solution: In this case, the Jacobian factor equals

$$\begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = |r \cos^2 \theta + r \sin^2 \theta| = |r| = r.$$

□

Theorem 2.12.4. Assume T is a C^1 -transformation with nonzero Jacobian matrix and that it maps a region R to a region S . Assume f is continuous on R and that T is bijective. Then

$$\iint_R f(x, y) dA = \int_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$

Now, if we had three variables instead and $x = x(u, v, w)$, $y = y(u, v, w)$, and $z = z(u, v, w)$, then the Jacobian of the transformation is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Solution: Left as an exercise for the student.

□

Example 2.12.5. Consider the change of coordinates given by $x = 3u$, $v = 2y$. What does a box in the uv -plane correspond to? For ease, pick the unit square with bottom left corner centered at the origin.

Solution: In the xy -plane one gets a box stretched horizontally by 3 and vertically by 2. □

Example 2.12.6. As general practice, consider the change of variables $u = x + y$ and $v = x - y$. What does a square box of length 1 with bottom left corner at the origin in the (uv) -plane correspond to?

Solution: Draw the lines $x + y = 1$, $x - y = 1$, $x + y = 0$, $x - y = 0$. Then look at the bounded region. □

Example 2.12.7. Compute the Jacobian of the transformation

$$x := \rho \sin \phi \cos \theta \quad \& \quad y = \rho \sin \phi \sin \theta \quad \& \quad z = \rho \cos \phi.$$

Example 2.12.8. Evaluate the integral

$$\iint_R (x + y) \sin(x - y) dA$$

where R is the region enclosed by the lines $y = x$, $y = x - 2$, $y = -x$, and $y = -x + 1$.

Solution: First off, one should use good coordinates: $u = x + y, v = x - y$.

Step two is to compute the Jacobian matrix. This requires rewriting x, y in terms of u, v . It is not hard to check that

$$x = \frac{u+v}{2} \quad \& \quad y = \frac{u-v}{2}$$

and so

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}.$$

Third, one should draw the region and then consider what happens under the transformation. For example, the line $y = x$ produces the line $u = x + y = 2x$ and $v = 0$. The line $y = -x$ produces the line $u = 0$ and $v = 2x$. The line $y = x - 2$ produces the line $u = x + y = 2x + 2$ and $v = 2$. The line $y = -x + 1$ produces the lines $u = x + y = 1$ and $v = x - y = x + x - 1 = 2x - 1$.

Meanwhile, the corner points of the original region were $(0, 0)$, $(1/2, 1/2)$, $(1, -1)$, and $(3/2, -1/2)$. So the corner points of the new rectangular region are at $(0, 0)$, $(2, 0)$, $(1, 2)$, and $(1, 0)$.

So the integral equals

$$\begin{aligned} \int_0^2 \int_0^1 u \sin(v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv &= \int_0^2 \int_0^1 \frac{1}{2} u \sin(v) du dv \\ &= \int_0^2 \frac{1}{4} \sin(v) dv \\ &= -\frac{1}{4} \cos(2) + \frac{1}{4} \\ &= \frac{1}{4}(1 - \cos(2)). \end{aligned}$$

□

Example 2.12.9. Rewrite $\iint_R \left(\frac{y-x}{y+x} \right)^2 dA$ in terms of $u = y - x$ and $v = y + x$ where R is the trapezoid bounded by the points $(0, 1)$, $(1, 0)$, $(2, 0)$, and $(0, 2)$ in the xy -plane.

Solution: First, we need to compute the Jacobian of the transformation. This means we need to solve for x and y in terms of u, v . It is not hard to see that $x = \frac{v-u}{2}$ and $y = \frac{u+v}{2}$.

The Jacobian factor is equal to

$$\begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}.$$

So the integrand can be changed into $\frac{1}{2} \frac{u^2}{v^2}$. But we need to figure out the bounds. What we have pictured for R is a trapezoid.

One should draw the picture. If one does, one can then set this integral up as

$$\int_1^2 \int_{-u}^u \frac{1}{2} u^2 v^{-2} dv du.$$

□

Example 2.12.10. Write down the change of coordinates of $\iint_D xy^3 dA$ where D is the region bounded by the curves $xy = 1$, $xy = 3$, $y = 2$, and $y = 6$ using the transformation $x = \frac{v}{6u}$ and $y = 2u$.

Solution: The student is encouraged to draw a picture. In any case, we have that

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{array}{cc} -\frac{v}{6u^2} & 2 \\ \frac{1}{6u} & 0 \end{array} \right| = \frac{1}{3u}.$$

Now we consider the region we get after we apply the change of coordinates.

The curve $y = 6$ corresponds to $u = 3$.

The curve $y = 2$ corresponds to $u = 1$.

The curve $xy = 1$ corresponds to $\frac{v}{6u} \cdot 2u = 1$ which means $\frac{v}{3} = 1$ and so $v = 3$.

The curve $xy = 3$ corresponds to $\frac{v}{6u} 2u = \frac{v}{3} = 3$ and so $v = 9$. So, we can rewrite this integral as

$$\int_3^9 \int_1^3 \frac{v}{6u} (2u)^3 \frac{1}{3u} du dv = \int_3^9 \int_1^3 \frac{4}{9} uv du dv = \cdots = 64.$$

□

2.13. Review.

2.13.1. *Request 1.* Here is a summary of formulas and theorems to remember off the top of your head.

- You will need to know how to use the multivariable chain rule e.g. if given $f(x, y, z)$, and that $x(s, t)$, $y(s, t)$, $z(s, t)$ are functions in s, t , then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s}.$$

- The **directional derivative** in the direction of the unit vector $\mathbf{u}$ of a function $f(x, y)$ is given by the formula

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

- **Theorem:** The direction $\mathbf{u}$ (i.e. unit vector) that maximizes $|D_{\mathbf{u}}f(x_0, y_0)|$ points in the same direction as $\nabla f(x_0, y_0)$ i.e. $\mathbf{u} = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$ (this gradient vector is nonzero of course).
- **Theorem:** Suppose you are given a level curve $f(x, y) = K$. Then the gradient ∇f is always orthogonal to the level curve.
- **Second Derivative Test:** If (x_0, y_0) is a **critical point** of $f(x, y)$, then the Hessian $D(x, y) = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ evaluated at the critical point tells you the following information: (a) if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$ then you have a local minimum, (b) if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$ then you have a local maximum, (c) if $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point, and (d) if $D(x_0, y_0) = 0$ then you know nothing from the test.
- **Method of Lagrange Multipliers:** If you wish to maximize or minimize $f(x, y)$ w.r.t a constraint $g(x, y) = C$ (assuming $g(x, y) = C$ produces a “compact” curve), then you can solve the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = C$ to get some set of points. The points that give the largest value give the maximum value. The points that give the smallest value give the minimum value.
- **Extreme Value Theorem:** A function $f(x, y)$ always attains a maximum or a minimum on a region $D \subseteq \mathbb{R}^2$ that is **closed** and **bounded**. You cannot drop either of these conditions.
- **Fubini’s Theorem:** If the bounds in the double and triple integrals are all constants, then you can change the order of integration freely and without changing the constants that appear.
- **Warning:** It is not true that one can always change the order of integration of any double or triple integral. However, in this class, we will only give problems where this is possible.
- For polar coordinates, the change of variables is given by $x = r \cos \theta$, $y = r \sin \theta$. The change of variables for double integrals gives

$$\iint_D f(x, y) dx dy = \iiint_D f(r \cos \theta, r \sin \theta) r dr d\theta.$$

You would need to do work to set-up the bounds of the integral

- For cylindrical coordinates, one just includes the z variable to get

$$\iiint_E f(x, y, z) dx dy dz = \iiint_E f(r \cos \theta, r \sin \theta, z) r dr d\theta dz.$$

- For spherical coordinates, the change of variables is $x = \rho \cos \theta \sin \phi$, $y = \rho \sin \theta \sin \phi$, $z = \rho \cos \phi$ and so

$$\iiint_E f(x, y, z) dx dy dz = \iiint_E f(\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi.$$

- **Warning:** You must never forget the Jacobian factor when you are doing a change of coordinates.
- For a transformation $T : uv\text{-plane} \rightarrow xy\text{-plane}$, the Jacobian factor is

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} \right|.$$

Then,

$$\iint_D f(x, y) dx dy = \iint_{T^{-1}(D)} f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

where $x = g(u, v)$ and $y = h(u, v)$.

- A vector field $\mathbf{F}(x, y)$ is **conservative** if there is a **potential function** $f(x, y)$ i.e. a function $f(x, y)$ such that $\nabla f = \mathbf{F}$.

2.13.2. Request 2.

Example 2.13.1. The solid that lies below the cone $z = -\sqrt{x^2 + y^2}$, and between the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 16$.

Solution: $V = \int_0^{2\pi} \int_{3\pi/4}^{\pi} \int_3^4 \rho^2 \sin \phi d\rho d\phi d\theta$. □

Example 2.13.2. Find the directional derivative of $f(x, y) = e^{x^2}y$ at the point $(1, 2)$ in the direction of the vector $\vec{v} = \langle -1, 3 \rangle$.

Solution: You want the unit vector $\mathbf{u} = \langle \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle$. Then you need the gradient $\nabla f = \langle 2xe^{x^2}y, e^{x^2} \rangle$ plugged in at $(1, 2)$ which is going to give you

$$D_{\mathbf{u}}f(1, 2) = \langle 4e, e \rangle \cdot \langle \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle = -\frac{e}{\sqrt{10}}.$$

□

Example 2.13.3. Write the system of Lagrange multipliers for $f(x, y, z) = xyz$ subject to the constraint $2x + y + 4z = 1$. Explain why none of the points you get for solving the system can give the absolute maximum or minimum.

Solution: The system of equations you get here is

$$\begin{aligned} yz &= 2\lambda \\ xz &= \lambda \\ xy &= 4\lambda \\ 2x + y + 4z &= 1. \end{aligned}$$

So now, I need to solve this system. Using the first two equations, I get that

$$yz = 2xz \Rightarrow z(y - 2x) = 0.$$

So there are two possibilities here (which are not necessarily exclusive).

Case 1. If $z = 0$. The above system of equations tells me that $\lambda = 0$ immediately. So $xy = 4\lambda = 0$. So one of x or y must be zero.

If $x = 0$, then the fourth equation forces me to have $y = 1$. So I get the point $(0, 1, 0)$.

If $y = 0$, then the fourth equation forces me to have $x = 1/2$. So I get the point $(1/2, 0, 0)$.

Case 2. If $y = 2x$, then I can deal with the case when $y = 0$ and when $y \neq 0$. If $y = 0$, then $x = 0$ and so the fourth equation forces $z = 1/4$ and I get the point $(0, 0, 1/4)$.

Now assume $y \neq 0$. Then $x \neq 0$ as well. The second and third equation give me

$$z = \frac{\lambda}{x} \quad \& \quad \lambda = \frac{xy}{4}.$$

Together, I get that

$$z = \frac{y}{4}.$$

Now, using the fourth equation, and the fact $2x = y$ and now $4z = y$, the third equation says $y + y + y = 1$ and so $y = \frac{1}{3}$. So, the point I get is $(1/6, 1/3, 1/12)$.

Caveat: We have now found a bunch of points from the system of Lagrange multipliers: $(0, 1, 0), (1/2, 0, 0), (0, 0, 1/4), (1/6, 1/3, 1/12)$. However, the plane $2x + y + 4z = 1$ is unbounded and one can actually show that there is no absolute maximum or absolute minimum in this case. $\square$

Example 2.13.4. Find the extremal values of $f(x, y) = xy + 2x$ subject to the condition that $x^2 + y^2 = 1$.

Solution: The system of equations from the method of Lagrange multipliers is

$$\begin{aligned} y + 2 &= \lambda 2x \\ x &= \lambda 2y \\ x^2 + y^2 &= 1. \end{aligned}$$

Now we deal with cases. If $y = 0$, then the second equation says that $x = 0$. But that implies first equation reads $2 = 0$ which is absurd. So $y \neq 0$.

Assume $y \neq 0$. Then $\lambda = \frac{x}{2y}$ by the second equation. Plugging this into the first equation, we get

$$y + 2 = \frac{x}{2y} 2x = \frac{x^2}{y} \Rightarrow y^2 + 2y = x^2.$$

Then, the third equation becomes

$$y^2 + 2y + y^2 = 1 \Rightarrow 2y^2 + 2y - 1 = 0 \Rightarrow y^2 + y - \frac{1}{2} = 0.$$

We can solve for y using the quadratic formula: $y = \frac{-1 \pm \sqrt{3}}{2}$. Now I need to solve for x . Using the third equation, we get

$$x^2 = 1 - y^2 = 1 - \left(\frac{-1 \pm \sqrt{3}}{2} \right)^2 = 1 - \frac{4 \mp 2\sqrt{3}}{4}.$$

Only the equation $x^2 = 1 - \frac{4-2\sqrt{3}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$ can be solved. In this case, $x = \pm \frac{3^{1/4}}{2^{1/2}}$. Keeping track of which solution for y this corresponded to, we get two points:

$$\left(\frac{3^{1/4}}{2^{1/2}}, \frac{-1 + \sqrt{3}}{2}\right) \quad \& \quad \left(-\frac{3^{1/4}}{2^{1/2}}, \frac{-1 + \sqrt{3}}{2}\right).$$

We leave determining which point produces the larger value of $f(x, y)$ to the reader to check.

□

Example 2.13.5. Consider the triple integral

$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{x^2+y^2}^{16} f(x, y, z) dz dy dx$$

and rewrite it so that the differentials are $dx dz dy$.

Solution: First, sketch a picture of this region. It's basically bounded by $z = 16, z = x^2 + y^2$ over a circle of radius 4.

One wants to get the bounding functions for the x coordinate. These come from the equation $z = x^2 + y^2$ and produce two equations $x = \pm\sqrt{z - y^2}$. After that, you need to consider the $x = 0$ plane to see what the bounds for $dy dz$ are. The answer is:

$$\int_{-4}^4 \int_{y^2}^{16} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f(x, y, z) dx dz dy$$

□

Example 2.13.6. Write down an integral for the volume of the solid above the plane $z = 0$, below the plane $z = x + y + 6$, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

Solution: Easiest to use cylindrical coordinates

$$\int_0^{2\pi} \int_1^3 \int_0^{r \cos \theta + r \sin \theta + 6} r dz dr d\theta.$$

□

Example 2.13.7. Is the vector field $\langle xy, x \rangle$ conservative?

Solution: The only way we have to do this so far is to integrate the two components w.r.t. the respective variable and see if you can actually solve the equations. In this case, integrating gives me

$$\frac{x^2}{2}y + h(y) \stackrel{?}{=} xy + g(x) \iff \left(\frac{x^2}{2} + x\right)y = \frac{x^2}{2}y + xy = g(x) - h(y).$$

There's no way to write a product of a function solely in variables x and solely in variables y as a sum of two functions with one solely in the variable x and the other solely in the variable y . □

2.13.3. Request 3.

Example 2.13.8. Compute $\frac{df}{dt}$ where $f(x, y) = x^2 + xy$, $x = \cos(s + t)$, and $y = \ln(st^2)$ and find $\frac{df}{dt}(1)$.

Example 2.13.9 (Second Derivative Test Example). Determine the critical points of the function $f(x, y) = x^3 - 2xy + 4y$ and classify the points.

Solution: $f_x(x, y) = 3x^2 - 2y$ and $f_y(x, y) = -2x + 4$ and we ask which points (x, y) make these zero. Clearly, $f_y(x, y) = 0$ if and only if $x = 2$. Then, $f_x(-2, y) = 12 - 2y = 0$ if and only if $y = 6$. So there is a single critical point $(2, 6)$.

Now we apply the second derivative test:

$$D(x, y) = \det \begin{pmatrix} 6x & -2 \\ -2 & 0 \end{pmatrix} = 0 - (-2)(-2) = -4.$$

So $D(2, 6) = -4 < 0$. So it is a saddle point. $\square$

Example 2.13.10. Rewrite the integral $\iint_T e^{x+y} dA$ where T is the triangle T with corners $(0, 0)$, $(1, 0)$, $(0, 1)$ by using the transformation $u = x + y$ and $v = x - y$.

Solution: One can rewrite $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Then the Jacobian of the transformation is $\frac{1}{2}$.

Now one must consider how to set-up the bounds for the new integral. We consider the lines defining the triangle T .

The first line $y = 0$ corresponds to the line $u = v$.

The second line $x = 0$ corresponds to the line $u = -v$ aka $v = -u$.

The third line $y = 1 - x$ aka $x + y = 1$ corresponds to the line $u = 1$.

So we can draw the picture and find that the new double integral is

$$\int_0^1 \int_{-u}^u \frac{1}{2} e^u dv du = \int_0^1 u e^u du$$

which one can easily integrate using integration by parts. We leave this to the reader. $\square$

Example 2.13.11. Evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region with corners at the points $(1, 0)$, $(2, 0)$, $(0, 2)$, $(0, 1)$.

Solution: The answer is $\frac{3}{2} \sin(1)$. One uses the change of coordinates $u = y + x$ and $v = y - x$. $\square$

Example 2.13.12. Use spherical coordinates to evaluate the triple integral

$$\iiint_E (x^2 + y^2 + z^2) \ln(x^2 + y^2 + z^2) dV$$

where E is the solid unit sphere centered at the origin.

Solution: This can be rewritten as

$$\int_0^\pi \int_0^{2\pi} \int_0^1 \rho^2 \ln(\rho^2) \rho^2 \sin(\phi) d\rho d\theta d\phi.$$

$\square$

Example 2.13.13. Consider the function $f(x, y) = 2x^2 + y^2 - 2x^2y + 6$. Classify its critical points and find all critical points.

Solution: There is a local minimum at $(0, 0)$ and saddle points at $(1, -1)$ and $(1, 1)$. $\square$

Example 2.13.14. Find the absolute maximum and absolute minimum of $f(x, y) = xy$ subject to the constraint $x^2 + 4y^2 = 2$.

Solution: The points one considers from the method of Lagrange multipliers are $(\pm 1, \pm \frac{1}{2})$. It is easy to determine which are maximums and minimums. $\square$

Example 2.13.15. Set-up a triple integral computing the volume of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 3$.

Solution: $\int_0^3 \int_0^{3-x} \int_0^{3-x-y} dz dy dx$. $\square$

Example 2.13.16. Set-up a triple integral computing the volume of the solid in the first octant that lies above the paraboloid $z = x^2 + y^2$ and below the paraboloid $z = 8 - x^2 - y^2$.

Example 2.13.17. Set-up a triple integral computing the volume of the solid that lies below the cone $z = -\sqrt{x^2 + y^2}$, and between the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 16$.

Solution: It is not hard to see that one can use

$$\int_{3\pi/4}^{\pi} \int_0^{2\pi} \int_3^4 \rho^2 \sin \phi d\rho d\theta d\phi.$$

$\square$

Example 2.13.18 (Question: When to use spherical versus when to use cylindrical coordinates?). The correct answer is to try both and see what works. The key sign for when to use spherical coordinates are when you see expressions of the form $x^2 + y^2 + z^2$ appear. The key sign for cylindrical is when you see something of the form $x^2 + y^2$.

Example 2.13.19. Set-up the integral $\iiint_E x^2 dV$ where E is the region inside both $x^2 + y^2 + z^2 = 36$ and $z = -\sqrt{3x^2 + 3y^2}$ in spherical coordinates.

Solution: $\int_{\frac{5\pi}{6}}^{\pi} \int_0^{2\pi} \int_0^6 \rho^4 \sin^3 \phi \cos^2 \theta d\rho d\theta d\phi$. Note that $x = \rho \cos \theta \sin \phi$.

The way to see this is to draw a picture and consider what the equation of the cone should translate to in spherical coordinates. If we rewrite this, we get

$$\rho \cos \phi = -\sqrt{3}(x^2 + y^2) = -\sqrt{3}\sqrt{\rho^2 \sin^2 \phi} = -\sqrt{3}\rho \sin \phi.$$

So I can solve for ϕ since the above becomes $\tan \phi = -\frac{1}{\sqrt{3}}$. In other words, $\phi = \frac{5\pi}{6}$. $\square$

Example 2.13.20. Set up the triple integral $\iiint_E dV$ where E is the solid outside of the cone $z^2 = x^2 + y^2$, inside the sphere $x^2 + y^2 + z^2 = 4$, and including the origin. Here, outside the cone means the region that contains the $z = 0$ plane removing the origin.

Solution: The integral in this case is

$$\int_{\pi/4}^{3\pi/4} \int_0^{2\pi} \int_0^2 \rho^2 \sin \phi d\rho d\theta d\phi.$$

$\square$

Example 2.13.21. Find the maximum and minimum value of $f(x, y) = xy^2 + x$ subject to the condition $x^2 + y^2 = 1$.

Solution: The system one gets from the method of Lagrange multipliers is

$$\begin{aligned}y^2 + 1 &= 2\lambda x \\ 2xy &= 2\lambda y \\ x^2 + y^2 &= 1.\end{aligned}$$

If $y = 0$, then $x = \pm 1$ by the third equation and we can always find a λ to solve the first equation. So $(\pm 1, 0)$ is a point that we must consider from the method of Lagrange multipliers.

If $y \neq 0$, then the second equation becomes $2x = 2\lambda$ and so $x = \lambda$. Plugging this into the first equation, we get

$$y^2 + 1 = 2x^2.$$

We can add 1 to the third equation to get $x^2 + y^2 + 1 = 2$ and using the above equation, get that

$$3x^2 = 2.$$

So, $x = \pm\sqrt{\frac{2}{3}}$. In that case, $y = \pm\sqrt{\frac{1}{3}}$ by the third equation. Altogether, we have six points to consider:

- (1) $(1, 0)$ gives $f(1, 0) = 1$,
- (2) $(-1, 0)$ gives $f(-1, 0) = -1$,
- (3) $(\sqrt{2/3}, \sqrt{1/3})$ gives the value $(\sqrt{2/3})(1/3) + (\sqrt{2/3})$
- (4) $(-\sqrt{2/3}, \sqrt{1/3})$ gives the value $-(\sqrt{2/3})(1/3) - (\sqrt{2/3})$,
- (5) $(\sqrt{2/3}, -\sqrt{1/3})$ gives the value $(\sqrt{2/3})(1/3) - (\sqrt{2/3})$
- (6) $(-\sqrt{2/3}, -\sqrt{1/3})$ gives the value $-(\sqrt{2/3})(1/3) - (\sqrt{2/3})$.

Now we need to determine the maximum and minimum value. We really only need to compare $\sqrt{2/3}(1/3) + \sqrt{2/3} = \frac{4}{3}\sqrt{\frac{2}{3}}$ against 1 to be able figure everything out. Thankfully, this is not hard to do:

$$\frac{4}{3}\sqrt{\frac{2}{3}} > 1 \iff \frac{16}{9} \cdot \frac{2}{3} > 1 \iff \frac{32}{27} > 1.$$

So what we get is that the maximum occurs at the points labeled (3) and (5) and the minimum occurs at the points labeled (4) and (6). $\square$

3. VECTOR CALCULUS

3.1. Vector Fields and Line Integrals.

Definition 3.1.1. A $2d$ -vector field on a region $D \subseteq \mathbb{R}^2$ is the association to each point $(x, y) \in D$ a vector $\langle P(x, y), Q(x, y) \rangle$. In other words,

$$\mathbf{F} = \mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle.$$

In other words, it is a vector function on D which is valued in vectors in $\mathbb{R}^2$. Consider going to [CalcPlot3D](#) and try drawing some vector fields. A $3d$ -vector field on a region D is defined analogously i.e. it is a vector function on $D \subseteq \mathbb{R}^3$ which is valued in vectors in $\mathbb{R}^3$.

Example 3.1.2. Draw the vector field given by

$$\mathbf{F}(x, y) = \vec{i} + x\vec{j}.$$

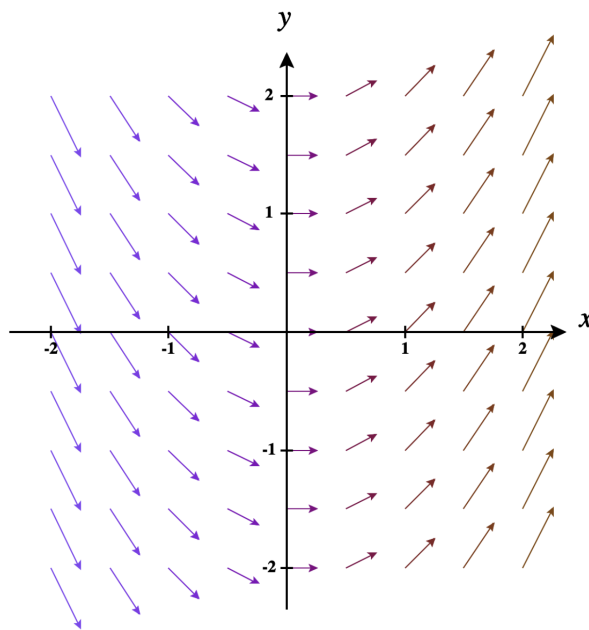


FIGURE 18. The vector field $\mathbf{F}(x, y) = \langle 1, x \rangle$.

Solution: See Figure 18.

□

Example 3.1.3. Recall that we defined the gradient of a function $f(x, y, z)$ to be $\nabla f := \langle f_x, f_y, f_z \rangle$. This takes a function and produces a vector field.

Draw the gradient of the following functions

- (1) $f(x, y, z) = x^2 + y^2 + z^2$,
- (2) $f(x, y, z) = 3 + x + y$,
- (3) $f(x, y, z) = xy$,
- (4) $f(x, y) = x^2 + 4y^2$.

Solution: We leave all four to the student to do as an exercise.

□

Example 3.1.4. Draw the gradient vector field of $f(x, y) = x^2 + 4y^2$. Draw its level curves. Compare them – what does this suggest about the gradient vector field?

Solution: This is left as an exercise for the student.

□

Definition 3.1.5. A vector field $\mathbf{F}$ is **conservative** if there is a function f such that $\nabla f = \mathbf{F}$. One calls f the **potential function** for $\mathbf{F}$.

Example 3.1.6. Is the vector field $\mathbf{F}(x, y) = x^2\vec{i} + y\vec{j}$ conservative?

Solution: Yes, this is a conservative vector field. We can take the potential function f to be $f(x, y) = \frac{x^3}{3} + \frac{y^2}{2}$.

□

Example 3.1.7. Is the vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$ conservative?

Solution: No, this is not a conservative vector field. If $f(x, y)$ were a potential function, then $f_x(x, y) = y$ and $f_y(x, y) = -x$. Integrating w.r.t. x and y respectively, we would need

$xy + f(y) = -xy + g(x)$. This can be written as the condition that there must exist functions $f(y), g(x)$ such that $2xy = g(x) - f(y)$. This is impossible to do. $\square$

Remark 3.1.8. One can think of conservative vector fields as those having some sort of “antiderivative”. We will make it clearer what we mean by this later.

Example 3.1.9. Is $\mathbf{F}(x, y, z) = \langle xy, x^2, 1 \rangle$ conservative?

Solution: No. If we integrated each component w.r.t. x, y, z respectively, we would need

$$\frac{x^2}{2}y + f(y, z) = x^2y + g(x, z) = z + h(x, y).$$

The first equality is impossible to solve for because the factor of $\frac{1}{2}$ for x^2y cannot arise from a function $g(x, z)$ depending solely on x, z . $\square$

We are now ready to discuss line integrals. So far we have focused on the “easiest” sort of integrals over a 1-dimensional domain. In particular, we’ve just been integrating over a straight line. What if we were integrating over a curve instead?

Definition 3.1.10. Let C be a smooth curve parameterized by the vector function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$. Then, the **line integral** where we integrate $f(x, y)$ over C is defined by

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)| dt.$$

As an example, if $f(x, y) = 1$, then this line integral gives you the arc length of the curve.

Example 3.1.11. Set-up the integral $\int_C y ds$ where C is the curve $x(t) = t, y(t) = \sqrt{t}$ for $2 \leq t \leq 6$. Do not try to evaluate the integral.

Solution: First off, $|\mathbf{r}'(t)| = |\langle 1, \frac{1}{2}t^{-1/2} \rangle| = \sqrt{1 + \frac{1}{4}t^{-1}}$. So,

$$\int_C y ds = \int_2^6 \sqrt{t} \sqrt{1 + \frac{1}{4}t^{-1}} dt = \dots = \frac{49}{6}.$$

We leave the evaluation of this integral to the reader. $\square$

Example 3.1.12. Evaluate $\int_C yx^2 ds$ where C is the curve $x(t) = \sin(t), y(t) = \cos(t)$ and $0 \leq t \leq \frac{\pi}{2}$.

Solution: Conveniently, $|\mathbf{r}'(t)| = 1$ in this case. So setting up this integral is quite easy:

$$\int_0^{\pi/2} \cos(t) \sin^2(t) dt = \dots = \frac{1}{3}.$$

$\square$

Example 3.1.13. Evaluate $\int_C (x + y^2) ds$ where C is the line segment from $(0, 0)$ to $(2, 5)$.

Solution: We can parameterize the line segment by $\mathbf{r}(t) = \langle 2t, 5t \rangle$ for $0 \leq t \leq 1$. Then, $|\mathbf{r}'(t)| = |\langle 2, 5 \rangle| = \sqrt{29}$. Then, the integral becomes

$$\int_0^1 (2t + (5t)^2) \sqrt{29} dt = \dots = \sqrt{29} \frac{28}{3}.$$

$\square$

Example 3.1.14. Evaluate $\int_C 1 ds$ in the following cases.

- a) $C : (\cos t, \sin t) \quad 0 \leq t \leq 2\pi$.
 b) $C : (\cos t, \sin t) \quad 0 \leq t \leq 4\pi$.

Solution: a) This computes the arc length of C . In that case, we just get the circumference of a circle which is 2π .

b) This is the arc length when we traverse the circle twice so the answer is 4π . $\square$

The moral of the previous example is that there is some ambiguity if one is clear with the notation " $\int_C \dots ds$ ". Does one mean to traverse the curve C once? Twice? If the curve is closed, then which orientation does one pick? Typically, problems will be phrased so that these ambiguities are made bare.

Example 3.1.15. Set up an integral representing $\int_C y^4 x ds$ where C is the right half of the circle $x^2 + y^2 = 16$ followed by the line segment $(0, 4)$ to $(2, 12)$. Note that you will need two integrals to do this.

Solution: This is left as an exercise for the student. $\square$

Question: As humans we favor the "easier to think about coordinate systems". Is there a way to rewrite line integrals so that the relationship to the x, y coordinate system is more prominent?

Answer: Yes. One could try to extract the change w.r.t. x and w.r.t. y separately.

Let C be a smooth curve. Then we can consider the integrals

$$\begin{aligned}\int_C f(x, y) dx &= \int_C f(x(t), y(t)) x'(t) dt \\ \int_C f(x, y) dy &= \int_C f(x(t), y(t)) y'(t) dt.\end{aligned}$$

Example 3.1.16. Evaluate $\int_C (x+3y) dx$ and $\int_C (x+3y) dy$ where C is the part of a parabola that connects $(1, 2)$ to $(2, 8)$.

In many applications, one can extract the change w.r.t. x and w.r.t. y . One writes

$$\int_C P(x, y) dx + Q(x, y) dy.$$

in these cases.

Remark 3.1.17. Note that the orientation of the curve matters. If you switch which direction you move along the curve C , your answer changes by a sign.

One can also compute line integrals in 3d-space. For example, let C be a curve in $\mathbb{R}^3$ parameterized by $x(t), y(t), z(t)$ where $a \leq t \leq b$, then

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$

Similarly, we can rewrite

$$\begin{aligned}& \int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz \\ &= \int_a^b P(x(t), y(t), z(t)) |x'(t)| dt + Q(x(t), y(t), z(t)) |y'(t)| dt + R(x(t), y(t), z(t)) |z'(t)| dt.\end{aligned}$$

Example 3.1.18. Setup $\int_C xyz dx + xy dy + z^2 dz$ for

$$C : x(t) = t, y(t) = t^2, z(t) = t^3, \quad 0 \leq t \leq 1$$

and evaluate it.

Solution: We shall set-up the integral but leave the evaluation to the student. In this situation we get

$$\int_0^1 t^6 dt + \int_0^1 t^3(2t) dt + \int_0^1 (t^3)^2(3t^2) dt.$$

□

Example 3.1.19. Compute the integral $\int_C x^2 ds$ where C is the circle with counterclockwise parameterization and is traversed once.

Solution: This is left as an exercise for the student.

□

We are now ready to introduce one level of complexity to the story. Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a vector field. Now imagine that C is a smoothly parameterized curve in $\mathbb{R}^2$. One can then ask about how much “work” is exerted on the curve as the curve moves through $\mathbb{R}^2$. This has a precise answer. Let $d\mathbf{r} = \langle dx, dy \rangle$. Then,

$$\mathbf{F}(x, y) \cdot d\mathbf{r} = P(x, y)dx + Q(x, y)dy.$$

In particular,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y)dx + Q(x, y)dy = \int_a^b F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Example 3.1.20. Suppose your curve C is a line segment from $(0, 0)$ to $(5, 0)$. That means you can parameterize C by $\langle t, 0 \rangle$ with $0 \leq t \leq 5$. Then consider the vector field $\mathbf{F}(x, y) = \langle -1, 0 \rangle$ i.e. each point has the vector $\langle -1, 0 \rangle$ associated to it.

a) What should $\int_C \mathbf{F} \cdot d\mathbf{r}$ equal?

b) What does one actually get if one computes the line integral?

Solution: We should expect the “work” to be -5 since we move along a line segment of length 5. Indeed,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^5 \langle -1, 0 \rangle \cdot \langle 1, 0 \rangle dt = \int_0^5 -1 dt = -5.$$

□

Example 3.1.21. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}(x, y) = \langle \cos x, \sin y \rangle$ and C is the curve given by $\mathbf{r}(t) = \langle t, t^2 \rangle$ with $0 \leq t \leq 2\pi$.

Solution: The integral can be set-up as

$$\int_0^{2\pi} \langle \cos(t), \sin(t^2) \rangle \cdot \langle 1, 2t \rangle dt = \int_0^{2\pi} \cos(t) + 2t \sin(t^2) dt.$$

We leave the evaluation of this integral to the reader.

□

The next definition relates this definition of a line integral of vector fields with line integrals of functions.

Theorem 3.1.22. Let $\vec{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ be the unit tangent vector. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \vec{T} ds.$$

Proof. $ds = |\mathbf{r}'(t)|dt$. □

3.2. Fundamental Theorem of Line Integrals. Let's begin with some warm-up problems since it has been some time.

Example 3.2.1 (Change of Variables). Rewrite the double integral

$$\iint_R \cos(8x + 3y) \sqrt{2x + y} dA$$

using the change of coordinates $u = 8x + 3y$ and $v = 2x + y$ and where R is the region bounded by the lines $x = -\frac{1}{2}y$, $x = -\frac{1}{2}y + 2$, $y = 2$, and $y = 4$.

Solution: First, we need to compute the Jacobian and this means we need to rewrite x, y in terms of u, v . One can check that

$$x = \frac{u - 3v}{2} \quad \& \quad y = -u + 4v.$$

So, the Jacobian equals

$$\left| \det \begin{pmatrix} 1/2 & -1 \\ -3/2 & 4 \end{pmatrix} \right| = \left| 2 - \frac{3}{2} \right| = \frac{1}{2}.$$

Next, we need to determine the corresponding region in the uv -plane. The line $x = -\frac{1}{2}y$ translates into $\frac{1}{2}v = x + \frac{1}{2}y = 0$ and so $v = 0$. The line $x = -\frac{1}{2}y + 2$ translates into $\frac{1}{2}v = x + \frac{1}{2}y = 2$ and so $v = 4$.

Meanwhile, the equation $y = 2$ translates into the condition that $u = 4v - 2$ and the equation $y = 4$ translates into the condition that $u = 4v - 4$.

Now we can rewrite the double integral as

$$\int_0^4 \int_{4v-4}^{4v-2} \frac{1}{2} \cos(u) \sqrt{y} du dv.$$

□

Example 3.2.2 (Spherical Coordinates). Set-up the triple integral $\iiint_E xz dV$ where E is the solid inside of $x^2 + y^2 + z^2 = 4$ and above $z = \sqrt{3x^2 + 3y^2}$ using spherical coordinates.

Solution: First off, the equation for the cone given translates into $\rho \cos \phi = \sqrt{3}\rho \sin \phi$. So, the equation for the cone here is that $\phi = \arctan(1/\sqrt{3})$. So, the triple integral transforms into

$$\int_0^{\arctan(1/\sqrt{3})} \int_0^{2\pi} \int_0^2 (\rho \cos \theta \sin \phi)(\rho \sin \theta \sin \phi) \rho^2 \sin(\phi) d\rho d\theta d\phi$$

This triple integral can be rewritten into the form

$$\int_0^{\arctan(1/\sqrt{3})} \int_0^{2\pi} \int_0^2 \rho^4 (\cos \theta \sin \theta) (\sin^3 \phi) d\rho d\theta d\phi.$$

□

Before introducing new materials, let us review what we have so far. We shall focus on the 2-dimensional situation.

- Vector fields: $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$,
- A conservative vector field $\mathbf{F}$ is one of the form $\nabla f = \mathbf{F}$ for some function $f(x, y)$.
- $\int_C f(x, y) ds$ called a “line integral of a function $f(x, y)$ over a curve”,
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$ called a “line integral of a vector field over a curve”,
- $\int_C P(x, y)dx + Q(x, y)dy$.
- $\int_C \mathbf{F} \cdot \mathbf{T} ds$

Example 3.2.3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle x^2, y^2 \rangle$ and C is the circle of radius 2 oriented counterclockwise traversed once.

Solution: The radius of circle 2 oriented counterclockwise traversed once is parameterized by

$$\langle 2 \cos(t), 2 \sin(t) \rangle \quad 0 \leq t \leq 2\pi.$$

In that case, we can rewrite the line integral as

$$\int_0^{2\pi} \langle 4 \cos^2(t), 4 \sin^2(t) \rangle \cdot \langle -2 \sin(t), 2 \cos(t) \rangle dt = \int_0^{2\pi} -8 \cos^2(t) \sin(t) + 8 \sin^2(t) \cos(t) dt.$$

We can break up the integral in two

$$- \int_0^{2\pi} 8 \cos^2(t) \sin(t) dt + \int_0^{2\pi} 8 \sin^2(t) \cos(t) dt$$

and evaluate them individually via u -substitution. We leave it to the student to check that the answer is 0. $\square$

Example 3.2.4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for both C ’s below, where

$$\mathbf{F} = \langle 3x^2y, 2x^3y \rangle$$

$$C_1 = \text{line from } (0, 0) \text{ to } (1, 4)$$

$$C_2 = \text{parabola from } (0, 0) \text{ to } (1, 4).$$

Solution: We leave this as an activity for the student. $\square$

Theorem 3.2.5 (Fundamental Theorem of Line Integrals). Let C be smoothly parameterized by $\mathbf{r}(t)$ with $a \leq t \leq b$. If ∇f is continuous on C , then

$$\int_C \nabla f(x, y) \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof. The theorem arises from the usual Fundamental Theorem of Calculus. Suppose C is parameterized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$. Then, we can rewrite the left hand side as

$$\int_a^b \langle f_x(x(t), y(t)), f_y(x(t), y(t)) \rangle \langle x'(t), y'(t) \rangle dt.$$

Evaluating this dot product gives me

$$\begin{aligned} \int_a^b [f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)] dt &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \end{aligned}$$

where the first equality is by the multivariable chain rule and the second is by the usual Fundamental Theorem of Calculus. $\square$

Remark 3.2.6. To use the theorem, one *must* be able to evaluate the values at the end point. This theorem also says that the integral calculus the “net change” of f ’s value from the starting to point versus the endpoint.

The most important application of this theorem is the fact that if C is **closed** (i.e. if $\mathbf{r}(a) = \mathbf{r}(b)$ for any parameterization $\mathbf{r}(t)$ with $a \leq t \leq b$), then the integral equals zero. The reason why this is so important is because it means that the path from $\mathbf{r}(b)$ to $\mathbf{r}(a)$ is **irrelevant**. Only the endpoints are important.

Remark 3.2.7. The Fundamental Theorem of Line Integrals holds in any dimension via the same proof.

Example 3.2.8. Find $\int_C \nabla f \cdot d\mathbf{r}$ where $f(x, y) = x^2 + 3xy^2$ and C is a nasty curve going from $(1, 1)$ to $(5, 3)$.

Solution: In this case, the Fundamental Theorem of Line Integrals says the curve itself doesn’t matter—just the endpoints. So we get

$$\int_C \nabla f \cdot d\mathbf{r} = f(5, 3) - f(1, 1) = 160 - 4 = 156.$$

□

Example 3.2.9. $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = \langle x, y \rangle$ and $C : \mathbf{r}(t) = \langle t^2 + 1, t^2 + 5 \rangle, 0 \leq t \leq 1$.

Solution: In this case, $\mathbf{F}$ is a conservative vector field with potential function $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Applying the Fundamental Theorem of Line Integrals and noting that the starting point and ending point of C are $(1, 5)$ and $(2, 6)$ respectively, we get

$$\frac{1}{2}2^2 + \frac{1}{2}6^2 - \frac{1}{2}1^2 - \frac{1}{2}5^2 = 7.$$

□

Remark 3.2.10. The main lesson of this section is that line integrals are more easily computable if the vector field is conservative.

Corollary 3.2.11. If $\mathbf{F}$ is a conservative vector field and C is a closed curve, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for any smooth parameterization $\mathbf{r}(t)$ with $a \leq t \leq b$ of C such that $\mathbf{r}(a) = \mathbf{r}(b)$.

Remark 3.2.12 (Important Notational Remark). For a closed curve C , the notation $\int_C \mathbf{F} \cdot d\mathbf{r}$ is ambiguous. For example, do we mean to traverse C only once? Perhaps twice? What if we traverse the curve C in a clockwise manner instead of a counterclockwise manner? To deal with this, we might write

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

to mean that C is traversed once in a counterclockwise manner and the starting point and the end point must coincide.

We now have some statements that follow from the Fundamental Theorem of Line Integrals. We also write down some definitions.

- (1) We say C is a **closed curve** if there is a parameterization $\mathbf{r}(t)$ with $a \leq t \leq b$ of C such that $\mathbf{r}(a) = \mathbf{r}(b)$.

- (2) We say C is **simple** if C has no self-intersections.
- (3) We say a line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent** if the value of the integral is independent of the path C taken i.e. any path from point $\mathbf{r}(a)$ to $\mathbf{r}(b)$ produces the same value for the integral.
- (4) A vector field $\mathbf{F}$ is conservative if and only if every line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent.
- (5) If C is a closed curve and $\mathbf{F}$ is conservative, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever you pick a parameterization $\mathbf{r}(t)$ satisfying $\mathbf{r}(b) = \mathbf{a}$.

The following four examples are left as an exercise to the student.

Example 3.2.13 (Conservativeness Matters). Let $\mathbf{F} = \langle y, -x \rangle$. Evaluate

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

for the two parameterizations

- (a) $\mathbf{r}(t) := \langle \cos(t), \sin(t) \rangle$ when $0 \leq t \leq 2\pi$,
- (b) $\mathbf{r}(t) := \langle \cos(t), \sin(t) \rangle$ when $0 \leq t \leq 4\pi$.

Notice that the answers differ. This has to do with the fact that $\mathbf{F}$ is not a conservative vector field.

Example 3.2.14. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ where $f(x, y) = x^3(3 - y^2) + 4y$ and $\mathbf{r}(t) = \langle 3 - t^2, 5 - t \rangle$ for $-2 \leq t \leq 3$.

Example 3.2.15. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is an ellipse given by the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and $\mathbf{F}$ is a conservative vector field. Here, assume that the ellipse is traversed once in a counter clockwise fashion with the initial and end points coinciding.

Example 3.2.16. Evaluate $\int_C \langle 1, 1, 1 \rangle \cdot d\mathbf{r}$ where C is a smooth curve from $(0, 0, 5)$ to $(0, 0, -3)$.

3.3. Green's Theorem. Let's just begin with a review problem.

Example 3.3.1. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle \sin(x) + x^2y, \frac{x^3}{3} \rangle$ and C is the curve given by the parameterization $\mathbf{r}(t) = \langle t^3 + 2t + 1, \frac{2}{\pi}t \cos(t) \rangle$ for $0 \leq t \leq \pi$. You do not need to write out an exact numerical answer nor simplify.

Solution: The vector field is conservative and a potential function is

$$f(x, y) = -\cos(x) + \frac{x^3}{3}y.$$

This makes it possible to use the Fundamental Theorem of Line Integrals. The starting point is $\mathbf{r}(0) = \langle 1, 0 \rangle$ while the end point is $\mathbf{r}(\pi) = \langle \pi^3 + 2\pi + 1, -2 \rangle$. So, we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\pi^3 + 2\pi + 1, -1) - f(1, 0).$$

□

In the previous lecture, we formulated some equivalent conditions for a vector field to be conservative on all of $\mathbb{R}^2$ or $\mathbb{R}^3$. That statement can be generalized.

Definition 3.3.2. A region D in the plane $\mathbb{R}^2$ is **open** if every point in D has an open neighborhood contained in D . A region D is **closed** if it contains all of its boundary points.

Theorem 3.3.3. Let D be an open connected region. Let $\mathbf{F}$ be a vector field on D . Then the following are equivalent

- (1) $\mathbf{F}$ is conservative on D ,
- (2) $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path for C a curve in D ,
- (3) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed curves C in D .

Theorem 3.3.4. If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ for sufficiently “nice” P, Q , and $\mathbf{F}$ is conservative, then

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Proof. If $\mathbf{F}$ is conservative, then that means $\mathbf{F} = \nabla f$ for some “nice function” f . Here, “nice” includes the requirement that the mixed partial derivatives exist and coincide. In particular, we then get that

$$\frac{\partial Q}{\partial x} = \frac{\partial^2}{\partial x \partial y} f = \frac{\partial^2}{\partial y \partial x} f = \frac{\partial P}{\partial y}.$$

The “nice” assumption ensures that the middle equality is true. □

Corollary 3.3.5. If $\mathbf{F}$ is a vector field where $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$, then $\mathbf{F}$ is **not** conservative.

Example 3.3.6. Show that the following vector fields are not conservative.

- $\mathbf{F}(x, y) = \langle 3x + y, 2y^2 \rangle$
- $\mathbf{F}(x, y) = \langle y, y \rangle$
- $\mathbf{F}(x, y) = \langle y, -x \rangle$.

Solution: This is an easy exercise left for the student to do. For example, the first vector field is not conservative since

$$\frac{\partial}{\partial x}(2y^2) = 0 \neq 1 = \frac{\partial}{\partial y}(3x + y).$$

□

Definition 3.3.7. An open connected region $D \subseteq \mathbb{R}^2$ is **simply connected** if it has no holes⁴. Since we are in no position to use a truly rigorous definition, we demonstrate this in a number of examples.

Example 3.3.8. The following regions are simply connected

- the open disk $\{(x, y) \mid x^2 + y^2 < 1\}$,
- the region $\mathbb{R}^2$.

On the other hand, the following regions are not simply connected

- the open disk $\{(x, y) \mid x^2 + y^2 < 1\} \setminus \{(0, 0)\}$,
- the annulus $\{(x, y) \mid \frac{1}{2} < x^2 + y^2 < 1\}$.

Theorem 3.3.9. If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a vector field on a simply connected region D . Then the following are equivalent

- $\mathbf{F}$ is a conservative vector field,
- $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D .

⁴The more rigorous mathematical definition is that $\pi_1(D) = 0$ and $\pi_0(D) = 0$.

Example 3.3.10. a) Check $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 2xy + 24x, x^2 + 16 \rangle$ is always independent of path for any oriented closed curve C .

b) Find the potential function ie. f st.

$$\nabla f = \mathbf{F}.$$

c) Evaluate the integral for C a path from $(0, 0)$ to $(1, 2)$.

We recall and introduce some conditions on curves C . First, we say C is **simple** if it has no self intersections. Second, we say that C is **positively oriented** if it is simple, closed, and oriented clockwise. The student is encouraged to draw pictures of these concepts.

Theorem 3.3.11 (Green's Theorem). Let C be a positive oriented simple closed curve and D is the open region bounded by C . Then,

$$\oint_C Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

The notation $\oint$ above indicates we are positively oriented whereas writing $\int$ would leave this ambiguous.

We will not provide a proof of Green's Theorem. The essential step to proving the theorem is showing $\oint_C P dx = \iint_D -\frac{\partial P}{\partial y} dA$ and $\oint_C Q dy = \iint_D \frac{\partial Q}{\partial x} dA$. Proving these equalities is easier when the region D is sufficiently nice i.e. when these double integrals can be set-up as iterated integrals.

Remark 3.3.12. One can compare this theorem with the Fundamental Theorem of Calculus. It is saying that the integral along the boundary is related to the integral on the interior via antidifferentiation/differentiation. Written more provocatively, Green's Theorem says

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \int_{\partial D} P dx + Q dy.$$

Remark 3.3.13. As a fun-fact, Green's Theorem is a special case of a more general phenomena which works in **all** dimensions. We will never work in dimension > 3 , however. The next week will focus on the dimension 2 “and 3” case.

Remark 3.3.14. It is actually a deep [theorem of Jordan](#) that a simple closed curve in $\mathbb{R}^2$ actually divides $\mathbb{R}^2$ into two connected regions with one region bounded and the other unbounded. Green's Theorem takes this as fact in its precise. One can remove this by asserting instead that “ C is a positive oriented simple closed curve which is the boundary of an open region D ”.

Example 3.3.15. $\oint_C xydy - y^2dx$ where C is the square w/ corners $(0, 0)$, $(0, 1)$, $(1, 1)$ and $(1, 0)$. Compute this line integral using the direct approach and then using Green's Theorem.

Solution: The direct approach is left for the student to do as an exercise (it is really annoying, please don't do it unless you want to practice computing line integrals!).

If one uses Green's Theorem, this becomes a straightforward double integral over a rectangular region. Indeed, Green's Theorem says this equals

$$\int_0^1 \int_0^1 \frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial y} (xy) dx dy = \int_0^1 \int_0^1 -x dx dy = -\frac{1}{2}.$$

□

Example 3.3.16. Evaluate $\oint_T x^4 dx + xy dy$ where T is the triangle whose vertices are the points $(0, 0)$, $(1, 0)$, $(0, 1)$.

Solution: Applying Green's Theorem, this equals

$$\int_0^1 \int_0^{1-x} (y - 0) dy dx = \int_0^1 \frac{1}{2}(1-x)^2 dx = \cdots = \frac{1}{6}.$$

□

Here's a very neat application of Green's Theorem. Suppose D is a simply connected region with a boundary curve C which is simple and positively oriented. Then,

$$\text{area}(D) = \iint_D dA = \int_C P(x, y)dx + Q(x, y)dy$$

where $\langle P(x, y), Q(x, y) \rangle$ is chosen so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. There are many options in this case e.g. $\langle 0, x \rangle$.

Example 3.3.17. Using the above technique, find the area of $\{(x, y) \mid x^2 + y^2 \leq 1\}$.

Solution: We can find the area by instead computing the line integral $\oint_C \langle 0, x \rangle \cdot d\mathbf{r}$ where C is the circle of radius 1. Parameterize the circle by $\langle \cos(t), \sin(t) \rangle$ for $0 \leq t \leq 2\pi$. Then, we get

$$\oint_C \langle 0, x \rangle \cdot d\mathbf{r} = \int_0^{2\pi} \langle 0, \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt = \int_0^{2\pi} \cos^2(t) dt.$$

This latter integral can be computed using the double-angle formula

$$\int_0^{2\pi} \cos^2(t) dt = \left(\frac{1}{2}t + \frac{1}{4}\sin(2t) \right) \Big|_0^{2\pi} = \pi.$$

□

Example 3.3.18. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle e^x + yx^2, e^y - y^2x \rangle$ and C is the circle $x^2 + y^2 = 4$ (pos. oriented and traversed once).

Solution: Evaluating the line integral directly is not worth the effort. We use Green's Theorem. The line integral then equals

$$\iint_D -y^2 - x^2 dA$$

where D is the disk of radius 2 centered at the origin. We can use polar coordinates to set this double integral up as

$$\int_0^{2\pi} \int_0^2 -r^2 \cdot r dr d\theta = 2\pi \int_0^2 -r^3 dr = 2\pi \left(-\frac{r^4}{4} \right) \Big|_0^2 = -8\pi.$$

□

The following point will never be tested on so we give no examples: once one knows how to use Green's Theorem for simply connected regions, it is clear what one must do for regions with holes – break it up into pieces. See the section titled “Extended Versions of Green's Theorem” in Stewart's textbook for more.

Example 3.3.19. Evaluate the integral

$$\oint_C yx^2 dx + (-x^2) dy$$

where C is simple counterclockwise oriented curve given by the line $x = 0$ and left half of the circle $x^2 + y^2 = 9$.

Solution: By Green's Theorem, we can rewrite this as

$$\iint_D -2x - x^2 dA$$

where D is the left half of the disk of radius 3 centered at the origin. One can use polar coordinates to set this double integral up as

$$\int_{\pi/2}^{3\pi/2} \int_0^3 (-2r \cos \theta - r^2 \cos^2 \theta) r dr d\theta = \dots = -\frac{81\pi}{8} + 3.$$

We leave it as an exercise for the student to evaluate this double integral. $\square$

Example 3.3.20. Evaluate

$$\oint_C (3 + e^{x^2}) dx + (\tan^{-1} y + 3x^2) dy$$

where C is the positively oriented curve given by the line segment from $(1, 0)$ to $(2, 0)$, the arc $\langle 2 \cos(t), 2 \sin(t) \rangle$ for $0 \leq t \leq \frac{\pi}{2}$, the line segment from $(0, 2)$ to $(0, 1)$, and the arc $\langle 2 \cos(t), 2 \sin(t) \rangle$ for $0 \leq t \leq \frac{\pi}{2}$.

Solution: The region D bounded is the portion of the annulus $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 2\}$ that lies in the first quadrant. First, we observe that

$$\frac{\partial Q}{\partial x} = 6x \quad \& \quad \frac{\partial P}{\partial y} = 0.$$

So by Green's Theorem and then rewriting in polar coordinates, we get

$$\iint_D 6x dA = \int_1^2 \int_0^{\pi/2} 6r^2 \cos \theta d\theta dr = \dots = 14.$$

$\square$

Example 3.3.21. Evaluate the line integral

$$\oint_C (e^{x^2+2} + y) dx + (2x + 3y) dy$$

where C is the closed curve given by the semicircle $y = \sqrt{1 - x^2}$ and the line segment from $(-1, 0)$ to $(0, 1)$.

Solution: By Green's Theorem, the line integral equals the double integral

$$\iint_D 2 - 1 dA = \iint_D 1 dA = \text{area}(D) = \frac{\pi}{2}.$$

Here, D is just half of the unit circle. $\square$

Remark 3.3.22. It is important that the student realizes the Green's Theorem can be used in either direction. Sometimes, you are given a line integral $\oint_C P dx + Q dy$ which is too difficult to evaluate directly and so the double integral $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$ is easier to compute. Sometimes, it is the other way around.

Example 3.3.23. Evaluate the line integral $\oint_C (xe^x + y) dx + (x + e^{\sin(y)}) dy$ where C the curve $\langle (\cos(t) \sin(t), \cos(t)) \rangle$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

Solution: Let D be the simple connected region bounded by the curve C . Then Green's Theorem says that the line integral equals

$$\iint_D (1 - 1) dA = \iint_D 0 dA = 0.$$

□

3.4. Curl and Divergence. Let's begin with a review problem from the first part of the course and a problem about Green's Theorem.

Example 3.4.1 (Second Derivative Test). Find and classify the critical points of the function $f(x, y) = 2x^3 + xy - y^2 + x + y$.

Solution: The partial derivatives are $f_x = 6x^2 + y + 1$, $f_y = x - 2y + 1$. Setting these two equal to zero, we see that $x = 2y - 1$ and $0 = 6x^2 + y + 1$. Use the first equation to write $y = \frac{1}{2}x + 1$ and plug this into the second equation to get

$$0 = 6x^2 + \frac{1}{2}x + 2.$$

Multiply this by 2 to get $0 = 12x^2 + x + 2$ and we can use the quadratic formula to solve this. The solutions are

$$x = \frac{-1 \pm \sqrt{1 - 4(2)(12)}}{24}.$$

These are not real solutions. So, this system of equations has no solutions. Therefore, $f(x, y)$ has no critical points. □

Example 3.4.2 (Review Problem for Green's Theorem). Evaluate $\oint_C (e^{x^2} + 4y) dx + (x + \cos(6y^3)) dy$, where C is the circle with radius $\sqrt{2}$ centered at the origin and oriented counterclockwise.

Solution: By Green's Theorem, if D is the disk centered at the origin with radius $\sqrt{2}$, then the line integral equals

$$\iint_D -3 dA = -3 \text{ area}(D) = -6\pi.$$

□

To prepare ourselves for Stokes' Theorem and Gauss's Theorem, we need to introduce some more concepts. In some sense, a lot of this is "just notation".

Definition 3.4.3. Let $\nabla := \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$. Let $\mathbf{F}(x, y, z) := \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ denote a vector field in $\mathbb{R}^3$. Then the **curl** is defined to be a the vector field

$$\nabla \times \mathbf{F} := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y, z) & Q(x, y, z) & R(x, y, z) \end{vmatrix}.$$

The notation $\text{curl}(\mathbf{F})$ is also used to denote the curl of a vector field.

Remark 3.4.4. The definition of the curl of a vector field as presented only makes sense for vector fields in $\mathbb{R}^3$. It is also important to note that the curl of a vector field is some new vector field.

Example 3.4.5. Evaluate the curl of the vector field $\mathbf{F} = \langle x, y, z \rangle$.

Solution: Following the definition,

$$\nabla \times \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\frac{\partial R}{\partial z} + \frac{\partial P}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

Therefore,

$$\text{curl}(\mathbf{F}) = \langle 0, 0, 0 \rangle.$$

This matches with our intuition that the curl vector field should describe how much “circular” motion there is in our original vector field. $\square$

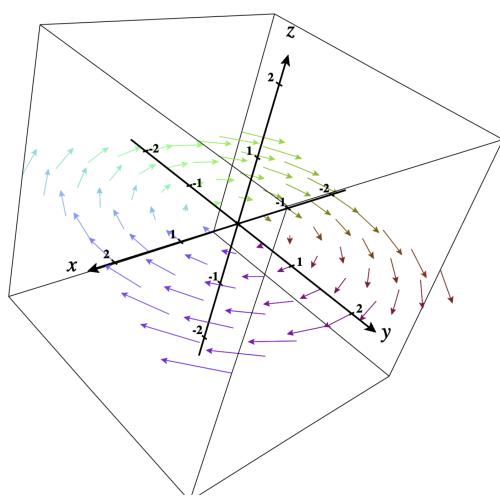
Example 3.4.6. Evaluate the curl of the following vector fields:

- (1) $\mathbf{F} = \langle x^2, 2xz, 2yz \rangle$,
- (2) $\mathbf{F} = \langle y, -x, 0 \rangle$,
- (3) $\mathbf{F} = \langle 3xyz^2, \sin z, xe^z \rangle$,
- (4) $\mathbf{F} = \langle x^2 + y^2, xyz, z \rangle$.

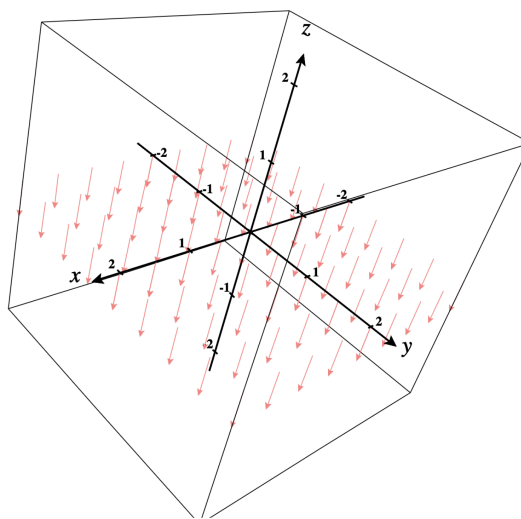
It is also possible to define the curl of a 2-dimensional vector field. The idea is to take the 2-dimensional vector field $\mathbf{F}(x, y) = \langle P, Q \rangle$ and then view it as a 3-dimensional vector field by setting $\mathbf{F}(x, y, z) := \langle P, Q, 0 \rangle$ and taking the curl of that. For example, if $\mathbf{F}(x, y) = \langle y, -x \rangle$ gives me

$$\text{curl}(\mathbf{F}) = -2\mathbf{k}.$$

One should view the curl as describing how much “circular” movement there is in a vector field.



(A) The vector field $\langle y, -x, 0 \rangle$.



(B) The curl of $\langle y, -x, 0 \rangle$ is $\langle 0, 0, -2 \rangle$.

Now we are ready to state the most important set of facts regarding the curl of a vector field. Recall that $\mathbf{F}$ is **conservative** if $\mathbf{F} = \nabla f$ for some f .

Theorem 3.4.7. If $\mathbf{F}$ is a conservative vector field, then $\text{curl}(\mathbf{F}) = \mathbf{0}$.

Proof. If $\mathbf{F}$ is a conservative vector field, then $\mathbf{F} = \langle f_x, f_y, f_z \rangle$ for some function $f(x, y, z)$. It then becomes a straightforward computation to see that $\text{curl}(\langle f_x, f_y, f_z \rangle) = 0$. For example, the first component is

$$\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} = f_{zy} - f_{yz} = 0.$$

The mixed partial derivatives coincide for all sufficiently nice functions so the last equality is true. $\square$

Theorem 3.4.8. If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\text{curl}(\mathbf{F}) = 0$, then $\mathbf{F}$ is conservative.

To state the third most important fact regarding curl, we need another definition.

Definition 3.4.9. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$. Then its **divergence** is defined by

$$\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The divergence of a vector field measures how much fluid “leaves” a point in $\mathbb{R}^3$ as given by the vector field. One might think of it as its “outgoingness”. For fluid flow, the divergence measures the tendency of a fluid to leave a point. If $\text{div} \mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.

Example 3.4.10. Compute the divergence of $\langle 0, 0, 1 \rangle = \mathbf{F}$. What does this say about the behavior of $\mathbf{F}$?

Solution: $\text{div} \mathbf{F} = 0$. There is no point in space where fluid coming out of the point. $\square$

Example 3.4.11. Compute the divergence of $\langle y, -x, 0 \rangle = \mathbf{F}$.

Solution: The divergence is also zero. $\square$

Example 3.4.12. Compute the divergence of $\langle x, y, z \rangle$.

Solution: The divergence is 3. If one looks at the origin, it appears as if there is fluid coming out of that point in space. $\square$

Example 3.4.13. Find $\text{div} \mathbf{F}$ if $\mathbf{F} = \langle x^2yz, y^2z, xy^2 \rangle$.

Solution: The divergence equals

$$\text{div} \mathbf{F} = (2xyz) + (2yz) + 0.$$

$\square$

Example 3.4.14. Find the divergence of $\mathbf{F}$ if $\mathbf{F} = \langle x + \cos x, \sin y + x, -\sin z \rangle$.

Solution: The divergence is equal to

$$\text{div} \mathbf{F} = (1 - \sin(x)) + (\cos(y)) + (-\cos(z)).$$

$\square$

Now we are ready for the main theorem which relates divergence and curl.

Theorem 3.4.15. $\text{div}(\text{curl}(\mathbf{F})) = 0$ for all vector fields $\mathbf{F}$ whose component functions are smooth.

Proof. This follows via a direct calculation. It is essential one use the statement “all mixed partial derivatives coincide” for the proof to work. $\square$

Using curl, we can rewrite Green's Theorem as the statement

$$(1) \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k} \, dA.$$

We can also relate this to divergence as well. Recall that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

If one draws the picture, it is clear that the LHS depended on the fact that we chose the tangent vector to the curve for the computation. On the otherhand, it is totally possible that one could have chosen the normal vector

$$\mathbf{n} := \left\langle \frac{-y^2(t)}{|\mathbf{r}(t)|}, \frac{x(t)}{|\mathbf{r}(t)|} \right\rangle.$$

In this case, one gets the answer

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div}(\mathbf{F}) \, dA.$$

Remark 3.4.16. One might try to ask the following question: “Which vector fields $\mathbf{F}$ (on $\mathbb{R}^3$) is the curl of another vector field?”. The answer is that if $\operatorname{div}(\mathbf{F}) = 0$ for $\mathbf{F}$ a vector field defined on all of $\mathbb{R}^3$, then $\mathbf{F} = \operatorname{curl}(\mathbf{G})$ for some vector field $\mathbf{G}$.

Questions of this nature are a tricky business and the statement in the previous paragraph is harder to prove than Theorem 3.4.15.

We list out some statements regarding curl and divergence (some already mentioned above). One does not need to know (6), (7), (8) off the top of one's head, but they are neat formulas to prove as an exercise.

- (1) $\operatorname{curl}(\nabla f) = \mathbf{0}$,
- (2) $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$,
- (3) curl is linear i.e. the curl of $a\mathbf{F} + b\mathbf{G}$ equals $a \operatorname{curl}(\mathbf{F}) + b \operatorname{curl}(\mathbf{G})$,
- (4) div is linear as well,
- (5) grad is linear,
- (6) $\operatorname{div}(f\mathbf{F}) = f \operatorname{div}(\mathbf{F}) + \mathbf{F} \cdot \nabla f$,
- (7) $\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + (\nabla f) \times \mathbf{F}$
- (8) $\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \operatorname{grad}(\operatorname{div} \mathbf{F}) - \nabla^2 \mathbf{F}$.

Example 3.4.17. Determine whether or not the following statements are true or false.

- If $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$, then $\mathbf{F} = \nabla f$ for a unique $f(x, y, z)$.
- $\operatorname{curl}(\operatorname{curl}(\mathbf{F})) = \mathbf{0}$ for any vector field $\mathbf{F}$ on $\mathbb{R}^3$.
- $\operatorname{grad}(f) = \mathbf{0}$ implies $f(x, y, z)$ is a constant function.
- If $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is path independent, then $\mathbf{F} = \operatorname{grad}(f)$ for some f .
- The sum of two conservative vector fields is also conservative.
- If $\operatorname{div}(\mathbf{F} + \mathbf{G}) = 0$, then $\operatorname{div}(\mathbf{F}) = 0$.

Remark 3.4.18 (Not for Students).

$$0 \rightarrow \mathbb{R} \rightarrow C^\infty(\mathbb{R}^3; \mathbb{R}) \xrightarrow{\operatorname{grad}} C^\infty(\mathbb{R}^3; \mathbb{R}^3) \xrightarrow{\operatorname{curl}} C^\infty(\mathbb{R}^3; \mathbb{R}^3) \xrightarrow{\operatorname{div}} C^\infty(\mathbb{R}^3; \mathbb{R}) \rightarrow 0$$

is exact.

3.5. Parametric Surfaces. So far, we have been parameterizing curves. Next, we shall parameterize surfaces. The idea is more or less the same but instead of one parameter t , we use two parameters u, v . Just as we used vector functions to parameterize curves, we shall use vector functions to parameterize surfaces

$$\mathbf{r}(u, v) := \langle x(u, v), y(u, v), z(u, v) \rangle.$$

One can also impose conditions on u, v e.g. $0 \leq u \leq 1, -2 \leq v \leq 2$.

Example 3.5.1. Parameterize the cylinder. $x^2 + y^2 = 16$.

Solution: We can use two variables θ, z to do this:

$$\mathbf{r}(\theta, z) := \langle 4 \cos(\theta), 4 \sin(\theta), z \rangle \text{ where } 0 \leq \theta \leq 2\pi, z \in \mathbb{R}$$

is a vector function in the variables θ, z . □

Example 3.5.2. Parameterize the sphere

$$x^2 + y^2 + z^2 = 81.$$

Solution: This is done via spherical coordinates. We write down a vector function in variables θ, ϕ below

$$\mathbf{r}(\theta, \phi) := \langle 9 \cos \theta \sin \phi, 9 \sin \theta \sin \phi, 9 \cos \phi \rangle \text{ where } 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$$

and it clearly parameterizes the sphere. □

Example 3.5.3. Parameterize the surface given by

$$x^2 + y + z = 3 \text{ where } 0 \leq x \leq 1, -1 \leq y \leq 1.$$

Solution: Rewrite the equation as $z = 3 - x^2 - y$. Then, the vector function

$$\mathbf{r}(x, y) = \langle x, y, 3 - x^2 - y \rangle \text{ for } 0 \leq x \leq 1, -1 \leq y \leq 1$$

parameterizes the desired surface. □

Example 3.5.4. Find a parametrization of the plane containing $\mathbf{a} = \langle 1, 0, -1 \rangle$ and $\mathbf{b} = \langle 0, 1, -1 \rangle$. Hint: Find the equation of the plane.

Many surfaces are obtained by taking a curve $y = f(x)$ in the xy -plane and then rotating it around the x -axis. One already saw this in single variable calculus where one used the washer method.

Here is how to parameterize such surfaces. First, take the curve $y = f(x)$ and rotate it around the x -axis. Then, the surface is parameterized by

$$\langle x, f(x) \cos \theta, f(x) \sin \theta \rangle$$

where the parameters are x and θ . Then, let $x \in \mathbb{R}$ vary and $0 \leq \theta \leq 2\pi$.

Example 3.5.5. Consider the curve $y = x$ in the xy -plane and rotated about the x -axis. This draws a curve. Write down the parameterization of this surface using the above formulae.

Solution: Consider

$$\mathbf{r}(x, \theta) := \langle x, x \cos \theta, \sin \theta \rangle \quad \text{where } x \in \mathbb{R}, \quad 0 \leq \theta \leq 2\pi.$$

□

Example 3.5.6 (Parameterization of the Solid Torus). Consider a solid torus centered at the origin radius R and tube having radius r . Then the torus can be parameterized by the vector function

$$\langle (R + r \cos(\theta)) \cos(\psi), (R + r \cos(\theta)) \sin(\psi), r \sin(\theta) \rangle$$

where $0 \leq \theta < 2\pi$, and $0 \leq \psi < 2\pi$.

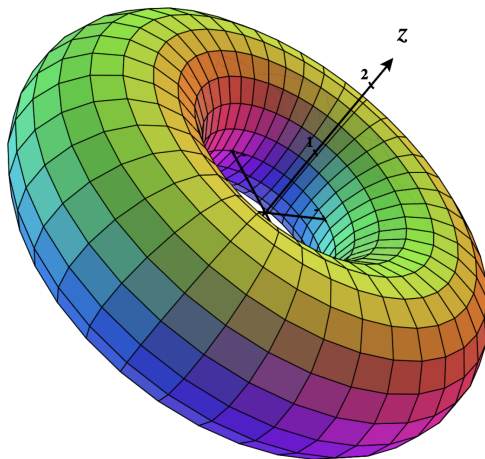


FIGURE 20. $\langle (2 + \cos(\theta)) \cos(\psi), (2 + \cos(\theta)) \sin(\psi), \sin(\theta) \rangle$.

Example 3.5.7. The third sort of surface one often faces is one cut out by $f(x, y) = z$. These are very easy to parameterize

$$\langle x, y, f(x, y) \rangle \quad \text{where } (x, y) \in \mathbb{R}^2.$$

For example, the equation $\frac{x^2}{4} + \frac{y^2}{9} = z$ is pictured in Figure 21.

Next, we talk about normal planes and tangent planes to surfaces. We already did this for surfaces given by an equation $z = f(x, y)$.

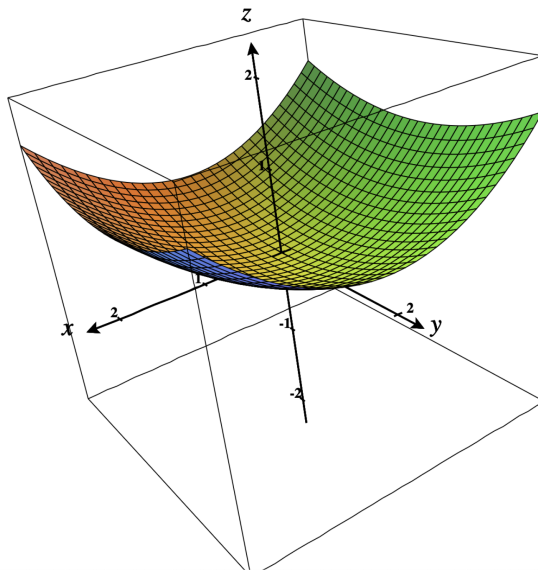
To formulate the correct answer for a parameterized surface, let us assume S is a surface parameterized by

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle.$$

Let

$$\begin{aligned} \mathbf{r}_u(u, v) &= \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \\ \mathbf{r}_v(u, v) &= \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle. \end{aligned}$$

Now if S is a **smooth surface**, then $\mathbf{r}_u \times \mathbf{r}_v \neq 0$ for all (u, v) in the domain used. If S is a smooth surface, then the tangent plane at the point $\mathbf{r}(u_0, v_0)$ has normal vector given by $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$.

FIGURE 21. $z = \frac{x^2}{4} + \frac{y^2}{9}$

The normal line through the point $\mathbf{r}(u_0, v_0)$ has line given pointing in the direction of $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$.

Example 3.5.8. Find an equation for the tangent plane to the surface S parameterized by $\mathbf{r}(u, v) = \langle 3u, 6u^2 - v^2, 2v \rangle$ at the point $(3, 5, 2)$.

Solution: The partial derivatives are $\mathbf{r}_u(u, v) = \langle 1, 6u, v \rangle$ and $\mathbf{r}_v(u, v) = \langle 0, 1, 1 \rangle$. The cross product with $v = 1$ equals $\langle 5, -1, 1 \rangle$. So, the tangent plane has equation given by

$$5(x - 3) + (-1)(y - 5) + (z - 2) = 0.$$

□

Example 3.5.9. Find an equation for the tangent plane to the surface S parameterized by $\mathbf{r}(u, v) = \langle u, 6u + v, v \rangle$ at the point $(0, 1, 1)$.

Example 3.5.10. Find an equation for the tangent plane to the surface S parameterized by $\mathbf{r}(u, v) = \langle 3u + 2v, 5u^3, v^2 \rangle$ at the point $(1, 5, 1)$.

With the normal vector to a point $\mathbf{r}(u_0, v_0)$ defined, we can write down an integral which computes the **surface area** of a surface. Assume S is parameterized by $\mathbf{r}(u, v)$ with $(u, v) \in D$. Assume S is covered once as (u, v) ranges over D . Then the **surface area** of S is given by the integral

$$\iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA.$$

Example 3.5.11. Find the surface area of a sphere of radius 1 centered at the origin. Hint: 4π is the answer. In general, the surface area of a sphere of radius r is $4\pi r$.

Example 3.5.12. Find the surface area of the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 0$.

3.6. Surface Integrals. The goal of the first part of this subsection is to define **surface integrals** i.e. an integral which integrates over a surface (as opposed to a curve or even a line).

Definition 3.6.1. Assume S is smoothly parameterized by $\mathbf{r}(u, v)$ with $(u, v) \in D \subseteq \mathbb{R}^2$ (the uv -plane). The surface integral of a function $f(x, y, z)$ over the surface S is given by

$$\iint_S f(x, y, z) d\mathbf{S} = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA.$$

Remark 3.6.2. One should regard surface integrals as a generalized of line integrals where one integrates over a surface instead of a curve.

- 1) $\iint_S 1 d\mathbf{S}$ computes the **surface area** of S .
- 2) This formula is similar to how

$$\int_C f(x, y, z) ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt.$$

- 3) Why does $|\mathbf{r}_u \times \mathbf{r}_v|$ show up?

Example 3.6.3. Evaluate $\iint_S ye^{x^2+y^2} d\mathbf{S}$ where S is given by $x^2 + z^2 = 9$ and $0 \leq y \leq 2$.

Solution: Use the parameterization

$$\langle 3 \cos \theta, y, 3 \sin \theta \rangle \text{ where } 0 \leq \theta \leq 2\pi, 0 \leq y \leq 2.$$

One can then show that

$$\iint_S ye^{x^2+z^2} dS = \iint_D ye^9 \cdot 3 d\theta dy = \int_0^{2\pi} \int_0^2 3ye^9 dy d\theta = 12\pi e^9.$$

□

Example 3.6.4. Evaluate $\iint_S (x^2 + y^2) d\mathbf{S}$ where

$$S := \{(x, y, z) \mid z = 2x + 3y + 5, 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Solution: The surface can be parameterized by $\mathbf{r}(u, v) = \langle u, v, 2u + 3v + 5 \rangle$ where $0 \leq u \leq 1$ and $0 \leq v \leq 1$. Then,

$$|\mathbf{r}_u \times \mathbf{r}_v| = |\langle 1, 0, 2 \rangle \times \langle 0, 1, 3 \rangle| = | \langle -2, -3, 1 \rangle | = \sqrt{4 + 9 + 1} = \sqrt{14}.$$

Then, we get

$$\iint_S (x^2 + y^2) d\mathbf{S} = \int_0^1 \int_0^1 (u^2 + v^2) \sqrt{14} du dv = \sqrt{14} \frac{2}{3}.$$

□

Example 3.6.5. Evaluate $\iiint_S \sqrt{1 + 4x^2 + 4y^2} d\mathbf{S}$ where

$$S = \left\{ (x, y, z) \mid \begin{array}{l} z = x^2 + y^2 \\ 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \end{array} \right\}.$$

Solution: This is left as an exercise for the student. $\square$

It is useful to know that if S is given by a function $z = g(x, y)$, then there is a formula one can always apply instead of having to compute the cross product each time.

If S given by $z = g(x, y)$, with (x, y) in D then

$$\iint_S f(x, y, z) d\mathbf{S} = \iint_D f(x, y, g(x, y)) \sqrt{1 + (g_x)^2 + (g_y)^2} dA.$$

One can take $\mathbf{r}(x, y) = \langle x, y, g(x, y) \rangle$ and compute the normal vector $\mathbf{r}_u \times \mathbf{r}_y$ quite easily:

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{pmatrix} = \langle -g_x, -g_y, 1 \rangle.$$

Example 3.6.6. Evaluate $\iint_S y d\mathbf{S}$ where S is the surface $z = x^2 + y^2$ with $0 \leq x \leq 1$ and $0 \leq y \leq 2$.

Solution: The above formula gives

$$\iint_S y d\mathbf{S} = \int_0^1 \int_0^2 y \sqrt{1 + 1 + 4y^2} dy dx = \dots = \frac{13\sqrt{2}}{3}.$$

$\square$

Just as we introduced vector fields to the study of line integrals, we too shall introduce vector fields to the surface integral story.

Let S be a surface. The **unit normal vector** at a point of S is the vector that is normal to the tangent vector at that point. An **orientation** of S is a continuous choice of unit tangent vectors to the surface. Not all surfaces can be oriented. For example, the Möbius band has no orientations. We say a surface is **orientable** if one can find an orientation for it.

Example 3.6.7. The following surfaces are orientable:

- the sphere,
- the flat plane given by $z = 0$ inside $\mathbb{R}^3$,
- all quadric surfaces that we discussed at the start of the course.

The following surfaces are not orientable:

- the Möbius band,
- there are many more out there which one can mention e.g. the Klein bottle, the real projective plane etc.. The student does not need to know this fact.

Definition 3.6.8. Let S be a surface parameterized by $\mathbf{r}(u, v)$. Then the outward orientation is given by

$$\mathbf{n}(u, v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

Using this, we can define surface integrals with respect to a vector field $\mathbf{F}$ on an orientable surface. If S is parameterized by $\mathbf{r}(u, v)$, then

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\mathbf{S} = \iint_S \mathbf{F} d\mathbf{S} := \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA.$$

One typically refers to the value of this integral as the **flux through the surface** S . It essentially measures the amount of fluid moving in or out of a surface. The notation $\iint_S \mathbf{F} d\mathbf{S}$

is often used when the unit normal vector is already specified in the problem or is clear from context.

If S is the graph of a function $f(x, y)$, then this integral can even be rewritten as

$$\iint_D -P \frac{\partial f}{\partial x} - Q \frac{\partial f}{\partial y} + R dA$$

where $\mathbf{F} = \langle P, Q, R \rangle$.

Example 3.6.9. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, for $\mathbf{F} = \langle 0, yz, z^2 \rangle$ where S is the surface $y^2 + z^2 = 1$, $z \geq 0$, $x = 0$, $x = 1$, the one gives S the outward orientation.

Solution: One can parameterizes S by $\mathbf{r}(u, v) = \langle u, \cos v, \sin v \rangle$ where $0 \leq u \leq 1$ and $0 \leq v \leq \pi$. Then, one can compute

$$\mathbf{n}(u, v) = \langle 0, -\cos v, -\sin v \rangle.$$

Note that this is the not the normal vector that gives the outward orientation (one can draw a picture). The negative of this vector does. So, the surface integral equals

$$\int_0^1 \int_0^\pi \langle 0, \cos(v) \sin(v), \sin^2(v) \rangle \cdot \langle 0, \cos(v), \sin(v) \rangle dv du = \int_0^1 \int_0^\pi \cos^2 v \sin v + \sin^3 v dv du.$$

We leave it as an exercise to check that this double integral equals 2. $\square$

3.7. Stokes' Theorem. Let's start with some warmup problems for today's lecture.

- (1) Parameterize the paraboloid $z = 4 - x^2 - y^2$ where $z \geq 0$.
- (2) Attempt to evaluate the line integral $\oint_C (e^{x^2} + 5x^4 y) dx + (\sin(y^2) + x^5) dy$ where C is the curve $\langle \cos(t), 2t^3 \rangle$ and $0 \leq t \leq 1$. What happens?
- (3) Evaluate the line integral $\oint_C (5x^2 + 2y) dx + (xy) dy$ where C is the positively oriented curve traversed once and given by the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$.
- (4) Determine which of the following are true or false:
 - $\operatorname{div}(f\mathbf{F}) = f \operatorname{div}(\mathbf{F}) + \operatorname{grad}(f) \cdot \mathbf{F}$.
 - $\oint_C \mathbf{F} \cdot d\mathbf{r} \geq 0$ if $\mathbf{F}$ is conservative.
- (5) Evaluate the line integral $\oint_C (3e^{x^2} + y) dx + (2e^y + \sin(y^2) + x) dy$ where C is the simple closed curve given $\frac{x^4}{3} + y^4 = 1$.
- (6) Set-up a triple integral which computes the volume of the tetrahedron with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 5, 0)$, and $(0, 0, 3)$.
- (7) Maximize the area of a rectangle inscribed inside the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

Solution:

- (1) A valid choice of parameterization is $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 4 - u^2 \rangle$ where $0 \leq u \leq 2$ and $0 \leq v \leq 2\pi$.
- (2) The curve is rather annoying – it is not closed for example. Thus, Green's Theorem is not applicable. So the reasonable approach is to find a potential function $f(x, y)$. One can check the partial derivatives and see that this is indeed a conservative vector field. However, finding a potential function explicitly is quite difficult because $\int e^{x^2} dx$ and $\int \sin(y^2) dy$ cannot be written in terms of elementary functions.

This does not mean one cannot compute this. Indeed, if we defined $f(x, y) = E(x) + S(y) + x^5 y$ where $E(x) = \int e^{x^2} dx$ and $S(y) = \int \sin(y^2) dy$ are *some* anti-derivative, then the fundamental theorem of line integrals says that the line integral

equals

$$E(\cos(1)) + S(2) + 2\cos(1)^5 - E(\cos(0)) - S(0) = 0.$$

One can actually ask a computer to do this computation – the answer is approximately 0.742169509953.

- (3) One can apply Green's Theorem to this line integral because C is actually a simple positively oriented curve. In that case, we are integrating over

$$\iint_D (y - 2) dA$$

where D is the region bounded by the ellipse. We leave it as an exercise to the student to evaluate this double integral. **Hint:** Use a change of variables $x = 2u$ and $y = 3v$. This basically transforms the ellipse into a disk which is easier to evaluate. The final answer should be -12π .

- (4) The answers are True and False respectively.

The first one is true basically by the product rule. The second is false which one can see by example.

- (5) The vector field involved is conservative. So, the line integral must be zero since we are integrating over a simple closed curve.

- (6) Let's write down an equation for the plane determining one of the tetrahedron's face: $\frac{1}{2}x + \frac{1}{5}y + \frac{1}{3}z = 1$. In that case, this face is given by the plane $z = 3 - \frac{3}{2}x - \frac{3}{5}y$.

A triple integral for the volume of the tetrahedron is then

$$\iiint_E dV = \int_0^3 \int_0^{3-\frac{3}{2}x} \int_0^{3-\frac{3}{2}x-\frac{3}{5}y} dz dy dx.$$

- (7) The goal is to maximize the function $f(x, y) = 4xy$ where (x, y) is a point on the ellipse with $x \geq 0$ and $y \geq 0$. The idea is to use Lagrange multipliers and compute

$$4y = \nabla 2x$$

$$4x = \nabla 2y$$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

□

We now have all the tools we need to state Stokes' Theorem.

Theorem 3.7.1 (Stokes' Theorem). Let S be an oriented piecewise smooth surface with boundary C which is a simple, closed, piecewise smooth curve with positive orientation (i.e. if you walk along the curve, the normal vector points up and the surface is to your left). Let $\mathbf{F}$ be a vector field whose components are differentiable on an open region containing S . Then,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Remark 3.7.2. Let me provide some comments on the philosophy of Stokes' Theorem and the impact it has on calculus as a whole.

One can rephrase the Fundamental Theorem of Calculus (I would wager, was how Newton originally thought of it anyway) as the statement that “the total change of a function / the flux” as related to the “behavior of the function at the boundary / total work down along

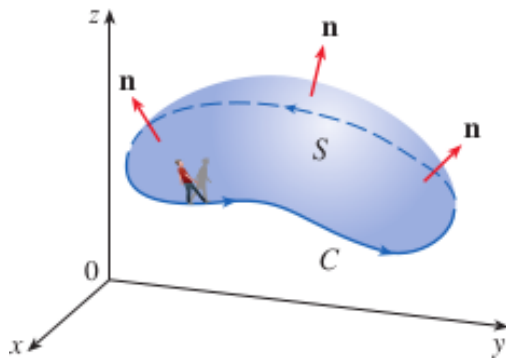


FIGURE 22. From Stewart's "Calculus 9th Ed."

the boundary". More precisely, the integral of "the derivative" over a region M should equal the integral of "the antiderivative" over ∂M .

The student does not need to know any of the following. In this language, one can then start to ask if there exists a "generalized Stokes Theorem" that works in all dimension (i.e. works for manifolds with boundary in general). The answer is yes, a generalized exists and is sometimes called the Stokes-Cartan Theorem: $\int_M d\omega = \int_{\partial M} \omega$ under certain assumptions. This generalized theorem implies all of the "theorems" we have seen/shall see in this course.

Remark 3.7.3. It is critical that the orientation for C matches the orientation determined by the unit normal vectors $\mathbf{n}$ for the surface.

Remark 3.7.4. It is critical that $\text{curl } \mathbf{F}$ appears in the surface integral one wants to apply Stokes' Theorem to. If one wishes to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ in general and S is some surface with boundary, then there is not much one can do in this class except a direct computation.

Remark 3.7.5. Stokes' Theorem implies Green's Theorem. This is precisely the content of Equation (1).

Example 3.7.6. Check Stokes' Theorem for

$$\mathbf{F} = y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}, \text{ where}$$

S is the cone $x^2 + y^2 = z^2$ laying above the xy -plane and the boundary is given by the circle $x^2 + y^2 = 1$ with $z = 1$.

Solution: The boundary curve is the circle parameterized by $\langle \cos(t), \sin(t), 1 \rangle$. So,

$$\int_0^{2\pi} \langle \sin(t), -\cos(t), 0 \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle dt = \int_0^{2\pi} -\sin^2(t) - \cos^2(t) dt = \int_0^{2\pi} -1 dt = -2\pi.$$

Meanwhile,

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \langle 0, 0, -2 \rangle$$

Now let's try to compute the surface integral. First off, we will use spherical coordinates to parameterize the surface. In that case, we get

$$\langle \rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi) \rangle$$

with the condition that $\rho^2 \sin^2(\phi) = \rho^2 \cos^2(\phi)$. In other words, $\sin(\phi) = \cos(\phi)$ and this occurs precisely when $\phi = \frac{\pi}{4}$. So altogether, the surface is parameterized by

$$\mathbf{r}(\rho, \theta) = \left\langle \frac{\sqrt{2}}{2}\rho \cos(\theta), \frac{\sqrt{2}}{2}\rho \sin(\theta), \rho \frac{\sqrt{2}}{2} \right\rangle \text{ where } 0 \leq \theta \leq 2\pi, 0 \leq \rho \leq \frac{2}{\sqrt{2}}.$$

We leave following claim to the student to verify

$$\mathbf{n} = \frac{\sqrt{3}}{2}\rho \left\langle -\frac{1}{2}\rho \cos(\theta), -\frac{1}{2}\rho \sin(\theta), \frac{1}{2}\rho \right\rangle.$$

Taking the dot product implies

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} d\mathbf{S} = \int_0^{\frac{2}{\sqrt{2}}} \int_0^{2\pi} -\rho d\theta d\rho = -2\pi \int_0^{2/\sqrt{2}} \rho d\rho = -2\pi.$$

□

Example 3.7.7. Use Stokes' Theorem to compute

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = \langle -y, x, z \rangle$ and S is the paraboloid $z = 5 - x^2 - y^2$ with $z \geq 1$. Assume the normal vector for the paraboloid is oriented in the positive z direction.

Solution: The boundary curve the paraboloid is the intersection of S with the plane $z = 1$. So, the curve is given by the conditions $1 = 5 - x^2 - y^2$ and $z = 1$. So, $x^2 + y^2 = 4$ and $z = 1$. This is precisely the circle C parameterized by

$$\mathbf{r}(t) = \langle 2 \cos(t), 2 \sin(t), 1 \rangle \text{ for } 0 \leq t \leq 2\pi$$

and our choice of parameterization is consistent with the conditions needed for Stokes' Theorem (i.e. C is positively oriented). Anyway, $\mathbf{r}'(t) = \langle -2 \sin(t), 2 \cos(t), 0 \rangle$ and so,

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} d\mathbf{r} \\ &= \int_0^{2\pi} \langle -2 \sin(t), 2 \cos(t), 1 \rangle \cdot \langle -2 \sin(t), 2 \cos(t), 0 \rangle dt \\ &= \int_0^{2\pi} 4 \sin^2(t) + 4 \cos^2(t) dt \\ &= \int_0^{2\pi} 4 dt = 8\pi. \end{aligned}$$

□

Example 3.7.8. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F} = z^2 \mathbf{i} + y^2 \mathbf{j} + x \mathbf{k}$$

and C is the triangle in space with vertex points

$$(1, 0, 0), (0, 1, 0), (0, 0, 1).$$

Hint: This triangle sits inside the plane

$$\begin{aligned} x + y + z &= 1, \text{ so} \\ z &= 1 - x - y. \end{aligned}$$

Solution: This is an example where using Stokes' Theorem can make the computation of a line integral easier. The answer equals $\int_0^1 \int_0^{-x+1} 2(1-x-y) - 1 dy dx = -\frac{1}{6}$. $\square$

Example 3.7.9. Use Stoke's Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle y, -x, yx^3 \rangle$ and S is the portion of the sphere of radius 4 sitting above the $z = 0$ plane and has normal vectors pointing upwards (which determines its orientation).

Solution: The circle C for the boundary is parameterized by $\mathbf{r}(t) = \langle 4 \cos(t), 4 \sin(t), 0 \rangle$ for $0 \leq t \leq 2\pi$. It is not hard to then see that

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \langle 4 \sin(t), -4 \cos(t), 256 \sin(t) \cos^3(t) \rangle \cdot \langle -4 \sin(t), 4 \cos(t), 0 \rangle \\ &= -16(\sin^2(t) - \cos^2(t)) = -16. \end{aligned}$$

Therefore,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -16 dt = -32\pi.$$

$\square$

Example 3.7.10. Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where

$$\mathbf{F} = \langle x^2 y^3 z, \sin(xyz), xyz \rangle$$

and S is the part of the cone $y^2 = x^2 + z^2$ that lies between the lines $y = 0$ and $y = 2$ oriented in the direction of the positive y -axis.

Example 3.7.11 (Invariance of Surface).⁵ Compute $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where

$$\mathbf{F} = \langle -yz, x, z^3 + z^2 + z \rangle$$

and S is the surface pictured in Figure 23.

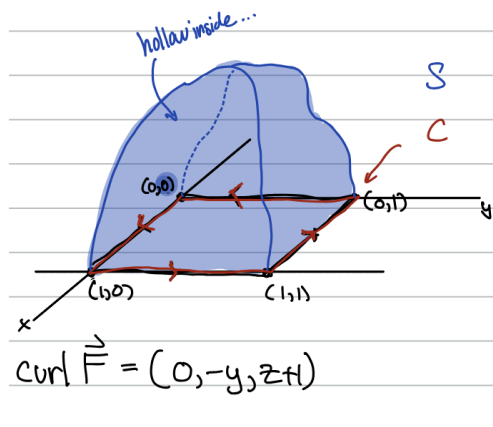


FIGURE 23. The only part of the surface sitting in the xy -plane is the square draw in red which is the surface's boundary.

⁵This is unlikely to be tested on any exam.

Solution: One can flatten the surface down to the xy -plane. So one is really just integrating over the solid square D . The normal vector points straight up. So we get (since $z = 0$)

$$\iint_D \langle 0, -y, 1 \rangle \cdot \langle 0, 0, 1 \rangle dA = \iint_D 1 dA = 1.$$

□

3.8. The Divergence Theorem (aka Gauss's Theorem). We also have enough to state Gauss's Theorem or more often referred to as the Divergence Theorem.

Theorem 3.8.1 (Divergence Theorem). Let E denote a solid in $\mathbb{R}^3$ and S denote the surface that is the boundary of E . Assume E is a simple solid region and that $\mathbf{F}$ is a vector field with continuous partial derivatives. Then,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV$$

where the orientation for S is the outward orientation (i.e. points away from E).

Remark 3.8.2. For exam questions and most exercises in this class, one will always use the outward orientation when applying the Divergence Theorem. Otherwise, one has to put a negative sign in front of the triple integral.

This theorem is also in the spirit of Stokes' Theorem. Since we do not talk about orientation of solids in this class, the student should treat this theorem as a blackbox.

Example 3.8.3. Evaluate $\iint_S \mathbf{F} d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS$ when $\mathbf{F} = \langle e^x \sin y, e^x \cos y, yz^2 \rangle$ and S is the box given by the planes

$$x = 0, x = 3, y = 0, y = 1, z = 0, z = 2.$$

Solution: Apply the Divergence Theorem to rewrite this as a triple integral. The divergence of the given vector field is

$$\operatorname{div} \mathbf{F} = e^x \sin y - e^x \sin y + 2zy = 2zy.$$

So,

$$\iiint_S \mathbf{F} d\mathbf{S} = \int_0^3 \int_0^1 \int_0^2 2zy dz dy dx = 6.$$

□

Example 3.8.4. Evaluate $\iint_S \mathbf{F} d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS$ when $\mathbf{F} = \langle x, y, z \rangle$ and S is the surface given by the cylinder $x^2 + y^2 = 1$ and planes $z = 2, z = 0$.

Solution: Again, you apply the Divergence Theorem. The divergence of the given vector field equals 3. So, we need to compute the triple integral

$$\iiint_E 3 dV$$

where E is the solid enclosed by the surface S . In that case, this becomes

$$\iiint_E 3 dV = \int_0^2 \int_0^{2\pi} \int_0^1 3r dr d\theta dz = \int_0^2 \int_0^{2\pi} \frac{3}{2} d\theta dz = 6\pi.$$

□

Example 3.8.5. Evaluate $\iint_S \mathbf{F} dS$ where $\mathbf{F}(x, y, z) = \langle x + 2y + \cos(z), x^3 + 2y + z^4, e^{x^2+y^2} + z \rangle$ and S is the outer surface of the solid sphere E of radius 1 centered at the origin.

Solution: Again, apply the Divergence Theorem. There really isn't much hope in computing this directly in a timely manner. This gives you

$$\iint_S \mathbf{F} \, dS = \iiint_E 1 + 2 + 1 \, dV = \iiint_E 4 \, dV = \int_0^\pi \int_0^{2\pi} \int_0^1 4\rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi = \frac{16\pi}{3}.$$

□

3.9. Review. Let us just do a bunch of problems. For each of the following, decide which of Green's Theorem, Stokes' Theorem, the Divergence Theorem, and the Fundamental Theorem of Line Integrals is most applicable.

- (1) $\int_C 2x^2y^2dx + 3x^3ydy$ where C is the that bounds the triangle with corner points $(0, 0)$, $(2, 2)$, $(0, 2)$ and is oriented counterclockwise.
- (2) $\int_C (3x + y)ds$ and C is a straight line from $(0, 0)$ to $(3, 2)$.
- (3) $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle y^2, y - 2x \rangle$ and C is a straight line from $(0, 0)$ to $(5, 1)$.
- (4) $\iint_S \mathbf{F} \cdot d\mathbf{S}$ outward through S which is the sphere of radius 1 centered at $(0, 0, 0)$ and

$$\mathbf{F} = \langle 2x + z^3e^y, y + \ln(x), z + x^2 \rangle.$$

- (5) $\oint_C ye^x dx + (e^x + 3e^y) dy$ and C is a nasty path from $(1, 1)$ to $(5, 3)$.
- (6) Verify Stokes's Theorem for $\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} d\mathbf{S}$ where S is the cone $z = \sqrt{x^2 + y^2}$ oriented downwards and out, and $\mathbf{F} = \langle -yz, xz, z^2 \rangle$.

Here are two common questions that are asked regarding this last week's lecture content.

- (1) How does one figure out the orientation for Stokes' Theorem?

So someone asked about "plugging into $r(t)$ and n " to make sure something matched up etc.

So here's the short story.

(a) Your normal vector depends on the parameterization and how you choose to compute it in the surface integral. For example, if you compute the normal vector for $r(u, v)$ using $r_u \times r_v$, then fine. But you should realize that $r_v \times r_u$ is also a normal vector and the two differ by a sign i.e. $r_u \times r_v = -(r_v \times r_u)$. So which do you pick? You need to pick the normal vector that matches up with the orientation that you are being told to use.

(b) You want the orientations to be compatible with the ones involved in your computations. For the most part, the orientation on the curve is induced by the orientation chosen by the normal vectors to your surface. But this in and of itself is not canonical. Does the normal vector indicate the counterclockwise orientation? Or does it indicate the clockwise orientation?

The counterclockwise is typically the one chosen, but there are people out there who use the other convention as well (rare but they exist). I don't really care which one you use as long as you are clear regarding it.

For the example from yesterday's class regarding the cone, technically the orientation clockwise is the correct one if one follows the right hand rule. But if you met someone who used the other rule, then the counterclockwise one is the one you would get.

The short answer is you just gotta say clearly what convention you are using and explain why you pick the orientations the way you do in the problem. Overall this is a very minor issue.

- (2) Someone asked about change of variables problems again. Let's just run through another example: $\iint_R x + 2y \, dA$ where $x = \frac{1}{2}(u - v)$ and $y = \frac{1}{4}(3u + v + 12)$ and R is the triangle with vertices $(0, 3)$, $(4, 1)$, and $(2, 6)$.

The transformed boundary curves are $v = 0$, $v = 2u - 8$, and $u = 0$.

The transformed integral is

$$\int_0^4 \int_{2u-8}^0 (u + 3) \, dv \, du$$

provided one computes the Jacobian and sees it is $\frac{1}{2}$.

The whole gimmick with these is that you need to sort of compute the boundary curves for the region and then shift it over.

- (3) Someone asked to review 2nd derivative test. Here's an exercise: find and classify $f(x, y) = -x^3 + 12xy - 8y^3$. The answers are there is a saddle point at $(0, 0)$ and a local maximum at $(2, 1)$.
- (4) Someone asked about the divergence theorem: does one need to worry about the orientation for the surface integral when it comes to the divergence theorem? The short answer is yes. You're always trying to use the outward flux so the orientation on the surface should point away from the solid.
- (5) Someone asked about finding limits.

For a lot of limits, you need to sort of think like a Calculus I student – especially if you're trying to decide if it exists.

For example, does the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y^2}{x^2 + y^2}$ exist? Yes it does. Because near the origin, the numerator behaves like x^5 while near the origin the denominator behaves more like x^2 . So you'd definitely expect it to be convergent to zero. The justification for this one is basically via Squeeze Theorem.

You can then do the same thing for stuff like $\lim_{(x,y) \rightarrow (0,0)} \sin^2(x) \frac{x^2 + y^2}{e^{x^2}}$. This limit exists and it is zero. Why? Because you either use Squeeze Theorem or just realize that everything is continuous and plugging in the origin is valid.

For most limit problems, what you should do is decide if it exists or not. You can do this by studying the behavior of the numerator and denominator separately. Or, you can start looking at what happens if you follow various paths.

For example,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x + y}{x^2 - y^2}$$

might scare people because they'll think it exists. But you can see that if you follow different paths $y = kx$ you get different answers.

- (6) Someone asked about line integrals and finding the bounds or even just parameterizing space curves. Way too open ended for there to be a good answer – the quick answer is that you imagine following the path given by your parameters t .

If I wanted to parameterize a circle of radius 4 going around the z -axis at $z = 3$? Try using polar coordinates $r(t) = \langle 4 \cos(t), 4 \sin(t), 3 \rangle$.

What about a line in space? Well, you got the formula of a line.

What about the intersection of two surfaces e.g. a tilted plane $3x + 2y + z = 0$ and the cylinder $4 = x^2 + y^2$. Well, you just rewrite the third coordinate in terms of x, y i.e. $z = -3x - 2y$ and then use

$$\langle 2 \cos(t), 2 \sin(t), -6 \cos(t) - 4 \sin(t) \rangle.$$

Now if you were being more difficult and wanted a plane $2x + 2y - z = 0$ to intersect $z = 4 - x^2 - y^2$, then you'd have way more work to do. The reason is that you'd be staring at some sort of tilted ellipse.

The short answer is yes there is a way to do it. The long answer is yes but no I will not do it because it is tedious. The idea is you would set the z -values equal to each other. Okay that gives you

$$2x + 2y = 4 - x^2 - y^2.$$

You can then solve the equation to get $x^2 + 2x + y^2 + 2y = 6$. Then this becomes $(x + 1)^2 + (y + 1)^2 = 6$. So you basically get some sort of circle here. Then you can parameterize as

$$\langle \sqrt{6} \cos(t) - 1, \sqrt{6} \sin(t) - 1, 4 - (\sqrt{6} \cos(t) - 1)^2 - (\sqrt{6} \sin(t) - 1)^2 \rangle.$$

