Notes for Graduate Real Analysis

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CHAPTER 1

Real Analysis: Modern Techniques and Their Applications

Abstract. This document consists of my study notes for Math 240ABC taken during the 2020-2021 academic school year at UCSD. Sprinkled across these notes include my own bits of exposition for myself and also an "appendix course notes" which detailed some problem solving strategies for myself. I do not expect these notes to be useful for anyone other than myself. These notes are likely riddled with mistakes (including some that I am aware of) and they are all my own fault.

1. Measures

1.1. Introduction.

Theorem 1.1. There does not exist a set function $\mu : \mathbb{R}^n \to [0, \infty]$ that is countablly additive over disjoint sets, preserved by translations, rotations, and reflections, and $\mu(Q) = 1$ for Q being the unit cube.

Proof. Assume $n = 1$ and the construction easily generalizes. Define $x \sim y$ on $[0, 1)$ when $x - y \in \mathbb{Q}$ and this is an equivalence relation. Let N be the subset consisting of one representative form each equivalence class. Define

$$N_r := \{x + r : x \in N \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in N \cap [1 - r, 1)\}$$

and these sets are disjoint i.e. $N_r \cap N_s = \emptyset$ when $r \neq s$. Also, $\mu(N) = \mu(N \cap [0, 1 - r)) + \mu(N \cap [1 - r, r)) = \mu(N_r)$. Then, $1 = \mu([0, 1)) = \sum_{r \in R} \mu(N_r)$ can never be satisfied. □

1.2. σ-algebras.

Definition 1.2. Fix X nonempty. An **algebra** of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X closed under finite unions and complements. A σ-algebra is an algebra that is also closed under countable unions. If $\mathcal{E} \subseteq P(X)$, then $\mathcal{M}(\mathcal{E})$ is the **σ-algebra generated by** $\mathcal{E}$ which is the intersection of all σ-algebras containing $\mathcal{E}$ (since intersections of σ-algebras are still σ-algebras). Clearly, $\mathcal{E} \subseteq \mathcal{F} \implies \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$. If X is a topological space, the Borel σ-algebra $\mathcal{B}_X$ is the σ-algebra generated by the open sets. The G_δ sets are countable intersections of open sets while F_σ are countable unions of closed sets. In the case of $X := \mathbb{R}$, $\mathcal{B}_\mathbb{R}$ can generated by the collection of open intervals, of closed intervals, of half-open intervals, of open rays, and of closed rays. On a product space, the **product σ-algebra** is the σ-algebra generated by $\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$ for $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$ projections. An **elementary family** is a collection $\mathcal{E} \subseteq P(X)$ that contains the empty set, is closed under finite intersections, and complements of elements are disjoint unions of members in the family. A technical lemma is that the collection of finite disjoint unions of members of $\mathcal{E}$ form an algebra.

1.3. Measures.

Definition 1.3. A **measure** on a **measurable space** $(X, \mathcal{M})$ is a set function $\mu : \mathcal{M} \to [0, \infty]$ s.t. $\mu(\emptyset) = 0$ and it is countably additive. The triple $(X, \mathcal{M}, \mu)$ is a **measure space**. A few properties that are useful of measures are "σ-finite", "semifinite", and "finite" which are omitted here. A result that holds true except on a set of measure zero is said to hold **almost everywhere** (a.e.). A set of μ-measure zero is called **μ-null**. A measure is **complete** if $\mu(E) = 0$ implies $\mu(F) = 0$ when $F \subseteq E$.

Theorem 1.4. A measure is monotonic, subadditive, continuous from below, and continuous from above.

Theorem 1.5 (Completion of measures)**.** Let $\mathcal{N}$ denote the set of measurable sets of measure zero. Define $\overline{\mathcal{M}} := \{E \cup F : E \in \mathcal{M},\ \exists N \in \mathcal{M} \text{ s.t. } F \subseteq N\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra and there is a unique extension $\overline{\mu}$ of μ onto $\overline{\mathcal{M}}$.

1.4. Outer Measures.

Definition 1.6. An **outer measure** on a nonempty set X is a set function $\mu^* : P(X) \to [0, \infty]$ that is zero for the emptyset, monotonic, and subadditive.

Theorem 1.7. Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\varnothing \in \mathcal{E}, X \in \mathcal{E}$, and $\rho(\varnothing) = 0$. For any $A \subset X$, define

$$\mu^*(A) = \inf\left\{\sum_1^\infty \mu\left(E_j\right) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j\right\}$$

Then μ^* is an outer measure.

Definition 1.8. If μ^* is an outer measure, a μ^*-measurable set E is a set s.t. for all A,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Theorem 1.9 (Carathéodory's Theorem)**.** If μ^* is an outer measure, the collection $\mathcal{M}$ μ^*-measurable sets forms a σ-algebra and μ^* restricted to $\mathcal{M}$ is a complete measure.

Definition 1.10. A **premeasure** satisfies the same conditions as a measure except the domain is only required to be an algebra.

Theorem 1.11. If μ_0 is a premeasure on $\mathcal{A}$ and μ^* is defined by

$$\mu^*(E) = \inf\left\{\sum_1^\infty \mu_0\left(A_j\right) : A_j \in \mathcal{A}, E \subset \bigcup_1^\infty A_j\right\},$$

then a. $\mu^* \mid \mathcal{A} = \mu_0$; b. every set in $\mathcal{A}$ is μ^* measurable.

Theorem 1.12. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the σ - algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0. Namely, $\mu = \mu^* \mid \mathcal{M}$ where μ^* is given above. If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < \infty$. If μ_0 is σ -finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.

1.5. Borel Measures on $\mathbb{R}$.

Theorem 1.13. If $F : \mathbb{R} \to \mathbb{R}$ is an increasing, right continuous function, then there is a unique Borel measure $\mu_F : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ s.t. $\mu_F((a, b]) = F(b) - F(a)$ and μ_F is unique up to a sum of a constant i.e. $\mu_{F+c} = \mu_F$ for $c \in \mathbb{R}$.

Conversely, any finite Borel measure μ that is finite on bounded Borel sets has $\mu = \mu_F$ where

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0, \\ 0 & \text{if } x = 0, -\mu((x, 0]) \quad \text{if } x < 0, \end{cases}$$

and F is increasing and right continuous.

Remark 1. This is a sort of representation theorem for Borel measures on $\mathbb{R}$. Given a right continuous increasing F, we can consider μ_F and then take the completion $\overline{\mu_F}$ and $\overline{\mu_F}$ is called the **Lebesgue-Stieltjes measure** associated to F.

The Lebesgue measure is defined to be the Lebesgue-Stieltjes measure associated to $F(x) = x$.

Lemma 1.14. For any $E \in \mathcal{M}_\mu$, $\mu(E) = \inf\{\sum_1^\infty \mu((a_j, b_j)) : E \subseteq \bigcup_1^\infty (a_j, b_j)\}$.

Theorem 1.15. If $E \in \mathcal{M}_\mu$, then E is both inner regular and outer regular. Furthermore, $E \subseteq \mathbb{R}$ satisfies $E \in \mathcal{M}_\mu$ iff $E = V \setminus N_1$ for some G_δ set V and $\mu(N_1) = 0$ iff $E = H \cup N_2$ where H is an F_σ-set and $\mu(N_2) = 0$.

Furthermore, if $\mu(E) < \infty$, then for any $\epsilon > 0$, there is a set A which is a finite union of open intervals s.t. $\mu(E \Delta A) < \epsilon$.

Theorem 1.16. The Lebesgue measure is translation invariant and preserves scaling for all Lebesgue measurable sets.

Theorem 1.17. Let C be the Cantor middle-thirds set. Then C is compact, nowhere dense, and totally disconnected and has no isolated points. Further, $m(C) = 0$ and C has the cardinality of the continuum.

Example 1.18. The **Cantor-Lebesgue function** is defined on p. 39. It is a continuous function that is not absolutely continuous. Furthermore, it is differentiable a.e. and its derivative is equal to 0 a.e. while it increases from zero to one.

2. Integration

2.1. Measurable Functions.

Definition 2.1. A mapping between two measure spaces is **measurable** if the preimage of every measurable set is measurable. This makes measure spaces into a category. It suffices to check measurability on a generating set if there exists such a set.

A **Lebesgue measurable** function $f : R \to \mathbb{C}$ is $(\mathcal{L}, \mathcal{B}_{\mathbb{C}})$-measurable.

A function is **measurable** on a set E if $f|_E$ is $\mathcal{M}_E$-measurable. The σ-algebra generated by $\{f_\alpha : X \to Y\}_{\alpha \in A}$ is the smallest σ-algebra s.t. all f_α are measurable.

Proposition 2.2. A function mapping into a product measure space is measurable iff its coordinates are measurable functions. Therefore, $f : X \to \mathbb{C}$ is measurable iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable. The sum and product of measurable functions $f, g : X \to \mathbb{C}$ are measurable. For $f_j : X \to \overline{\mathbb{R}}$ measurable, the supremum, infimum, limit superior, limit inferior, and limit (if it exists) are all measurable. The max and min of two such functions are also measurable. This result holds readily for $\mathbb{C}$ under appropriate generalizations.

Theorem 2.3.

(1) For every $f \in L^+$, the set of positive measurable functions, there exists a sequence $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$ of simple functions that converge pointwise to f and uniformly on any set on which f is bounded.
(2) For $f : X \to \mathbb{C}$ measurable, there is a sequence of simple functions s.t. $0 \leq |\phi_1| \leq \cdots \leq |f|$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

Theorem 2.4. If X is a measure space, let $\overline{X}$ be the completion, and f on $\overline{X}$ is measurable iff there exists g measurable on X s.t. $f = g$ $\overline{\mu}$-a.e..

2.2. Integration of Nonnegative Functions.

Theorem 2.5 (Monotone Convergence Theorem)**.** If $\{f_j\} \subseteq L^+$ is a monotonically increasing a.e. sequence and $f = \lim_j f_j = \sup_j f_n j$, then $\int f = \lim_j \int f_j$. Consequently, finite and countable sums of L^+ functions commute with the integral.

Proposition 2.6. If $f \in L^+$, then $\int f = 0$ iff $f = 0$ a.e. and so, f_n increases to f a.e. implies $\int f = \lim \int f_n$.

Lemma 2.7 (Fatou's Lemma)**.** If $\{f_n\} \subseteq L^+$, then

$$\int (\liminf f_n) \leq \liminf \int f_n.$$

2.3. Integration of Complex Functions.

Theorem 2.8 (Dominated Convergence Theorem)**.** Let $\{f_n\}$ be a sequence in L^1 s.t. $f_n \to f$ a.e. and there exists $g \in L^1 \cap L^+$ s.t. $|f_n| \leq g$ a.e. for all n. Then $f \in L^1$ and $\int f = \lim_n \int f_n$.

Theorem 2.9. The set of integrable simple functions are dense in $L^1(\mu)$. If μ is Lebesgue-Stieltjes measure on $\mathbb{R}$, the sets in the definition of the simple function can be taken to be finite unions of open intervals. In the same situation, there is a continuous g vanishing outside a bounded interval s..t $\int |f - g| d\mu < \epsilon$.

2.4. Modes of Convergence.

Proposition 2.10.

(1) Convergence in L^1 implies convergence in measure.
(2) A Cauchy in measure sequence $\{f_n\}$ converges to some measurable function f in measure and there is a subsequence $f_{n_j} \to f$ a.e.. If $f_n \to g$ in measure as well, then $g = f$ a.e..
(3) If $f_n \to f$ in L^1, then some subsequence converges a.e. to f.
(4) (Egorov's Theorem). On a finite measure space, if $f_1, f_2, \ldots, f : X \to \mathbb{C}$ are measurable and $f_n \to f$ a.e., then for $\epsilon > 0$, there exists $E \subseteq X$ s.t. $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. That is, $f_n \to f$ almost uniformly.

2.5. Product Measures.

Theorem 2.11 (Fubini-Tonelli Theorem)**.** Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.

a. (Tonelli) If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and

$$\int f d(\mu \times \nu) = \int \left[\int f(x,y) d\nu(y) \right] d\mu(x) = \int \left[\int f(x,y) d\mu(x) \right] d\nu(y).$$

b. (Fubini) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X, f^y \in L^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x d\nu$ and $h(x) = \int f^y d\nu$ are in $L^1(\mu)$ and $L^1(\nu)$, respectively, and the equation above holds.

Theorem 2.12 (Fubini-Tonelli for Completed Measures)**.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be complete, σ-finite measure spaces, and let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$. If f is $\mathcal{L}$ -measurable and either $(a) f \geq 0$ or $(b) f \in L^1(\lambda)$, then f_x is $\mathcal{N}$ -measurable for a.e. x and f^y is $\mathcal{M}$ -measurable for a.e. y, and in case (b) f_x and f^y are also integrable for a.e. x and y. Moreover, $x \mapsto \int f_x d\nu$ and $y \mapsto \int f^y d\mu$ are measurable, and in case (b) also integrable, and

$$\int f d\lambda = \iint f(x,y) d\mu(x) d\nu(y) = \iint f(x,y) d\nu(y) d\mu(x).$$

2.6. The n-dimensional Lebesgue Integral. The outer regularity and inner regularity of the one dimensional Lebesgue measure translates well to $m := m^n$ on $\mathbb{R}^n$. One can also approximate finite measure sets by collections of disjoint rectangles.

Theorem 2.13. The Lebesgue measure is translation invariant. If $f \in L^1(m)$ or $f \geq 0$, then $\int f \circ \tau_a dm = \int f dm$ where $\tau_a(x) = x + a$.

Theorem 2.14 (Invariance under Linear Maps)**.** Suppose that Ω is an open set in $\mathbb{R}^n$ and $G : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism. a. If f is a Lebesgue measurable function on $G(\Omega)$, then $f \circ G$ is Lebesgue measurable on Ω. If $f \geq 0$ or $f \in L^1(G(\Omega), m)$, then

$$\int_{G(\Omega)} f(x) dx = \int_\Omega f \circ G(x) \left| \det D_x G \right| dx$$

b. If $E \subset \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and $m(G(E)) = \int_E \left| \det D_x G \right| dx$.

c. As a consequence, the Lebesgue measure is invariant under rotation.

Theorem 2.15 (Invariance under Differetiable Maps)**.** Suppose that Ω is an open set in $\mathbb{R}^n$ and $G : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism. a. If f is a Lebesgue measurable function on $G(\Omega)$, then $f \circ G$ is Lebesgue measurable on Ω. If $f \geq 0$ or $f \in L^1(G(\Omega), m)$, then

$$\int_{G(\Omega)} f(x) dx = \int_\Omega f \circ G(x) \left| \det D_x G \right| dx$$

b. If $E \subset \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and $m(G(E)) = \int_E \left| \det D_x G \right| dx$.

2.7. Integration in Polar Coordinates.

3. Signed Measures and Differentiation

3.1. Signed Measures.

Definition 3.1. A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ s.t. $\nu(\emptyset) = 0$, ν assumes at most one of $\pm\infty$, and if $\{E_j\}$ is a disjoint sequence, then

$$\nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$$

and the sum converges absolutely if $|\nu(\bigcup_1^\infty E_j)| < \infty$. In particular, every measure is a signed measure. If μ_1, μ_2 are measures on $(X, \mathcal{M})$ where one of them is finite, then $\nu = \mu_1 - \mu_2$ is a signed measure on X. If μ is a measure on $\mathcal{M}$ and $f : X \to \overline{\mathbb{R}}$, and one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite (such f is called **extended μ-integrable**, then $\nu(E) = \int_E f d\mu$ for $E \in \mathcal{M}$ defines a signed measure on X.

Definition 3.2. Let ν be a signed measure. A set $E \in \mathcal{M}$ is **positive** for ν if $\nu(F) \geq 0$ for all $F \subseteq E$ and $F \in \mathcal{M}$. Similarly for **negative** and **null** are similar.

Example 3.3. If $\nu(E) = \int_E f d\mu$, then ν is positive, negative, and null for E if $f \geq 0$, $f \leq 0$, $f = 0$ μ-a.e. on E respectively.

Theorem 3.4 (Hahn-Decomposition)**.** If ν is a signed measure, there exists positive and negative sets P, N respectively s.t. $P \cup N = X$, $P \cap N = \emptyset$ and it is unique up to symmetric difference i.e. if P', N' is another pair, $P \Delta P'$ and $N \Delta N'$ are ν-null.

Definition 3.5. If μ, ν are signed measures, then they are **mutually singular** if there exists $E, F \in \mathcal{M}$ s.t. $E \cup F = X$, $E \cap F = \emptyset$, and $\mu(E) = \nu(F) = 0$. Denote $\mu \perp \nu$ for this relation.

Theorem 3.6 (Jordan Decomposition)**.** If ν is a signed measure, there exist unique positive measures ν^+, ν^- (called **positive** and **negative variations** of ν) s.t. $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. We call $|\nu| = \nu^+ + \nu^-$ the **total variation** of ν.

Definition 3.7. A signed measure ν is **finite** or **σ-finite** if $|\nu|$ is finite or σ-finite.

3.2. Lebesgue Radon-Nikodym Theorem.

Definition 3.8. We say ν is **absolutely continuous** w.r.t μ where ν is signed and μ is positive when $\mu(E) = 0 \implies \nu(E) = 0$. This relation is denoted $\nu \ll \mu$. If $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

Theorem 3.9. If ν is a finite signed measure and μ a positive measure, then $\nu \ll \mu$ iff for all $\epsilon > 0$, there exists $\delta > 0$ s.t. $|\nu(E)| < \epsilon$ when $\mu(E) < \delta$.

Lemma 3.10. Let ν and μ be finite measures on $(X, \mathcal{M})$. Either $\nu \perp \mu$ or there exists $\epsilon > 0$ and $E \in \mathcal{M}$ s.t. $\mu(E) > 0$ and $\nu \geq \epsilon\mu$ on E.

Theorem 3.11 (Lebesgue-Radon-Nikodym Theorem)**.** All measures are on $(X, \mathcal{M})$. Let ν be a σ-finite signed measure, μ a σ-finite positive measure. Then there exists a unique σ-finite signed λ, ρ s.t.

$$\lambda \perp \mu,\ \rho \ll \mu,\ \&\ \nu = \lambda + \rho.$$

Also, there exists an **extended μ-integrable** function $f : X \to \mathbb{R}$ s.t. $d\rho = f d\mu$ and any such f are equal μ-a.e..

Definition 3.12. In the preceding theorem, $\nu = \lambda + \rho$ where $\lambda \perp \mu$ and $\rho \ll \mu$ is the **Lebesgue decomposition** of ν w.r.t μ.

If $\nu \ll \mu$, then $d\nu = f d\mu$ for some f. We call f the **Radon-Nikodym derivative** of ν w.r.t μ and often written $\frac{d\nu}{d\mu} = f$ since $d\nu = \frac{d\nu}{d\mu} d\mu$.

Proposition 3.13. Assume ν is a σ-finite signed measure and μ, λ are σ-finite measures on $(X, \mathcal{M})$ s.t. $\nu \ll \mu$ and $\mu \ll \lambda$.

a. If $g \in L^1(\nu)$, then $g(d\nu/d\mu) \in L^1(\mu)$ and
$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu.$$
b. We have $\nu \ll \lambda$ and
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda - a.e..$$

Sketch. For a., show we can assume $\nu \geq 0$. Then, show that the result holds for characteristic functions and extend to general $g \in L^1(\nu)$ by approximations. For b., use a. with $g = \chi_E(d\nu/d\mu)$.

□

Corollary 3.13.1. If $\mu \ll \lambda$ and $\lambda \ll \mu$, then $(d\lambda/d\mu)(d\mu/d\lambda) = 1$ a.e. w.r.t. either λ or μ.

Example 3.14. Let μ be the Lebesgue measure and ν the point mass at 0 on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then $\nu \perp \mu$. However, there is no Radon-Nikodym Derivative.

Proposition 3.15. If $\{\mu_i\}_{i=1}^n$ are measures, then there is a measure μ s.t. $\mu_i \ll \mu$ for all i, namely $\sum_1^n \mu_i$.

3.3. Complex Measures.

Definition 3.16. A **complex measure** on $(X, \mathcal{M})$ is a $\nu : \mathcal{M} \to \mathbb{C}$ s.t.

- $\nu(\emptyset) = 0$
- if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$ and the series converges absolutely.

Remark 2. Infinite values are not allowed for complex measures. Then, a positive measure is a complex measure iff it is a finite positive measure.

If ν is a complex measure, the real and imaginary parts are ν_r and ν_i respectively and these are signed measures taking on finite values only. If μ is a complex measure, then we say $\nu \perp \mu$ iff $\nu_a \perp \mu_b$ for $a, b = i, r$.

If $f \in L^1(\mu)$, then $d\nu = f d\mu$ defines a complex measure.

Theorem 3.17 (Lebesgue-Radon-Nikodym Theorem)**.** If ν is a complex measure and μ is a σ -finite positive measure on $(X, \mathcal{M})$, there exist a complex measure λ and an $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$. If also $\lambda' \perp \mu$ and $d\nu = d\lambda' + f' d\mu$ then $\lambda = \lambda'$ and $f = f' \mu$ -a.e..

Definition 3.18. The **total variation** of a complex measure ν is the defined by $d|\nu| = |f| d\mu$ where $d\nu = f d\mu$ and μ is a positive measure. This is independent of μ and f chosen.

Take $\mu = |\nu_r| + |\nu_i|$ and then the Lebesgue-Radon-Nikodym Theorem asserts there is an $f \in L^1(\mu)$ s..t $d\nu = f d\mu$ and then set $d|\nu| = |f| d\mu$.

Remark 3. This agrees with the definition for signed measures ν when ν is finite. Just take $d\nu = (\chi_P - \chi_N) d|\nu|$ where P, N are a Hahn-Decomposition and $|\chi_P - \chi_N| = 1$.

Proposition 3.19. Let ν be a complex measure.

(1) $|\nu(E)| \leq |\nu|(E)$ for $E \in \mathcal{M}$,

(2) $\nu \ll |\nu|$ and $\frac{d\nu}{d|\nu|}$ has absolute value 1 ν-a.e.,
(3) $L^1(\nu) = L^1(|\nu|)$ and if $f \in L^1(\nu)$, then $|\int f d\nu| \leq \int |f| d|\nu|$.

Proposition 3.20. If ν_1, ν_2 are complex measures, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

3.4. Differentiation on Euclidean Space. We work with $(\mathbb{R}^n, \mathcal{L}, m)$ in this section.

Lemma 3.21 (Vitali Covering Lemma)**.** Suppose U is a union of open balls in a set $\mathcal{C}$ of open balls of $\mathbb{R}^n$. If $c < m(U)$, then there exists finitely many disjoint balls B_i in $\mathcal{C}$ s.t. $\sum_1^k m(B_j) > \frac{c}{3^n}$.

Proof. Since $c < m(U)$, choose a compact $K \subseteq U$ s.t $m(K) > c$ and $K \leq U$. Then there is a finite subcover of K by $A_1, \ldots, A_m$ from $\mathcal{C}$. Let B_1 be the ball with largest radius from among the A_i. Then choose B_2 to be the ball with largest radius disjoint from B_1 and B_3 with largest radius disjoint from B_1, B_2. Repeat this until the list is exhausted. By construction, if A_i is not one of the B_j's, then there is a j in which $A_i \cap B_j = \emptyset$ and A_i has radius at most that of B_j. Then $A_i \subseteq B_j^*$ were B_j^* is the ball with center that of the center of B_j except with radius three times larger than B_j. Then $K \subseteq \bigcup_1^k B_j^*$. But then,

$$c < m(K) \leq \sum_1^k m(B_j^*) = 3^n \sum_1^j m(B_j).$$

$\square$

Definition 3.22. A measurable $f : \mathbb{R}^n \to \mathbb{C}$ is **locally integrable** if $\int_K |f(x)| dx < \infty$ for all bounded measurable sets $K \subseteq \mathbb{R}^n$. Let L^1_{loc} denote the space of locally integrable functions and

$$A_r f(x) := \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) dy \qquad f \in L^1_{loc}.$$

The **Hardy-Littlewood maximal function** is

$$Hf(x) := \sup_{r>0} A_r |f|(x).$$

Theorem 3.23. There exists $C > 0$ s.t. for all $f \in L^1$ and $\alpha > 0$,

$$\lambda_{Hf}(\alpha) := m(\{x : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int |f(x)| dx.$$

This is an example of a weak-type estimate.

Theorem 3.24. If $f \in L^1_{loc}$, then $\lim_{r \to 0} A_r f(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$.

Definition 3.25. A family $\{E_r\}_{r>0}$ of Borel subets of $\mathbb{R}^n$ **shrinks nicely** to x if $E_r \subseteq B_r(x)$ and there is an $\alpha > 0$ s.t. $m(E_r) > \alpha m(B_r(x))$ and this is independent of r.

Theorem 3.26 (Lebesgue Differentiation Theorem)**.** Given $f \in L^1_{loc}$. Then for all $f \in L_f$, all E_r shrinks nicely to x, and so a.e. x,

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0 \qquad \& \qquad \lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x).$$

Remark 4. A little slogan: " L^1_{loc} functions have full measure Lebesgue sets".

Definition 3.27. A Borel measure ν on $\mathbb{R}^n$ is **regular** if it is finite on compact sets and outer regular on Borel sets. A signed or complex Borel measure ν is **regular** if $|\nu|$ is.

Example: $f \in L^+(\mathbb{R}^n)$ yields fdm regular iff $f \in L^1_{loc}$.

Theorem 3.28. If ν is a regular signed or complex Borel measure on $\mathbb{R}^n$ and $d\nu = d\lambda + fdm$ its Lebesgue-Radon-Nikodym decomposition, then for m-a.e. $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every family E_r shrinking nicely to x.

Remark 5. This is the most crucial differentiation result of the section. In particular, it generalizes the usual differentiation rules of undergraduate analysis. Slogan: "differentiation can be viewed as differentiation of measures fdm w.r.t to sets that shrink nicely".

3.5. Functions of Bounded Variation.

Theorem 3.29. If $F : \mathbb{R} \to \mathbb{R}$ is increasing, then its set of discontinuities is countable and $F'(x)$ exists for a.e. $x \in \mathbb{R}$ and $F'(x) = G'(x)$ for a.e. $x \in \mathbb{R}$ where $G(x) := F(x+)$ for all x. Also, we have

$$F(b) - F(a) \geq \int_a^b F'(x)dx \quad \forall a < b.$$

Proof. **(a)** As F is increasing, the intervals $(F(x-), F(x+))$ are disjoint and for $|x| < N$, they lie in the interval $(F(-N), F(N))$. So,

$$\sum_{|x|<N} [F(x+) - F(x-)] \leq F(N+) - F(-N) < \infty$$

which shows that the set points of discontinuity with $x \in (-N, N)$ is countable. Since this is true for all N, this holds for all $\mathbb{R}$.

(b) Observe G is increasing and right continuous and $G = F$ except where F is discontinuous. Also,

$$G(x+h) - G(x) = \begin{cases} \mu_G((x, x+h]) & h > 0 \\ -\mu_G((x+h, x]) & h < 0 \end{cases}$$

and the families of intervals $\{(x-r, x]\}$ and $\{(x, x+r]\}$ shrink nicely as $r = |h| \to 0$. We know μ_G is regular and applying Theorem 3.22 of Folland, $G'(x)$ exists for a.e. x.

Next, we show if $H = G - F$, then H' exists and equals zero a.e.. Enumerate the points at which $H \neq 0$ as $\{x_j\}$. Then $H(x_j) > 0$ and there are countably many such points. Set $\mu = \sum_j H(x_j)\delta_j$ where δ_j is the point mass at x_j. Then μ is finite on compact sets and regular.

Also, $\mu \perp m$ since $m(E) = \mu(E^c) = 0$ where $E = \{x_j\}$. Then,

$$\left|\frac{H(x+h) - H(x)}{h}\right| \leq \frac{H(x+h) + H(x)}{|h|} \leq \frac{4\mu((x - 2|h|, x + 2|h|))}{4|h|} \to 0$$

as $h \to 0$ for a.e. x by Theorem 3.22. So $H' = 0$ a.e. and we are done.

(c) Then $\forall a < b$,

$$\int_a^b F'(x)dx = \int_a^b G'(x)dx = \int_a^b \frac{d\mu_G}{dm} dm \leq \mu_G((a,b)) = G(b-) - G(a) \leq F(b) - F(a).$$

$\square$

Remark 6. Positive measures on $\mathbb{R}$ are related to increasing functions and complex measures are related to functions of bounded relations.

Definition 3.30. The **total variation** of $F : \mathbb{R} \to \mathbb{C}$ is

$$T_F(x) := \sup \left\{ \sum_{n=1}^{N} |F(x_n) - F(x_{n-1})| \mid N \in \mathbb{N}, x_0 < x_1 < \cdots < x_N = x \right\} \forall x.$$

If $T_F(\infty) := \lim_{x\to\infty} T_F(x) < \infty$, then F has bounded variation, and the class of all such F is BV (or $BV(\mathbb{R})$) If $F\chi_{[a,b]} \in BV$, then $F \in BV(a,b)$ and $T_{F\chi_{[a,b]}}(b) - T_{F\chi_{[a,b]}}(a)$ is the total variation of F on $[a,b]$.

Example 3.31. (1) If $F : \mathbb{R} \to \mathbb{R}$ is bounded and increasing, then $F \in BV$.
(2) If $F, G \in BV$, then $aF + bG \in BV$ for $a, b \in \mathbb{C}$.
(3) If F is differentiable and F' is bounded, then $F \in BV([a,b])$ by the MVT.
(4) If $F(x) = \sin x$, then $F \in BV[a,b]$ from the above.
(5) If $F(x) = x\sin(x^{-1})$ and $F(0) = 0$, then $F \notin BV([a,b])$ for if $0 \in [a,b]$.

Lemma 3.32. If $F : \mathbb{R} \to \mathbb{R}$ is BV, then $T_F \pm F$ are both increasing (nondecreasing).

Theorem 3.33. a. $F \in BV$ iff $\operatorname{Re} F \in BV$ and $\operatorname{Im} F \in BV$.

b. If $F : \mathbb{R} \to \mathbb{R}$, then $F \in BV$ iff F is the difference of two bounded increasing functions; for $F \in BV$ these functions may be taken to be $\frac{1}{2}(T_F + F)$ and $\frac{1}{2}(T_F - F)$.

c. If $F \in BV$, then $F(x+) = \lim_{y\searrow x} F(y)$ and $F(x-) = \lim_{y\nearrow x} F(y)$ exist for all $x \in \mathbb{R}$, as do $F(\pm\infty) = \lim_{y\to\pm\infty} F(y)$.

d. If $F \in BV$, the set of points at which F is discontinuous is countable.

e. If $F \in BV$ and $G(x) = F(x+)$, then F' and G' exist and are equal a.e.

Proof. We prove c., d., and e., as the others are proven in Folland. From here on, take the real and imaginary parts of F to assume F is real valued. Now, $F(x+) = \lim_{y\searrow x} F(y)$ is clear since F is a sum of two increasing functions. Similarly for the statement regarding $F(x-)$. The statements for $F(\pm\infty)$ follows from the fact that the function is increasing and bounded.
d. If F BV, assuming F is real-valued, then the fact that F can be written as a difference of increasing functions means its points of discontinuity is the union of the set of discontinuities of the summands which are countable. So, the set of discontinuities of F is also.
e. This is immediate from writing F as a sum of increasing functions and applying the first theorem. □

Remark 7. Part a. and b. lead to a connection between BV and the space of complex Borel measures on $\mathbb{R}$ which we soon discuss. Notice the similarity with the Hahn-Jordan Decomposition of a complex measure.

Definition 3.34. If $F : \mathbb{R} \to \mathbb{R}$, then $\frac{1}{2}(T_F \pm F)$ are the **positive/negative variation function** of F and $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$ is the **Jordan decomposition** of F.

Remark 8. Since $x^\pm = \frac{1}{2}(|x| \pm x)$, for any $F \in BV$ and $F : \mathbb{R} \to \mathbb{R}$, we have $\forall x \in \mathbb{R}$

$$\frac{1}{2}(T_F \pm F)(x) = \sup \left\{ \sum_{n=1}^{N} [F(x_n) - F(x_{n-1})]^\pm \mid x_0 < x_1 < \cdots < x_N = x \right\} \pm \frac{1}{2}F(-\infty)$$

(because the formula for $T_F(x)$ can require $x_0 = -\infty$.)

Definition 3.35. The **normalized bounded variation** is given by

$$NBV := \{F \in BV \mid F \text{ right-continuous and } F(-\infty) = 0\}$$

Remark. If $F \in BV$, then $G(x) := F(x+) - F(-\infty) \in NBV$ and $G' = F'$ a.e. by theorem 3.27(a, b, d) (because if $F = F_1 - F_2$ and F_1, F_2 are increasing and bounded, then $G(x) = F_1(x+) - F(-\infty) - F_2(x+)$) By theorem 3.27, $G \neq F - F(-\infty)$ at countably many points.

Lemma 3.36. If $F \in BV$, then $T_F(-\infty) = 0$. If F is also right continuous, then so is T_F

Theorem 3.37. If μ is a complex Borel measure on $\mathbb{R}$ and $F(x) = \mu((-\infty, x])$, then $F \in NBV$. Conversely, if $F \in NBV$, there is a unique complex Borel measure μ_F such that $F(x) = \mu_F((-\infty, x])$; moreover, $|\mu_F| = \mu_{T_F}$

Remark 9. The theorem essentially gives a correspondence between NBV and complex Borel measures on $\mathbb{R}$ and the total variation function gives the total variation of the complex Borel measure.

Proof. ($\implies$) Decompose into positive and negative parts of the real and imaginary part. Then $F_j^{\pm}(x) = \mu_j^{\pm}((-\infty, x])$ satisfies $F_j^{\pm}(-\infty) = 0$, and $F_j^{\pm}(\infty) = \mu_j^{\pm}(\mathbb{R}) < \infty$ where $j = 1, 2$ in $\mu = \mu_1^+ - \mu_1^- + i(\mu_2^+ - \mu_2^-)$. Applying Theorem 3.27(a,b) of Folland, $F = F_1^+ - F_1^- + i(F_2^+ - F_2^-)$ is in NBV.

($\impliedby$): By Theorem 3.27 and Lemma 3.28 of Folland, any $F \in NBV$ is in the decomposition $F_j^{\pm}$. Theorem 1.16 of Folland tells us each $F_j^{\pm}$ gives rise to measures $\mu_j^{\pm}$ and $F(x) = \mu_F((-\infty, x])$ where $\mu_F = \mu_1^+ - \mu_1^- + i(\mu_2^+ - \mu_2^-)$. $\square$

The next question we answer is "which NBV functions correspond to μ s.t. $\mu \perp m$ and $\mu \ll m$?" since we want to apply the Lebesgue-Radon-Nikodym Theorem.

Proposition 3.38. If $F \in NBV$, then $F' \in L^1(m)$. Moreover, $\mu_F \perp m$ iff $F' = 0$ a.e., and $\mu_F \ll m$ iff $F(x) = \int_{-\infty}^x F'(t)dt$.

Definition 3.39. A function $F : \mathbb{R} \to \mathbb{C}$ is called **absolutely continuous** if for $\epsilon > 0$, there exists $\delta > 0$ s.t. for any finite set of disjoint intervals (a_j, b_j), one has

$$\sum_1^N (b_j - a_j) < \delta \qquad \implies \qquad \sum_1^N |F(b_j) - F(a_j)| < \epsilon.$$

One says F is **absolutely continuous on** $[a, b]$ if this holds when the intervals are contained in $[a, b]$.

The above condition obviously implies uniform continuity when $N = 1$. Conversely, if F is everywhere differentiable and F' is bounded, then F is absolutely continuous since $|F(b_j) - F(a_j)| \leq (\max |F'|)(b_j - a_j)$ by the MVT.

Proposition 3.40. If $F \in NBV$, then F is absolutely continuous iff $\mu_F \ll m$.

Corollary 3.40.1. If $f \in L^1(m)$, then the function $F(x) = \int_{-\infty}^x f(t)dt$ is in NBV and is absolutely continuous, and $f = F'$ a.e. Conversely, if $F \in NBV$ is absolutely continuous, then $F' \in L^1(m)$ and $F(x) = \int_{-\infty}^x F'(t)dt$

Proof. Proposition 3.30 and 3.32 are from Folland. If $f \in L^1(m)$, then $F(x)$ is in NBV by Proposition 3.30 since $\mu_F \ll m$ by definition. Meanwhile, it is absolutely continuous by

Proposition 3.32 and $f = F'$ a.e. by Proposition 3.30. The second follows from the "iff" aspect of the propositions. $\square$

Lemma 3.41. If F is absolutely continuous on $[a,b]$, then $F \in BV([a,b])$

Theorem 3.42 (The Fundamental Theorem of Calculus for Lebesgue Integrals.)**.** If $-\infty < a < b < \infty$ and $F : [a,b] \to \mathbb{C}$, TTFAE:

a. F is absolutely continuous on $[a,b]$.
b. $F(x) - F(a) = \int_a^x f(t)dt$ for some $f \in L^1([a,b],m)$.
c. F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b],m)$, and $F(x) - F(a) = \int_a^x F'(t)dt$.

Proof. (a) $\implies$ (c): Assuming WLOG that $F(a) = 0$ by subtracting by a constant, we then set $F(x) = 0$ for $x < a$ and $F(x) = F(b)$ for $x > b$. This means $F \in NBV$ by the preceding lemma and (c) follows from Corollary 3.33 of Folland.

(c) $\implies$ (b) is trivial.

(b) $\implies$ (a) set $f(t) = 0$ on $[a,b]^c$ and then apply Corollary 3.33 of Folland to deduce that $F(x) - F(a) = \int_{-\infty}^x f(t)dt - \int_{-\infty}^a f(t)dt = \int_a^x f(t)dt$. $\square$

Remark 10. The Cantor-Lebesgue function F is an example of a continuous a.e. differentiable function with $F' = 0$ a.e. that is not absolutely continuous on $[a,b]$. Hence, c. fails for F since the integral is zero while $F(1) - F(0) = 1$. This shows that the condition $F(x) - F(a) = \int_a^x F'(t)dt$ is essential for absolute continuity and vice-versa.

Definition 3.43. If $F \in NBV$, denote the integral of a function g with respect to the measure μ_F by $\int gdF$ or $\int g(x)dF(x)$. Such integrals are called **Lebesgue-Stieltjes** integrals.

Theorem 3.44. If F and G are in NBV and at least one of them is continuous, then for $-\infty < a < b < \infty$

$$\int_{(a,b]} FdG + \int_{(a,b]} GdF = F(b)G(b) - F(a)G(a).$$

Proof. Since F, G are linear combinations of increasing NBV functions, it suffices to assume F and G are increasing. Suppose G is continuous and let $\Omega = \{(x,y) : a < x \leq y \leq b\}$. Apply Fubini-Tonelli to compute $\mu_F \times \mu_G(\Omega)$ in two ways and we get, the second equality uses the fact that $G(x) = G(x-)$,

$$\mu_F \times \mu_G(\Omega) = \int_{(a,b]} \int_{a,y]} dF(x)dG(y) = \int_{(a,b]} FdG - F(a)[G(b) - G(a)]$$
$$\mu_F \times \mu_G(\Omega) = \int_{(a,b]} \int_{[x,b]} dG(y)dF(x) = G(b)[F(b) - F(a)] - \int_{(a,b]} GdF.$$

Subtracting the two equations gives the desired result. $\square$

4. Point-Set Topology

4.1. Topological Spaces and Continuous Maps.

Definition 4.1. Let X be a nonempty set. A **topology** on X is a family $\mathcal{T}$ of subsets of X that contains $\emptyset$ and X that is closed under finite intersection and arbitrary unions. The pair $(X, \mathcal{T})$ is called a **topological space**. The elements of $\mathcal{T}$ are called **open sets**. The complements of open sets are **closed sets**. We summarize the key definitions.

(1) The **interior** of a set $A \subseteq X$ is the union A° of all open sets contained in A.

(2) The **closure** $\overline{A}$ is the intersection of all closed sets containing A.
(3) The **boundary** of a set A is $\partial A = \overline{A} \setminus A^\circ$.
(4) A set A is **dense** if $\overline{A} = X$.
(5) A set A is **nowhere dense** if $\overline{A}^\circ = \emptyset$.
(6) A neighborhood of a point x is a set A s.t. $x \in A^\circ$.
(7) A point x is an accumulation point of A if $A \cap (U \setminus \{x\}) \neq \emptyset$ for all neighborhoods U of x.
(8) If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say $\mathcal{T}_1$ is **courser** than $\mathcal{T}_2$ and $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ (also resp. weaker and stronger).
(9) The topology generated by a subset $\mathcal{E} \subseteq \mathcal{P}(X)$ is the unique weakest topology containing the set.
(10) A **neighborhood base** for $\mathcal{T}$ at $x \in X$ is a family $\mathcal{N} \subseteq \mathcal{T}$ s.t. $x \in V$ for all $V \in \mathcal{N}$ and if $x \in U \in \mathcal{T}$, then there exists $V \in N$ s.t. $V \subseteq U$.
(11) A **base** for $\mathcal{T}$ is a family $\mathcal{B} \subseteq \mathcal{T}$ that contains a neighborhood base for $\mathcal{T}$ for each $x \in X$.
(12) A topological space (TS) is **first countable** if there is a countable neighborhood base for $\mathcal{T}$ at every point.
(13) A TS is **second countable** if it has a countable base.
(14) A TS is **separable** if X has a countable dense subset.
(15) A sequence $\{x_j\}$ in a TS converges to x if for all neighborhoods $U \ni x$, there exists N s.t. for all $j \geq N$, we have $x_j \in U$.
(16) A function f between two topological spaces is continuous iff the preimage of an open set is open. It is continuous at a point x iff every neighborhood of $f(x)$ has preimage that is the neighborhood of x.
(17) A function f between topological spaces is a homeomorphism if f is bijective, f is continuous, and f^{-1} is continuous.
(18) Given a family $f_\alpha : X \to Y_\alpha$, the **weak topology** is the weakest topology on X that makes the functions in the family $\{f_\alpha\}$ continuous.
(19) A function between topological spaces f is an **embedding** if f is injective and it is a homeomorphism onto its image which is given the relative topology.

Proposition 4.2. We summarize some standard facts about topological spaces.

(1) If $\mathcal{T}$ is a topology on X and $\mathcal{E} \subset \mathcal{T}$, then $\mathcal{E}$ is a base for $\mathcal{T}$ iff every nonempty $U \in \mathcal{T}$ is a union of members of $\mathcal{E}$.
(2) If $\mathcal{E} \subset \mathcal{P}(X)$, in order for $\mathcal{E}$ to be a base for a topology on X it is necessary and sufficient that the following two conditions be satisfied:
  a. each $x \in X$ is contained in some $V \in \mathcal{E}$;
  b. if $U, V \in \mathcal{E}$ and $x \in U \cap V$, there exists $W \in \mathcal{E}$ with $x \in W \subset (U \cap V)$.
(3) If $\mathcal{E} \subset \mathcal{P}(X)$, the topology $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ consists of $\varnothing, X$, and all unions of finite intersections of members of $\mathcal{E}$.

Theorem 4.3. Every subspace of a second countable space is also second countable.

Proof. Let X be second countable and $Y \subseteq X$ a subspace. Let $\mathcal{B}$ be a countable basis for X. Then define $\mathcal{C} := \{B \cap Y \mid B \in \mathcal{B}\}$ and since $\mathcal{B}$ is countable, $\mathcal{C}$ is also countable. Every open set in X is the union of finite intersections of elements of $\mathcal{B}$. Suppose $V \subseteq Y$ were open. Then by definition, $V := U \cap Y$ for some open set $U \subseteq X$. So, $U := \bigcup_\alpha \bigcap_{i \in I_\alpha} B_i$

with $B_i \in \mathcal{B}$. Then, $U \cap Y := \left(\bigcup_\alpha \bigcap_{i \in I_\alpha} B_i\right) \cap Y = \bigcup_\alpha \bigcap_{i \in I_\alpha} (B_i \cap Y)$ which shows that Y is the union of finite intersections of elements of $\mathcal{C}$. $\square$

Theorem 4.4. A subspace of a separable metric space is separable.

PROOF. A metric space is separable iff it is second countable. A subspace of a second countable space is necessarily second countable. So, a subspace of a metric is necessarily separable. $\square$

Theorem 4.5. Every second countable space is separable.

PROOF. There is a countable basis and choosing a point from each set in the basis, we get a countable dense set. $\square$

Theorem 4.6. If X is first countable and $A \subset X$, then $x \in \overline{A}$ iff there is a sequence in A converging to x.

PROOF. For the forward direction, use the definition of first countability to construct the sequence. For the converse, if $x \notin \overline{A}$, then we have a contradiction since the sequence is in A and converges. $\square$

Definition 4.7. Let X be a topological space and we say X is T_i for $i = 1, 2, 3, 4$ respectively when

(1) $i = 0$ if for points $x \neq y$, there exists an open set containing x but not y or vice versa.
(2) $i = 1$ if for $x \neq y$, there is an open set containing x but not y.
(3) $i = 2$ if we can separate distinct points by disjoint open sets.
(4) $i = 3$ if we can separate closed sets from points by open sets and X is T_1.
(5) $i = 4$ if we can separate disjoint closed sets by open sets and X is T_4.

Theorem 4.8. A space is T_1 iff $\{x\}$ is closed for any point x in the space.

Remark 11. The importance of this fact is that it shows $T_4 \implies T_3$ and $T_3 \implies T_2$.

Theorem 4.9. If X_α and T are topological spaces, $X = \prod_\alpha X_\alpha$, then $f : X \to Y$ is continuous iff $\pi_\alpha \circ f$ is continuous for all α.

Proposition 4.10. If X is a topological space, A is a nonempty set, and $\{f_n\}$ is a sequence in X^A, then $f_n \to f$ in the product topology iff $f_n \to$ f pointwise.

Proposition 4.11. If X is a topological space, $BC(X)$ is a closed subspace of $B(X)$ in the uniform metric; in particular, $BC(X)$ is complete.

Definition 4.12. We write $BC(X, F)$, $C(X, F)$ for the space of bounded continuous and continuous functions from X to F where $F \in \{\mathbb{R}, \mathbb{C}\}$. The uniform norm on $B(X)$ is defined by $\|f\|_u = \sup\{|f(x)| : x \in X\}$.

Lemma 4.13 (Urysohn's Lemma)**.** Let X be normal, and A and B disjoint closed subsets of X. There exists $f \in C(X, [0, 1])$ s.t. $f = 0$ on A and $f = 1$ on B.

Lemma 4.14 (Tietze Extension Theorem)**.** Let X be a normal space. If A is a closed subset of A and $f \in C(A, [a, b])$, there exists $F \in C(X, [a, b])$ s.t. $F|_A = f$.

Corollary 4.14.1. If X is normal and $A \subseteq X$ is closed with $f \in C(A)$, then there exists $F \in C(X)$ s.t. $F|_A = f$.

Corollary 4.14.2. A metric space is compact iff all continuous real-valued functions on it are bounded.

PROOF. The forward direct follows from how compact sets are preserved by continuous functions and compacts subsets of $\mathbb{R}$ are bounded.

For the other direction, we prove the contrapositive. Suppose X is a noncompact metric space. So, it is not sequentially compact and there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ with no convergent subsequence. Let $\mathcal{S} := \{x_1, x_2, x_3, \dots\}$ and we claim $\mathcal{S}$ is closed in X. Indeed, any convergent sequence in $\mathcal{S}$ must be eventually constant and hence, has limit inside $\mathcal{S}$. Let $f : X \to \mathbb{R}$ be defined on $\mathcal{S}$ by $f(x_n) = n$ and since $\mathcal{S}$ is discrete, f is continuous on $\mathcal{S}$. Since X is normal, the Tietze Extension Theorem gives an $F : X \to \mathbb{R}$ continuous s.t. $F|_{\mathcal{S}} = f$. But that means F is unbounded on X as desired. $\square$

Remark 12. It is useful to note that metric spaces are always normal. Therefore, it is always possible to apply the conclusions of these theorems to spaces such as L^p for $1 \le p \le \infty$.

4.2. Nets.

Definition 4.15. A **directed set** is a set A with a binary relation $\le$ s.t.

(1) $\forall a \in A$, $a \le a$;
(2) $\forall a, b, c \in A$ s.t. $a \le b$ and $b \le c$, $a \le c$;
(3) $\forall a, b \in A$, $\exists c \in A$ s.t. $a \le c$ and $b \le c$.

Definition 4.16. A net in a set X is a mapping $a \mapsto x_a$ from a directed set A into X. Denote such mappings by $\langle x_\alpha \rangle_{\alpha\in A}$ for ease.

Example 4.17. Here are some examples of nets.

(1) $\mathbb{N}$ with $j \le k$ iff $j \le k$ in the usual sense.
(2) $\mathbb{R} \setminus \{a\}$ with $x \le y$ iff $|x - a| \ge |y - a|$ in the usual sense.
(3) The set of all partitions of $[a, b]$ ordered by $\{x_i\} \le \{y_j\}$ iff $\max(x_j - x_{j-1}) \ge \max(y_k - y_{k-1})$.
(4) The set of $\mathcal{N}$ of neighborhoods of $x \in X$ with $U \le V$ iff $U \supseteq V$ which is said to be ordered by reverse inclusion.
(5) The product $A \times B$ has $(a, b) \le (a', b')$ iff $a \le a'$ and $b \le b'$. This is the standard way to form a directed set $A \times B$ from two directed sets.

Definition 4.18. If X is a topological space, $E \subseteq X$, then a net $\langle x_\alpha \rangle$ is **eventually in** E if $x_a \in E$ for all $a \ge a_0$ for some a_0. A point $x \in X$ is the **limit point** of $\langle x_\alpha \rangle$ iff for every neighborhood $U \ni x$, $\langle x_\alpha \rangle$ is eventually in U. A net is **frequently in** E if for all a, there is a $b \ge a$ for which $x_b \in E$. A **cluster point** x is a point s.t. for all neighborhood $U \ni x$, $\langle x_\alpha \rangle$ is frequently in U.

Proposition 4.19. If $E \subset X$ is a topological space with $x \in X$, then x is an accumulation point of E iff there is a net $E \setminus \{x\}$ that converges to x and $x \in \overline{E}$ iff there is a net in E that converges to x.

Definition 4.20. A subnet of a net $\langle x_\alpha \rangle$ is a net $\langle y_\beta \rangle$ with map $\beta \mapsto \alpha_\beta$ from B to A s.t.

(1) $\forall \alpha_0 \in A$, $\exists \beta_0$ s.t. $\alpha_\beta \ge \alpha_0$ whenever $\beta \ge \beta_0$;
(2) $y_\beta = x_{\alpha_\beta}$.

4.3. Compact Spaces.

Definition 4.21. A **precompact** set is a subset of a TS whose closure is compact. A set is **compact** if every open cover of the set has a finite subcover. The **finite intersection property** of a family of subsets means the finite intersections sets in the family are always nonempty.

Proposition 4.22. The following is a brief recollection of standard and **useful** results.

(1) Closed subsets of compact spaces are compact.
(2) Compact subsets of Hausdorff spaces are necessarily closed.
(3) Continuous image of compact sets are compact i.f. if X is compact, then $f(X)$ is compact when $f : X \to Y$ is continuous.
(4) $BC(X) = C(X)$ whenever X is compact.

Theorem 4.23. A topological space is compact iff every family of closed sets satisfying the finite intersection property has nonempty intersection.

PROOF. Take the complement and use the definitions. □

Theorem 4.24. In a Hausdorff space, one can separate compact sets and points.

Theorem 4.25. Every compact Hausdorff space is normal.

Theorem 4.26. Every second countable space is a Lindelöf space.

PROOF. Let $\{U_i\}_{i\in I}$ be an open cover. For each basis B_n, let V_n be any U_i containing B_n if such a B_n exists. If not, choose nothing. This gives a countable subcollection $\{V_n\}$ of the cover. We claim this countable subcollection covers X.

Assume not and there is an $x \in X \setminus V_n$ for all n. Since the U_i's cover X, $x \in U_j$ for some $j \in I$. Then, there is an $x \in B_n \subset U_j$ by definition of the basis. Since $B_n \subset U_i$, it was already selected among the V_n and hence, $x \in V_n$ for some n. Contradiction. □

Proposition 4.27. A second countable, regular space is normal.

PROOF. Let A, B be disjoint closed sets. For each $a \in A$, choose a neighborhood U of x that does not intersect V by regularity. With $a \in A$ and $A \setminus U$ closed, choose a neighborhood V of a s.t. V does not intersect $A\setminus U$. Then, $V \subset U$ and the closure of V is actually contained in U. Then, choose an element of the basis W s.t. $a \in W \subset V$. Do this for every $a \in A$ to get a covering $\{U_a\}_{a\in A}$. Since the space is second countable, it is Lindelöf so there is a countable subcover say $\{U_n\}$.

Do the same with B to get a countable cover $\{V_n\}$. Set $U := \bigcup_{n\in\mathbb{N}} U_n$ and $V := \bigcup_{n\in\mathbb{N}} V_n$. These sets may not be disjoint, but clearly $U \supseteq A$ and $V \supseteq B$. To make these sets disjoint, define $U_n' := U_n \setminus \bigcup_{i=1}^n \overline{V_n}$ and $V_n' := V_n \setminus \bigcup_{i=1}^n \overline{U_n}$. In this way, V_n' does not intersect any of the U_n' and clearly, U_n' and V_n' are open since they are the set difference of an open set by a closed set. These the U_n' form a covering for A because every $x \in U_n'$ for some n and x is in none of the $\overline{V_n}$ by construction. This completes the proof. □

Theorem 4.28. Any continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

PROOF. Use the fact that the continuous image of a compact set is compact (this gives that the inverse map is continuous). □

Theorem 4.29. TFAE for any topological space:

(1) X is compact,
(2) every net in X has a convergent subnet,
(3) every net has a cluster point.

Definition 4.30. A TS is **countably compact** if every countable open cover has a finite subcover. A TS is **sequentially compact** if every sequence has a convergent subsequence.

4.4. Locally Compact Hausdorff Spaces.

Definition 4.31. A **locally compact** space is one in which every point has a compact neighborhood. Let LCH mean locally compact Hausdorff.

Proposition 4.32. If X is an LCH space, $U \subset X$ open, and $x \in U$, there is a compact neighborhood N of x s.t. $N \subset U$.

PROOF. [1] First, assume $\overline{U}$ is compact since we can always choose a compact neighborhood $K \ni x$ and then replace U by $(U \cap K)^{\circ}$.

Second, since the space is Hausdorff, we can separate compact sets and points. In particular, there open sets $V \ni x$ and $W \ni \partial U$ that are disjoint. Therefore, $V \subseteq U$. Taking the closure $\overline{V}$, since $\overline{V} \subseteq \overline{U}$ is now a closed subset of a compact set, $\overline{V}$ is compact. □

Proposition 4.33. If X is an LCH space, $K \subset U \subset X$ with K compact and U open, then there is a precompact open V s.t. $K \subset V \subset \overline{V} \subset U$.

PROOF. Cover K by taking a compact neighborhood V_x for every $x \in X$. Then V_x° forms a open cover of K so there is a finite subcover $K \subseteq \bigcup_1^n V_n^{\circ}$. Then, the closure is a finite union of compact sets in U so the closure is compact. This completes the proof. □

Theorem 4.34 (Urysohn's Lemma for LCH Spaces)**.** If X is an LCH space, $K \subset U \subset X$ where K is compact and U open, then $\exists f \in C(X, [0,1])$ s.t. $f = 1$ on K and $f = 0$ outside a compact subset of U.

Corollary 4.34.1. An LCH space is completely regular.

PROOF. Clearly LCH implies T_1. Let $A \subset X$ be closed and $x \notin A$. The point x has a compact neighborhood K disjoint from A. Apply Urysohn's Lemma on that compact neighborhood so we have an $f(x) = 1$ and $f = 0$ outside a compact subset of an open set of K. □

Theorem 4.35 (Tietze Extension Theorm for LCH Spaces)**.** If X is an LCH space, $K \subset X$ is compact and $f \in C(K)$, there exists an $F \in C(X)$ s.t. $F|_K = f$ and f may be taken to vanish outside a compact subset of X.

PROOF. There exists an open precompact $V \supseteq K$. Define f on ∂V by $f|_{\partial V} = 0$. Since $\overline{V}$ is normal, the usual Tietze Extension shows there exists $F \in C(\overline{V})$ s.t. $F|_K = f$ and $F|_{\partial V} = 0$. Define $F|_{\overline{V}^c} = 0$ and this is the desired $F \in C_c(X)$. □

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[1] An incorrect proof: Let $x \in U \subseteq X$. Choose a compact neighborhood $M \ni x$ and consider $(M \cap U)^{\circ}$. Then there is an open set $B \subset (M \cap U)^{\circ}$ that contains x. Taking the closure $\overline{B} =: N$, which is a closed subset of the compact set M, this means $x \in N \subseteq U$ is our desired subset.

Definition 4.36. The **support** of a function $f \in C(X)$ is the closure over the set $f^{-1}(\{0\}^c) =:$ $\operatorname{supp}(f)$. If $\operatorname{supp}(f)$ is compact, we say f is **compactly supported**. The space of $f \in C(X)$ that are compactly supported is denoted $C_c(X)$. If $\forall \epsilon > 0$, $\overline{\{x : |f(x)| \geq \epsilon\}}$ is compact, then f **vanishes at infinity**. Let $C_0(X)$ be the space of $f \in C(X)$ that vanishing at infinity.

Remark 13. If X is Hausdorff, $C_c(X) \subseteq C_0(X) \subseteq BC(X)$. Compactness gives the reverse.

Proposition 4.37. If X is an LCH space, then $C_0(X)$ is the closure of $C_c(X)$ in the supremum metric.

PROOF. If $f_n \to f$ uniformly on X and $f \in C_c(X)$, then if $\epsilon > 0$, let n be s.t. $\|f_n - f\|_u < \epsilon$. Then, $\overline{f^{-1}(B_\epsilon(0)^c)} \subseteq \overline{f_n^{-1}(\{0\}^c)}$ and compact. Therefore, $f_n \in C_0(X)$.

If $f \in C_0(X)$ and $n \in \mathbb{N}$, then $K_n := f^{-1}(B_{\frac{1}{n}}(0)^c)$ is compact. Choose $g_n \in C_c(X)$ s.t. $g_n \equiv 1$ on K_n and $0 \leq g_n \leq 1$ by Urysohn's Lemma. So then, $fg_n \in C_c(X)$ and equals f on K_n. Hence, $\|fg_n - f\|_u \leq \frac{1}{n}$. Hence, $f \in \overline{C_c(X)}$. ☐

Theorem 4.38. If X is LCH and $E \subseteq X$, then E is closed iff $E \cap K$ is closed for every compact set $K \subseteq X$.

PROOF. ($\implies$): Is immediate since $E \cap K$ is an intersection of closed sets.

($\impliedby$): If E is not closed, there is an $x \in \overline{E} \setminus E$. Then, there is a compact neighborhood $K \ni x$. Then, x is an accumulation point for the intersection $K \cap E$ (because $x \in \overline{E}$ and K compact so $x \in \overline{E \cap K}$), but $x \notin K \cap E$ since $x \notin E$. ☐

Definition 4.39. Let X be a noncompact LCH space. Let ∞ be a point that is not in X and $X^* := X \cup \{\infty\}$ with $\mathcal{T}$ being the collection of subsets U of X^* s.t. either U is an open subset of X or $\infty \in U$ and U^c is a compact subset of X.

The space X^* is called the **one-point compactification** or **Alexandroff compactification** of X.

Proposition 4.40. If X, X^*, and $\mathcal{T}$ are as above, then $(X^*, \mathcal{T})$ is a compact Hausdorff space, and the inclusion map $i : X \to X^*$ is an embedding. Moreover, if $f \in C(X)$, then f extends continuously to X^* iff $f = g + c$ where $g \in C_0(X)$ and c is a constant, in which case the continuous extension is given by $f(\infty) = c$.

Definition 4.41. The topology of uniform convergence on $\mathbb{C}^X$ is generated by sets

$$\left\{g \in \mathbb{C}^X : \sup_{x \in X} |g(x) - f(x)| < n^{-1}\right\} \quad \left(n \in \mathbb{N}, f \in \mathbb{C}^X\right).$$

The **topology of uniform convergence on compact sets** is generated by the sets

$$\left\{g \in \mathbb{C}^X : \sup_{x \in K} |g(x) - f(x)| < n^{-1}\right\} \quad \left(n \in \mathbb{N}, f \in \mathbb{C}^X, K \subset X \text{ compact }\right).$$

Definition 4.42. A TS is **σ-compact** if it is a countable union of compact sets.

Proposition 4.43. If X is a σ -compact LCH space, there is a sequence $\{U_n\}$ of precompact open sets such that $\bar{U}_n \subset U_{n+1}$ for all n and $X = \bigcup_1^\infty U_n$.

Proposition 4.44. If X is a σ -compact LCH space and $\{U_n\}$ is as in the preceding proposition, then for each $f \in \mathbb{C}^X$ the sets

$$\left\{g \in \mathbb{C}^X : \sup_{x \in \bar{U}_n} |g(x) - f(x)| < m^{-1}\right\} \quad (m, n \in \mathbb{N})$$

form a neighborhood base for f in the topology of uniform convergence on compact sets. Hence this topology is first countable, and $f_j \to f$ uniformly on compact sets iff $f_j \to f$ uniformly on each $\bar{U}_n$.

Definition 4.45. If X is a TS and $E \subseteq X$, a **partition of unity on** E is a collection of functions $h_\alpha \in C(X, [0,1])$ s.t. each $x \in X$ has a neighborhood on which only finitely many h_α are nonzero and $\sum_{\alpha \in A} h_\alpha(x) = 1$.

A POU is **subordinate** to an open cover of E if for each α, there is an open set U in the cover s.t. $\mathrm{supp}(h_\alpha) \subseteq U$.

Theorem 4.46. If X is an LCH space, K compact subset of X, and $\{U_j\}_{j=1}^n$ an open cover for K, then there is a POU on K subordinate to the open cover consisting of compactly supported functions.

4.5. Two Compactness Theorems.

4.6. Stone-Weierstrass Theorem.

Definition 4.47. Let X be a topological space and $\mathcal{A} \subset BC(X, \mathbb{R})$ or $BC(X)$. Then $\mathcal{A}$ is an **algebra** if it is a real/complex vector subspace of $BC(X, \mathbb{R})/BC(X)$ s.t. $fg \in \mathcal{A}$ when $f, g \in \mathcal{A}$. In particular, if $\mathcal{A}$ is an algebra, so is $\overline{\mathcal{A}}$.

It **separates points** if for all x, y distinct, there exists $f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$.

Theorem 4.48 (Stone-Weierstrass)**.** Let X be a compact Hausdorff space and $\mathcal{A} \subseteq C(X, \mathbb{R})$ a closed subalgebra. Then either $\mathcal{A} = C(X, \mathbb{R})$ or $\mathcal{A} = \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$ for some $x_0 \in X$.

Proof. We need a few lemmas.

Lemma 4.49. The only subalgebras of $C(\{x, y\}, \mathbb{R}) \cong \mathbb{R}^2$ are $\{(0,0)\}$, and linear spans of $(a, 0)$, $(0, b)$, (c, c) where $a, b, c \in \mathbb{R}$.

Next, we need to approximate certain functions well.

Lemma 4.50. $\forall \epsilon > 0, \exists P_\epsilon \in \mathbb{R}[x]$ s.t. $P_\epsilon(0) = 0$ and $\||x| - P(x)\|_u \leq \epsilon$ where the supremum norm is over $[-1, 1]$.

Lemma 4.51. Assume X is a compact Hausdorff space. If $\mathcal{A} \subseteq C(X, \mathbb{R})$ is a closed subalgebra of $C(X, \mathbb{R})$, then $|f|$, $\min\{f, g\}$, and $\max\{f, g\}$ are in $\mathcal{A}$ for all $f, g \in \mathcal{A}$.

Lemma 4.52. Assume X is a compact Hausdorff space. If $\mathcal{A}$ is a closed subalgebra $f \in C(X, \mathbb{R})$ s.t. for all $x, y \in X$, there exists $g_{xy} \in \mathcal{A}$ s.t. $f = g_{xy}$ at x and y, then $f \in \mathcal{A}$.

The proof is immediate from these lemmas. Consider $\mathcal{A}_{xy} = \{(f(x), f(y)) : f \in \mathcal{A}\}$ and notice $\mathcal{A}_{xy}$ is a subalgebra of $\mathbb{R}^2$. There are five cases by the first lemma. If $\mathcal{A}_{xy} = \mathbb{R}^2$, then $\mathcal{A} = C(X, \mathbb{R})$ by the last lemma above. Otherwise, there is an $x_0 \in X$ s.t. $\mathcal{A}_{x_0 y} := \{(0, a) : a \in \mathbb{R}\}$ for some $y \in \mathbb{R}$ since $\mathcal{A}$ separates points (this rules out one case in the first lemma and our current case is essential the same as another case in the lemma). Then every function $f \in \mathcal{A}$ has $f(x_0) = 0$ and x_0 is unique. Then, $\mathcal{A} := \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$. $\square$

Theorem 4.53. Let X be a compact Hausdorff space and $\mathcal{A} \subseteq C(X)$ be a closed subalgebra of $C(X)$ that separates points and closed under complex conjugation. Then either $\mathcal{A} = C(X)$ or there exists x_0 s.t. $\mathcal{A} := \{f \in C(X) : f(x_0) = 0\}$.

Proof. Apply the real version using the fact that if $\mathcal{A}_\mathbb{R}$ is the set of real valued parts of $f \in \mathcal{A}$, then $\mathcal{A}_\mathbb{R}$ is also the set of imaginary valued parts and $\mathcal{A} = \mathcal{A}_\mathbb{R} + i\mathcal{A}$. ☐

Example 4.54. All polynomials in e^{ix} form a subalgebra of $C([-\pi,\pi])$ where $-\pi \sim \pi$. However, its closure does not contain e^{-ix}.

Example 4.55. One can apply the preceding theorem to $\mathcal{A} := \left\{\sum_{n=-N}^{N} a_n e^{inx} : N \in \mathbb{N}, a_n \in \mathbb{C}\right\}$. Then $\overline{\mathcal{A}}$ is a closed subalgebra and is in $C([-\pi,\pi])$ where $-\pi \sim \pi$ are identified. Then, this is dense in $L^2[-\pi,\pi]$ and then $\{e^{inx} : n \in \mathbb{Z}\}$ forms an orthonormal basis of $L^2[-\pi,\pi]$ with inner product $\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\overline{g}dx$.

There is one more "general" version of the SW Theorem where the space is noncompact and locally compact Hausdorff. However, we omit this and refer to [1].

Example 4.56. The SW Theorem in the usual sense for $X = \mathbb{R}$ can be applied with polynomials. That is, one can approximate continuous functions using polynomials (including constant polynomials) and even better, approximate general L^p functions on $\mathbb{R}$ since continuous functions in L^p are dense in L^p.

Theorem 4.57 (Standard Version of SW)**.** Suppose $\mathcal{B}$ is a subalgebra of $C(X,\mathbb{R})$ separating points which contains the constant functions. Then $\mathcal{B}$ is dense in $C(X,\mathbb{R})$.

5. Elements of Functional Analysis

5.1. Normed Vector Spaces. Throughout, fix K as either $\mathbb{R}$ or $\mathbb{C}$ and X a vector space over K.

Definition 5.1. A **seminorm** $\|\cdot\| : X \to [0,\infty)$ on K satisfies the first two conditions and a seminorm is a **norm** if it satisfies the third condition:

(1) $\|x+y\| \le \|x\| + \|y\|$
(2) $\|\lambda x\| = |\lambda|\|x\|$
(3) if $\|x\| = 0$, then $x = 0$.

The space $(X,\|\cdot\|)$ with a norm is a **normed vector space** (NVS). The function $\rho(x,y) = \|x-y\|$ naturally defines a metric and hence, a topology on X called the **norm topology**. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent iff they generate the same norm topology. That is, iff there exists $C_1, C_2 > 0$ s.t.

$$C_1\|x\|_1 \le \|x\|_2 \le C_2\|x\|_1$$

A complete normed vector space is called a **Banach space** (BS).

Theorem 5.2. A NVS X is complete iff every absolutely convergent series in X converges.

Proof. See Theorem 5.1 of [1]. ☐

Definition 5.3. Let $L(X,Y)$ denote the space of **bounded linear maps** $T : X \to Y$. A **bounded linear map** $T : X \to Y$ is a linear map s.t. $\exists C \ge 0$ s.t. $\forall x \in X$, $\|Tx\| \le C\|x\|$. The space $L(X,Y)$ has an **operator norm** defined by

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\} = \inf\{C : \|Tx\| \le C\|x\| \text{ for all } x\}.$$

If $T \in L(X,Y)$, T is **invertible**, or an **isomorphism**, if T is bijective and T^{-1} is bounded. If $\|Tx\| = \|x\|$ for all $x \in X$, then T is called an **isometry**.

Theorem 5.4. If X and Y are NVS and $T : X \to Y$ are linear maps, TFAE:

(1) T is continuous;
(2) T is continuous at 0;
(3) T is bounded.

Example 5.5. Consider $\ell^2(\mathbb{N})$ and the operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by $Te_n = \frac{1}{n}e_n$. This is a bounded operator because $\|T\| = 1$. However, its inverse T^{-1} which is defined by $e_n \mapsto ne_n$ is *not bounded* because $\|Te_n\| = n\|e_n\|$ and as $n \to \infty$, it becomes clear that this linear operator is not bounded.

Theorem 5.6. If Y is complete, so is $L(X,Y)$.

PROOF. Let $\{f_n\}$ be Cauchy in $L(X,Y)$. Then $\{f_n x\}$ is Cauchy in Y for every x since

$$\|f_n x - f_m x\| \le \|f_n - f_n\|\|x\| < \epsilon\|x\|.$$

Thereforem, $\{f_n x\}$ converges. Define $f(x) = \lim_n f_n(x)$. Then,

$$f(ax+by) = a\lim_n f(x) + b\lim_n f(y) = af(x) + bf(y)$$

for all $a, b \in \mathbb{C}$ and $x, y \in X$ so $f : X \to Y$ is linear. For all x, we have

$$\|f(x)\| = \lim_{n\to\infty} \|f_n(x)\| \le \lim_{n\to\infty} \|f_n\|\|x\|$$

and $\lim_{n\to\infty} \|f_n\| < \infty$ because

$$|\|f_n\| - \|f_m\|| \le \|f_n - f_m\|$$

shows $\|f_n\|$ is Cauchy in $\mathbb{C}$ and hence, converges. □

Theorem 5.7. The closed unit ball of a normed linear space X is compact iff X is finite dimensional.

PROOF. The closed unit ball in a finite dimensional linear space is compact by a HW in 240B. We show that $B := B_1(0)$ fails to be compact in infinite dimensional space. If $\dim X = \infty$, we can find a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ satisfying $\|x_n - x_m\| \ge \frac{1}{2}$ when $n \ne m$ and this sequence has no convergent subsequence. Therefore, B would fail to be compact. Choose an $x_1 \in B$. For an $n \in \mathbb{N}$, suppose we chose $\{x_1, \ldots, x_n\}$ already. Then, we choose an x_{n+1} by letting $M := \operatorname{span}(x_1, \ldots, x_n)$ and choose x_{n+1} to be the unit vector s.t.

$$\|x_{n+1} + M| \ge 1 - \frac{1}{2} = \frac{1}{2}$$

because M is a proper closed subspace of X. □

5.2. Linear Functionals.

Fix X to be a vector space and $K \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 5.8. A **linear functional** is a linear map $f : X \to K$. If X is a NVS, then $L(X,K)$ is the space of bounded linear functionals which is called the **dual space** of X. It is a Banach space with the operator norm.

Proposition 5.9. Let X be over $\mathbb{C}$ and f a complex linear functional on X and $u := \operatorname{Re} f$. Then u is a real linear functional and $f(x) = u(x) - iu(ix)$ for $x \in X$.

Conversely, if u is a real linear functional on X and $f : X \to \mathbb{C}$ is defined by $f(x) = u(x) - iu(ix)$, then f is complex linear. Furthermore, if X is normed, $\|u\| = \|f\|$.

PROOF. If f is complex linear, then u is real linear and Im $(f) = -\text{Re}\,[if(x)] = -u(ix)$. Hence, $f(x) = u(x) - iu(ix)$.

Conversely, if u is real linear and $f(x) = u(x) - iu(ix)$, then f is linear over $\mathbb{R}$. Also,

$$f(ix) = u(ix) - iu(-x) = u(ix) + iu(x) = if(x)$$

so f is linear over $\mathbb{C}$.

Suppose X were normed. Then $|u(x)| \leq |\text{Re}\, f| \leq |f(x)|$ which means $\|u\| \leq \|f\|$. For the other inequality, if $f(x) \neq 0$, let $\alpha = \overline{\text{sgn}(f)}$. Then,

$$|f(x)| = \alpha f(x) = f(\alpha x) = u(\alpha x) \quad \text{since } f(\alpha x) \text{ is real.}$$

Hence, $|f(x)| \leq \|u\|\|\alpha x\| = \|u\|\|x\|$. Hence, $\|f\| \leq \|u\|$. ☐

Definition 5.10. If X is a vector space on $\mathbb{R}$, a **sublinear functional** on X is a $p : X \to \mathbb{R}$ s.t.

$$p(x + y) \leq p(x) + p(y) \qquad \& \qquad p(\lambda x) = \lambda p(x)$$

for $x, y \in X$ and $\lambda \geq 0$.

Theorem 5.11 (Hahn-Banach)**.** If X is a vector space over $\mathbb{R}$, p a sublinear functional on X, M a subspace of X, and f a linear functional on M s.t. $f(x) \leq p(x)$ for $x \in M$, then there exists a linear functional F on X s.t. $F(x) \leq p(x)$ for all $x \in X$ and $F|_M = f$.

PROOF. First, we extend f to a linear functional g on $M + \mathbb{R}x$ for $x \in X \setminus M$ s.t. $g(y) \leq p(y)$ on $M + \mathbb{R}x$. If $y_1, y_2 \in M$,

$$f(y_1) + f(y_2) = f(y_1 + y_2) \leq p(y_1 + y_2) \leq p(y_1 - x) + p(x + y_2)$$

and so,

$$f(y_1) - p(y_1 - x) \leq p(x + y_2) - f(y_2).$$

Hence,

$$\sup\{f(y) - p(y - x) : y \in M\} \leq \alpha \leq \inf\{p(x + y) - f(y) : y \in M\}$$

and where we chose α to be any number satisfying the inequality. Define $g : M + \mathbb{R}x \to \mathbb{R}$ by $g(y + \lambda x) = f(y) + \lambda\alpha$. Then g is linear and $g|_M = f$. Also, $g(y) \leq p(y)$ for $y \in M$. If $\lambda > 0$ and $y \in M$, one has

$$g(y + \lambda x) = \lambda[f(y/\lambda) + \alpha] \leq \lambda[f(y/\lambda) + p(x + (y/\lambda)) - f(y/\lambda)] = p(y + \lambda x)$$

and if $\lambda < 0$, one has

$$g(y + \lambda x) = \mu[f(y/\mu) - \alpha] \leq \mu[f(y/\mu) - f(y/\mu) + p((y/\mu) - x)] = p(y + \lambda x).$$

Hence, $g(z) \leq p(z)$ for $z \in M + \mathbb{R}x$. Note that the inequality follows form the definition of α.

The same reasoning can be applied to any linear extension F of f with $F \leq p$ on its domain. The domain of a maximal linear extension F s.t. $F \leq p$ must be the whole space. The family of linear extensions F of f satisfying $F \leq p$ can be partially ordered by inclusion.

The union of any increasing family of subspaces is again a subspace and the relation $F \leq p$ is preserved. Invoke Zorn's lemma to complete the proof. ☐

Remark 14. In the case of a finite dimensional or even countable dimensional vector space over $\mathbb{R}$, it is possible to not rely on the Axiom of Choice. However, in the general case where the dimension is far too large to work with, the Axiom of Choice is necessary.

Theorem 5.12 (Complex Hahn-Banach)**.** Let X be a complex vector space, p a seminorm on X, M a subspace of X, and f a complex linear functional on M s.t. $|f(x)| \leq p(x)$ for $x \in M$. Then there exist a complex linear functional F on X s.t. $|F(x)| \leq p(x)$ for $x \in X$ and $F|_M = f$.

PROOF. Let $u = \text{Re } f$. Apply the preceding theorem to get a real linear extension U of u to X s.t. $|U(x)| \leq p(x)$ for $x \in X$. Let $F(x) = U(x) - iU(ix)$. Then F is a complex linear extension of f and if $\alpha = \overline{\text{sgn } F(x)}$, we have

$$|F(x)| = \alpha F(x) = F(\alpha x) = U(\alpha x) \leq p(\alpha x) = p(x)$$

where we use the seminorm condition p for the last equality. □

Example 5.13. The normed vector space $\ell^2(\mathbb{R})$ does not have a countable basis. In fact, it has an "uncountable dimension". On the other hand, $L^2(\mathbb{R})$ has countable dimension because it has a countable orthonormal basis. See here.

Theorem 5.14. Let X be a normed vector space.
a. If M is a closed subspace of X and $x \in X \backslash M$, there exists $f \in X^*$ such that $f(x) \neq 0$ and $f \mid M = 0$. In fact, if $\delta = \inf_{y \in M} \|x - y\|$, f can be taken to satisfy $\|f\| = 1$ and $f(x) = \delta$
b. If $x \neq 0 \in X$, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.
c. The bounded linear functionals on X separate points.
d. If $x \in X$, define $\widehat{x} : X^{**} \to \mathbb{C}$ by $\widehat{x}(f) = f(x)$. Then the map $x \mapsto \widehat{x}$ is a linear isometry from X into X^{**} (the dual of X^*).

PROOF. **a.** Define $f(y + \lambda x) = \lambda\delta$ on $M + \mathbb{C}x$. Then $f(x) = \delta$ and $f|_M = 0$ and $|f(y+\lambda x)| = |\lambda|\delta \leq |\lambda|\|\lambda^{-1}y + x\| = \|y + \lambda x\|$. Apply the Hahn-Banach theorem with $p(x) = \|x\|$ and $M + \mathbb{C}x$ in place of M.
b. Apply a. with $M = \{0\}$
c. If $x \neq y$, there is an $f \in X^*$ s.t. $f(x - y) \neq 0$.
d. $\hat{x}$ is immediately a linear functional by definition and $x \mapsto \hat{x}$ is linear. Also,

$$|\hat{x}(f)| = |f(x)| \leq \|f\|\|x\| \qquad \Longrightarrow \qquad \|\hat{x}\| \leq \|x\|$$

and applying (b)

$$\|x\| = |f(x)| = |\hat{x}(f)| \leq \|\hat{x}\|\|f\| = \|\hat{x}\|$$

which means $\|x\| \leq \|\hat{x}\|$. □

Remark 15.
(1) There are enough bounded linear functionals to separate points and closed subspaces.
(2) One can require the linear functional to take a center at its norm and be bounded above in the operator norm by 1.
(3) There are enough bounded linear functionals to separate points. So we can distinguish points using functionals.
(4) One can identify the space with its image in X^{**} under the $\widehat{\ }$-map. So studying a space can just entail that we study the dual space.

Proposition 5.15. Let X be a NVS and $x \in X$ s.t. $\widehat{x}(f) = 0$ for all $f \in X^*$. Then $x = 0$.

PROOF. If $x \neq 0$, then we can find a bounded linear functional s.t. $f(x) \neq f(0)$. Then, $\widehat{x}f = f(x) \neq 0$ which is a contradiction. □

Proposition 5.16. Let X be a nonempty normed vector space. Suppose there does not exist two distinct points s.t. there is a bounded linear functional that separates them. Then, X is a single-point space.

PROOF. If there existed two distinct points, then the preceding theorem yields a contradiction. $\square$

Example 5.17. It is true (but not obvious, see Exercise 5.17 of [1]) that a linear functional on a NVS $f : X \to \mathbb{C}$ is continuous iff $f^{-1}(\{0\})$.

An example of such a linear functional is $f : C([0,1]) \to \mathbb{C}$ defined by $f(g) = g(\frac{1}{2})$ where $C([0,1])$ has the L^1-norm. The space $f^{-1}(\{0\})$ is not closed because it is possible to construct a convergent sequence of functions g_n s.t. $f(g_n) = 0$ but g_n does not have converge in $f^{-1}(\{0\})$.

Furthermore, $C([0,1])$ with the L^1-norm is an example of a NVS that is not a Banach Space.

Example 5.18. If $X = \ell^\infty(\mathbb{C})$, then there is an $L \in X^*$ s.t. $\|L\| = 1$ and if $x \in X$ s.t. $\lim_{n\to\infty} x_n = x$, then $L(x) = a$.

First, define $Y \subseteq X$ by

$$Y := \{x = (x_1, x_2, x_3, \dots) \in X : \lim_{n\to\infty} x_n \text{ exists in } X\}.$$

Clearly, Y is a subspace. Now define a linear functional $f : Y \to \mathbb{C}$ by

$$f(x) = \lim_{n\to\infty} x_n.$$

It is clearly linear and continuous. Then $f(x)$ is dominated by the sublinear functional $p(x) = \limsup_{n\to\infty} x_n$. If $x = (a, a, a, \dot{)}$ is a constant sequence, we have $f(x) = a$ so that means $|f(x)| = a = \sup_n x_n$. This implies $\|f\| \geq 1$. Also, $\|f(x)\| = |\lim_{n\to\infty} x_n| \leq \sup_n x_n$ which means $\|f\| \leq 1$. Altogether, $\|f\| = 1$.

Applying the Hahn-Banach Theorem, there is a linear functional $L : X \to \mathbb{C}$ s.t. $L = f$ on Y and $\|L\| = \|f\| = 1$.

5.3. The Baire Category Theorem and its Consequences.

Definition 5.19. A set $E \subset X$ is of the **first category** if E is a countable union of nowhere dense sets. We say E is of the **second category** otherwise.

In modern terms, E is **meager** if it is of the first category. The complement of a meager set is called **residual**.

Remark 16. There exist meagre sets whose complement has Lebesgue measure zero. Hint: Construct the set of Lebesgue measure zero first and then show the complement is meagre. Take E to be the intersection of sets $\bigcup_{k=1}^\infty (r_k - \frac{1}{2^k n}, r_k + \frac{1}{2^k n})$ where one enumerates over the rationals $\{r_k\}$.

Theorem 5.20 (**The Baire Category Theorem**)**.** Let X be a complete metric space.

a. If $\{U_n\}_1^\infty$ is a sequence of open dense subsets of X, then $\bigcap_1^\infty U_n$ is dense in X.

b. X is not a countable union of nowhere dense[2] sets.

[2]A set A is nowhere dense if $\overline{A}^\circ = \emptyset$.

PROOF. a. It suffices to show

$$\forall \text{ nonempty and open } V \subseteq X,\ V \cap \bigcap_{n=1}^{\infty} U_n \neq \emptyset.$$

Since V is open, there exists an $x_1 \in X$ and $r_1 \in (0,1)$ s.t. $\overline{B_{r_1}(x)} \subseteq U_1 \cap V$. Then, $\exists x_2 \in X, r_2 \in (0,1/2)$ s.t. $\overline{B_{r_2}(x_2)} \subseteq B_{r_1}(x) \cap U_2$ and inductively, we construct a sequence $\{x_n\}$ that is Cauchy (since $r_n \to 0$). Then, there exists a limit $x := \lim x_n \in \bigcap_{m=1}^{\infty} \overline{B_{r_m}(x_m)} \subseteq (\bigcap_{m=1}^{\infty} U_n) \cap V$. This completes the first claim.
b. If E_n is a sequence of nowhere dense sets in X, then $\{\overline{E_n}^c\}$ is a sequence of open dense sets. Then, $\bigcap_{n=1}^{\infty} \overline{E}^c \neq \emptyset$ which means $\bigcup_{n=1}^{\infty} E_n \subset \bigcap_{n=1}^{\infty} \overline{E_n} \neq X$. □

Remark 17. It is useful to know that $\overline{X \setminus A} = X \setminus A^\circ$ and $(X \setminus A)^\circ = \overline{X \setminus A}$ so,

$$X \setminus \overline{E}^\circ = X \iff \overline{(X \setminus \overline{E})} = X \iff X \setminus \overline{E} \text{ is dense in } X.$$

Remark 18. The theorem is purely topological and hence, applies to any space that is homeomorphic to a complete metric space.

Remark 19. The theorem can be used to prove existence results. First, one shows that objects having a certain property exist by showing that the set of objects *not* having the property is meager.

Definition 5.21. If X and Y are TS, then $f : X \to Y$ is **open** if $\forall U$ open in X, $f(U)$ open in Y.

Remark 20. If X, Y are normed vector spaces, then a linear map $f : X \to Y$ is open iff $\exists r > 0$ s.t. $f(B_1(0)) \supseteq B_r(0)$.
Check: Let U be open in X and we show $f(U)$ is open in Y. It suffices to show every point $f(x) := y \in f(U)$ is contained in a ball inside $f(U)$. By linearity, we can translate $f(U)$ so that the point $f(x) = 0$. Then, $x = 0$. Since U is open, there is a ball $B_s(0) \subseteq U$ and by scaling, we want to show $f(B_1(0))$ contains a ball about the origin. Altogether, it suffices to show there is an $r > 0$ s.t. $B_r(0) \subseteq f(B_1(0))$.

Theorem 5.22 (The Open Mapping Theorem). If X and Y are Banach spaces, and $T \in L(X,Y)$ is surjective, then T is open.

PROOF. It suffices to show $T(B_1(0)$ contains a ball about 0 in Y. Write $X := \bigcup_{n=1}^{\infty} B_n(0)$ and since T is surjective, $Y = \bigcup_{n=1}^{\infty} T(B_n(0))$. Since Y is complete and $y \mapsto ny$ is a homeomorphism which sends $T(B_1(0))$ to $T(B_n(0))$, Baire's Theorem implies $T(B_1(0))$ is not nowhere dense[3]

So, there is a $B_{4r}(y_0) \subset \overline{T(B_1(0))}$. Then pick $y_1 = Tx_1 \in T(B_1(0))$ s.t. $\|y_1 - y_0\| < 2r$. Then,

$$B(2r, y_1) \subset B(4r, y_0) \subset \overline{T(B_1(0)}$$

so if $\|y\| < 2r$, we get

$$y = -Tx_1 + (y + y_1) \in \overline{T(-x_1 + B_1(0))} \subset \overline{T(B_2(0))}.$$

Dividing by two, we conclude that there is an $r > 0$ s.t. if $\|y\| < r$, then $y \in \overline{T(B_1(0))}$. The issue is that $\overline{T(B_1(0))}$ is not the set we want to be contained in. The next part of the proof deals with that.

[3]Joke: somewhere dense.

Since T commutes with dilations, it follows that if $\|y\| < r2^{-n}$, then $y \in \overline{T(B_{2^{-n}})}$. Suppose $\|y\| < r/2$. Then we can find $x_1 \in B_{1/2}$ s.t. $\|y - Tx_1\| < r/4$. Repeat this process to get $\|y - \sum_1^n Tx_j\| < r2^{-n-1}$ and since X is complete, the series $\sum_1^\infty x_n$ converges to some x. But then $\|x\| < \sum_1^\infty 2^{-n} = 1$ and $y = Tx$ which means $x \in B_1(0)$ and $y \in T(B_1(0))$. Hence, $T(B_1(0))$ contains all y s.t. $\|y\| < r/2$ i.e. $B_{r/2}(0) \subset T(B_1(0))$ and we are done. □

Corollary 5.22.1. If X and Y are Banach spaces and $T \in L(X, Y)$ is bijective, then T is an isomorphism i.e. $T^{-1} \in L(X, Y)$.

PROOF. If T is bijective, the continuity of T^{-1} is equivalent to T being an open map. The Open Mapping Theorem gives us the result in this case. □

Corollary 5.22.2. Let X be a Banach space and let $X_1, X_2 \subset X$ be two closed linear subspaces such that $X = X_1 \oplus X_2$. Then there exists a constant $c \geq 0$ such that

$$\|x_1\| + \|x_2\| \leq c \|x_1 + x_2\|$$

for all $x_1 \in X_1$ and all $x_2 \in X_2$.

PROOF. Define a map $T : X_1 \times X_2 \to X$ by $(x_1, x_2) \mapsto x_1 + x_2$ and $X_1 \times X_2$ has norm defined by $\|(x_1, x_2)\| = \|x_1\| + \|x_2\|$. Then T is clearly linear and bounded. Proving the inequality entails that we show T^{-1} is bounded. By our hypotheses, the map T is bijective. Therefore, the preceding corollary implies $T^{-1} \in L(X, Y)$. □

Definition 5.23. If X and Y are normed spaces and T is a linear map from X to Y, then the **graph** of T is the subspace

$$\Gamma(T) := \{(x, y) \in X \times Y : Tx = y\} \subseteq X \times Y$$

We say $T : X \to Y$ is **closed** if $\Gamma(T)$ is a closed subspace of $X \times Y$.

Remark 21. This definition is not the same as the definition of a closed map in topology. A closed map in topology requires no structure on the map. In our current context, T is a linear map.

Theorem 5.24 (Closed Graph Theorem). If X and Y are Banach spaces and $T : X \to Y$ is a closed linear map, then $T \in L(X, Y)$ i.e. T is bounded.

PROOF. Let π_X, π_Y be the projection maps of $\Gamma(T)$ onto X and Y. Then $\pi_X \in L(\Gamma(T), X)$ and $\pi_Y \in L(\Gamma(T), Y)$.

Because X and Y are complete, so is $X \times Y$. Therefore, $\Gamma(T)$ is complete because it is closed in $X \times Y$ by hypothesis. The map π_X is a bijection from $\Gamma(T)$ to X (this is an obvious bijection by definition of $\Gamma(T)$) and by the preceding corollary π_X^{-1} is bounded.

But then $T = \pi_Y \circ \pi_X^{-1}$ is bounded. □

Remark 22. The continuity of a linear map of Banach spaces $T : X \to Y$ means if $x_n \to x$, then $Tx_n \to Tx$ (sequential definition of continuity). On the other hand, closedness means if $x_n \to x$, and $Tx_n \to y$, then $y = Tx$. Therefore, the Closed Graph Theorem says

"to show continuity, assume $Tx_n \to y$ and show $y = Tx$ where $x_n \to x$".

Failure of the conclusion occurs if completeness of X and Y are dropped.

Theorem 5.25 (Uniform Boundedness Principle). Suppose X and Y are normed vector spaces and $A \subseteq L(X, Y)$.

a. If $\sup_{T \in A} \|Tx\| < \infty$ for all x in some nonmeager subset of X, then $\sup_{T \in A} \|T\| < \infty$.
b. If X is a Banach space and $\sup_{T \in A} \|Tx\| < \infty$ for all $x \in X$, then $\sup_{T \in A} \|T\| < \infty$.

PROOF. We frequently use b. so we do not prove a.. Let

$$E_n := \{x \in X : \sup_{T\in A} \|Tx\| \leq n\} = \bigcap_{T\in A} \{x \in X : \|Tx\| \leq n\}.$$

The RHS shows that E_n is the intersection of closed sets so E_n is closed. By the Baire Category Theorem, there is some $\overline{E_n}$ that is not nowhere dense. Then, there is an $x \in X$ and $r > 0$ s.t. $\overline{B_r(x)} \subseteq E_n$. Then for $x' \in \overline{B_r(0)}$ and for all $T \in A$,

$$\|Tx'\| \leq \|Tx' + Tx\| + \|Tx\| \leq 2n$$

Dividing by r, we have $\|Tx'\| \leq \frac{2n}{r}$ for all $\|x'\| = 1$. Since T was arbitrary, $\sup_{T\in A} \|T\| \leq \frac{2n}{r}$ as desired. □

Remark 23. The results can be summarized or categorized as results which show when a linear map is bounded. The Open Mapping Theorem can be used to show that the inverse map is bounded, the Closed Graph Theorem can be used to show when the map itself is bounded, and the Uniform Boundedness Principle can be used to show that some collection of linear maps is uniformly bounded. It is crucial that we notice that we actually uniformly bound the operators in the Uniform Boundedness Principle. The value is in knowing that we can bound all of the linear maps in A with a single constant.

5.4. Hilbert Spaces.

Definition 5.26. Let H be a complex vector space and an **inner product** $\langle \cdot, \cdot \rangle$ on H is a map $H \times H \to \mathbb{C}$ s.t. it is linear in the first coordinate, complex symmetric $\langle x, y\rangle = \overline{\langle y, x\rangle}$, and $\langle x, x\rangle \in (0, \infty)$ for all nonzero $x \in \mathcal{H}$. The conditions show that the inner product is conjugate linear in the second coordinate. We call H with an inner product an **inner product space**. There is a norm defined by $\|x\| = \sqrt{\langle \cdot, \cdot \rangle}$ for any inner product space.

Example 5.27. If one considers the space of sequences of scalars in $\mathbb{C}$, then for $\alpha \in \mathbb{Z}$, we have an inner product

$$\langle \{a_n\}, \{b_n\}\rangle = \sum_{n=1}^{\infty} n^\alpha a_n \overline{b_n}$$

Theorem 5.28 (Schwarz Inequality)**.** $|\langle x, y\rangle| \leq \|x\|\|y\|$ for all x, y and equality occurs iff x and y are linearly dependent.

Theorem 5.29. The function $x \mapsto \|x\|$ is a norm on H.

Definition 5.30. An inner product space complete w.r.t its induced norm is a **Hilbert space**.Fix H a Hilbert space.

Theorem 5.31. If $x_n \to x, y_n \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

PROOF. First, notice

$$\langle x_n, y_n\rangle = \langle x_n - x + x, y_n\rangle = \langle x, y_n\rangle + \langle x_n - x, y_n\rangle.$$

This means we can write

$$\begin{aligned} 0 &\leq |\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x, y_n\rangle + \langle x_n - x, y_n\rangle - \langle x, y\rangle| = |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle| \\ &\leq \|x_n - x\|\|y_n\| + \|y_n - y\|\|x\| \to 0 \end{aligned}$$

as $n \to \infty$ because $\|y_n\| \to \|y\| < \infty$. □

Theorem 5.32. For $x, y \in H$,

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|).$$

PROOF. Add $\|x \pm y\|^2 = \|x\|^2 \pm 2\text{Re}\ \langle x, y\rangle + \|y\|^2$. $\square$

Theorem 5.33 (Pythagorean Theorem)**.** If $x_1, \ldots, x_n \in H$ are pairwise orthogonal elements for $j \neq k$,

$$\left\|\sum_1^n x_j\right\|^2 = \sum_1^n \|x_j\|^2.$$

Theorem 5.34. If M is a closed subspace of H, then $H = M \oplus M^\perp$ and $x \in H$ can always be uniquely written as $x = y + z$ where y and z have minimal distance to x. The element $z = Px$ is often defined as the projection of x onto M.

Remark 24. This result gives us the **orthogonal projection** onto M. See Exercise 5.58. It is crucial to recognize that this uniqueness part allows one to quickly determine the orthogonal projection of an element.

Theorem 5.35 (Riesz-Fréchet)**.** If $f \in H^*$, there is a unique $y \in H$ s.t. $f(x) = \langle x, y\rangle$ for all $x \in X$. In particular, Hilbert spaces are reflexive and are naturally isomorphic to H^*.

PROOF. Uniqueness of y is immediate.

Given $f \in H^*$ and assume $f \neq 0$. Then, $M := \{x \in H : f(x) = 0\}$ is a closed subspace of H. Apply the preceding theorem to find $H = M \oplus M^\perp$. Then, let $z \in M^\perp$ s.t. $\|z\| = 1$. For any $u = zf(x) - f(z)x \in M$, we have

$$0 = \langle u, z\rangle = \|z\|^2 f(x) - f(z)\langle x, z\rangle = f(x) - \langle x, \overline{f(z)}z\rangle.$$

So, $\overline{f(z)}z$ is our desired $y \in H$. Notice that we can write $x := \frac{zf(x)}{f(z)} - \frac{u}{f(z)}$ and this means $f(x) = \langle x, \overline{f(z)}z\rangle$. $\square$

Definition 5.36. An **orthonormal subset** of H is a set of linearly independent vectors with norm 1.

Any linearly independent set of vectors can be made into an orthonormal set by the Gram-Schmidt process.

Theorem 5.37 (Gram-Schmidt Process)**.** Refer to any standard linear algebra text for more.

Given a sequence $\{x_n\}$. Set $u_1 = \frac{x_1}{\|x_1\|}$. Then, set $v_2 := x_2 - \langle x_1, u_1\rangle u_1$ and let $u_2 = \frac{v_2}{\|v_2\|}$. Inductively, let u_n be the normalization of $x_n - \sum_1^{n-1}\langle x_i, u_i\rangle u_1$.

Example 5.38. Consider $L^2([-1,1])$. Apply the Gram-Schmidt procedure on $1, x, x^2$. Then the orthonormal basis is $\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \sqrt{\frac{5}{2}}(x^2 - \frac{1}{3})$.

Theorem 5.39 (Bessel's Inequality)**.** If $\{u_\alpha\}$ is an orthonormal set in H, then for any $x \in H$,

$$\sum_{\alpha \in A} |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2.$$

In particular, $\{\alpha : \langle x, u_\alpha\rangle \neq 0\}$ is countable.

Theorem 5.40. If $\{u_\alpha\}_{\alpha \in A}$ is an orthonormal set in H, TTFAE:

(1) If $\langle x, u_\alpha\rangle = 0$ for all α, then $x = 0$.

(2) (Parseval's Identity) $\|x\|^2 = \sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2$ for all $x \in H$.
(3) For each $x \in H$, $x = \sum_\alpha \langle x, u_\alpha\rangle u_\alpha$ where the sum on the right has countably many nonzero terms and converges in the norm independent of the ordering of the terms.

PROOF. (2) implies (1) is obvious.
(1) implies (3) follows by letting α_k be an enumeration for which $\langle x, u_\alpha\rangle \neq 0$. Then, Bessel's inequality shows $\sum_{\alpha_k} |\langle x, u_{\alpha_k}| \leq \|x\|^2$ converges and the Pythagorean theorem says

$$\|\sum_n^m \langle x, u_{\alpha_k} u_{\alpha_k}\| = \sum_n^m |\langle x, u_{\alpha_k}\rangle|^2 \to 0$$

as $m, n \to \infty$. Or in particular, letting $m \to \infty$ first and then letting $n \to \infty$. We deduce that the series is absolutely convergent and converges by completeness of H. If $y = x - \sum_k \langle x, u_{\alpha_k}\rangle u_{\alpha_k}$, then $\langle y, u_\alpha\rangle = 0$ for all α. Hence, $y = 0$.
(3) implies (2) by Bessl's inequality, we have the obvious inequality. So, we show the other inequality. The other inequality is immediate by consider $\|x\|^2 = \langle x, x\rangle$ and (3) shows us that x is just the sum. Use the Dirac δ-function to simplify the work. Alternatively,

$$\|x\|^2 - \sum_1^n |\langle x, u_{\alpha_j}\rangle^2 = \|x - \sum_1^n \langle x, u_{\alpha_j} u_{\alpha_j}\|^2 \to 0$$

as $n \to \infty$. $\square$

Definition 5.41. An orthonormal set satisfying any of the preceding three equivalent conditions is called an **orthonormal basis**. This is *not* the same notion as a basis of a vector space.

Example 5.42. For example, let $H = \ell^2(A)$ for some set A. For every $\alpha \in A$, let $e_\alpha \in \ell^2(A)$ be defined by $e_\alpha(\beta) = 0$ when $\beta \neq \alpha$ and $e_\alpha(\alpha) = 1$. Then $\{e_\alpha\}_{\alpha\in A}$ is orthonormal and it is certainly complete because $\langle f, e_\alpha\rangle = f(\alpha)$.

Theorem 5.43. Every Hilbert space has an orthonormal basis.

Theorem 5.44. A Hilbert space H is separable iff it has a countable orthonormal basis. In which case, every orthonormal basis for H is countable.

Example 5.45. If $\{f_n\} \subseteq L^2[0,1]$ s.t. for all $g \in L^2[0,1]$, we have $\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = 0$, then $\lim_{n\to\infty} \|f_n\|^2 = 0$. Indeed, $L^2[0,1]$ has an inner product defined by $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dt$ and so, the condition implies that $\lim_{n\to\infty} f_n = 0$ for all n (just replace f by f_n and g by elements in the orthonormal basis and use continuity of the inner product). Therefore, by continuity of the norm, $\lim_{n\to\infty} \|f_n\|_2 = 0$.

Definition 5.46. A map $U : H_1 \to H_2$ of two Hilbert spaces is a **unitary map** $H_1 \to H_2$ if it is an invertible linear map that preserves inner product i.e. $\langle Ux, Uy\rangle_2 = \langle x, y\rangle_1$ for all $x, y \in H_1$.

Proposition 5.47. If $\{e_\alpha\}$ is an orthonormal basis for X, then the correspondence $x \mapsto \widehat{x}$ defined by $\widehat{x} = \langle x, u_\alpha\rangle$ is a unitary map from H to $\ell^2(A)$.

Example 5.48. Define $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $Tf(x) = f(|x|)$. It is easy to see that this is a well-defined linear operator:

$$\|Tf(x)\|_2^2 = \int_{\mathbb{R}} |Tf(x)|^2 dx = \int_{\mathbb{R}} |f(|x|)|^2 dx = \int_0^\infty |f(x)|^2 dx < \infty$$

so $Tf(x) \in L^2(\mathbb{R})$. Next, it is continuous because

$$\|Tf(x)\|_2 = \left(\int_0^\infty 2|f(x)|^2\right)^{1/2} \le \sqrt{2}\|f\|_2$$

which means $\|T\| \le \sqrt{2}$. This is actually an equality because when $f(x)$ is taken to be zero for all $x < 0$, the inequality above is an equality. This means $\|Tf(x)\|_2 = \sqrt{2}\|f\|_2$ for $f(x) = 0$ when $x < 0$ which means $\|T\| \ge \sqrt{2}$. The kernel consists of functions that are a.e. zero on $\mathbb{R}$ and are measurable square-integrable functions. The range is the space of square-integrable even functions on $\mathbb{R}$. One can also compute the adjoint for the operator:

$$\begin{aligned}\langle Tf, g\rangle &= \int_{\mathbb{R}} Tf(x)\overline{g(x)}dx = \int_{\mathbb{R}} f(|x|)\overline{g(x)}dx = \int_0^\infty f(x)\overline{g(x)}dx + \int_{-\infty}^0 f(-x)\overline{g(x)}dx \\ &= \int_0^\infty f(x)\left(\overline{g(x) + g(-x)}\right)dx = L(f, T^*g).\end{aligned}$$

That is,

$$T^*g = \begin{cases} g(x) + g(-x) & x \ge 0 \\ 0 & x < 0 \end{cases}$$

5.5. Topological Vector Spaces.

Definition 5.49. A **topological vector space** (TVS) is a vector space over K which is endowed with a topology s.t. the maps $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda x$ are continuous from $X \times X$ and $K \times X$ to X respectively.

Definition 5.50. A TVS is **locally convex** if there is a base for the topology consisting of convex sets.

Theorem 5.51. Let $\{p_\alpha\}_{\alpha\in A}$ be a family of seminorms on a vector space X. If $x \in X$, $\alpha \in A$, and $\epsilon > 0$, let

$$U_{x\alpha\epsilon} := \{y \in X : p_\alpha(y - x) < \epsilon\},$$

and let $\mathcal{T}$ be the topology generated by the sets $U_{x\alpha\epsilon}$

a. For each $x \in X$, the finite intersections of the sets $U_{x\alpha\epsilon}(\alpha \in A, \epsilon > 0)$ form a neighborhood base at x.

b. If $\langle x_i\rangle_{i\in I}$ is a net in X, then $x_i \to x$ iff $p_\alpha(x_i - x) \to 0$ for all $\alpha \in A$.

c. $(X, \mathcal{T})$ is a locally convex topological vector space.

Theorem 5.52. Suppose X and y are vector spaces with topologies defined, respectively, by the families $\{p_\alpha\}_{\alpha\in A}$ and $\{q_\beta\}_{\beta\in B}$ of seminorms, and $T : x \to y$ is a linear map. Then T is continuous iff for each $\beta \in B$ there exist $\alpha_1, \ldots, \alpha_k \in A$ and $C > 0$ such that $q_\beta(Tx) \le C\sum_1^k p_{\alpha_j}(x)$.

Proposition 5.53. Let X be a vector space equipped with the topology defined by a family $\{p_\alpha\}_{\alpha\in A}$ of seminorms.

a. X is Hausdorff iff for each $x \neq 0$ there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$.

b. If X is Hausdorff and A is countable, then X is metrizable with a translation-invariant metric (i.e., $\rho(x, y) = \rho(x + z, y + z)$ for all $x, y, z \in \mathcal{X}$).

Definition 5.54. A net $\langle x_i\rangle_{i\in I}$ in a topological vector space X is **Cauchy** if $\langle x_i - x_j\rangle_{(i,j)\in I\times I}$ converges to zero.

In this situation, X is called **complete** if every Cauchy net converges.

Definition 5.55. A complete Hausdorff topological vector space whose topology is defined by a countable family of seminorms is called a **Fréchet space**.

Example 5.56. Let X be an LCH space. Then $\mathbb{C}^X$ with the topology of uniform convergence is defined by the family of seminorms $p_k(x) := \sup_{x\in K} |f(x)|$ where K ranges over compact sets.

If X is σ-compact, we can take this to be a countable family of seminorms by choosing the compact sets $\overline{U_n}$.

Example 5.57. The space $L^1_{loc}(\mathbb{R}^n)$ with the countable family of seminorms $p_k(f) = \int_{|x|\leq k} |f(x)|dx$

Example 5.58. The seminorms $p_k(f) = \sup_{x\in[0,1]} |f^{(k)}|$ makes $\mathbb{C}^\infty([0,1])$ into a Fréchet space.

Definition 5.59. Let X be a normed vector space. The weak topology generated by X^* is called **the weak topology** on X. Convergence w.r.t to the weak topology is known as **weak convergence**. That is, $\langle x_\alpha\rangle \to x$ weakly in X iff $f(x_\alpha) \to f(x)$ for all $f \in X^*$.

The **basis of the weak topology** on X is given by

$$U_{V_i,f_i} := \bigcap_{i=1}^{n} f_i^{-1}(V_i) \qquad V_i \subseteq \mathbb{C}, f_i \in X^*.$$

The weak topology on X^* is generated by $\widehat{X} \subseteq X^{**}$ which is called the **weak* topology**.

Example 5.60. In an infinite dimensional Hilbert space, any orthonormal sequence converges weakly to zero. Indeed, a sequence in a Hilbert space $\{x_n\}$ converges weakly to x iff for all $z \in H$,

$$\langle x_n, z\rangle \to \langle x, z\rangle.$$

Example 5.61. Consider the basis e_n of ℓ^2. Then it converges weakly, but it does not converge strongly.

Remark 25. A few basic properties can be immediately derived.

1. The weak limit of a sequence or net is necessarily unique.
2. Subsequences of weakly convergent sequences also converge weakly to the same limit.
3. For sequences, convergence in the norm topology implies weak convergence and to the same limit. This is an immediate consequence of the fact that the weak topology is weaker than the norm topology.
4. In finite dimensions, weak convergence is the same as strong convergence (see next remark).
5. Weak convergence respects sums, scalar multiplication, bounded linear operators, and weakly closed subspaces contain weak limits of its sequences.

Remark 26. Our usual notion of convergence in a normed space is referred to as **strong convergence**.

Definition 5.62. If X, Y are Banach spaces, then the topology on $L(X,Y)$ generated by evaluation maps $T \mapsto Tx$ is called the **strong operator topology**.

In this case, $T_\alpha \to T$ **strongly** iff $T_\alpha x \to Tx$ in the norm topology of Y for each $x \in X$.

The **basis of the strong operator topology** is given by sets of the form, for $T \in L(X,Y)$,

$$U_\epsilon(T) := \{T' \in L(X,Y) : \|T'x - Tx\| < \epsilon\}.$$

Definition 5.63. If X, Y are Banach spaces, then the topology on $L(X,Y)$ generated by linear functionals $T \mapsto f(Tx)$ for $x \in X, f \in Y^*$ is called the **weak operator topology** on $L(X,Y)$.

In this case, $T_\alpha \to T$ **weakly** iff $T_\alpha x \to Tx$ in the weak topology of Y for each $x \in X$.

The **basis for the weak operator topology** is given by sets of the form, for $T \in L(X,Y)$,

$$U_{f_1,\dots,f_n,\epsilon}(T) := \{T' \in L(X,Y) : \|f_i T' x - f_i T x\| < \epsilon, 1 \le i \le n\}.$$

Remark 27. There is a hierarchy in terms of the topologies defined on $L(X,Y)$:

$$\textit{norm topology} \supseteq \textit{strong topology} \supseteq \textit{weak topology}$$

Similarly, on X^*,

$$\textit{norm topology} \supseteq \textit{strong topology} \supseteq \textit{weak topology} \supseteq \textit{weak}^* \textit{ topology.}$$

Example 5.64. We consider examples with $\ell^2(\mathbb{N})$ and with elements denoted by $a := (a_1, a_2, \dots)$.

The operators $T_n a := (a_n, a_{n+1}, \dots)$ converge strongly to some operator T and we note that T is *not* zero.

On the other hand, the operators $S_n a := (0_1, \dots, 0_n, x_1, x_2, \dots)$ do not converge strongly to some S. However, they do converge weakly to some S. See Exercise 5.64 of [1] to see these examples in more detail.

Proposition 5.65. Let $\{T_n\}_{n=1}^\infty \subset L(X,Y)$, $\sup_n \|T_n\| < \infty$, and $T \in L(X,Y)$. If

$$\|T_n x - Tx\| \to 0 \qquad \forall x \in D, D \text{ dense subset of } X,$$

then $T_n \to T$ strongly.

Theorem 5.66 (Alaoglu's Theorem)**.** If X is a normed vector space, the closed unit ball $B^* = \{f \in X^* : \|f\| \le 1\}$ in X^* is compact in the weak* topology.

Remark 28. "On a bounded subset of X^*, the weak*-topology can be generated by countably many seminorms if it is separable. Therefore, it is metrizable." Zlátos's proof / hint: Define seminorms $p_x(f) = |f(x)|$ on X^*. One can show that $f_n \to f$ pointwise iff $p_x(f_n - f) \to 0$ for each x. Now use this fact to show the result for separable case above.

Remark 29. The theorem does not show X^* is locally compact in the weak* topology.

Corollary 5.66.1. If X is a NVS, then there exists a compact Hausdorff K s.t. X is linearly isomorphic to a linear subspace normed by the maximum norm.

Proof. Let K be closed unit ball with weak*-topology. Then K compact by Alaoglu's Theorem and Hausdorff by the Hahn-Banach Theorem. Now, define $\Phi : X \to C(K)$ by $K|_K$ where $JLX \to X^{**}$ defined by $x \mapsto \widehat{x}$. It is an isometry (clearly). $\square$

Corollary 5.66.2. If X is a reflexive Banach space, then every sequence in X has a weakly convergence subsequence.

6. L^p-spaces

6.1. Basic Theory of L^p-spaces.

Definition 6.1. Let $L^p(X, \mathcal{M}, \mu)$ denote the space of measurable $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$. We always take $0 < p \leq \infty$ and the case of $p = \infty$ has norm defined by

$$\|f\|_\infty := \inf\{C : \mu(|f|^{-1}((C, \infty)) = 0\}.$$

It follows that for every $\epsilon > 0$ the set $E_\epsilon := \{x \in X : |\|f\|_\infty - \epsilon| < |f(x)|\}$ has nonzero measure.

Remark 30. The infinity norm coincides with the supremum norm under certain conditions. The key difference is dependent on the underlying vector space. For example, if the underlying vector space consists of continuous functions, then they are the same.

Theorem 6.2 (Young's Inequality)**.** If $a, b \geq 0$ and $0 < \lambda < 1$, then

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda b)$$

with equality iff $a = b$.

Theorem 6.3 (Hölder's Inequality)**.** If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (calling p and q conjugates), and if $f, g : X \to \mathbb{C}$ are measurable, then

$$\|fg\|_1 = \|f\|_p \|g\|_q.$$

If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and equality occurs iff $a|f|^p = b|g|^q$ for some constants a, b not both equal to zero.

Theorem 6.4 (Minkowski's Inequality)**.** If $1 \leq p \leq \infty$ and $f, g \in L^p$, then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_q.$$

In particular, L^p is a Banach space.

Corollary 6.4.1. Let $p^{-1} + q^{-1} = 1$. Let $\{f_n\} \subseteq L^p$ and $\{g_n\} \subseteq L^q$ where $f_n \to f$ and $g_n \to g$ w.r.t the respective norms. Then, $f_n g_n \to fg \in L^1$.

Proof. By Hölder's inequality, $\{f_n g_n\} \subseteq L^1$. To show $f_n g_n \to fg$ in L^1, let $\epsilon > 0$. Then,

$$\|f_n g_n - fg\|_1 \leq \|f_n g_n - f_n g\|_1 + \|f_n g - fg\|_1 \leq \|f_n\|_p \|g_n - g\|_q + \|f_n - f\|_p \|g\|_q$$

and as $n \to \infty$, notice $\|f_n\|_p \to \|f\| < \infty$, $\|g_n - g\|_q \to 0$, and $\|f_n - f\|_p \to 0$. $\square$

Theorem 6.5. The set of simple functions is dense in $L^p(\mu)$ for $1 \leq p < \infty$. In particular, we require the simple functions to have characteristic functions on sets with finite measures.

Theorem 6.6. "If $0 < p < q < r \leq \infty$, then $L^p \cap L^r \subset L^q$ and $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$ where $\lambda \in (0, 1)$ is defined as the convex conjugate: $\frac{1}{q} = \lambda \frac{1}{p} + (1 - \lambda) \frac{1}{r}$." The inequality is actually an equality when $r = \infty$ if $|f| = \|f\|_\infty$ a.e.. The inequality is actually an equality if Hölder's inequality holds for $\|f\|_1 \leq \|f\|_{p/\lambda q} \|f\|_{r/(1-\lambda)q}$.

Theorem 6.7. If $0 < p < q < r \leq R$, then $L^q \subset L^p + L^r$.

Proof. If $f \in L^q$, then let $f = f\chi_E + f\chi_F$ where $E := |f|^{-1}(1, \infty)$ and $F = E^c$. $\square$

Theorem 6.8. If $0 < p < q < r \leq \infty$, then $L^q \cap L^r \subset L^q$ where $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$ and

$$\lambda = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}} \qquad \& \qquad q^{-1} = \lambda p^{-1} + (1 - \lambda) r^{-1}.$$

Theorem 6.9. If A is a set and $0 < p < q \leq \infty$, then $l^p(A) \subset l^q(A)$ and $\|f\|_q \leq \|f\|_p$.

Theorem 6.10. If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p(\mu) \supset L^q(\mu)$ and $\|f\|_p \leq \|f\|_q \mu^(X)^{\frac{1}{p}-\frac{1}{q}}$.

6.2. Examples of L^p-spaces.

Example 6.11. The space $\ell^2(\mathbb{N})$ has orthonormal basis $e_i := (0, \dots, 1, 0, \dots)$ with 1 in the ith coordinate.

Example 6.12. For $p \geq 1$, we have $-\log x \in L^p((0,1))$ but $-\log x \notin L^\infty((0,1))$.

Example 6.13. Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = e^{-x}$. Then

$$\int_{(0,\infty)} |e^{-px}| dx \int_0^\infty \frac{1}{e^{px}} dx = \frac{-e^{-px}}{p} |_0^\infty = \frac{1}{p}.$$

Therefore, it is in L^p for $1 \leq p < \infty$.

Example 6.14. An obvious example of a Cauchy sequence in L^∞ is just a sequence $f_n(x) = \frac{1}{n}$ with $L^\infty(m)$.

Example 6.15. For what values of p is $\frac{1}{\sqrt{n}\log n}$ and ℓ^p sequence? Notice that we need

$$\sum_{n=2}^{\infty} \frac{1}{(\sqrt{n}\log n)^p} < \infty.$$

We can use the integral comparison test.

6.3. The Dual of L^p.

We assume p, q are conjugate exponents in this entire section.

Definition 6.16. If $g \in L^q(\mu)$, let $\phi_g(f) = \int f g d\mu$ and this is a bounded linear functional due to Hölder's inequality and $\|\phi_g\| \leq \|g\|_q$.

In the case of $p = q = 2$, then $L^2(\mu)$ is a Hilbert space and every bounded linear function is of the form $\langle \cdot, g \rangle = \int \cdot \overline{g} d\mu$. So in this situation, identify $g \mapsto \phi_{\overline{g}}$ as the natural identification of $L^2(\mu)$ and $(L^2(\mu))^*$.

Proposition 6.17. If p, q are conjugate exponents and $1 \leq q < \infty$ and if $g \in L^q$, then

$$\|g\|_q = \|\phi_g\| = \sup \left\{ \left| \int fg \right| : \|f\|_p = 1 \right\}$$

If μ is semifinite, this result holds also for $q = \infty$.

Theorem 6.18. Let p, q be conjugate. Let g be a measurable function on X and $fg \in L^1$ for all $f \in \Sigma$, the space of simple functions that vanish outside sets of finite measure. Then

$$M_q(g) = \sup \left\{ \left| \int fg \right| : f \in \Sigma \ \& \ \|f\|_p = 1 \right\}$$

is finite.

Also, if either $S_g = \{x : g(x) \neq 0\}$ is σ-finite or μ is semifinite, then $g \in L^q$ and $M_q(g) = \|g\|_q$.

PROOF. The case for $q < \infty$ is dealt with in [1]. If $q = \infty$, for all $\epsilon > 0$, let $E_\epsilon := g^{-1}(\mathbb{C} \setminus B_{M_\infty(g)+\epsilon}(0))$. If $\mu(E_\epsilon) > 0$, by σ-finiteness or semifiniteness, we can choose a subset A s.t. $\mu(A) \in (0, \infty)$. Then

$$M_\infty(g) \geq \int \overline{\operatorname{sgn} g}\chi_A g d\mu \|\overline{\operatorname{sgn} g}\chi_A\|_1^{-1} \geq \mu(A) \cdot (M_\infty(g) + \epsilon)\mu(A)^{-1} = M_\infty(g) + \epsilon$$

which is a contradiction. Therefore, $\|g\|_\infty \leq M_\infty(g)$ and Hölder's inequality gives the opposite inequality. ☐

Remark 31. We'll show in the next result that $(L^p(\mu))^* = L^q(\mu)$ if $p \in (1, \infty)$ or if $p = 1$ and the measure is σ-finite. See p. 191 of [1] for examples $p = 1$ and μ not being σ-finite does not yield the equality. If $p = \infty$, then $L^1(\mu)$ is clearly isometrically embedded into $(L^\infty(\mu))^*$. In particular, by Proposition 6.13. In Problem 3 on Midterm, one shows that $(L^\infty(\mathbb{R}^n))^* \neq L^1(\mathbb{R}^n)$ when μ is the Lebesgue measure. If one takes $\phi_\lambda(f) := \int f d\lambda$ for all $f \in C_c(\mathbb{R}^n)$ (where λ is a complex Borel measure on $\mathbb{R}^n$), then the Hahn-Banach Theorem extends this to a bounded linear functional on all of $L^\infty(\mathbb{R}^n)$. Taking $\lambda = \delta_0$ (the Dirac-δ measure at zero) shows the above.

Theorem 6.19. Let p, q be conjugate exponents. If $1 < p < \infty$ for all $\phi \in (L^p)^*$, there exists $g \in L^q$ s.t. $\phi(f) = \int fg$ for all $f \in L^p$. Hence, L^q is isometrically isomorphic to $(L^p)^*$.
The same conclusion is true when $p = 1$ so long as μ is σ-finite.

PROOF. We already know that $L^q \to (L^p)^*$ defined by $g \mapsto \phi_g$ is injective by one of the preceding results. So, we need to show surjectivity i.e. give any bounded linear functional, there is a g in which ϕ_g is precisely that given bounded linear functional.
Case 1: Suppose μ is finite so all simple functions are in $L^p(\mu)$. Given $\phi \in (L^p)^*$ and E a measurable set. Define $\nu(E) := \phi(\chi_E)$. For any disjoint sequence $\bigcup_1^\infty E_j = E$, we have

$$\left\| \chi_E - \sum_1^n \chi_{E_j} \right\|_p = \mu\left(\bigcup_{n+1}^\infty E_j\right)^{1/p} \to 0 \qquad n \to \infty$$

where we use $p < \infty$. So, ϕ is linear and continuous and therefore $\nu(E) = \sum \phi(\chi_{E_j}) = \sum \nu(E_j)$. Clearly $\nu(\emptyset) = 0$. Altogether, this shows ν is a complex measure.
If $\mu(E) = 0$, then $\chi_E = 0$ as an element of $L^p(\mu)$ so $\nu(E) = 0$. Therefore, $\nu \ll \mu$.
Apply the Radon-Nikodym Theorem to deduce that

$$\exists g \in L^1(\mu) \ s.t., \ \phi(\chi_E) = \nu(E) = \int_E g d\mu \qquad \forall E.$$

Therefore, $\phi(f) = \int fg d\mu$ for all simple functions f. Also, $|\int fg| \leq \|phi\|\|f\|_p$ since it is a bounded linear functional so $g \in L^q$ by the preceding theorem. At last, we deduce from density of simple functions that $\phi(f) = fg$ for all $f \in L^p$.
Case 2: Assume μ is σ-finite and E_n is an increasing sequence of finite nonzero measure sets and $X = \bigcup_{n=1}^\infty E_n$. The preceding argument shows

$$\forall n, \exists g_n \in L^q(E_n), \ s.t., \forall f \in L^p(E_j), \ \phi(f) = \int fg \ \& \ \|g_n\|_q \leq \||\phi| L^p(E_n)\| \leq \|\phi\|.$$

The function g is unique modulo null sets so $g_n = g_m$ a.e. on E_n when $n < m$. Set $g = g_n$ on E_n. By MCT, $\|g\|_q \leq \lim \|g_n\|_q \leq \|phi\|$ so $g \in L^q$.

Also, if $f \in L^p$, the DCT applied to $f\chi_{E_n} \to f$ in the L^p norm shows $\phi(f) = \lim \int_{E_n} fg = \int fg$.

This shows the result when μ is σ-finite and $p \in [1, \infty)$.

Case 3: Let μ arbitrary and $p > 1$ and $q < \infty$. For each σ-finite set E, there is an a.e.-unique $g_E \in L^p(E)$ s.t. $\phi(f) = \int fg_E$ for all $f \in L^p(E)$ and $\|g_E\|_q \leq \|phi\|$.

If F σ-finite and contains E, then $g_F = g_E$ a.e. on E so that $\|g_F\|_q \geq \|g_E\|_p$. Let M be the supremum of $\|g_E\|_q$ ranging over all σ-finite sets E. Note $M \leq \|\phi\|$ by this.

Choose E_n sequence s.t. $\|g_{E_n}\|_q \to M$. Set $F = \bigcup_1^\infty E_n$. Then F is σ-finite and $\|g_F\|_q \geq \|g_{E_n}\|_q$. Whence, $\|g_F\|_q = M$.

If A is σ-finite and contains F, we have

$$\int |g_F|^q + \int |g_{A\setminus F}|^q = \int |g_A|^q \leq M^q = \int |g_F|^q$$

so $g_{A\setminus F} = 0$ and $g_A = g_F$ a.e. (using fact $q < \infty$).

If $f \in L^p$, we know $A = F \cup \{x : f(x) \neq 0\}$ is σ-finite (it is a union of two σ-finite sets and the latter set is σ-finite since $\int |f|^p < \infty$) so $\phi(f) = \int fg_A = \int fg_F$. This means we just take $g = g_F$.

Thus, $\phi = \phi_g$ as desired. ☐

Corollary 6.19.1. If $1 < p < \infty$, then L^p is reflexive.

6.4. Some Useful Inequalities.

Theorem 6.20 (Chebyshev's Inequality)**.** If $f \in L^p(0 < p < \infty)$, then for any $\alpha > 0$,

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left[\frac{\|f\|_p}{\alpha}\right]^p.$$

Corollary 6.20.1. If $f \in L^p$, then the set $\{x : |f| \neq 0\}$ is σ-finite.

Proof. This is an immediate consequence of a similar result in Chapter 2 of [1]. However, we can derive this from Chebyshev's inequality. Define $E_n := \{x : |f(x)| > n\}$ and $n \in \mathbb{N}_0$. Then,

$$\mu(E_n) \leq \frac{\|f\|_p^p}{n^p} < \infty$$

for all n and therefore, $\{x : |f| \neq 0\} := \bigcup_{n \in N_0} E_n$. ☐

Theorem 6.21. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ -finite measure spaces, and let K be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$. Suppose that there exists $C > 0$ such that $\int |K(x, y)| d\mu(x) \leq C$ for a.e. $y \in Y$ and $\int |K(x, y)| d\nu(y) \leq C$ for a.e. $x \in X$, and that $1 \leq p \leq \infty$. If $f \in L^p(\nu)$, the integral

$$Tf(x) = \int K(x, y) f(y) d\nu(y)$$

converges absolutely for a.e. $x \in X$, the function Tf thus defined is in $L^p(\mu)$, and $\|Tf\|_p \leq C\|f\|_p$

Proof. We prove the result for $p \in (1, \infty)$. Let $q = \frac{p}{p-1}$. Apply Hölder's inequality to

$$|K(x, y) f(y)| = |K(x, y)|^{1/q} (|K(x, y)|^{1/p} |f(y)|) \qquad \Longrightarrow \qquad \int |K(x, y) f(y)| d\nu(y) \leq C^{1/q} \left[\int |K(x, y)||f$$

for a.e. $x \in X$. Apply Tonelli's Theorem when integrating w.r.t to x to obtain

$$\int \left[\int |K(x,y)f(y)|d\nu(y)\right]^p d\mu(x) \leq C^{p/q} \int\int |K(x,y)||f(y)|^p d\nu(y)d\mu(x) \leq C^{(p/q)+1}\|f\|_p^p.$$

The last integral is finite so $K(x,\cdot)f \in L^1(\nu)$ for a.e. x so that Tf is well-defined a.e. and taking pth roots, we get the desired inequality from above.

Assume $p = 1$. Checking well-definedness and absolutely convergent a.e. is easy. Then,

$$\left|\int Tf d\mu\right| \leq \int\int |K(x,y)||f(y)|d\nu d\mu = \int |f(y)| \int |K(x,y)|d\mu d\nu \leq C\|f\|_1$$

by an application of Fubini's Theorem. We used $\int |K(x,y)|d\mu(x) \leq C$ a.e. $y \in Y$ here.

Similarly for $p = \infty$, it is easy to check well-definedness and absolutely convergent a.e.. Then,

$$\|Tf\|_\infty \leq \|f\|_\infty \left\|\int K(\cdot,y)d\nu\right\|_\infty \leq \|f\|_\infty\|C\|_\infty = C\|f\|_\infty.$$

We used the second inequality with $\leq C$ in the theorem statement here. □

Theorem 6.22 (Minkowski's Inequality for Integrals)**.** Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ -finite measure spaces, and let f be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$.

a. If $f \geq 0$ and $1 \leq p < \infty$, then

$$\left[\int \left(\int f(x,y)d\nu(y)\right)^p d\mu(x)\right]^{1/p} \leq \int \left[\int f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y)$$

b. If $1 \leq p \leq \infty$, $f(\cdot,y) \in L^p(\mu)$ for a.e. y, and the function $y \mapsto \|f(\cdot,y)\|_p$ is in $L^1(\nu)$, then $f(x,\cdot) \in L^1(\nu)$ for a.e. x, the function $x \mapsto \int f(x,y)d\nu(y)$ is in $L^p(\mu)$, and

$$\left\|\int f(\cdot,y)d\nu(y)\right\|_p \leq \int \|f(\cdot,y)\|_p d\nu(y).$$

Proof. (a) If $p = 1$, then we get equality from Tonelli's Theorem. For $p \in (1,\infty)$, let $q = \frac{p}{p-1}$ and $g \in L^q$. By Tonelli's and Hölder's,

$$\int \left[\int f(x,y)d\nu(y)\right] |g(x)|d\mu(x) = \int\int f(x,y)|g(x)|d\mu d\nu \leq \|g\|_q \int \left[\int f(x,y)^p d\mu\right]^{1/p} d\nu.$$

Apply Theorem 6.14 of [1] to finish off the proof.
(b) For $p < \infty$, the assertion follows from (a) and Fubini's Theorem (replace f with $|f|$). For $p = \infty$, use monotonicity of the integral. □

Theorem 6.23. Let $K : (0,\infty)^2 \to \mathbb{C}$ be a measurable function s.t. $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$ for all $\lambda > 0$ (this is called (-1)-homogeneity) and $\int_0^\infty |K(x,1)|x^{-1/p}dx = C < \infty$ for some $p \in [1,\infty]$, and let q be the conjugate exponent to p. For $f \in L^p$ and $g \in L^q$, let

$$Tf(y) = \int_0^\infty K(x,y)f(x)dx, \quad Sg(x) = \int_0^\infty K(x,y)g(y)dy.$$

Then Tf and Sg are defined a.e. and $\|Tf\|_p \leq C\|f\|_p$ and $\|Sg\|_q \leq C\|g\|_q$.

Remark 32. This theorem is not as valuable as it may appear. The theorem shows that K induces a linear operator $L^p \to L^p$ and $L^q \to L^q$.

Remark 33. Some easy examples of $K(x,y)$ satisfying the hypothesis of (-1)-homogeneity are $\frac{1}{x}+\frac{1}{y}$, $\frac{1}{x+y}$, and $\frac{1}{x}+\frac{1}{y}+\frac{x}{y^2}$.

Corollary 6.23.1. Let

$$Tf(y) = y^{-1}\int_0^y f(x)dx, \quad Sg(x) = \int_x^\infty y^{-1}g(y)dy$$

Then for $1 < p \le \infty$ and $1 \le q < \infty$

$$\|Tf\|_p \le \frac{p}{p-1}\|f\|_p, \quad \|Sg\|_q \le q\|g\|_q.$$

This result is a special case of **Hardy's Inequality**.

PROOF. Define $K(x,y) = \frac{1}{y}\chi_E(x,y)$ where $E := \{(x,y) : x < y\}$. Then apply the preceding result. □

6.5. Distribution Functions and Weak L^p.

Definition 6.24. If f is a measurable function on $(X,\mathcal{M},\mu)$, we define its **distribution function**

$$\lambda_f : (0,\infty) \to [0,\infty] \text{ by } \lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\}).$$

Proposition 6.25. a. λ_f is decreasing and right continuous.
b. If $|f| \le |g|$, then $\lambda_f \le \lambda_g$.
c. If $|f_n|$ increases to $|f|$, then λ_{f_n} increases to λ_f.
d. If $f = g+h$, then $\lambda_f(\alpha) \le \lambda_g\left(\frac{1}{2}\alpha\right) + \lambda_h\left(\frac{1}{2}\alpha\right)$.

Remark 34. Since λ_f is decreasing but right continuous, it defines a negative Borel measure on $(0,\infty)$ by $\nu((a,b]) = \lambda_f(b) - \lambda_f(a)$.

The following theorem gives the relationship between integrals of $|f|$ on X to Borel measures defined by distribution functions.

Theorem 6.26. If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and ϕ is a nonnegative Borel measurable function on $(0,\infty)$, then

$$\int_X \phi \circ |f| d\mu = -\int_0^\infty \phi(\alpha)d\lambda_f(\alpha).$$

Remark 35. The important aspect of the preceding theorem is when applied to $\phi(\alpha) = \alpha^p$. It gives,

$$\int_X |f|^p d\mu = -\int_0^\infty \alpha^p d\lambda_f(\alpha).$$

If we integrate by parts for the RHS then we obtain the following result. We need to verify that $\alpha^p\lambda_f(\alpha) \to 0$ as $\alpha \to 0$ and $\alpha \to \infty$, however. The key point of the preceding result is that it converts integrals over arbitrary measure spaces to one with signed Lebesgue measures. The following result gives a way to show that a function is in L^p provided we have a strong bound on the distribution function.

Proposition 6.27. If $0 < p < \infty$, then

$$\int |f|^p d\mu = p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha.$$

Proof. Professor Zlatos's has a proof that does not use integration by parts. To integrate by parts, we would still need to check some hypotheses. Nonetheless,

$$\int_0^\infty \alpha^p d\lambda_f(\alpha) = (\alpha^p \lambda_f(\alpha)|_\infty - (0^p\lambda_f(0)) - \int_0^\infty \lambda_f(\alpha)d\alpha = -\int_0^\infty \lambda_f(\alpha)d\alpha.$$

Multiplying by -1 gives the result. All that is left is to show $\lim_{\alpha\to\infty}\alpha^p\lambda_f(\alpha) = 0$ since we cannot evaluate at infinity. This part of the argument follows by dealing with simple functions by proving the result when $\lambda_f(\alpha) = \infty$ for some $\alpha > 0$ and then the case where this does not occur. Apply the MCT with Proposition 6.22c. $\square$

Definition 6.28. If $f : X \to \mathbb{C}$ measurable and $0 < p < \infty$, define

$$[f]_p = \left(\sup_{\alpha>0} \alpha^p\lambda_f(\alpha)\right)^{1/p}$$

and define **weak** L^p to be the set of all f s.t. $[f]_p < \infty$. However, $[\cdot]_p$ is *not a norm.* However, it is a *topological vector space.*

Remark 36. The condition $[f]_p < \infty$ is equivalent to requiring there exists $C > 0$ s.t. $\lambda_f(t) \leq (C/t)^p$ for all t. In this case, the best possible constant that can be obtained is $[f]_p$. The following inequality which is a consequence of Chebyshev's is often very useful

$$\lambda_f(\alpha) \leq \frac{\|f\|_p^p}{\alpha^p}.$$

Remark 37. We first have

$$L^p \subset \text{ weak } L^p \qquad \& \qquad [f]_p \leq \|f\|_p.$$

On the other hand, if $\lambda_f(\alpha)$ is replaced by $([f]_p/\alpha)^p$ in the integral $p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha$, we obtain $\int_0^\infty \alpha^{-1}d\alpha$.

Example 6.29. The function $x^{-1/p}$ is weak L^p *but* not in L^p (on $(0,\infty)$ with Lebesgue measure). In some sense, weak L^p is *almost* L^p and the case of $x^{-1/p}$ is a border case.

Theorem 6.30. If f is a measurable function and $A > 0$, let $E(A) = \{x : |f(x)| > A\}$, and set

$$h_A = f\chi_{X\setminus E(A)} + A(\operatorname{sgn} f)\chi_{E(A)}, \quad g_A = f - h_A = (\operatorname{sgn} f)(|f| - A)\chi_{E(A)}$$

Then

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A), \quad \lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \text{if } \alpha < A \\ 0 & \text{if } \alpha \geq A \end{cases}$$

6.6. Interpolation of L^p-spaces.

Lemma 6.31 (The Three Lines Lemma.)**.** Let ϕ be a bounded continuous function on the strip $0 \leq \operatorname{Re} z \leq 1$ that is holomorphic on the interior of the strip. If $|\phi(z)| \leq M_0$ for $\operatorname{Re} z = 0$ and $|\phi(z)| \leq M_1$ for $\operatorname{Re} z = 1$, then $|\phi(z)| \leq M_0^{1-t}M_1^t$ for $\operatorname{Re} z = t$ $0 < t < 1$

Theorem 6.32 (The Riesz-Thorin Interpolation Theorem)**.** Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces and $p_0, p_1, q_0, q_1 \in [1, \infty]$. If $q_0 = q_1 = \infty$, suppose also that ν is semifinite. For $0 < t < 1$, define p_t and q_t by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

If T is a linear map from $L^{p_0}(\mu)+L^{p_1}(\mu)$ into $L^{q_0}(\nu)+L^{q_1}(\nu)$ such that $\|Tf\|_{q_0} \leq M_0\|f\|_{p_0}$ for $f \in L^{p_0}(\mu)$ and $\|Tf\|_{q_1} \leq M_1\|f\|_{p_1}$ for $f \in L^{p_1}(\mu)$, then

$$\|Tf\|_{q_t} \leq M_0^{1-t}M_1^t\|f\|_{p_t} \text{ for } f \in L^{p_t}(\mu), 0<t<1.$$

Example 6.33. If $T : L^1(-\pi,\pi)) \to \ell^\infty(\mathbb{Z})$ is the Fourier series operator $(Tf)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}f(x)dx$, then we know that $\|T\|_{L^1\to\ell^\infty} = 1 = \|T\|_{L^2\to\ell^2}$. So, we obtain an infinite family of bounds

$$\|T\|_{L^p\to\ell^q} \leq 1$$

when $p \in [1,2]$ and $q = p/(p-1) \in [2,\infty]$. This latter inequality is called the (Hausdorff-Young inequality).

When applying the theorem, the easiest cases are to apply it with $1,2,\infty$.

Remark 38. If $M(t) := \|T\|_{L^{p_t}(\mu)\to L^{q_t}(\nu)}$, then theorem says $M(t) \leq M(0)^{1-t}M(1)^t$ for all $t \in (0,1)$.

The log statement of the $M(t)$ is the key point of the theorem so we state this as a separate result. See p. 202 of [1].

Definition 6.34. Let T be a map from some vector space $\mathcal{D}$ of measurable functions on $(X,\mathcal{M},\mu)$ to the space of all measurable functions on $(Y,\mathcal{N},\nu)$

- T is called **sublinear** if $|T(f+g)| \leq |Tf|+|Tg|$ and $|T(cf)| = c|Tf|$ for all $f,g \in \mathcal{D}$ and $c>0$

- A sublinear map T is **strong type** $(p,q)(1 \leq p,q \leq \infty)$ if $L^p(\mu) \subset \mathcal{D}, T$ maps $L^p(\mu)$ into $L^q(\nu)$, and there exists $C>0$ such that $\|Tf\|_q \leq C\|f\|_p$ for all $f \in L^p(\mu)$

- A sublinear map T is **weak type** $(p,q)(1 \leq p \leq \infty, 1 \leq q < \infty)$ if $L^p(\mu) \subset \mathcal{D}, T$ maps $L^p(\mu)$ into weak $L^q(\nu)$, and there exists $C>0$ such that $[Tf]_q \leq C\|f\|_p$ for all $f \in L^p(\mu)$. Also, we shall say that T is weak type (p,∞) iff T is strong type (p,∞)

Theorem 6.35 (The Marcinkiewicz Interpolation Theorem)**.** Suppose that $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ are measure spaces; p_0,p_1,q_0,q_1 are elements of $[1,\infty]$ such that $p_0 \leq q_0$ $p_1 \leq q_1$, and $q_0 \neq q_1$; and

$$\frac{1}{p} = \frac{1-t}{p_0}+\frac{t}{p_1} \text{ and } \frac{1}{q} = \frac{1-t}{q_0}+\frac{t}{q_1}, \quad \text{where } 0<t<1.$$

If T is a sublinear map from $L^{p_0}(\mu)+L^{p_1}(\mu)$ to the space of measurable functions on Y that is weak types (p_0,q_0) and (p_1,q_1), then T is strong type (p,q). More precisely, if $[Tf]_{q_j} \leq C_j\|f\|_{p_j}$ for $j=0,1$, then $\|Tf\|_q \leq B_p\|f\|_p$ where B_p depends only on p_j,q_j,C_j in addition to p; and for $j=0,1, B_p\,|p-p_j|$ (resp. B_p) remains bounded as $p \to p_j$ if $p_j<\infty$ (resp. $p_j=\infty$).

Corollary 6.35.1. Suppose $1 \leq p < r < q \leq \infty$. Let T be a sublinear operator defined on L^p+L^q.

(1) If T is of weak-type (p,p) and T is a bounded operator on L^∞, then T is a bounded operator on L^r.

(2) If $q<\infty, T$ is of weak-type (p,p), and T is of weak-type $q-q$, then T is a bounded operator on L^r.

Corollary 6.35.2. There is a constant $C>0$ such that if $1<p<\infty$ and $f \in L^p(\mathbb{R}^n)$ then

$$\|Hf\|_p \leq C\frac{p}{p-1}\|f\|_p.$$

Remark 39. It is useful to note that Hf is not a bounded linear operator on L^1 so this is the reason we require $1 < p$ in the theorem.

Theorem 6.36. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ -finite measure spaces, and $1 < q < \infty$. Let K be a measurable function on $X \times Y$ such that, for some $C > 0$ we have $[K(x, \cdot)]_q \leq C$ for a.e. $x \in X$ and $[K(\cdot, y)]_q \leq C$ for a.e. $y \in Y$. If $1 \leq p < \infty$ and $f \in L^p(\nu)$, the integral

$$Tf(x) = \int K(x,y)f(y)d\nu(y)$$

converges absolutely for a.e. $x \in X$, and the operator T thus defined is weak type $(1, q)$ and strong type (p, r) for all p, r such that $1 < p < r < \infty$ and $p^{-1} + q^{-1} = r^{-1} + 1$. More precisely, there exist constants B_p independent of K such that

$$[Tf]_q \leq B_1 C \|f\|_1, \quad \|Tf\|_r \leq B_p C \|f\|_p \quad \left(p > 1, r^{-1} = p^{-1} + q^{-1} - 1 > 0\right).$$

Remark 40. We make some additional remarks about the relation of the two interpolation theorems. First off, the Riesz-Thorin has better estimates, requires less hypotheses on the estimates, but more from the operator T. On the other hand, the Marcinkiewics Theorem requires less hypotheses on the operator, but the estimates are weak-type estimates which are stronger and also, the estimates obtained are not as good as the one in the Riesz-Thorin Theorem.

7. Radon Measures

7.1. Positive Linear Functionals on $C_c(X)$. Throughout this whole section, we will fix X an LCH space.

Definition 7.1. If $I : C_c(X) \to \mathbb{C}$ is a linear functional. then I is **positive** if $I(f) \geq 0$ whenever $f \geq 0$.

Proposition 7.2. If I is a positive linear functional on $C_c(X)$, for each $K \subseteq X$ compact, there is a constant C_K s.t. $|I(f)| \leq C_K \|f\|_u$ for all $f \in C_c(X)$ s.t. $\mathrm{supp}(f) \subseteq K$.

Definition 7.3. Let μ be a Borel measure on X and E a Borel subset of X. Then μ is **outer regular** on E if

$$\mu(E) := \inf\{\mu(U) : U \supseteq E, U \text{ is open}\}$$

and **inner regular** on E if

$$\mu(E) := \sup\{\mu(K) : K \subseteq E, K \text{ is compact}\}.$$

We say μ is **regular** when μ is outer and inner regular on all Borel sets.

Remark 41. The Lebesgue measure is regular on $\mathcal{B}_{\mathbb{R}}$. The measure μ defined by $\mu(\emptyset) = 0$, $\mu(\mathbb{R}) = \infty$, and $\mu(\{x\}) = 0$ is outer regular, but not inner regular. The measure ν defined by $\mu(\emptyset) = 0$, $\mu(\{1\}) = 0$, $\mu(\{A\}) = \infty$ for $A \in P(\mathbb{R}) \setminus \{\{1\}\}$ is inner regular, but not outer regular.

Definition 7.4. A **Radon measure** on X is a Borel measure finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

Remark 42. The Lebesgue and Dirac measures are Radon on $\mathbb{R}$ but the counting measure is not.

Definition 7.5. Write $f \prec U$, where $f \in C_c(X)$ and $U \subseteq X$ is open, to mean $0 \leq f \leq 1$ and $\mathrm{supp}(f) \subseteq U$.

Theorem 7.6 (Riesz Representation Theorem)**.** If I is a positive linear functional on $C_c(X)$, there is a unique Radon measure μ on X such that $I(f) = \int f d\mu$ for all $f \in C_c(X)$. Moreover, μ satisfies

$$\mu(U) = \sup\{I(f) : f \in C_c(X), f \prec U\}$$

for all open $U \subset X$ and

$$\mu(K) = \inf\{I(f) : f \in C_c(X), f \geq \chi_K\}$$

for all compact $K \subset X$.

Remark 43. This is known as the Riesz–Markov–Kakutani representation theorem. The theorem also gives a way to construct new Radon measures. Let me demonstrate this with μ being the Lebesgue measure on $\mathbb{R}$. If we define I by $f \mapsto \int fgd\mu$ for some $g \geq 0$, the theorem asserts that there is a unique Radon measure ν s.t. $I(f) = \int fd\nu$ for all $f \in C_c(X)$ with ν characterized by the two equations in the theorem. *Note the representation $I(f) = \int fd\mu$ only applies to $f \in C_c(X)$.*

Definition 7.7. Let $\mathcal{B}_X^0$ denote the σ-algebra generated by $C_c(X)$ i.e. the smallest σ-algebra s.t. every $f \in C_c(X)$ is measurable. Elements of $\mathcal{B}_X^0$ are called **Baire sets**.

7.2. Regularity and Approximation Theorems.

Proposition 7.8. Radon measures are inner regular on σ-finite sets.

Proof. Prove for finite measure sets first, say $\mu(E) < \infty$. Choose open set $U \supseteq E$ s.t. $\mu(U) < \mu(E) + \epsilon$. Then choose compact set K s.t. $\mu(K) > \mu(U) - \epsilon$ and $K \subseteq U$. Then, $\mu(K) > \mu(E) - \epsilon$. Now, $\mu(U \setminus E) < \epsilon$ so choose an open set $V \supseteq U \setminus E$ s.t. $\mu(V) < 2\epsilon$. Then set $K := F \setminus V$. Then,

$$\mu(K) = \mu(F) - \mu(F \cap V) > \mu(E) - \epsilon - \mu(V) > \mu(E) - \epsilon - 2\epsilon = \mu(E) - 3\epsilon$$

and this gives inner regularity.

Suppose $\mu(E) = \infty$. From σ-finiteness, choose a sequence of E_j s.t. $\mu(E_j) \to \infty$ (just take the sequence of finite unions). Then for all $N \in \mathbb{N}$, there exists a j s.t. $\mu(E_j) > N$. Then, there exists a compact set $K_j \subseteq E_j$ s.t. $\mu(E_j) - \epsilon < \mu(K_j)$. Therefore, $N - \epsilon < \mu(K_j)$. For N large and $\epsilon > 0$ small, we obtain a sequence of $K_j \subseteq E_j \subseteq E$ s.t. $\mu(K_j) \to \infty$. So $\mu(E)$ is inner regular. □

Corollary 7.8.1. Every σ-finite Radon measure is regular. If X is σ-compact, then every Radon measure is regular.

Proposition 7.9. Let μ be a σ-finite Radon measure on X and $E \in \mathcal{B}_X$.

(1) For $\epsilon > 0$, there exists an open U and closed F s.t. $F \subseteq E \subseteq U$ and $\mu(U \setminus F) < \epsilon$.
(2) There exists a F_σ-set A and G_δ-set B s.t. $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$.

Proof. (1) Let $E = \bigcup_1^\infty E_n$ be a union of disjoint finite measure sets. Choose $U_j \supseteq E_j$ open s.t. $\mu(U_j) < \mu(E_j) + \frac{\epsilon}{2^j}$. Set U as the union of the U_j and so, $\mu(U) \leq \mu(E) + \epsilon$. Choose an open $V \supseteq E^c$ s.t. $\mu(V) \leq \mu(E^c) + \epsilon$. Then,

$$\mu(U \setminus V^c) = \mu(U \setminus E) + \mu(E \setminus V^c) < \epsilon + \mu(E \setminus V^c) = \epsilon + \mu(V \setminus E^c) < 2\epsilon.$$

(2) Choose open and closed U_j and W_j s.t. $W_j \subseteq E \subseteq U_j$ and $\mu(U_j \setminus E_j) < \frac{1}{j}$. Let $B := \bigcap_1^\infty U_j$ and $A := \bigcup_1^\infty W_j$. Then B is G_δ and A is F_σ while for all $j \in \mathbb{N}$,

$$\mu(B \setminus A) \leq \mu(U_j \setminus A) \leq \mu(U_j \setminus W_j) < \frac{1}{j}.$$

So $\mu(B \setminus A) = 0$. Furthermore, $A \subseteq E \subseteq B$. $\square$

Remark 44. Every finite positive Borel measure ν is Radon iff for all $E \in \mathcal{B}_X$ and $\epsilon > 0$, there is an open set U and compact K s.t. $K \subseteq E \subseteq U$ and $\nu(U \setminus K) < \epsilon$. This from the two preceding propositions.

Theorem 7.10. If X is an LCH space where every open set is σ-compact, then every Borel measure on X that is finite on compact sets is regular and hence, Radon.

Remark 45. The space $\mathbb{R}^n$ is second countable and LCH. The theorem says that the only thing preventing a Borel measure from being a Radon measure is whether or not it is finite on compact sets. By definition, Radon measures are Borel measures and finite on compact sets. So the theorem says, under some assumptions on X, a measure is a Radon measure iff it is Borel and finite on compact sets.

Proposition 7.11. If μ is a Radon measure, then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < \infty$.

Proof. Suffices to show for any Borel set E of finite measure, χ_E can be approximated by $C_c(X)$ (since simple functions are dense in L^p). Let $\epsilon > 0$. Since E is of finite measure, there is are compact and open K, U s.t. $K \subseteq E \subseteq U$ s.t. $\mu(U \setminus K) < \epsilon$. Then apply Urysohn's Lemma to find an $f \in C_c(X)$ s.t. $f|_K = 1$ and $\text{supp}(f) \subseteq U$. Then, $\|\chi_E - f\|_p^p = \int_X |\chi_E - f|^p d\mu \leq \mu(U \setminus K) < \epsilon$. $\square$

Theorem 7.12 (Lusin's Theorem)**.** Let μ be a Radon measure on X and $f : X \to \mathbb{C}$ a measurable function with vanishing outside a set of finite measure. Then for any $\epsilon > 0$, there exists $\phi \in C_c(X)$ s.t. $\phi = f$ except on a set of measure $< \epsilon$. If f is bounded, ϕ can be assumed to satisfy $\|\phi\|_u \leq \|f\|_u$.

Definition 7.13. Let $f : X \to (-\infty, \infty]$ and f is **lower semi-continuous** if $f^{-1}((a, \infty])$ is open for all $a \in \mathbb{R}$.

Let $g : X \to [-\infty, \infty)$ and g is **upper semi-continuous** if $f^{-1}([-\infty, a))$ is open for all $a \in \mathbb{R}$.

Proposition 7.14. Let X be a topological space.

a. If U is open in X, then χ_U is LSC.
b. If f is LSC and $c \in [0, \infty)$, then cf is LSC.
c. If $\mathcal{G}$ is a family of LSC functions and $f(x) = \sup\{g(x) : g \in 9\}$, then f is LSC.
d. If f_1 and f_2 are LSC, so is $f_1 + f_2$
e. If X is an LCH space and f is LSC and nonnegative, then $f(x) = \sup\left\{g(x) : g \in C_c(X), 0 \leq g \leq f\right\}$.

Proposition 7.15. Let $\mathcal{G}$ be a family of nonnegative LSC functions on an LCH space X that is directed by $\leq$ (that is, for every $g_1, g_2 \in \mathcal{G}$ there exists $g \in \mathcal{G}$ such that $g_1 \leq g$ and $g_2 \leq g$). Let $f = \sup\{g : g \in \mathcal{G}\}$. If μ is any Radon measure on X, then $\int f d\mu = \sup\left\{\int g d\mu : g \in \mathcal{G}\right\}$

Corollary 7.15.1. If μ is Radon and f is nonnegative and LSC, then $\int f d\mu = \sup\left\{\int g d\mu : g \in C_c(X), 0 \leq\right.$

Proposition 7.16. If μ is a Radon measure and f is a nonnegative Borel measurable function, then $\int f d\mu = \inf\left\{\int g d\mu : g \geq f \text{ and } g \text{ is } LSC\right\}$. If $\{x : f(x) > 0\}$ is σ -finite, then $\int f d\mu = \sup\left\{\int g d\mu : 0 \leq g \leq f \text{ and } g \text{ is } USC\right\}$.

7.3. The Dual of $C_0(X)$.

Remark 46. Recall for an LCH space X, $C_0(X)$ is the uniform closure of $C_c(X)$. Any function $I(f)=\int f d\mu$ with μ Radon defined on $C-c(X)$ extends to $C_0(X)$ continuously iff it is bounded w.r.t. the uniform norm. Because

$$\mu(X)=\sup\left\{\int f d\mu : f\in C_c(X),\ 0\le f\le 1\right\},$$

and $|\int f d\mu|\le\int|f|d\mu$, this happens when $\mu(X)<\infty$ and this also gives $|\mu(X)|=\|I\|$. Essentially, positive bounded linear functionals on $C_0(X)$ are given by integration against finite Radon measures.

Lemma 7.17. If $I\in C_0(X,\mathbb{R})^*$, there exist positive functionals $I^{\pm}\in C_0(X,\mathbb{R})^*$ such that $I=I^+-I^-$

Definition 7.18. Any $I\in C_0(X)^*$ is uniquely determined by its restriction to $C_(X,\mathbb{R})$ and the restriction J has $J=J_1+iJ_2$ for real linear functionals J_i. Then, there are finite Radon measures μ_i for $i=1,2,3,4$ s.t. $I(f)=\int f d\mu$ and $\mu=\mu_1-\mu_2+i(\mu_3-\mu_4)$. A **signed Radon measure** is a signed Borel measure whose posiutive and negative variations are Radon. A **complex Radon measure** is a complex Borel measure whose real and imaginary parts are signed Radon measures. Let $M(X)$ be the **space of complex Radon measures** and $\|\mu\|=|\mu|(X)$ defines a norm on $M(X)$ by the next result.

Proposition 7.19. If μ is a complex Borel measure, then μ is Radon iff $|\mu|$ is Radon. Moreover, $M(X)$ is a vector space and $\mu\mapsto\|\mu\|$ is a norm on it.

Proof. Let $\epsilon>0$ and E Borel. Suppose $K\subseteq E\subseteq K$ and $|\mu|(U\setminus K)<\epsilon$. Then $\nu_j(U\setminus K)<\epsilon$ for all $j=1,2,3,4$ in the decomposition $\mu=\mu_1-\mu_2+i(\mu_3-\mu_4)$. Conversely, if there are $\mu_j(U_j\setminus K_j)<\epsilon$, then taking $\bigcap_{j=1}^4 U_j$ and $\bigcup_{j=1}^4 K_j$, we get $\mu(U\setminus K)<4\epsilon$. The same idea shows $M(X)$ is closed under addition and scalar multiplication. Also, $\|\cdot\|$ being a norm follows from Proposition 3.14 of Folland. ☐

Theorem 7.20 (The Riesz Representation Theorem.)**.** Let X be an LCH space, and for $\mu\in M(X)$ and $f\in C_0(X)$ let $I_\mu(f)=\int f d\mu$. Then the map $\mu\mapsto I_\mu$ is an isometric isomorphism from $M(X)$ to $C_0(X)^*$.

Proof. Suppose $\mu\in M(X)$. We know $|\int f d\mu|\le\int|f|d|\mu|\le\|f\|_u\|\mu\|$ so $I_\mu\in C_0(X)^*$ and $\|I_\mu\|\le\|\mu\|$. By the Radon-Nikodym Theorem, $h:=d\mu/d|\mu|$ exists and $|h|=1$. By Lusin's Theorem, there exists $f\in C_c(X)$ s.t. $\|f\|_u=1$ and $f=\overline{h}$ except on some set E with $|\mu|(E)<\epsilon$. Then,

$$\|\mu\|=\int|h|^2d|\mu|=\int\overline{h}d\mu=|\int f d\mu|+\int(f-\overline{h})d\mu|\le|\int f d\mu|+2|\mu|(E)<|\int f d\mu|+\epsilon\le\|I_\mu\|+\epsilon.$$

☐

Corollary 7.20.1. If X is a compact Hausdorff space, then $C(X)^*$ is isometrically isomorphic to $M(X)$

Remark 47. One can embed $L^1(\mu)$ into $M(X)$ by $f\mapsto\nu_f$ where $d\nu_f=f d\mu$. Then $\|\nu_f\|=\|f\|_1$ and the range consists of those $\nu\in M(X)$ s.t. $\nu\ll\mu$. The most important example of this is $L^1(m)\subseteq M(\mathbb{R}^n)$.

Definition 7.21. The **vague topology** on $M(X)$ is the weak* topology on $M(X) = C_0(X)^*$. This means $\mu_\alpha \to \mu$ in weak* iff $\int f d\mu_\alpha \to \int f d\mu$ for all $f \in C_0(X)$. So $\mu_\alpha \to \mu$ weakly iff $\int f d\mu_\alpha \to \int f d\mu$ for all $f \in C_0(X)$.

Remark 48. Every $f \in L^1(\mu)$ defines a Radon measure $d\nu_f = f d\mu$. This yields an isometric embedding of $L^1(\mu)$ into $M(X)$ whose range are those $\nu \in M(X)$ s.t. $\nu \ll \mu$.

Proposition 7.22. Suppose $\mu, \mu_1, \mu_2, \ldots \in M(\mathbb{R})$, and let $F_n(x) = \mu_n((-\infty, x])$ and $F(x) = \mu((-\infty, x])$.

a. If $\sup_n \|\mu_n\| < \infty$ and $F_n(x) \to F(x)$ for every x at which F is continuous, then $\mu_n \to \mu$ vaguely.

b. If $\mu_n \to \mu$ vaguely, then $\sup_n \|\mu_n\| < \infty$. If, in addition, the μ_n 's are positive, then $F_n(x) \to F(x)$ at every x at which F is continuous.

7.4. Products of Radon Measure.

Theorem 7.23. a. $\mathcal{B}_X \otimes \mathcal{B}_Y \subset \mathcal{B}_{X\times Y}$

b. If X and Y are second countable, then $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X\times Y}$

c. If X and Y are second countable and μ and ν are Radon measures on X and Y, then $\mu \times \nu$ is a Radon measure on $X \times Y$.

Proposition 7.24. Let $\mathcal{P}$ be the vector space spanned by the functions $g \otimes h$ with $g \in C_c(X), h \in C_c(Y)$. Then $\mathcal{P}$ is dense in $C_c(X \otimes Y)$ in the uniform norm. More precisely, given $f \in C_c(X \times Y), \epsilon > 0$, and precompact open sets $U \subset X$ and $V \subset Y$ containing $\pi_X(\mathrm{supp}(f))$ and $\pi_Y(\mathrm{supp}(f))$, there exists $F \in \mathcal{P}$ such that $\|F - f\|_u < \epsilon$ and $\mathrm{supp}(F) \subset U \times V$.

Proposition 7.25. Every $f \in C_c(X \times Y)$ is $\mathcal{B}_X \otimes \mathcal{B}_Y$ -measurable. Moreover, if μ and ν are Radon measures on X and Y, then $C_c(X \times Y) \subset L^1(\mu \times \nu)$, and

$$\int f d(\mu \times \nu) = \iint f d\mu d\nu = \iint f d\nu d\mu \quad (f \in C_c(X \times Y))$$

Lemma 7.26. a. If $E \in \mathcal{B}_{X\times Y}$, then $E_x \in \mathcal{B}_Y$ for all $x \in X$ and $E^y \in \mathcal{B}_X$ for all $y \in Y$.

b. If $f : X \times Y \to \mathbb{C}$ is $\mathcal{B}_{X\times Y}$ -measurable, then f_x is $\mathcal{B}_Y$ -measurable for all $x \in X$ and f^y is $\mathcal{B}_X$ -measurable for all $y \in Y$.

Proposition 7.27. Let μ and ν be Radon measures on X and Y. If U is open in $X \times Y$, then the functions $x \mapsto \nu(U_x)$ and $y \mapsto \mu(U^y)$ are Borel measurable on X and Y, and

$$\mu \widehat{\times} \nu(U) = \int \nu(U_x)\, d\mu(x) = \int \mu(U^y)\, d\nu(y).$$

Lemma 7.28. If $f \in C_c(X \times Y)$ and μ, ν are Radon measures on X, Y, then the functions $x \mapsto \int f_x d\nu$ and $y \mapsto \int f^y d\mu$ are continuous.

Proof. Prove the result for f_x and the one for f^y follows similarly. It suffices to show for any $x_0 \in X$ and $\epsilon > 0$, there exists a neighborhood $U \ni x_0$ s.t. $\|f_x - f_{x_0}\|_u < \epsilon$ for $x \in U$ since

$$\left| \int f_x - f_{x_0} \right| d\nu \leq \|f_x - f_{x_0}\| \nu(\pi_Y(\mathrm{supp}(f))) < \epsilon \nu(\pi_Y(\mathrm{supp}(f)))$$

For each $y \in \pi_Y(\mathrm{supp}(f))$, there exist neighborhoods U_y, V_y of x_0, y s.t. if $(x, z) \in U_y \times V_y$, we have $|f(x_0, y) - f(x, z)| < \frac{1}{2}\epsilon$. Since the V_y must cover $\pi_Y(\mathrm{supp}(f))$, we may choose a finite subcover V_{y_i} for $i = 1, \ldots, n$ and then set $U := \bigcap_{i=1}^m U_{y_i}$. Now we verify that U is the desired set. Clearly, $x_0 \in U$ since U_{y_i} are neighborhoods of x_0. Next, if $x \in U$, then with $z \in V_{y_i}$ for some i and our estimate,

$$\begin{aligned}\|f_x - f_{x_0}\|_u = \sup\{|f_x(y) - f_{x_0}(y)| : y \in \pi_Y(\operatorname{supp}(f))\} &= \sup\{|f(x,y) - f(x_0,y)| : y \in \bigcup_{i=1}^n V_{y_i}\} \\ \leq \sup\{|f(x,y) - f(x_0,z)| : y \in \bigcup_{i=1}^n V_{y_i}\} &+ \sup\{|f(x_0,z) - f(x_0,y)| : y \in \bigcup_{i=1}^n V_{y_i}\} < \epsilon.\end{aligned}$$

□

Theorem 7.29. Suppose that μ and ν are σ -finite Radon measures on X and Y. If $E \in \mathcal{B}_{X\times Y}$, then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ (which make sense by Lemma 7.23) are Borel measurable on X and Y, and

$$\mu \widehat{\times} \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$$

Moreover, the restriction of $\mu \widehat{\times} \nu$ to $B_X \otimes B_Y$ is $\mu \times \nu$.

Theorem 7.30 (The Fubini-Tonelli Theorem for Radon Products.)**.** Let μ and ν be σ -finite Radon measures on X and Y, and let f be a Borel measurable function on $X \times Y$. Then f_x and f^y are Borel measurable for every x and y. If $f \geq 0$, then $x \mapsto \int f_x d\nu$ and $y \mapsto \int f^y d\mu$ are Borel measurable on X and Y. If $f \in L^1(\mu \hat{x} \nu)$, then $f_x \in L^1(\nu)$ for a.e. x, $f^y \in L^1(\mu)$ for a.e. y, and $x \mapsto \int f_x d\nu$ and $y \mapsto \int f^y d\mu$ are in $L^1(\mu)$ and $L^1(\nu)$. In both cases, we have

$$\int f d(\mu \widehat{\times} \nu) = \iint f d\mu d\nu = \iint f d\nu d\mu.$$

Theorem 7.31. Suppose that, for each $\alpha \in A, \mu_\alpha$ is a Radon measure on the compact Hausdorff space X_α such that $\mu_\alpha(X_\alpha) = 1$. Then there is a unique Radon measure μ on $X = \prod_{\alpha \in A} X_\alpha$ such that for any $\alpha_1, \ldots, \alpha_n \in A$ and any Borel set E in $\prod_1^n X_{\alpha_j}$

$$\mu\left(\pi^{-1}_{(\alpha_1,\ldots,\alpha_n)}(E)\right) = \left(\mu_{\alpha_1} \widehat{\times} \cdots \widehat{\times} \mu_{\alpha_n}\right)(E).$$

8. Elements of Fourier Analysis

We follow Chapter 8 of [1]. The lecture notes here are also helpful. This whole section works over $\mathbb{R}^n$ and omissions of the ambient space should be assumed to be in $\mathbb{R}$ and n is fixed as the dimension of the space.

8.1. Preliminaries. For this entire section, we work on $\mathbb{R}^n$, any measures are assumed to be the Lebesgue measure, the L^p spaces are w.r.t to the measurable subsets of $\mathbb{R}^n$ and the Lebesgue measures and

$$x \cdot y = \sum_1^n x_j y_j, \qquad |x| = \sqrt{x \cdot x}$$

are the dot products and Euclidean norm on $\mathbb{R}^n$.

The **multi-index** notation $\alpha = (\alpha_1, \ldots, \alpha_n)$ is used for convenience. For example, $|\alpha| = \sum_1^n \alpha_j$ and $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$ where $\partial_i = \frac{\partial}{\partial x_i}$. For $x \in \mathbb{R}^n$, we write $x^\alpha = \prod_1^n x_j^{\alpha_j}$.

The space $C_c^\infty(U)$ is usually referred to the space of test functions over U and $C_c^\infty(U)$ is the space of smooth functions whose support is compact and contained in U.

Lemma 8.1. There exists $\phi \in C_c^\infty$ s.t. $0 \leq \phi \leq 1$ and $\text{supp}(\phi) = \overline{B_n}$, the closed unit ball in $\mathbb{R}^n$.

PROOF. One defines $\psi(x) = \eta(1 - |x|^2)$ which is $\exp[(|x| - 1)^{-1})]$ whne $|x| < 1$ and 0 when $|x| \geq 1$. More details are in Exercise 8.3. $\square$

Definition 8.2. The **Schawrtz space** $\mathcal{S}$ is the vector space of $f \in C^\infty$ s.t. $\|f\|_{(N,\alpha)} < \infty$ for all $(N, \alpha) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n_{\geq 0}$ endowed with a family of seminorms,

$$\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N (|\partial^\alpha f(x)|.$$

Example 8.3. For all α, $f_\alpha(x) = x^\alpha e^{-|x|^2} \in \mathcal{S}$.

Example 8.4. If $f \in \mathcal{S}$, then $\partial^\alpha f \in L^p$ for all α and all $p \in [1, \infty]$.

Theorem 8.5. $\mathcal{S}$ is a complete space i.e. a Fréchet space wit its topology induced by its seminorms.

Definition 8.6. We define $\tau_y f(x) = f(x-y)$ as the translation operator and f is **uniformly continuous** if $\|\tau_y f - f\|_u \to 0$ as $y \to 0$.

Lemma 8.7. If $f \in C_c(\mathbb{R}^n)$, then f is uniformly continuous.

Lemma 8.8. If $1 \leq p < \infty$, then translation is continuous in the L^p norm.

Definition 8.9. We say $f : \mathbb{R}^n \to \mathbb{C}$ is periodic if $f(x + k) = f(x)$ for $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$. Every periodic function is determined by its value on $Q = [-\frac{1}{2}, \frac{1}{2})^n$.

Periodic functions can be viewed as functions on the torus $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n = \mathbb{R}^n/\mathbb{Z}^n$. Note that $\mathbb{T}^n$ is a compact Hausdorff space and is identifiable with the unit sphere in $\mathbb{C}^n$.

We regard $m(\mathbb{T}^n) = 1$ by identifying $\mathbb{T}^n$ with the unit cube.

8.2. Convolutions.

Definition 8.10. If f, g are measurable functions on $\mathbb{R}^n$, then

$$f * g(x) = \int_{\mathbb{R}} f(x - y) g(y) dy$$

where the integral exists. It is not guaranteed that the integral exists for all x, but one can guarantee that it does exist a.e. under certain conditions.

Theorem 8.11 (Young's Convolution Inequality)**.** Assume $1 \leq p \leq \infty$. Let $f \in L^1$, $g \in L^p$. Then, $f * g \in L^p$ and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

PROOF. The proof is an easy consequence of Minkowski's Inequality, but we provide a proof using Hölder's inequality here.

The case of $p = \infty$ is clear. For the general case, first, let q s.t. $\frac{1}{q} + \frac{1}{p} = 1$ and then,

$$\begin{aligned}
|f * g(x)| &\leq \int |f(x - y) g(y)| dy = \int |f(x - y)^{1/q} |f(x - y)|^{1/p} |g(y)| dy \\
&\leq \|f^{\frac{1}{q}}\|_q \left(\int |f(x - y)| |g(y)|^p dy \right)^{1/p} \\
&= \|f\|_1^{1/q} \left(\int |f(x - y)| g(y)|^p dy \right)^{1/p} < \infty
\end{aligned}$$

because $f \in L^1$ and $g \in L^p$ Next,

$$\begin{aligned}
\|f * g\|_p &\leq \left(\int \left(\int |f(x-y)|g(y)|dy\right)^p dx\right)^{1/p} \leq \left(\int \|f\|_1^{p/q} \int |f(x-y)||g(y)|^p dydx\right)^{1/p} \\
&= \|f\|_1^{1/q} \left(\int \int |f(x-y)||g(y)|^p dydx\right)^{1/p} \\
&= \|f\|_1^{1/q} \left(\int |g(y)|^p \int |f(x-y)|dxdy\right)^{1/p} \\
&= \|f\|_1^{1/q} \left(\int |g(y)|^p \|f\|_1 dy\right)^{1/p} \\
&= \|f\|_1^{1/q} \|f\|_1^{1/p} \|g\|_p = \|f\|_1 \|g\|_p.
\end{aligned}$$

□

Theorem 8.12. Assume all integrals exists. Then,

(1) $f * g = g * f$,
(2) $(f * g) * h = f * (g * h)$,
(3) for $z \in \mathbb{R}^n$, $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$,
(4) If $A = \overline{\{x + y : x \in \operatorname{supp}(f), y \in \operatorname{supp}(g)\}}$, then $\operatorname{supp}(f * g) \subset A$.

Proposition 8.13. If p, q are conjugate exponents, $f \in L^p$, and $g \in L^q$, then $f * g(x)$ exists for every x, $f * g$ is bounded and uniformly continuous, and

$$\|f * g\|_u \leq \|f\|_p \|g\|_q. \tag{1}$$

If $1 < p < \infty$, then $f * g \in C_0(\mathbb{R}^n)$.

Proposition 8.14. Suppose $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

(1) (**Young's Inequality, General Form**) if $f \in L^p, g \in L^q$, then $f * g \in L^r$ and
$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \tag{2}$$
(2) Let $p, q > 1$ and $r < \infty$. If $f \in L^p$ and $g \in$ weak L^q, then $f * g \in L^r$ and
$$\|f * g\|_r \leq C_{pq} \|f\|_p [g]_q \tag{3}$$
where C_{pq} is independent of f, g.
(3) Let $p = 1$ and $r = q > 1$. If $f \in L^1$ and $g \in$ weak L^q, then $f * g \in$ weak L^q and $[f * g]_q \leq C_q \|f\|_1$ where C_q is independent of f and g.

Proposition 8.15. If $f \in L^1$, $g \in C^k$, and $\partial^\alpha g$ is bounded for $|\alpha| \leq K$, then $f * g \in C^k$ and $\partial^\alpha (f * g) = f * (\partial^\alpha g)$ for $|\alpha| \leq k$.

Proposition 8.16. If $f, g \in \mathcal{S}$, then $f * g \in \mathcal{S}$.

Definition 8.17. If ϕ is any function $\mathbb{R}^n$ and $t > 0$, we set

$$\phi_t(x) := t^{-n} \phi(t^{-1} x). \tag{4}$$

Observe that if $\phi \in L^1$, then

$$\int \phi_t = \int \phi(t^{-1} x) t^{-n} dx = \int \phi(y) dy = \int \phi. \tag{5}$$

Furthermore, the mass of ϕ_t becomes concentrated at the origin as $t \to 0$.

Theorem 8.18. Suppose $\phi \in L^1$ and $\int \phi(x)dx = a$.

(1) if $f \in L^p$ $(1 \leq p < \infty)$, then $f * \phi_t \to af$ in L^p as $t \to 0$;
(2) if f is bounded and uniformly continuous, then $f * \phi_t \to af$ uniformly as $t \to 0$;
(3) if $f \in L^\infty$ and f is continuous on an open set U, then $f * \phi_t \to af$ uniformly on compact subsets of U as $t \to 0$.

Theorem 8.19. Let $|\phi(x)| \leq C(1+|x|)^{-n-\epsilon}$ for some $C, \epsilon > 0$ and $\int \phi(x)dx = a$. If $f \in L^p$ $(1 \leq p \leq \infty)$, then $f * \phi_t(x) \to af(x)$ as $t \to 0$ for every x in the Lebesgue set of f. In particular, for a.e. x, and for all x at which f is continuous.

Theorem 8.20. C_c^∞ (and also $\mathcal{S}$) is dense in L^p for $1 \leq o < \infty$ and in C_0.

Theorem 8.21 (C^∞ Urysohn Lemma)**.** If $K \subseteq \mathbb{R}^n$ is compact and U is an open set containing K, then there exists $f \in C_c^\infty$ s.t. $0 \leq f \leq 1$, $f = 1$ on K, and $\operatorname{supp}(f) \subseteq U$.

8.3. The Fourier Transform.

Theorem 8.22. If $\phi : \mathbb{R}^n \to \mathbb{C}$ (resp. $\mathbb{T}^n$) is measurable s.t. $\phi(x+y) = \phi(x)\phi(y)$ and $|\phi| = 1$, there exists $\xi \in \mathbb{R}^n$ (resp. $\mathbb{T}^n$) s.t. $\phi(x) = e^{2\pi i \xi \cdot x}$.

Theorem 8.23. Let $E_\kappa(x) := e^{2\pi i \kappa \cdot x}$. Then $\{E_\kappa : \kappa \in \mathbb{Z}^n\}$ forms an orthonormal basis for $L^2(\mathbb{T}^n)$.

Definition 8.24. If $f \in L^2(\mathbb{T}^n)$, its **Fourier transform** $\widehat{f}$ is a function on $\mathbb{Z}^n$ defined by

$$\widehat{f}(\kappa) = \langle f, E_\kappa \rangle = \int_{\mathbb{T}^n} f(x) e^{-2\pi i \kappa \cdot x} dx \tag{6}$$

and the series

$$\sum_{\kappa \in \mathbb{Z}^n} \widehat{f}(\kappa) E_\kappa \tag{7}$$

is the **Fourier series** of f.

Theorem 8.25 (Hausdorff-Young inequality)**.** If $1 \leq p \leq 2$ and q is the conjugate exponent to p and $f \in L^p(\mathbb{T}^n)$, then $\widehat{f} \in \ell^q(\mathbb{Z}^n)$ and $\|\widehat{f}\|_q \leq \|f\|_p$.

Theorem 8.26. Let $f, g \in L^1(\mathbb{R}^n)$.

(1)
(2)
(3)
(4)
(5)
(6)

Corollary 8.26.1. $\mathcal{F}$ maps the Schwartz class $\mathcal{S}$ continuous into itself.

Proposition 8.27. If $f(x) = e^{-\pi a |x|^2}$ for $a > 0$, then $\widehat{f}(\xi) = a^{-n/2} e^{-\pi |\xi|^2 / a}$.

Definition 8.28. If $f \in L^1$, define

$$f^\vee(x) = \widehat{f}(-x) = \int f(\xi) e^{2\pi i \xi \cdot x} d\xi. \tag{8}$$

Lemma 8.29. If $f, g \in L^1$, then $\int \widehat{f} g = \int f \widehat{g}$.

Theorem 8.30 (Fourier inversion)**.** If $f \in L^1$ and $\widehat{f} \in L^1$, thne f agrees a.e. with a continuous function f_0 and $(\widehat{f})^\vee = \widehat{(f^\vee)} = f_0$.

Corollary 8.30.1. If $f \in L^1$ and $\widehat{f} = 0$, then $f = 0$ a.e.. Also, $\mathcal{F}$ is an isomorphism of $\mathcal{S}$ onto itself.

Theorem 8.31 (Plancherel Theorem)**.** If $f \in L^1 \cap L62$, then $\widehat{f} \in L^2$ and $\mathcal{F}|_{L^1 \cap L^2}$ extends uniquely to a unitary isomorphism on L^2.

Theorem 8.32 (Hausdorff-Young inequality)**.** Let $1 \leq p \leq 2$ and q be a conjugate exponent to p. If $f \in L^p(\mathbb{R}^n)$, then $\widehat{f} \in L^q(\mathbb{R}^n)$ and $\|\widehat{f}\|_q \leq \|f\|_p$.

Theorem 8.33.

Theorem 8.34 (Poisson summation formula)**.** If $f \in C(\mathbb{R}^n)$ satisfies $|f(x)| \leq C(1+|x|)^{-n-\epsilon}$ and $|\widehat{f}(\xi)| \leq C(1+|\xi|)^{-n-\epsilon}$ for some $C, \epsilon > 0$. Then,

$$\sum_{k \in \mathbb{Z}^n} f(x+k) = \sum_{\kappa \in \mathbb{Z}^n} \widehat{f}(\kappa) e^{2\pi i \kappa \cdot x} \tag{9}$$

where both series converge absolutely and uniformly on $\mathbb{T}^n$. Taking $x = 0$,

$$\sum_{k \in \mathbb{Z}^n} f(k) = \sum_{\kappa \in \mathbb{Z}^n} \widehat{f}(\kappa). \tag{10}$$

8.4. Summation of Fourier Integrals and Series.

8.5. Pointwise Convergence of Fourier Series.

8.6. Fourier Analysis of Measure.

8.7. Applications to Partial Differential Equations.

9. Elements of Distribution Theory

9.1. Distributions.

9.2. Compactly supported, Tempered, and Periodic Distributions.

9.3. Sobolev Spaces.

10. Topics in Probability Theory

10.1. Basic Concepts.

10.2. The Law of Large Numbers.

10.3. The Central Limit Theorem.

10.4. Construction of Sample Spaces.

10.5. The Wiener Process.

11. More Measures and Integrals

12. Appendix Course Notes for UCSD Math 240

Approximate: It is easier to prove results for simpler classes of objects. Here is a list of approximation results to consider.

(1) Theorem 2.26 of [1] - L^1 functions can be approximated using simple functions and if working on $\mathbb{R}$, one can approximate L^1 functions using $C_c(\mathbb{R})$ functions.
(2) Theorem 2.40 of [1] - Lebesgue measure is inner regular and outer regular and can approximated using rectangles.
(3) Proposition 6.7 of [1] - simple functions with finite support are dense in L^p for $1 \le p \le \infty$.
(4) Theorem 3.14 of [2] - for LCH spaces, $C_c(X)$ is dense in L^p for $1 \le p < \infty$.
(5) Theorem 5.27 and Proposition 5.28 of [1] - Hilbert spaces have an orthonormal basis and every element of the space is a countable sum.
(6) Section 5.3 of [1] - the big four theorems of Functional Analysis are Baire's Category Theorem, Closed Graph, Open Mapping, and Uniform Boundedness.
(7) Use the *epsilon of room* technique - for example, to show that $x \le y$, it is sufficient to show for all $\epsilon > 0$, one has $x \le y + \epsilon$ and let $\epsilon \to 0$. This is a dumb example, of course.
(8) Exercise 5.12 of [1] - The Riesz Lemma allows one to find an unit vector x s.t. $\|x + M\| \ge 1 - \epsilon$ for M a proper closed subspace. It is a fundamental result that is needed to show that the unit ball is not compact in infinite dimensional normed vector spaces.
(9) Chapter 4 of [1] - Whenever you need to define a continuous function on a space, *especially* a metric space, one of the extension theorems will certainly be helpful.
(10) Section 5.2 of [1] - The Hahn-Banach Theorem is invaluable for defining linear functionals fulfilling certain properties. For example, creating bounded linear functionals.
(11) Proposition 5.17 of [1] - A useful result to show $T_n \to T$ strongly.
(12) Stone-Weierstrass is often useful to show that some general subclass of functions can approximate another class of functions very well. Be careful int he complex case.
(13) If a function vanishes outside some finite measure set, then one can always approximate it by a $C_c(X)$ function. See Lusin's Theorem 7.10.
(14) Theorem 1.14 is a useful reduction tool for results about measures. For example, the Lebesgue measure.

Epsilon and Inequality Tricks: There are a number of "tricks" in analysis that are standard techniques. Stuff like the $\epsilon/3$ proof are rather easy to recognize, but others are not so much.

(1) ϵ of room - The idea is to prove the result for all $\epsilon > 0$. For example, $|z - y| < \epsilon$ for all $\epsilon > 0$ implies $z = y$ for real numbers. If $\|f\|_\infty < \infty$, then the set of x such that $\|f\|_\infty - \epsilon < f(x)$ is of positive measure. An ϵ of room often gives enough room to approximate stuff.
(2) $\limsup$ tactics - Limits may not exist in general and writing lim willy-nilly might break the inequality. It often safer to write $\limsup$ or $\liminf$ because those are guaranteed to exist.

(3) General inequalities to always smash - whenever there is an assumption of a function f being in L^p for $p \in [1, \infty]$ and you have an integral, just try smashing through it with Hölder's inequality.

Similarly, triangle inequality, Schwarz inequality, Minkowski's inequality, Chebyshev's inequality, and etc. It is easier to just apply them and see what you get. If it gets you what you need, check the hypotheses and you are good to go.

(4) Advice for Minkowski's Inequality - It is most useful in showing boundedness of integral operators.
(5) Advice for L^p inequalities - the assumption on p are often very helpful. For example, if $p = \infty$, then it might be a good idea to bound $|f|$ above by $\|f\|_\infty$. If $p \in [1, \infty]$, then Hölder's inequality might be helpful for an upper bound.
(6) If one wants to adjust a function to get, say equality in the triangle inequality, it never hurts to see if one multiply by sgn of the function to normalize.
(7) Suppose one has a function $|f(x)|$. One can decompose $|f(x)|$ as $\int_0^\infty \chi_{y<|f(x)|} dy = |f(x)|$. The spirit of his is in Section 6.3.
(8) Two of the best estimates one has on Hilbert spaces are Bessel's Inequality and the Schwarz Inequality.

Constructions: There are many result which shows when one can "construct" a certain object. Some results do not seem "obvious" to employ in this way so we describe a few methods.

(1) A few obvious choices for constructing continuous functions are the Tietze and Urysohn Theorems.
(2) For constructing bounded linear functionals, the Hahn-Banach is the most useful.
(3) For constructing functions in an L^p space, it is possible to define a bounded linear functional and use $(L^p)^* \cong L^q$ for q conjugate to p.
(4) To extend a bounded linear functional on a normed vector space, the Hahn-Banach Theorem can be used and the extended bounded linear functional will have the same norm.
(5) Suppose X is a topological vector space with topology defined by a family of seminorms. If X is Hausdorff and the collection of seminorms is countable, it is possible to metrize the space. See Section 5.4 of [1].
(6) For constructing extensions of linear operators on L^p, it may be more possible to construct an extension on the simple functions and then extend linearly and taking limits.
(7) The Riesz-Fischer Theorem is a valuable tool for constructing vectors satisfying a given criterion in a Hilbert Space.
(8) It is possible to construct bounded linear functionals on C_c or C_0 using the Riesz-Representation Theorems. The first in Chapter 7 finds Radon measures satisfying a relation with a bounded linear functional while the second can be used to find complex measures and/or the desired bounded linear functional on C_0.

Specific:

(1) To check if a Borel measure on $\mathbb{R}^n$ is Radon (or any LCH second countable space), just check finiteness on compact sets as in Theorem 7.8 of Folland.
(2) If you have to "generalize" or "modify" a specific result in the book for your needs on the qualifying exam, it is likely not the way to go about doing it.

(3) *Try* to write out the measure of sets as the integral over *some* characteristic function. It opens up measure theory to integration theory.
(4) For showing well-definedness everywhere for integral operators, note that sets of measure zero do not matter.
(5) When trying to evaluate integrals, it does not hurt to draw the region one is integrating over. Sometimes, differences of integrals can cancel out.
(6) Given a signed measure, always decompose into positive and negative parts for ease.
(7) Integrating a given integral may open up the door to the Fubini-Tonelli Theorem.
(8) Use sequences as often as possible to avoid issues with measures and countability conditions.
(9) $\limsup E_n$ is set where x in infinitely many E_n while $\liminf$ is where x is in all but finitely many.
(10) Split up integrals whenever possible to deal with the different situations that can occur for the integrand. Divide and conquer.
(11) To show the weak convergence of some sequence, proof by contradiction may work.
(12) Draw pictures!

Nontrivial Facts to be Aware of

(1) A convex bounded closed subset of a reflexive space is weakly compact. This follows from Alaoglu's Theorem, Hahn-Banach, and the fact that convex closed sets are weakly closed.
(2) Any open subset of $\mathbb{R}$ is a countable union of open intervals.
(3) Over $\mathbb{R}$, one can write $m(E) = m(E \cap A^c) + m(E \cap A)$ when A, E measurable.
(4) A standard trick is as follows. Suppose we know $m(E) > 0$. Then one can choose an r large enough s.t. $m(E \cap B_r(0)) > 0$ as well. Even better, one can always decompose the range of a measurable function.

Examples and Counterexamples

(1) Any of the Cantor set or Cantor-(insert) example. For example, the Cantor-Lebesgue function is has derivative a.e. equal to zero, but it increases from zero to one.
(2) e_n converges weakly to zero in $\ell^2(\mathbb{N})$, but it does not converge in the norm.
(3) When working on $L^p[0,1]$, then function $f(x) = 1/x^r$ is in L^p iff $pr < 1$. For example, $f = 1/\sqrt{x}$ is in L^p for all $[1,2)$ but not in L^p for $p \in [2, \infty)$.
(4) The function $-\log x$ belongs to $L^p((0,1))$ for $p \geq 1$, but is not in L^∞.
(5) Exercise 5.6 of [1] gives examples of functions that are in L^p spaces for certain values of p.
(6) Consider $L^3(\mathbb{R})$. Some basic examples of functions in this space are $\frac{1}{x^{r/3}}$ for $r > 3$, any simple functions with finite support, $\frac{1}{e^x}$, and continuous functions with compact support.
(7) The subspace of polynomials of $L^2([-1,1])$ has an orthonormal basis given by $f_0(x) = \frac{1}{\sqrt{2}}$, $f_1(x) = \sqrt{\frac{3}{2}}x$, $f_2(x) = \sqrt{\frac{8}{45}}(x^2 - 1/3)$.
(8) There can exist functions in $L^1(m)$ which satisfy strange properties. For example, $\lim_{x\to\infty} f(x) \neq 0$ or even the fact that $\lambda_{f\chi_I}(\alpha) > 0$ for all α and all intervals of I of $\mathbb{R}$.

CHAPTER 2

Exercises: Real Analysis - Modern Techniques and Their Applications by Folland

1. Measures

Folland Exercise 1.1 A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is called a ring if it is closed under finite unions and differences (i.e., if $E_1, \ldots, E_n \in \mathcal{R}$, then $\bigcup_1^n E_j \in \mathcal{R}$, and if $E, F \in \mathcal{R}$, then $E \backslash F \in \mathcal{R}$). A ring that is closed under countable unions is called a σ -ring.

a. Rings (resp. σ -rings) are closed under finite (resp. countable) intersections.

b. If $\mathcal{R}$ is a ring (resp. σ -ring), then $\mathcal{R}$ is an algebra (resp. σ -algebra) iff $X \in \mathcal{R}$.

c. If $\mathcal{R}$ is a σ -ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ -algebra.

d. If $\mathcal{R}$ is a σ -ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ -algebra.

Proof. **(a)** This is immediate from the fact that $A \cap B = (A \cup B) \setminus [(B \setminus A) \cup (A \setminus B)]$ for any sets $A, B \subseteq X$.
(b) This follows immediately from the definitions of an algebra and part a..
(c) The latter collection is clearly a σ-algebra since we only need to see that it is closed under complements.
(d) Follows from the definition. $\square$

Folland Exercise 1.2 Complete the proof of Proposition 1.2 .

Proof. This is a trivial verification. $\square$

Folland Exercise 1.3

Proof. (a) Since $\mathcal{M}$ is nonempty, choose a nonempty $E_1 \subseteq \mathcal{M}$. Then, consider the σ-algebra $\mathcal{M}_{E_1} = \{E \cap E_1^c : E \in \mathcal{M} \text{ and } E_1 \neq E\}$. Checking that this is a σ-algebra is easy.

We can assume that $\mathcal{M}_{E_1}$ contains infinitely many sets. For if not, then we can take E_1^c in place of E_1. Then, $\mathcal{M}_{E_1^c}$ contains infinitely many sets because only finitely many intersect E_1 and there are infinitely many elements in $\mathcal{M}$.

Now repeat the process and choose an $E_2 \in \mathcal{M}_{E_1}$. Then, consider $\mathcal{M}_{E_2} = \{E \cap E_2^c : E \subseteq \mathcal{M}_{E_1} \text{ and } E \neq E_2\}$. If $\mathcal{M}_{E_2}$ contains only finitely many sets, we can take E_2^c in place of $_2$. So $\mathcal{M}_{E_2}$ has infinitely many elements. Repeating this process inductively gives a sequence of nonempty sets $\{E_n\}_{n=1}^\infty$. The sequence is disjoint because at each stage, we made sure that E_n was disjoint to all E_i for $i < n$.

(b) By (a), choose an infinite sequence $\{E_n\}_{n=1}^\infty$ of disjoint sets. Consider indexing sets $I \subseteq \mathbb{N}$. Because $\mathcal{M}$ is a σ-algebra, the sets $\bigcup_{i \in I} E_i$ are in $\mathcal{M}$. Because the E_i's are disjoint, $\bigcup_{i \in I} E_i \neq \bigcup_{j \in J} E_j$ if $J \neq I$ for every indexing set J. Thus, $\mathcal{M}$ contains a collection of sets and such a collection has cardinality card $\mathcal{P}(\mathbb{N})$. So, the cardinality of $\mathcal{M}$ is at least that of card $\mathcal{P}(\mathbb{N})$. Thus, card $\mathcal{M} \geq \mathfrak{c}$.

$\square$

Folland Exercise 1.4 An algebra $\mathcal{A}$ is a σ -algebra iff $\mathcal{A}$ is closed under countable increasing unions (i.e., if $\{E_j\}_1^\infty \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$, then $\bigcup_1^\infty E_j \in \mathcal{A}$)

Proof. ($\implies$): Is immediate from definition.

($\impliedby$): One only needs to show $\mathcal{A}$ is closed under countable unions and since $\mathcal{A}$ is closed under finite unions, just take the countable increasing unions over N with the finite unions $\bigcup_1^N E_j$. $\square$

Folland Exercise 1.5 If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ -algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. (Hint: Show that the latter object is a σ -algebra.)

Proof. The latter object is clearly a σ-algebra because it is clearly closed under complements and countable unions (take $\mathcal{E}$ to be the collection of sets E_n and then $\bigcup_1^\infty E_n$ is clearly in the latter object).

Clearly, the latter object is contained in $\mathcal{E}$. The other containment follows since every element of $\mathcal{M}$ is contained in the latter object by definition of a "σ-algebra generated by a collection". $\square$

Folland Exercise 1.6 Complete the proof of Theorem 1.9.

Proof. Relatively straightforward. $\square$

Folland Exercise 1.7

Proof. First,

$$\sum_1^n a_j\mu_j(\emptyset) = \sum_1^\infty 0 = 0.$$

Second, let $\{E_k\}$ be a sequence of disjoint sets. Then,

$$\sum_1^n a_j\mu_j(\cup_{k=1}^\infty E_k) = \sum_1^n a_j \sum_{k=1}^\infty (\mu(E_k)) = \sum_{k=1}^\infty \sum_1^n a_j\mu(E_k).$$

Note that we can interchange the sums for the last inequality because we have a finite sum and an infinite sum as well as the fact that the indices don't depend on each other. $\square$

Folland Exercise 1.8 If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu\left(\liminf E_j\right) \leq \liminf \mu\left(E_j\right)$. Also, $\mu\left(\limsup E_j\right) \geq \limsup \mu\left(E_j\right)$ provided that $\mu\left(\bigcup_1^\infty E_j\right) < \infty$.

Proof. First,

$$\mu(\liminf E_j) \leq \sup_k \mu\left(\bigcap_{n\geq k} E_n\right) = \sup_k \inf_{n\geq k} \mu(E_n).$$

Second,

$$\mu(\limsup E_j) \geq \inf_k \mu\left(\bigcup_{n\geq k} E_n\right) \geq \inf_k \sup_{n\geq k} \mu(E_n) = \limsup \mu(E_n)$$

and the first inequality is permitted because of the hypothesis $\mu(\bigcup_1^\infty E_j) < \infty$. □

Folland Exercise 1.9 If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

PROOF. First off,

$$\begin{aligned}\mu(E) &= \mu(((E \cup F) \setminus F) \cup (E \cap F)) = \mu((E \cup F) \setminus F) + \mu(E \cap F) \\ &\geq \mu(E \cup F) - \mu(F) + \mu(E \cap F)\end{aligned}$$

which implies

$$\mu(E) + \mu(F) \geq \mu(E \cup F) + \mu(E \cap F).$$

Secondly,

$$\begin{aligned}\mu(E) + \mu(F) - \mu(E \cup F) &\leq \mu((E \cap F) \cup (E \cap F^c)) + \mu(F) - \mu(E \cup F) \\ &\leq \mu(E \cap F) + \mu(E \cap F^c) + \mu(F) - \mu(E \cup F) \\ &\leq \mu(E \cap F)\end{aligned}$$

where the last inequality follows by

$$\mu(E \cap F^c) + \mu(F) \leq \mu(E \cup F).$$

□

Folland Exercise 1.10

PROOF. First, $\mu_E(\emptyset) = \mu(\emptyset \cap E) = 0$. Second, let $\{E_k\}$ be a disjoint sequence of measurable sets. Then the $E_j \cap E$ are disjoint and

$$\mu_E(\bigcup_{j=1}^\infty E_j) = \mu(\bigcup_{j=1}^\infty E_j \cap E) = \sum_{j=1}^\infty \mu(E_j \cap E) = \sum_{j=1}^\infty \mu_E(E_j).$$

□

Folland Exercise 1.11 A finitely additive measure μ is a measure iff it is continuous from below as in Theorem 1.8c. If $\mu(X) < \infty, \mu$ is a measure iff it is continuous from above as in Theorem 1.8 d.

PROOF. For the first statement, it is necessary to show that a measure preserves countable unions of disjoint sets. If $\{E_n\}$ is a disjoint sequence of measurable sets, then one applies continuity from below with $F_k := \sum_1^k E_n$.

The method of proof for the second is similar given that $\mu(X) < \infty$. □

Folland Exercise 1.13 Every σ -finite measure is semifinite.

PROOF. □

Folland Exercise 1.12

PROOF. (a) Assume $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$. Then,

$$\mu(E) = \mu(((E \Delta F) \setminus F) \cup (E \cap F)) = \mu((E \Delta F) \setminus F) + \mu(E \cap F) = \mu(E \cap F).$$

The second equality is true because the sets are disjoint and last equality is because $\mu(E \Delta F) = 0$. By a dual argument, $\mu(F) = \mu(E \cap F)$. Hence, $\mu(E) = \mu(F)$.

(b) We verify the definition of an equivalence relation.

i. First, $E \sim E$ is clear since $\mu(E\Delta E) = \mu(\emptyset) = 0$.
ii. Suppose $E \sim F$. Then $F \sim E$ since $E\Delta F = F\Delta E$.
iii. Suppose $E \sim F$ and $F \sim G$. By properties of symmetric difference, $E\Delta G = (E\Delta F)\Delta(F\Delta G)$. So,

$$\mu(E\Delta G) = \mu((E\Delta F)\Delta(F\Delta G)) \le \mu(E\Delta F) + \mu(F\Delta G) = 0$$

where the inequality follows by monotonicity.

(c) Certainly, $\rho(E, E) = 0$ because $E\Delta E = \emptyset$ and so, $\mu(E\Delta E) = 0$. Similarly, since $E\Delta F = F\Delta E$, we have $\rho(E, F) = \rho(F, E)$. We have

$$\rho(E, G) = \mu(E\Delta G) \le \mu(E\Delta F) + \mu(F\Delta G) = \rho(E, F) + \rho(F, G)$$

as desired.

□

Folland Exercise 1.14 If μ is a semifinite measure and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$

PROOF. □

PROOF. Assume μ is semifinite and $\mu(E) = \infty$. Consider the set

$$K := \{D \mid \text{there exists } F \subseteq E \text{ s.t. } D < \mu(F) < \infty\}.$$

The set is nonempty since it contains 0 due to the semifinitness of μ. It suffices to show that $\sup K = \infty$ because if $C > 0$ is given, there is some $B > C$ and $B \in K$. Hence, there exists $F \subseteq E$ s.t. $B < \mu(F) < \infty$ and so, $C < \mu(F) < \infty$.

Proceed by contradiction and let $\sup K = c$ for some $c < \infty$. We can partially order K by $\le$ and because $c < \infty$, there is an upper bound for each chain. Indeed, given a chain

$$D_1 \le D_2 \le \dots,$$

we get a sequence of F_i s.t. $D_i < \mu(F_i) < \infty$ and we can take c s.t. $c < \mu(C) < \infty$ for some C to be the upper bound. By Zorn's Lemma, there exists a maximal value b and an F s.t. $b < \mu(F) < \infty$ and $F \subseteq E$. Then, $\mu(E \cap F^c) = \infty$. So, we can choose a $G \subseteq E \cap F^c$ s.t. $0 < \mu(G) < \infty$. Choose an ϵ s.t. $0 < \epsilon < \mu(G)$. Because $\mu(F \cup G) = \mu(F) + \mu(G)$, we have

$$b + \epsilon < \mu(F \cup G) < \infty.$$

Since $F \cup G \subseteq E$, this contradicts the maximality of b. Contradiction. □

Folland Exercise 1.15

PROOF. (a) We show that μ_0 is a measure. First, $\mu_0(\emptyset)$. Given a sequence of disjoint sets $\{E_n\}$. Then,

$$\begin{aligned}\mu_0(\cup_{n=1}^{\infty} E_n) &= \sup\{\mu(F) : F \subset \cup_{n=1}^{\infty} E_n \text{ and } \mu(F) < \infty\} \\ &= \sup\{\sum_{n=1}^{\infty} \mu(F \cap E_n) : F \subset \cup_{n=1}^{\infty} E_n \text{ and } \mu(F) < \infty\} = \sum_{n=1}^{\infty} \mu_0(E_n).\end{aligned}$$

Suppose not and that μ_0 is not a semifinite measure. Let $\mu_0(E) = \infty$. Then there won't exist any subset $F \subset E$ s.t. $0 < \mu(F) < \infty$. Thus, either $\mu(F) = 0$ for all $F \subset E$ or $\mu(F) = \infty$ for all $F \subset E$. In the first case, we get $\mu_0(F) = 0$ which is a contradiction. In the second case, $\mu_0(F) = 0$ since there are no nonempty subsets of finite measure and so, we have a contradiction.

(b) Assume μ is semifinite.

Let E be a set and assume $\mu(E)$ is finite. Then, $\mu_0(E) = \mu(E)$ by the definition of μ_0. The nontrivial case is when $\mu(E) = \infty$. By Exercise 1.14, for any $C > 0$, we can find a $F \subset E$ s.t. $C < \mu(F) < \infty$. In this case, $\mu_0(E) = \infty$ since the least upper bound cannot be finite. Thus, $\mu(E) = \mu_0(E)$.

(c) We construct a $\nu : \mathcal{M} \to \{0, \infty\}$ s.t. $\mu = \mu_0 + \nu$. We define ν as follows.

If $E \in \mathcal{M}$ is a σ-finite set w.r.t μ, then we take $\nu(E) = 0$. If $E \in \mathcal{M}$ is not σ-finite w.r.t μ, then $\nu(E) = \infty$. This is certainly well-defined.

We check that ν is a measure. Certainly, $\nu(\emptyset) = 0$ since $\emptyset$ is not σ-finite. Also, given a disjoint sequence of sets E_j, we have two cases. The first case is where $\cup_{j=1}^{\infty} E_j$ is σ-finite. In which case, $\nu(\cup_{j=1}^{\infty} E_j) = 0$ and since each E_j will be σ-finite, $\nu(E_j) = 0$. Thus, $\nu(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \nu(E_j)$. The second case is where $\cup_{j=1}^{\infty} E_j$ is not σ-finite. Then $\nu(\cup_{j=1}^{\infty} E_j) = \infty$ and there is some E_j which can't be written as a union of sets of finite measure. So, $\nu(E_j) = \infty$. Thus, $\nu(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \nu(E_j)$ since both sides are equal to infinity.

We now verify that $\mu = \mu_0 + \nu$. Given $E \in \mathcal{M}$. We have two cases: the case where E is σ-finite and the case where E is not. If E is σ-finite, then we can write $E = \cup_{j=1}^{\infty} E_j$ where $\mu(E_j) < \infty$. By out definition, we get $\nu(E) = 0$. WLOG, we can assume the union is disjoint. Thus,

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \mu_0(E_j) = \mu_0(E) + \nu(E).$$

For the case where E is not σ-finite, we get $\mu(E) = \infty$ and $\nu(E) = \infty$ and so,

$$\mu(E) = \mu_0(E) + \nu(E)$$

since we use the convention that $\infty + \infty = \infty$.

□

Folland Exercise 1.16 Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is called **locally measurable** if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\tilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subset \tilde{\mathcal{M}}$; if $\mathcal{M} = \tilde{\mathcal{M}}$, then μ is called **saturated**.

a. If μ is σ -finite, then μ is saturated.

b. $\mathcal{M}$ is a σ -algebra.

c. Define $\tilde{\mu}$ on $\tilde{\mathcal{M}}$ by $\tilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ otherwise. Then $\tilde{\mu}$ is a saturated measure on $\tilde{\mathcal{M}}$, called the **saturation** of μ.

d. If μ is complete, so is $\tilde{\mu}$.

e. Suppose that μ is semifinite. For $E \in \tilde{\mathcal{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) : A \in \mathcal{M}$ and $A \subset E\}$. Then μ is a saturated measure on $\tilde{\mathcal{M}}$ that extends μ.

f. Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and $\mathcal{M}$ the σ -algebra of countable or co-countable sets in X. Let μ_0 be counting measure on $\mathcal{P}(X_1)$, and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathcal{M}$ $\tilde{\mathcal{M}} = \mathcal{P}(X)$, and in the notation of parts (c) and (e), $\tilde{\mu} \neq \underline{\mu}$.

Proof. Omitted. The exercise itself is not too valuable to do and is relatively straightforward. □

Folland Exercise 1.17

PROOF. By subadditivity, we have $\mu^*(E \cap (\cup_{j=1}^\infty A_j)) \leq \sum_{j=1}^\infty \mu^*(E \cap A_j)$. So it suffices to prove the other inequality.

Let $\epsilon > 0$. Choose a sequence of sets $\{B_j\}$ in the algebra s.t. $A_j \subseteq \bigcup_{j=1}^\infty B_j$, and so $E \cap \left(\bigcup_{j=1}^\infty A_j\right) \subseteq \bigcup_{k=1}^\infty B_j$. We can choose the sequence of sets $\{B_j\}$ so that

$$\mu^*\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right)\right) + \epsilon \geq \mu^*\left(\bigcup_{j=1}^\infty B_j\right).$$

So, we have

$$\mu^*\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right)\right) + \epsilon \geq \mu^*\left(\bigcup_{k=1}^\infty B_k\right) \geq \mu^*\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right)\right) \geq \sum_{j=1}^\infty \mu^*(E \cap A_j).$$

where the second inequality follows from monotonicity and the last follows from subadditivity. Since $\epsilon > 0$ was arbitrary, we get the desired inequality. □

Folland Exercise 1.18

(a) This follows from the definition. We can choose sets A_j in the algebra s.t. $E \subseteq \cup_{j=1}^\infty A_j$ and $\mu_0(\cup_{j=1}^\infty A_j) \leq \mu^*(E) + \epsilon$. Then, $\mu^*(\cup_{j=1}^\infty A_j) \leq \mu^*(E) + \epsilon$. Thus, $A = \cup_{j=1}^\infty A_j$ is our desired set.

(b) ($\implies$): We can find covering $B_n \in \mathcal{A}_\sigma$ where $B_n = \cup_{k=1}^\infty B_{n,k}$ and $B_{n,k} \in \mathcal{A}$ s.t. $E \subseteq B_n$ and $\mu^*(B_n) \leq \mu^*(E) + \frac{1}{n}$. Let $B = \cap_{n=1}^\infty B_n$. Since E is μ^*-measurable, we have

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \setminus E) = \mu^*(E) + \mu^*(B \setminus E)$$

and so, $\mu^*(B \setminus E) = \mu^*(B) - \mu^*(E)$. Since $\mu^*(B) - \mu^*(E) \leq \frac{1}{n}$ for every $n \in \mathbb{N}$, we have $\mu^*(B \setminus E) = 0$ as desired.

($\impliedby$): Suppose there exists a $B \in \mathcal{A}_{\sigma\delta}$ where $E \subseteq B$ and $\mu^*(B \setminus E) = 0$. To see that E is μ^*-measurable, notice that since $\mu^*(B \setminus E) = 0$, we know that $B \setminus E$ is μ^*-measurable. Hence, B is μ^*-measurable. Because $\mathcal{M}^*$ is σ-algebra, $B \setminus (B \setminus E)$ is also μ^*-measurable. But $B \setminus (B \setminus E))$ is just E.

(c) If μ_0 is σ-finite, we take $X = \cup_{k=1}^\infty E_k$ s.t. $\mu_0(E_k) < \infty$ for all k. Applying (b) for the case with E replaced by $E \cap E_k$, we know that $E \cap E_k$ is μ^*-measurable iff there exists $B_k \in \mathcal{A}_{\sigma\delta}$ with $E \cap E_k \subset B$ and $\mu^*(B_k \setminus (E \cap E_k)) = 0$. Thus, $E = \cup_{k=1}^\infty E_k$ is μ^*-measurable iff there exists $B = \cup_{k=1}^\infty B_k$ s.t. $\mu^*(B \setminus E) = 0$.

Folland Exercise 1.19

PROOF. ($\implies$): Assume E is μ^*-measurable. We have

$$\mu^*(X) = \mu^*(E \cap X) + \mu^*(X \cap E^c) \qquad \text{and so} \qquad \mu^*(X) = \mu^*(E) + \mu^*(E^c).$$

Since X is in the algebra, we know that $\mu_0(X) = \mu^*(X)$. From the above, $\mu_*(E) = \mu^*(E)$ as desired.

($\impliedby$): We take the approach hinted at by Folland and make use of Exercise 18. Suppose $\mu^*(E) = \mu_*(E)$. This implies that $\mu_0(X) = \mu^*(E) + \mu^*(E^c)$. So, we can always write $\mu^*(E^c) = \mu_0(X) - \mu^*(E)$. In particular, if we have an element B in the algebra $\mathcal{A}$, we can always write

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c).$$

This fact will be used repeatedly.

By Exercise 18.b, it suffices to show that there exists some $B \in \mathcal{A}_{\sigma\delta}$ s.t. $E \subset B$ and $\mu^*(B \setminus E) = 0$.

By Exercise 18.a, we find sets $B_j \in \mathcal{A}_\sigma$ s.t. $E \subseteq B_j$ and $\mu^*(B_j) \leq \mu^*(E) + \frac{1}{j}$. So,

$$\mu^*(E^c) = \mu^*(E^c \cap B_j) + \mu^*(E^c \cap B_j^c) = \mu^*(E^c \cap B_j) + \mu^*(B_j^c) = \mu^*(B \setminus E) + \mu^*(B_j^c)$$

and

$$\mu^*(E^c) = \mu_0(X) - \mu^*(E) = \mu^*(B_j) + \mu^*(B_j^c) - \mu^*(E).$$

Using the above equations we have

$$\mu^*(B \setminus E) + \mu^*(B_j^c) = \mu^*(B_j) + \mu^*(B)j^c) - \mu^*(E).$$

Rewriting the above equation, we have $\mu^*(B_j \setminus E) = \mu^*(B_j) - \mu^*(E)$. Because of the choice of B_j, we have

$$\mu^*(B_j \setminus E) \leq \frac{1}{j}.$$

Let $B = \cap_{j=1}^\infty B_j$, we have $B \in \mathcal{A}_{\sigma\delta}$ and $\mu^*(B \setminus E) = 0$. By Exercise 18.b, E is μ^*-measurable. □

Folland Exercise 1.22

Proof. (a) Proposition 1.10, 1.11, and 1.13 show that μ coincides with the completion $\overline{\mu}$ on the domain of the $\overline{\mu}$. Let $F \subset E$ where $\mu^*(E) = 0$ and E is μ^*-measurable. By Exercise 18.b, choose a $B \in \mathcal{A}_{\sigma\delta}$ s.t. $\mu^*(B \setminus E) = 0$ and $E \subset B$. Then,

$$\begin{aligned}\mu^*(B \setminus F) &= \mu^*((B \setminus E) \cup (E \setminus F)) \\ &\leq \mu^*(B \setminus E) + \mu^*(E \setminus F) \\ &\leq \mu^*(B \setminus E) + \mu^*(E) \\ &= 0.\end{aligned}$$

By Exercise 18.b once more, F is μ^*-measurable. Hence, $\overline{\mu}$ is the completion of μ.

(b) Omitted.

(c) Omitted. □

Folland Exercise 1.23 Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a < b \leq \infty$

a. $\mathcal{A}$ is an algebra on $\mathbb{Q}$. (Use Proposition 1.7.)

b. The σ -algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

c. Define μ_0 on $\mathcal{A}$ by $\mu_0(\varnothing) = 0$ and $\mu_0(A) = \infty$ for $A \neq \varnothing$. Then μ_0 is a premeasure on $\mathcal{A}$, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Proof.

a. Clearly, $\mathcal{A}$ forms an elementary family. Checking the remaining requirements to apply Proposition 1.17 is trivial.
b. The containment $\mathcal{E}(\mathcal{A}) \subseteq \mathcal{P}(\mathbb{Q})$ is trivial so we show the other containment. Indeed, any element of $\mathcal{P}(\mathbb{Q})$ is a countable union of countable intersections of elements of form $(a, b] \cap \mathbb{Q}$.
c. Verifying that μ_0 is a premeasure is trivial. To see that there are more than one extensions on $\mathcal{P}(\mathbb{Q})$, just notice that the counting measure is a possible extension and the measure μ defined by $\mu(E) = \infty$ if $E \neq \emptyset$ and $\mu(E) = 0$ if $E = \emptyset$ is another possible extension.

□

Folland Exercise 1.24

PROOF. (a) We have two set equalities where we use $A \cap E = B \cap E$

$$A \cap B^c \cap E = B \cap B^c \cap E = \emptyset$$
$$B \cap A^c \cap E = A \cap A^c \cap E = \emptyset.$$

From the first, equation, we know that $(A \setminus B) \subseteq E^c$ and so, $(A \setminus B)^c \supseteq E$. From the second, $(B \setminus A) \subseteq E^c$ and so, $(B \setminus A)^c \supseteq E$.

Using the above conclusions, the fact that $\mu^*(E) = \mu^*(X)$, and monotonicty, we have two equalities

$$\mu^*(X) = \mu(A \setminus B) + \mu((A \setminus B)^c) \geq \mu(A \setminus B) + \mu^*(E) = \mu(A \setminus B) + \mu^*(X)$$
$$\mu^*(X) = \mu(B \setminus A) + \mu((B \setminus A)^c) \geq \mu(B \setminus A) + \mu^*(E) = \mu(B \setminus A) + \mu^*(X).$$

The first line gives us $0 \geq \mu(A \setminus B)$ and the second line gives us $0 \geq \mu(B \setminus A)$. Thus, $\mu(A \setminus B) = 0\mu(B \setminus A)$.

At last, we can use the preceding fact to get $\mu(A) = \mu(B)$. Using addivity of measures,

$$\mu(A) = \mu(A) + \mu(B \setminus A) = \mu(A \cup B) = \mu(B) + \mu(A \setminus B) = \mu(B)$$

and so, we conclude that $\mu(A) = \mu(B)$.

(b) First, we show that $\mathcal{M}_E$ is a σ-algebra.

Suppose $\{E_j\}$ is a countable sequence of sets in $\mathcal{M}_E$. Let $E_j = A_j \cap E$ for some $A_j \in \mathcal{M}$. Then, $\cup_{j=1}^{\infty} E_j = \cup_{j=1}^{\infty} A_j \cap E = (\cup_{j=1}^{\infty} A_j) \cap E$, and since $\cup_{j=1}^{\infty} A_j \in \mathcal{M}$, we conclude that $\cup_{j=1}^{\infty} E_j \in \mathcal{M}_E$ as desired.

Next, we show that ν is a measure on $\mathcal{M}_E$. By (a), this is well-defined so we just check the definition. Clearly, $\nu(\emptyset) = 0$ since $\mu(\emptyset) = 0$. Now, suppose $\{E_j\}$ is a disjoint sequence of sets in $\mathcal{M}_E$. Then, $E_j = A_j \cap E$ for $A_j \in \mathcal{M}$. We may assume the A_j are disjoint because $\cup_{j=1}^{\infty} A_j \cap E = \cup_{j=1}^{\infty} F_j \cap E$ where $F_k = A_k \setminus \cup_{j=1}^{k-1} A_k$ and (a). Thus,

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \nu(A_j)$$

as desired.

□

Folland Exercise 1.25 Complete the proof of Theorem 1.19 .

PROOF. Relatively straightforward. □

Folland Exercise 1.26 Prove Proposition 1.20. (Use Theorem 1.18.)

PROOF. By Theorem 1.18, choose an open set U s.t. $E \subseteq U$ and $\mu(U) < \mu(E) + \frac{\epsilon}{2}$. Note that this means $\mu(U \setminus E) < \frac{\epsilon}{2}$. Then, $U := \bigcup_{k=1}^{\infty} I_k$ is a union of disjoint open intervals. Next, let $N \in \mathbb{N}$ be sufficiently large s.t. $\mu\left(U \setminus \bigcup_{k=1}^{N} I_k\right) < \frac{\epsilon}{2}$. Then take $A := \bigcup_{k=1}^{N} I_k$ and we have

$$\mu(E \Delta A) \leq \mu(E \setminus A) + \mu(A \setminus E) \leq \mu(E \setminus A) + \mu(U \setminus E)$$
$$\leq \mu(E \setminus A) + \epsilon \leq \mu(U \setminus A) + \frac{\epsilon}{2} < \epsilon.$$

□

Proof. By Theorem 1.18, choose an open set U which we can write as a union of disjoint open intervals $U = \bigcup_{k=1}^{\infty} I_k$ s.t. $E \subset U$ and $\mu(U) \le \mu(E) + \frac{\epsilon}{2}$.

Since μ is a measure,

$$\sum_{k=1}^{\infty} \mu(I_k) \le \mu(E) + \frac{\epsilon}{2}.$$

Additionally, we have

$$\sum_{k=1}^{\infty} \mu(I_k \setminus E) = \mu(U \setminus E) \le \frac{\epsilon}{2}.$$

Since the infinite series is finite, choose $N \in \mathbb{N}$ sufficiently large s.t.

$$\sum_{k=N}^{\infty} \mu(I_k) < \frac{\epsilon}{2}.$$

Let $A = \bigcup_1^N I_k$. So,

$$\begin{aligned}
\mu(E\Delta A) &= \mu(E \setminus A) + \mu(A \setminus E) \le \mu(U \setminus A) + \mu(A) - \mu(E) \\
&= \mu\left(\sum_{k=N}^{\infty} I_k\right) + \mu(A) - \mu(E) < \frac{\epsilon}{2} + \mu(A) - \mu(E) \\
&\le \frac{\epsilon}{2} + \sum_{k=1}^{N} \mu\left(I_k\right) - \mu(E) \le \frac{\epsilon}{2} + \sum_{k=1}^{\infty} \mu(I_k) - \mu(E) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

So, $\mu(E\Delta A) < \epsilon$ as desired. □

Folland Exercise 1.28

Proof. There are four equalities we need to check. We use the fact that F is right continuous and increasing to conclude that $\mu_F((a,b]) = \mu_F(b) - \mu_F(a)$. Lastly, we use continuity from above and so, $\mu_F(\cap_{n=1}^{\infty}(a - \frac{1}{n}, b]) = \lim_{n\to\infty} \mu((a - \frac{1}{n}, b])$ repeatedly.

(1) First,

$$\mu_F((a - \frac{1}{n}, a]) = F(a) - F(a-)$$

and letting $n \to \infty$ gives $\mu_F(\{a\}) = F(a) - F(a-)$.

(2) Second,

$$\mu_F((a - \frac{1}{n}, b - \frac{1}{n}]) = F(b - \frac{1}{n}) - F(a - \frac{1}{n})$$

and letting $n \to \infty$ gives $\mu_F(([a,b)) = F(b-) - F(a-)$.

(3) Third,

$$\mu_F((a - \frac{1}{n}, b]) = F(b) - F(a - \frac{1}{n})$$

and letting $n \to \infty$ gives $F(b) - F(a-)$.

(4) Lastly,
$$\mu((a, b-\frac{1}{n}]) = F(b-\frac{1}{n}) - F(a)$$
and letting $n \to \infty$ gives $F(b-) - F(a)$.

$\square$

Folland Exercise 1.30 If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$

Proof. It suffices to assume $m(E) < \infty$ by semifiniteness of the Lebesgue measure. Suppose not and that there exists $\alpha \in (0,1)$ such that $m(E \cap I) \geq \alpha m(I)$ for all open intervals I.

By Theorem 1.18, choose an open set U s.t $U \supset E$ and $m(U) < m(E) + \epsilon$ where $\epsilon > 0$. So, $m(U \setminus E) < \epsilon$. Assume $\bigcup_{k=1}^{\infty} I_k = U$ where the I_k's are disjoint open intervals since U is an open set. Using the fact that $E \cap I_k$ is disjoint from $E^c \cap I_k$ and the hypothesis,
$$m(I_k) = m(E \cap I_k) + m(E^c \cap I_k) \leq \alpha m(I_k) + m(I_k \setminus E).$$
Subtracting $\alpha m(I_k)$ from both sides and dividing by $1-\alpha$,
$$m(I_k) \leq \frac{1}{1-\alpha} m(I_k \setminus E).$$
Now,
$$\begin{aligned} m(E) \leq m(U) = m\left(\bigcup_{k=1}^{\infty} I_k\right) &\leq \sum_{k=1}^{\infty} m(I_k) \leq \sum_{k=1}^{\infty} \frac{1}{1-\alpha} m(I_k \setminus E) \\ &= \frac{1}{1-\alpha} m\left(\bigcup_{k=1}^{\infty} I_k \setminus E\right) = \frac{1}{1-\alpha} m(U \setminus E) < \frac{\epsilon}{1-\alpha}, \end{aligned}$$
where we used subadditivity, our hypothesis, additivity of m, and our choice of U. So, $m(E) < \frac{\epsilon}{1-\alpha}$. Since $\epsilon > 0$ arbitrary, we have $m(E) \leq 0$. Contradiction.

$\square$

Folland Exercise 1.31 If $E \in \mathcal{L}$ and $m(E) > 0$, the set $E - E = \{x - y : x, y \in E\}$ contains an interval centered at 0. (If I is as in Exercise 30 with $\alpha > \frac{3}{4}$, then $E - E$ contains $\left(-\frac{1}{2}m(I), \frac{1}{2}m(I)\right)$.)

Proof. The proof is actually easier to do with knowledge of convolutions and so, we take this approach instead. This assumes the results of Chapter 2 and Chapter 8 for those who are unfamiliar with convolutions.

Define
$$\chi_E * \chi_{-E}(x) := \int_{\mathbb{R}} \chi_E(y)\chi_{-E}(x-y)dy =: f.$$
We know that $f(x)$ is continuous (see Chapter 8) and
$$f(0) = \int_{\mathbb{R}} \chi_E(y)\chi_{-E}(-y)dy = \int_R \chi_E(y)^2 dy = m(E)^2 > 0.$$
By continuity, there is an interval $(-\epsilon, \epsilon)$ for $\epsilon > 0$ sufficiently small in which $I := (-\epsilon, \epsilon) \subseteq f^{-1}((0, \infty))$ which is an open set. We claim $I \subseteq E - E$.

Let $x \in I$. Because $f(x) \in (0,\infty)$ and

$$f(x) = \int_{\mathbb{R}} \chi_E(y)\chi_{-E}(-y)dy = m\left(\{y \in \mathbb{R} : y \in E, x - y \in -E\}\right) > 0.$$

So, there is a $y \in \{y \in \mathbb{R} : y \in E, x - y \in -E\}$ since the set cannot be nonempty. This means $y - x \in E$ and therefore,

$$x = y - (y - x) \in E - E.$$

We conclude that $I \subseteq E - E$. □

Folland Exercise 1.32 Suppose $\{\alpha_j\}_1^\infty \subset (0,1)$

a. $\prod_1^\infty (1-\alpha_j) > 0$ iff $\sum_1^\infty \alpha_j < \infty$. (Compare $\sum_1^\infty \log(1-\alpha_j)$ to $\sum \alpha_j$.

b. Given $\beta \in (0,1)$, exhibit a sequence $\{\alpha_j\}$ such that $\prod_1^\infty (1-\alpha_j) = \beta$.

Proof. Omitted. The proof itself is not very insightful and requires only knowledge of undergraduate analysis. □

Folland Exercise 1.33 There exists a Borel set $A \subset [0,1]$ such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0,1]$. (Hint: Every subinterval of [0,1] contains Cantor-type sets of positive measure.)

Proof. Enumerate all intervals (q,r) where $q, r \in \mathbb{Q} \cap [0,1]$ and form the Fat Cantor set by removing the middle thirds, then the middle ninths of the remaining intervals, then the middle 27ths of the remaining intervals, and so forth. Then, let A be the union all of those Fat Cantor sets that are obtained intersected with $[0,1]$.

It is clear that $m(A \cap I) > 0$ because I contains some interval $(q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n})$ and therefore, one of the Fat Cantor sets we constructed, but $m(I) > m(A \cap I)$. □

2. Integration

Folland 2.1 Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable iff $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in \mathcal{M}$, and f is measurable on Y.

Proof. The only if direction is easy so suppose $f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \in \mathcal{M}$ and f measurable on Y. Then f is measurable because measurable subset of $\overline{\mathbb{R}}$ is in the σ-algebra generated by $\mathcal{B}_{\mathbb{R}} \cup \{\pm\infty\}$. □

Folland Exercise 2.2

Proof. Throughout this proof, we make use of Exercise 1 to show measurability.

(a) We show fg is measurable. Let $Y = (fg)^{-1}(\mathbb{R})$. Certainly, fg is measurable on Y by Proposition 2.6. Next,

$$(fg)^{-1}(\infty) = (f^{-1}(\infty) \cap g^{-1}(\infty)) \cup (f^{-1}(\infty) \cap g^{-1}(\mathbb{R})) \cup (f^{-1}(\mathbb{R}) \cap g^{-1}(\infty)) \in \mathcal{M}.$$

Then, set

$$M_1 = f^{-1}(-\infty) \cap g^{-1}((0,\infty]) \in \mathcal{M},$$
$$M_2 = [f^{-1}((0,\infty]) \cap g^{-1}(-\infty)] \in \mathcal{M},$$
$$M_3 = f^{-1}(\infty) \cap g^{-1}([-\infty,0) \in \mathcal{M},$$
$$M_4 = f^{-1}(-\infty,0)) \cup g^{-1}(-\infty) \in \mathcal{M}$$

and we are able to conclude that each of the sets are measurable since they are the intersection or union of measurable sets.

So, $(fg)^{-1}(-\infty) = M_1 \cup M_2 \cup M_3 \cup M_4$ which is measurable since it is the union of measurable sets. This completes the proof.

(b) Let

$$h(x) := \begin{cases} a & \text{if } f(x) = g(x) = \pm\infty \\ f(x) + g(x) & \text{if not.} \end{cases}$$

Certainly, $h^{-1}(\mathbb{R}) \in \mathcal{M}$ since $h^{-1}(\mathbb{R}) = (f+g)^{-1}(\mathbb{R}) \in \mathcal{M}$. The last two cases follow as so,

$$h^{-1}(\infty) = (f^{-1}(\infty) \cap g^{-1}((-\infty,\infty])) \cup (f^{-1}((-\infty,\infty]) \cap g^{-1}(\infty)) \in \mathcal{M}$$

and

$$h^{-1}(-\infty) = (f^{-1}(-\infty) \cap g^{-1}([-\infty,\infty)) \cup (f^{-1}([-\infty,\infty)) \cap g^{-1}(-\infty)) \in \mathcal{M}.$$

We are able to conclude that $h^{-1}(\infty)$ is measurable since each of the sets involved are measurable. Simiarly, $h^{-1}(-\infty)$ is measurable since it is the union of sets which are intersections of measurable functions.

□

Folland Exercise 2.3

Proof. Let $\{f_n\}$ be a sequence of measurable functions on X. We show that the set

$$L = \{x : \lim f_n(x) \text{ exists}\}$$

is a measurable set.

Note that $\lim f_n(x)$ exists iff $\limsup f_n(x) = \liminf f_n(x)$ and neither are equal to $\pm\infty$. Both the the limit superior and limit inferior of measurable functions are measurable by Proposition 2.7. Define,

$$h(x) := \begin{cases} \limsup f_n(x) - \liminf f_n(x) & \text{if } \limsup f_n(x) = -\liminf f_n(x) \neq \pm\infty \\ 1 & \text{if not} \end{cases}$$

and h is a measurable function by the preceding exercise. Since $\{0\}$ is in the σ-algebra of $\overline{\mathbb{R}}$, and $L = h^{-1}(0)$ we know L is a measurable set. □

Folland Exercise 2.4

Proof. If $a \in \mathbb{R}$, then choose $r_n < a$ rational s.t. $r_n \to a$. Then, $(a,\infty] = \bigcap_{n=1}^{\infty}(r_n,\infty]$. Then use Proposition 1.2, Proposition 2.3, and Exercise 2.1. □

Folland Exercise 2.5

Proof. We know f measurable on X iff every $f^{-1}(E)$ is measurable when E is measurable iff $(f^{-1}(E) \cap A) \cup f^{-1}(E) \cap B$ is measurable for E measurable iff $f^{-1}(E) \cap A$ and $f^{-1}(E) \cap B$ measurable for E measurable iff $f|_A$ and $f|_B$ are measurable. □

Folland Exercise 2.6

Proof. Consider the family $\{\chi_x\}_{x \in N}$ of functions from $\mathbb{R}$ to $\overline{\mathbb{R}}$ where N is the Lebesgue nonmeasurable set described in Section 1.1. Then, $\sup\{\chi_x\} = \chi_N$ which is not a measurable function since $\chi_N^{-1}(\{1\}) = N$ is not measurable. □

Folland Exercise 2.7

PROOF. We define $f(x)$ as follows:

$$f(x) := \inf_{r\in\mathbb{Q},\ x\in E_r} r$$

where the infimum is taken over all possible values of $r \in \mathbb{Q}$ satisfying the condition that $x \in E_r$. By the completeness of $\mathbb{R}$, such an infimum exists and so, our function is well-defined.

First, we show that $f(x) \leq \alpha$ for all $x \in E_\alpha$. Let $x \in E_\alpha$. Because $E_\alpha \subseteq E_\beta$ for any $\beta > \alpha$ by the hypothesis, and since $x \in E_\alpha$, we conclude that $x \in E_s$ for any $s \in \mathbb{Q}$ s.t. $s > \alpha$. So, we know that $f(x) \leq s$ for any $s > \alpha$. Letting $s \to \alpha$, we conclude that $f(x) \leq \alpha$.

Second, we show that $f(x) > \alpha$ for all $x \in E_\alpha^c$. Since $E_\alpha \subseteq E_\beta$ for any $\beta > \alpha$, we know that $E_\beta^c \subseteq E_\alpha^c$ for any $\beta > \alpha$. By the definition of $f(x)$, since $x \in E_\alpha^c$, and we know that $x \in E_\beta^c$ for any $\beta > \alpha$, we know that $f(x) \geq \beta$ for any $\beta > \alpha$. Letting $\beta \to \alpha$, we have $f(x) \geq \alpha$ for all $x \in E_\alpha^c$.

Third, we check that $f(x)$ never attains the values $+\infty$ or $-\infty$. Certainly, $f(x) \neq \infty$ since there is no point that lies in every E_r as $r \to \infty$. Secondly, $f(x) \neq -\infty$ since $\bigcap_{\alpha\in\mathbb{R}} E_\alpha = \emptyset$ and so, there is not point that lies in the intersection of every E_r.

Finally, we will show that $f(x)$ is measurable. To do this, note that we do not need to consider $(r, \infty]$ when using Exercise 2.4. Since $f : X \to \mathbb{R}$, we can use Exercise 2.1 and conclude that it suffices to show that $f^{-1}((r, \infty))$ is measurable for each $r \in \mathbb{Q}$. Because (r, ∞) is measurable iff $(-\infty, r]$ is measurable, it suffices to show that $f^{-1}((-\infty, r])$ is measurable for every $r \in \mathbb{Q}$. By the definition of $f(x)$, we have $f^{-1}((-\infty, r]) = E_r$ for every $r \in \mathbb{Q}$. We show this explicitly. The inclusion $E_r \subseteq f^{-1}((-\infty, r])$ follows from the fact that $f(x) \leq r$ for every $x \in E_r$ and the other inclusion follows from the fact that if $s < r$, then $E_s \subset E_r$ and hence, $f^{-1}((-\infty, s]) \subseteq E_r$ for every $s < r$ and $s \in \mathbb{Q}$. Then,

$$f^{-1}((-\infty, r]) = \bigcup_{s<r, s\in\mathbb{Q}} f^{-1}((-\infty, s]) \bigcup_{s<r, s\in\mathbb{Q}} (E_s) \subseteq E_r$$

Thus, $f^{-1}((-\infty, r]) = E_r$ and since for each $r \in \mathbb{Q}$, E_r is measurable, we conclude that $f(x)$ is a measurable function.

□

Folland Exercise 2.8 If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

PROOF. WLOG, assume $f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing. Then, $f^{-1}((a, b))$ is always an interval (possibly open, half-open, or closed) of $\mathbb{R}$ which are all measurable. Since the open intervals (a, b) for $a, b \in \mathbb{R}$ generate $\mathcal{B}_\mathbb{R}$, we deduce that f is $(\mathcal{B}_\mathbb{R}, \mathcal{B}_\mathbb{R})$-measurable. □

Folland Exercise 2.9 Let $f : [0, 1] \to [0, 1]$ be the Cantor function (§1.5), and let $g(x) = f(x) + x$.

a. g is a bijection from [0,1] to $[0, 2]$, and $h = g^{-1}$ is continuous from [0,2] to $[0, 1]$.

b. If C is the Cantor set, $m(g(C)) = 1$

c. By Exercise 29 of Chapter 1, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.

d. There exist a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

PROOF. (a) We show that g is a bijection from $[0, 1]$ to $[0, 2]$.

Suppose $g(x) = g(y)$. Using the fact that f is monotonically increasing, we must have $x = y$. If not, we may assume WLOG that $x < y$. Then, we get $f(x) \leq f(y)$ and so,

$$f(x) + x < f(y) + y$$

which contradicts $g(x) = g(y)$.

To show surjectivity, we note that f is a surjection We will use the Intermediate Value Theorem to show surjectivity. The domain of $g(x)$ is $[0, 1]$. Evaluating the functions, we know $g(0) = f(0) + 0 = f(0) = 0$ and $g(1) = f(1) + 1 = 2$. Lastly, we know $f(x)$ is continuous and so, $g(x)$ is continuous. Thus, the Intermediate Value Theorem shows that if $w \in [0, 2]$ there is a $y \in [0, 1]$ s.t. $g(y) = w$. Thus, $g(x)$ is a surjection from $[0, 1]$ to $[0, 2]$.

Now, $h := g^{-1}$ is continuous since the inverse of an increasing, bijective, and continuous function exists and is always continuous. Alternatively, we apply Theorem 4.17 of Baby Rudin[1] since $[0, 1]$ is compact, g is bijective and continuous.

(b) To show that $m(g(C)) = 1$, notice that $m(g([0, 1])) = m([0, 2]) = 2$ and that $m(g([0, 1]/C)) = m([1, 2])$. Then,

$$m(g(C)) = m(g([0, 1])) - m(g([0, 1] \setminus C)) = 2 - 1 = 1.$$

(c) From the assumptions, $g(B) = A \subseteq C$. Since $m(C) = 0$, we have that $m(A) = 0$. By the completeness of the Lebesgue measure, B is Lebesgue measurable

Additionally, B is not Borel measurable. For if B is Borel measurable, g continuous, and $g^{-1}(A) = B$ implies that A is a Borel measurable set. So, A Lebesgue measurable. Contradiction.

(d) Let $F := \chi_B$ and $G = g^{-1}$ on $[0, 2]$ and extend G to all of $\mathbb{R}$ by zero. These functions satisfy the conditions since g^{-1} is continuous and B is Lebesgue measurable by (c). We have,

$$(F \circ G)^{-1}(\{1\}) = G^{-1}(F^{-1}(\{1\})) = G^{-1}((B) = g(B) = A.$$

However, $\{1\}$ is measurable meanwhile A is not and so, $F \circ G$ is not Lebesgue measurable. $\square$

Folland Exercise 2.10

PROOF.

(a) Assume μ complete. Let f be measurable, $g = f$ a.e. and we claim that g is measurable.

Let $E = \{x : f(x) \neq g(x)\}$ which has measure zero. For any $a \in [0, \infty]$,

$$g^{-1}((a, \infty]) = f^{-1}((a, \infty]) \cup F$$

where F is the subset $E \cap g^{-1}((a, \infty])$. Then F is a subset of measure zero and by completeness, it is measurable. Since $g^{-1}((a, \infty])$ is a union of measurable sets, it is measurable. Since a was arbitrary, g is measurable.

For the converse, let $E \in \mathcal{M}$. Suppose $\mu(E) = 0$ and $F \subseteq E$. We wish to show that $F \in \mathcal{M}$. Let $f := \chi_E$ and $g := \chi_F$. Then, $g^{-1}([1, \infty]) \in \mathcal{M}$ because $f = g$ a.e. and by the assumption, g is measurable. But $g^{-1}([1, \infty])$ is just F. So, $F \in \mathcal{M}$ and μ is complete.

[1] The theorem states that a continuous $1 - 1$ mapping of a compact metric space X onto a metric space Y has a continuous inverse.

(b) Assume μ is a complete measure, f_n measurable for all $n \in \mathbb{N}$, and $f_n \to f$ μ-a.e.. By Proposition 2.7, $f^* := \limsup_{n\to\infty} f_n(x)$ is measurable. Since $f_n \to f$ μ-a.e., $f^* = f$ μ-a.e.. Since f^* is measurable and $f^* = f$ μ-a.e., part (a) shows that f is measurable.

For the converse, suppose statement (b) is true. Let $F \subset E$ where E has measure zero. Define a sequence by $f_n := 0$ for all $n \in \mathbb{N}$. Let $f := \chi_F$ and so, $f_n \to f$ μ-a.e. since f is nonzero on a subset of measure zero. By (b), f is measurable and so, $f^{-1}(\{1\}) = A$ is measurable as desired.

□

Folland Exercise 2.11
Folland Exercise 2.12

Proof. For the first statement, if the set $E := \{x : f(x) = \infty\}$ had a non-null measure, then the integral $\int f$ would not be finite. Indeed, if it had a non-null measure,

$$\int_X f \geq \int_E f = \mu(E) \cdot \infty = \infty.$$

For the second statement, consider measurable sets of the form $f^{-1}((n-1, n])$ for $n \in \mathbb{N}$. Then, the set $\{x : f(x) > 0\}$ is just $\bigcup_{n=1}^{\infty} f^{-1}((n-1, n])$. Each $f^{-1}((n-1, n])$ is of finite measure because we had $\int f < \infty$. Indeed, we have that

$$(n-1)\mu(f^{-1}((n-1, n]) \leq \int_{f^{-1}((n-1,n])} f \leq \int_X f < \infty$$

and so, $\mu(f^{-1}((n-1, n]))$ is finite. Hence $\{x : f(x) > 0\}$ is σ-finite. □

Folland Exercise 2.13

Proof. Let $E \in \mathcal{M}$. Throughout, when we write $\int_E g$, we mean $\int_X g\chi_E d\mu$. By Fatou's Lemma,

$$\int_E f = \int_E \liminf f_n \leq \liminf \int_E f_n.$$

So, it suffices that we show

$$\limsup \int_E f_n \leq \int_E f$$

since

$$\int_E f \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E f$$

implies that $\int_E f = \lim \int_E f_n$.

We have the following inequalities,

$$\begin{aligned}\int_E f = \int_X f - \int_{X\setminus E} f = \int_X f - \int_{X\setminus E} \liminf f_n \\ \geq \int_X f - \liminf \int_{X\setminus E} f_n \geq \limsup \left(\int_X f_n - \int_{X\setminus E} f_n\right) \\ \geq \limsup \int_E f_n.\end{aligned}$$

Thus, $\int_E f \geq \limsup \int_E f$ as desired.

The result is not necessarily true if $\int f = \lim \int f_n = \infty$. Let us work with the Lebesgue measure and $X = \mathbb{R}$. Let, $f_n := \chi_{(0,\infty)} + n\chi_{(-\frac{1}{n},0]}$ for $n \in \mathbb{N}$. Then, $f_n \to f$ where $f = \chi_{(0,\infty)}$ and

$$\int_{\mathbb{R}} f = \infty \qquad \text{and} \qquad \int_{\mathbb{R}} f_n = \infty.$$

Let $E = (-\infty, 0)$ (which is a measurable set). Then,

$$\int_E f = \int_E \chi_{(0,\infty)} = 0 \qquad \text{and} \qquad \int_E f_n = \int_E \chi_{(0,\infty)} + n\chi_{(-\frac{1}{n},0]} = \int_E n\chi_{(-\frac{1}{n},0]} = n\frac{1}{n} = 1.$$

So $\int_E f_n = 1$ for all n and $\lim_{n\to\infty} \int_E f_n = 1$. So, $\int_E f = 0 \neq 1 = \lim_{n\to\infty} \int_E f_n$.

□

Folland Exercise 2.14

Proof. We check that λ is a measure on $\mathcal{M}$. First,

$$\lambda(\emptyset) = \int_\emptyset f d\mu = \int_X f\chi_\emptyset d\mu \leq \mu(\emptyset) \sup_{x\in X} f = 0$$

and so $\lambda(\emptyset) = 0$. Suppose we have a disjoint sequence of sets $\{E_n\}_{n=1}^\infty$. Let $E = \bigcup_{n=1}^\infty E_n$. Then,

$$\begin{aligned}\lambda(E) &= \int_E f d\mu = \int_X f\chi_E d\mu = \int_X \sum_{n=1}^\infty f\chi_{E_n} d\mu \\ &= \sum_{n=1}^\infty \int_X f\chi_{E_n} d\mu = \sum_{n=1}^\infty \int_X f\chi_{E_n} d\mu = \sum_{n=1}^\infty \lambda(E_n)\end{aligned}$$

where we used Theorem 2.15 to justify the interchange of the integral with the infinite sum.

Now for the second statement that $\int g d\lambda = \int g f \mu$. Consider the case when g is simple and $g = \sum_{n=1}^m a_n \chi_{E_n}$. Then $\int g d\lambda = \int g f d\mu$ follows by linearity of the integral;

$$\begin{aligned}\int g d\lambda &= \int \sum_{n=1}^m a_n \chi_{E_n} d\lambda = \sum_{n=1}^m \int a_n \chi_{E_n} d\lambda = \sum_{n=1}^m a_n \lambda(E_n) \\ &= \sum_{n=1}^m a_n \int_{E_n} f d\mu = \int \sum_{n=1}^m a_n \chi_{E_n} f d\mu = \int g f \mu.\end{aligned}$$

For the statement when $g \in L^+$, choose an sequence of increasing simple functions $\{\phi_n\}_{n=1}^\infty$ whose pointwise limit is g. We may make such a choice by Theorem 2.10. By what we have shown,

$$\int \phi_n d\lambda = \int \phi_n f d\mu.$$

Since $\phi_n, f \in L^+$, the product $\phi_n f$ is in L^+. Also, $\phi_n f \to g f$ pointwise. Applying the Monotone Convergence Theorem to the sequence $\{\phi_n f\}_{n=1}^\infty$,

$$\int g f d\mu = \lim_{n\to\infty} \int \phi_n f d\mu.$$

By what we have shown, the right hand side is equal to $\lim_{n\to\infty}\int gd\lambda$. Thus,

$$\int gfd\mu = \int gd\lambda$$

as desired. $\square$

Folland Exercise 2.15 If $\{f_n\}\subset L^+, f_n$ decreases pointwise to f, and $\int f_1<\infty$, then $\int f=\lim\int f_n$.

PROOF. The sequence $\{f_1-f_n\}$ is a monotonically increasing sequence and so, applying MCT to get $\lim_n\int f_1-f_n=\int f_1-\lim_n f_n$ gives the desired result given that $\int f_1<\infty$. $\square$

Folland Exercise 2.16

PROOF. Since $f\in L^+$, Theorem 2.10 says we can choose an increasing sequence of simple function $\{\phi_n\}_{n=1}^\infty$ s.t. $0\le\phi_n\le f$ and $\phi_n\to f$ a.e.. There exists an $N\in\mathbb{N}$ sufficiently large s.t. $\int f-\epsilon<\int\phi_N$. This follows from the fact that we can choose ϕ_N so that $\left|\int\phi_N-\int f\right|<\epsilon$ by using Corollary 2.17.

Let $\phi_N=\sum_{n=1}^m a_n\chi_{E_n}$. Let $E=\bigcup_{n=1}^m E_n$. In this case, E is measurable. Also, we show $\mu(E)<\infty$. Suppose not, and then we have

$$\int\phi_N=\int_E\phi_N\ge a_M\mu(E)=\infty$$

where $a_M:=\max_{1\le n\le m}a_n$. But this contradicts the assumption that $\int f<\infty$ since

$$0\le\phi_N\le f\quad\Longrightarrow\quad\infty\le\int\phi_N\le\int f<\infty.$$

We conclude that $\mu(E)<\infty$.

Because $0\le\phi_N\le f$, we have

$$\int_X\phi_N\le\int_X f.$$

So,

$$\int_X f-\epsilon<\int_X\phi_N=\int_E\phi_N\le\int_E f,$$

as desired. $\square$

Folland Exercise 2.17 Assume Fatou's lemma and deduce the monotone convergence theorem from it.

PROOF. The proof is more or less immediate. Given the fact that

$$\int f=\int\liminf f_n\le\liminf\int f_n\le\limsup\int f_n\le\int\limsup f_n=\int f.$$

$\square$

Folland Exercise 2.18 Fatou's lemma remains valid if the hypothesis that $f_n\in L^+$ is replaced by the hypothesis that f_n is measurable and $f_n\ge -g$ where $g\in L^+\cap L^1$. What is the analogue of Fatou's lemma for nonpositive functions?

PROOF. First, $f_n + g \geq 0$ so applying Fatou's lemma,

$$\int \liminf_{n\to\infty}(f_n+g) = \liminf \int (f_n+g) \qquad \Longrightarrow \qquad \int \liminf f_n + \int g \leq \liminf \int f_n + \int g.$$

Subtracting both sides by $\int g$, since it is finite, gives the desired inequality. The analogue for Fatou's lemma for $\{f_n\} \in -L^+$ is given by

$$\int \limsup f_n = \int -\liminf(-f_n) = -\int \liminf(-f_n) \geq -\liminf \int -f_n = \limsup \int f_n$$

where we apply Fatou's lemma and use properties of lim inf and lim sup as needed. Therefore, if $\{f_n\} \subseteq -L^+$, we get

$$\limsup \int f_n \leq \int \limsup f_n.$$

$\square$

Folland Exercise 2.19

PROOF. (a) Let $\epsilon > 0$. Suppose $\mu(X) < \infty$ and that $f_n \to f$ uniformly where $f_n \in L^1(\mu)$. For n sufficiently large, we have $|f(x) - f_n(x)| < \epsilon$ for all $x \in X$. So, since $\mu(X) < \infty$,

$$\left|\int_X f - \int_X f_n\right| = \left|\int_X f - f_n\right| \overset{(2.22)}{\leq} \int_X |f - f_n| < \mu(X)\epsilon < \infty.$$

Since $\epsilon > 0$ was arbitrary, we have $\int_X f_n \to \int_X f$ as $n \to \infty$.

Then,

$$\int_X |f| = \int_X |f + f_n - f_n| \leq \int_X |f - f_n| + \int_X |f_n| < \int_X \epsilon + \int_X |f_n| < \infty$$

and we conclude that $f \in L^1(\mu)$.

(b) Both conclusions can fail if $\mu(X) = \infty$.

To see that the first conclusion fails, for $X = \mathbb{R}$ and μ the Lebesgue measure, let $f_n := \frac{1}{x}\chi_{[0,n)}$. Also, $f_n \to f$ where $f = \frac{1}{x}\chi_{[0,\infty)}$. We show that $f_n \to f$ uniformly. Let $\epsilon > 0$ and take $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \epsilon$. Suppose $n \geq N$ and let $x \in \mathbb{R}$. Then we have

$$|f_n(x) - f(x)| = \left|\frac{1}{x}\chi_{[0,n)} - \frac{1}{x}\chi_{[0,\infty)}\right| \leq \frac{1}{N} < \epsilon$$

where we get the second to last inequality by considering the case where $x > n$ and $x \leq n$ separately and noting that we still have a term less than or equal to $\frac{1}{N}$. Notice that $\int_{\mathbb{R}} f_n = \int_0^n \frac{1}{x}dx < \infty$ where we can evaluate the integral using the Riemann integral. So, $\{f_n\} \subset L^1(\mu)$. On the other hand, Riemann integration shows that $\int_{\mathbb{R}} f = \int_0^\infty \frac{1}{x} = \infty$ and so, $f \notin L^1(\mu)$.

To see that the second conclusion fails, for $X = \mathbb{R}$ and μ the Lesbesgue measure, we consider $f_n = \frac{1}{n}\chi_{[0,n)}$. We show that f_n is uniformly convergent. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \epsilon$. Let $n \geq N$. Then for any $x \in \mathbb{R}$, we have

$$|f_n(x) - f(x)| = \left|\frac{1}{n}\chi_{[0,n)}\right| \leq \frac{1}{N} < \epsilon.$$

We have $\int_X f_n = \frac{1}{n}\mu([0,n)) = 1$ for all $n \in \mathbb{N}$. So, $\lim_{n\to\infty} \int_X f_n = 1$. However $f_n \to 0$ which implies $\int_X \lim f_n = 0$.

□

Folland Exercise 2.20 (A generalized Dominated Convergence Theorem) If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \le g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$. (Rework the proof of the dominated convergence theorem.)

PROOF. The proof can be found within the proof of **Folland Exercise 6.10**. □

Folland Exercise 2.21 Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$. (Use Exercise 20.)

PROOF. For the forward direction, apply Folland Exercise 2.20 with $|f_n| \le |f_n - f| + |f|$. For the other direction, apply Folland Exercise 2.20 with $|f_n - f| \le |f_n| + |f|$. □

Folland Exercise 2.22 Let μ be counting measure on $\mathbb{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorems as statements about infinite series.

PROOF. Fatou's lemma says $\sum_{n\in\mathbb{N}} \liminf_m a_{n,m} \le \liminf_m \sum_{n\in\mathbb{N}} a_{n,m}$. The statement is similar for the Monotone Convergence Theorem and the Dominated Convergence Theorem. The DCT is more or less a comparison test. □

Folland Exercise 2.24 Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $(X, \overline{\mathcal{M}}, \bar{\mu})$ be its completion. Suppose $f : X \to \mathbb{R}$ is bounded. Then f is $\overline{\mathcal{M}}$ -measurable (and hence in $L^1(\bar{\mu})$) iff there exist sequences $\{\phi_n\}$ and $\{\psi_n\}$ of $\mathcal{M}$ -measurable simple functions such that $\phi_n \le f \le \psi_n$ and $\int (\psi_n - \phi_n)\, d\mu < n^{-1}$. In this case, $\lim \int \phi_n d\mu = \lim \int \psi_n d\mu = \int f d\bar{\mu}$.

PROOF. Fix a bounded function $f : X \to \mathbb{R}$ on a measure space $(X, \mathcal{M}, \mu)$ where $\mu(X) < \infty$ and consider the completion $(X, \overline{\mathcal{M}}, \overline{\mu})$. Let $M = \sup_{x\in X} f(x)$.

($\implies$) We may also prove the result for the case where $f : X \to [0, \infty)$ since we can split f into its positive and negative parts. Since f is $\overline{\mathcal{M}}$-measurable, there is a $\mathcal{M}$-measurable function g such that $f = g$ μ-a.e.. Define

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k} + 0\chi_{F_n} \quad \& \quad \psi_n = \sum_{k=0}^{2^{2n}-1} (k+1)2^{-n}\chi_{E_n^k} + M\chi_{F_n}.$$

where

$$E_n^k = g^{-1}((k2^{-n}, (k+1)2^{-n}]) \quad \& \quad F_n = g^{-1}((2^n, M]).$$

Certainly, $\phi_n \le g \le \psi_n$ from this construction and these functions are $\mathcal{M}$-measurable. We now define

$$\tilde{\phi}_n = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k\setminus N} + M\chi_{F_n\setminus N} \quad \& \quad \tilde{\psi}_n = \sum_{k=0}^{2^{2n}-1} (k+1)2^{-n}\chi_{E_n^k\setminus N} + M\chi_{F_n\setminus N}.$$

where N is the μ-null where f differs from g. This ensures that $\tilde{\phi}_n \le f \le \tilde{\psi}_n$ occurs by removing the set where f differs from g and where $\phi_n > f$ or $\psi_n < f$ occurs. Since N is a μ-null set, the simple functions $\tilde{\phi}_n$ and $\tilde{\psi}_n$ are $\mathcal{M}$-measurable.

We now show that $\int \tilde{\psi}_n - \tilde{\phi}_n d\mu < \frac{1}{n}$. It suffices to show that $\int |\tilde{\psi}_n - \tilde{\phi}_n| \to 0$ because we can then choose a subsequence of $\{\tilde{\psi}_n - \tilde{\phi}_n\}$ so that $\int \tilde{\psi}_n - \tilde{\phi}_n d\mu < \frac{1}{n}$.

We have

$$\int_X |\tilde{\psi_n} - \tilde{\phi_n}| = \int_X \sum_{k=0}^{2^{2n}-1} 2^{-n}\chi_{E_n^k \setminus N} = \sum_{k=0}^{2^{2n}-1} \int_X 2^{-n}\chi_{E_n^k \setminus N} \tag{11}$$

$$= \sum_{k=0}^{2^{2n}-1} \frac{\mu(E_n^k \setminus N)}{2^n} \leq \sum_{k=0}^{2^{2n}-1} \frac{\mu(X)}{2^n} \tag{12}$$

where we used the definition and the fact $\psi_n \geq \phi_n$ for the first equality, interchanged the integral with the sum by linearity for the second equality, and used monotonicity for the final inequality. Now, since $\mu(X) < \infty$ and letting $n \to \infty$ implies that the RHS approaches zero, we have our desired conclusion.

($\Longleftarrow$) Assume there exists sequences $\{\phi_n\}$ and $\{\psi_n\}$ of $\mathcal{M}$-measurable simple functions s.t. $\phi_n \leq f \leq \psi_n$ and $\int(\psi_n - \phi_n)d\mu < \frac{1}{n}$. For each $\psi_n = \sum_{k=1}^m a_k\chi_{E_k}$ we may replace any terms $a_k > M$ with M.

We claim that f is $\overline{\mathcal{M}}$-measurable. Let $\psi = \lim_{n\to\infty}\psi_n$ and $\phi = \lim_{n\to\infty}\phi_n$. In this case, since ϕ is bounded above by M and is limits of $L^1(\mu)$ functions, we know that ϕ is in $L^1(\mu) \subset L^1(\overline{\mu})$. Now,

$$\int(\psi_n - f)d\mu + \int(f - \phi_n)d\mu = \int(\psi_n - f + f - \phi_n) = \int(\psi_n - \phi_n)d\mu \to 0$$

and so, ψ and ϕ differ from f on a μ-null set. Also from the above, ψ and ϕ differ on a μ-null set. So, $\psi \in L^1(\mu)$ as well. Also, we have $\int|\psi_n - f|d\mu = \int(\psi_n - f)d\mu \to 0$ and $\int|f - \phi_n|d\mu = \int(f - \phi_n)d\mu \to 0$. Thus, we know $\int|\psi - f|d\mu = 0$ and $\int|\phi - f|d\mu = 0$. Because $\phi, \psi, f \in L^1(\mu) \subset L^1(\overline{\mu})$, Proposition 2.23 says that $f = \phi$ μ-a.e.. In particular, $f = \phi$ $\overline{\mu}$-a.e. and since ϕ is $\overline{\mu}$-measurable, Proposition 2.11 shows that f is $\overline{\mu}$-measurable.

□

Exercise 2.25 Let $f(x) = x^{-1/2}$ if $0 < x < 1, f(x) = 0$ otherwise. Let $\{r_n\}_1^\infty$ be an enumeration of the rationals, and set $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$

a. $g \in L^1(m)$, and in particular $g < \infty$ a.e.

b. g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.

c. $g^2 < \infty$ a.e., but g^2 is not integrable on any interval.

Proof. Fix $f(x) = x^{-1/2}$, for $0 < x < 1$ and $f(x) = 0$ otherwise. Fix $\{r_n\}_{n=1}^\infty$ an enumeration of the rationals and let $g(x) := \sum_{n=1}^\infty 2^{-n} f(x - r_n)$.

(a) We justify the an interchange of the sum and integral by showing that the conditions of Theorem 2.25 are fulfilled. The functions $f(x - r_n)$ are in L^1 since

$$\int_{\mathbb{R}} f(x - r_n) = \int_{r_n}^{r_n+1} f(x - r_n) = \int_{r_n}^{r_n+1} \frac{1}{\sqrt{x - r_n}} = 2$$

where we used the Riemann integral to evaluate the last integral and the first equality follows from the definition of $f(x)$.

Next, we verify that $\sum_1^\infty \int|f_n| < \infty$. We have

$$\sum_1^\infty \int_{\mathbb{R}} \left|2^{-n}f(x - r_n)\right| = \sum_1^\infty \int_0^1 \left|2^{-1}\frac{1}{\sqrt{x}}\right| = \sum_1^\infty 2^{-n}2 = \sum_1^\infty \frac{1}{2^{n-1}} = 2 < \infty$$

where we use the fact that the $f(x)$ is zero when $x \notin (0,1)$ to simplify the evaluation of the integral in the second equality and evaluated the integral as a Riemann integral in the third equality.

Now, $g < \infty$ a.e. is certainly true by Exercise 2.12 which states that $\{x : g(x) = \infty\}$ is a null set when the integral is finite.

(b) We show that $g(x)$ is unbounded on every interval. WLOG, let us assume that we have an open interval I. Let $M > 0$. Choose $y \in I$ irrational and $r_n \in I$ rational so that

$$0 < y - r_n < \frac{1}{9M^2 2^{2n}}.$$

The choice above is possible for if y irrational, then there is always a rational r_n sufficiently close to it. Now,

$$g(y) \geq 2^{-n} f(y - r_n) = 2^{-n}\sqrt{9M^2 2^{2n}} = 2^{-n}(3M2^n) = 3M > M. \tag{13}$$

Since M can be arbitrarily large, the RHS is arbitrarily large. Note that we can still make our choice of r_n so that we have $0 < y - r_n < \frac{1}{9M^2 2^{2n}}$. Thus, we conclude that $g(x)$ is unbounded on every interval.

Let $y \in \mathbb{R}$ be arbitrary. To show discontinuity, fix $\epsilon = 1$. Let $\delta > 0$. Let y be a given point and consider the interval $I = (y - \delta, y + \delta)$. It suffices to show discontinuity at y in this interval because our choice of $\delta > 0$ and $y \in \mathbb{R}$ were arbitrary.

Then for any $x \in I$, we can take M to be sufficiently large, say $M := g(y) + 1$. Thus,

$$|g(x) - g(y)| > |M - g(y)| = |g(y) + 1 - g(y)| = 1.$$

Since y and $\delta > 0$ were arbitrary, we see that $g(x)$ is discontinuous at every point for which $g(x) < \infty$.

We now show that $g(x)$ is still unbounded and discontinuous at every point even after a modification of a Lebesgue null set. Indeed, if we make a modification of $g(x)$ by a Lebesgue null set, we can still always find an irrational point y in (13). This is because in any interval, the set of irrational numbers is uncountable and the set of irrational numbers on any interval (a, b) has measure

$$m((a,b) \cap (\mathbb{R} \setminus \mathbb{Q})) = m((a,b)) - m((a,b) \cap \mathbb{Q}) = m((a,b)).$$

Thus, after a change of Lebesgue null set, we can still find irrational points so that we can repeat the proof as above.

(c) We know that $g < \infty$ a.e. and so, except on the same set of measure zero, $g^2 < \infty$.

To see that g^2 is not integrable on any interval, it suffices to show it is not integrable on $(0,1)$. This is because we can prove the result for any interval by taking an appropriate change of coordinates. From here on, we redefine g as $g\chi_{\mathbb{R}\setminus N}$ where N is the set of measure zero on which g^2 is not finite. We have,

$$g^2 = \left(\sum_1^\infty 2^{-n} f(x - r_n)\right)^2 \geq \sum_1^\infty 2^{-2n} f(x - r_n)^2$$

since the RHS has less terms than g^2 and all terms in the series is positive. Then,

$$\int_0^1 |g^2(x)|dx \geq \int_0^1 \left|\sum_1^\infty 2^{-2n} f(x-r_n)^2\right| dx \tag{14}$$

$$= \int_0^1 \sum_1^\infty 2^{-2n} f(x-r_n)^2 dx \tag{15}$$

$$= \sum_1^\infty 2^{-2n} \int_0^1 f(x-r_n)^2\, dx \tag{16}$$

$$\geq \frac{1}{4^M} \int_0^1 f(x)^2 dx \tag{17}$$

$$= \frac{1}{4^M} \int_0^1 \frac{1}{x} dx \geq \infty. \tag{18}$$

We got (16) by using Theorem 2.15 and linearity of the integral. We got (17) considering $M \in \mathbb{N}$ for which $r_M = 0$.

Then, (18) follows evaluating the integral as a Riemann integral $\lim_{r\to 0^+} \int_r^1 \frac{1}{x} dx$ and we are allowed to do so because $\frac{1}{x}$ is continuous every except at 0. Indeed, $\{0\}$ is a set of Lebesgue measure zero and we use Theorem 2.28b.

Thus, we have shown that g^2 is not an $L^1((0,1))$ function and hence, it is not integrable on any interval.

□

Exercise 2.26 If $f \in L^1(m)$ and $F(x) = \int_{-\infty}^x f(t)dt$, then F is continuous on $\mathbb{R}$.

Proof. The following proof uses a well-known fact that if $f \in L^1(\mu)$, then for $\epsilon > 0$, there exists a $\delta > 0$ s.t. $\left|\int_E f d\mu\right| < \epsilon$ whenever $\mu(E) < \delta$.

Given $\epsilon > 0$ and $x \in \mathbb{R}$. Choose $\delta > 0$ according to the above. Then, for all y s.t. $|x-y| < \delta$, we find that $|F(x) - F(y)| \leq \left|\int_y^x f(x)dx\right| < \epsilon$. This means F is continuous on $\mathbb{R}$. □

Exercise 2.27 Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$.

a. $\sum_1^\infty \int_0^\infty |f_n(x)|\, dx = \infty$.
b. $\sum_1^\infty \int_0^\infty f_n(x)dx = 0$
c. $\sum_1^\infty f_n \in L^1([0,\infty), m)$, and $\int_0^\infty \sum_1^\infty f_n(x)dx = \log(b/a)$.

Proof. □

Exercise 2.28 Compute the following limits and justify the calculations:

a. $\lim_{n\to\infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n)dx$.
b. $\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n}\, dx$.
c. $\lim_{n\to\infty} \int_0^\infty n \sin(x/n) \left[x(1 + x^2)\right]^{-1} dx$.
d. $\lim_{n\to\infty} \int_a^\infty n(1 + n^2x^2)^{-1}\, dx$. (The answer depends on whether $a > 0$, $a = 0$, or $a < 0$. How does this accord with the various convergence theorems?)

Proof. (a) First, we will show that $f_n(x) = \frac{1}{(1+\frac{x}{n})^n} \sin(\frac{x}{n}) \in L^1$.

Fix an integer $n \geq 2$ (note we can still apply the DCT later since we can just take $\{f_{n+1}(x)\}_{n\in\mathbb{N}}$ as our sequence). We will show that $\int_0^\infty |f_n(x)|dx < \infty$.

Then we use the Binomial Theorem to get

$$\begin{aligned}|f_n(x)| &= \left|\frac{1}{(1+\frac{x}{n})^n}\sin(\frac{x}{n})\right| \leq \frac{1}{(1+\frac{x}{n})^n} = \frac{1}{1+\binom{n}{1}\frac{x}{n}+\binom{n}{2}\left(\frac{x}{n}\right)^2+\cdots+\left(\frac{x}{n}\right)^n} \\ &\leq \frac{1}{1+x+\frac{n(n-1)}{2}\left(\frac{x}{n}\right)^2} = \frac{1}{1+x+\frac{n-1}{2n}x^2} \leq \frac{1}{1+\frac{n-1}{2n}x^2} \\ &\leq \frac{1}{1+\frac{1}{4}x^2}.\end{aligned}$$

Now, integrating both sides gives

$$\begin{aligned}\int_0^\infty |f_n(x)|dx &\leq \int_0^\infty \frac{1}{1+\frac{1}{4}x^2}dx = \lim_{r\to\infty}\int_0^r \frac{1}{1+\frac{1}{4}x^2}dx \\ &= \lim_{r\to\infty}\ \arctan\left(\frac{x}{2}\right)\Big|_0^r = \lim_{r\to\infty}\arctan(\frac{r}{2}) - \arctan(0) \\ &= \pi < \infty.\end{aligned}$$

We could integrate the latter function because $\frac{1}{1+\frac{1}{4}x^2}$ is continuous and hence, Riemann integration techniques are applicable. Because n was arbitrary, we conclude that $f_n(x) \in L^1$ for all $n \in \mathbb{N}$.

Set $g(x) := \frac{1}{1+\frac{1}{4}x^2}$. What we have done above shows that $g(x)$ is in L^1. Next, $\lim_{n\to\infty} f_n(x) = 0$ for all x because

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{1}{(1+\frac{x}{n})^n}\sin\left(\frac{x}{n}\right) = \left(\lim_{n\to\infty}\frac{1}{(1+\frac{x}{n})^n}\right)\left(\lim_{n\to\infty}\sin\left(\frac{x}{n}\right)\right) = \frac{1}{e^x}\cdot 0 = 0.$$

Since $g(x)$ is integrable, $|f_n| \leq g$, and $f_n \to 0$ a.e., we can apply the Dominated Convergence Theorem. So,

$$\lim_{n\to\infty}\int_0^\infty \frac{1}{(1+\frac{x}{n})^n}\sin\left(\frac{x}{n}\right)dx = \int_0^\infty \lim_{n\to\infty} f_n(x)dx = \int_0^\infty 0dx = 0.$$

(b) Let $f_n(x) = \frac{(1+nx^2)}{(1+x^2)^n}$. We show that $f_n(x) \in L^1([0,1])$. Let $n \in \mathbb{N}$. We use Bernoulli's Inequality, which states

$$(1+x)^n \geq 1+nx$$

for $x \geq -1$ and $n \in \mathbb{N}$ and it is an immediate consequence of the Binomial Theorem. Then,

$$|f_n(x)| = \left|\frac{1+nx^2}{(1+x^2)^n}\right| \leq \left|\frac{1+nx^2}{1+nx^2}\right| = 1.$$

So, $\int_0^1 |f_n(x)| \leq \int_0^1 1 < \infty$ and $f_n(x) \in L^1([0,1])$ for all $n \in \mathbb{N}$.

Set $g(x) := 1$. Then $|f_n(x)| \leq g(x)$ and we apply the Dominated Convergence Theorem. We compute the limit. Because $1 < 1+x^2$ and since the exponential is in the denominator and grows faster than the numerator which grows linearly, we conclude that

$$f = \lim_{n\to\infty}\frac{1+nx^2}{(1+x^2)^n} = 0.$$

Thus,

$$\lim_{n\to\infty}\int_0^1 (1+nx^2)(1+x^2)^{-n}dx = \int_0^1 0dx = 0.$$

(c) We have

$$n\sin(x/n)\frac{1}{x(1+x^2)} \le \frac{x}{n}n\frac{1}{x(1+x^2)} = \frac{1}{1+x^2}$$

and

$$\int_0^\infty \left|\frac{1}{1+x^2}\right| dx = \int_0^1 \frac{1}{1+x^2}dx + \int_1^\infty \frac{1}{1+x^2}dx \le \int_0^1 1dx + \int_1^\infty \frac{1}{x^2}dx < \infty.$$

By the Dominated Convergence Theorem, we can interchange the integral with the limit. Doing so, we see that the integral is equal to

$$\int_0^\infty \lim_n n\sin(x/n)[x(1+x^2)]^{-1}dx = \int \frac{1}{1+x^2}dx = \frac{\pi}{2}$$

because $\lim_{n\to 0}\frac{1}{n}\sin(nx) = \lim_{n\to\infty} n\sin(x/n) = x$.

(d) Omitted.

□

Folland Exercise 2.29 Show that $\int_0^\infty x^n e^{-x}dx = n!$ by differentiating the equation $\int_0^\infty e^{-tx}dx = 1/t$. Similarly, show that $\int_{-\infty}^\infty x^{2n}e^{-x^2}dx = (2n)!\sqrt{\pi}/4^n n!$ by differentiating the equation $\int_{-\infty}^\infty e^{-tx^2}dx = \sqrt{\pi/t}$(see Proposition 2.53).

Proof. Omitted. □

Folland Exercise 2.30 Show that $\lim_{k\to\infty}\int_0^k x^n\left(1-k^{-1}x\right)^k dx = n!$.

Proof. Omitted. □

Folland Exercise 2.31 Derive the following formulas by expanding part of the integrand into an infinite series and justifying the term-by-term integration. Exercise 29 may be useful. (Note: In (d) and (e), term-by-term integration works, and the resulting series converges, only for $a > 1$, but the formulas as stated are actually valid for all $a > 0$.)

a. For $a > 0$, $\int_{-\infty}^\infty e^{-x^2}\cos axdx = \sqrt{\pi}e^{-a^2/4}$.

b. For $a > -1$, $\int_0^1 x^a(1-x)^{-1}\log xdx = \sum_1^\infty (a+k)^{-2}$.

c. For $a > 1$, $\int_0^\infty x^{a-1}\left(e^x-1\right)^{-1}dx = \Gamma(a)\zeta(a)$, where $\zeta(a) = \sum_1^\infty n^{-a}$.

d. For $a > 1$, $\int_0^\infty e^{-ax}x^{-1}\sin xdx = \arctan\left(a^{-1}\right)$.

e. For $a > 1$, $\int_0^\infty e^{-ax}J_0(x)dx = \left(s^2+1\right)^{-1/2}$, where $J_0(x) = \sum_0^\infty (-1)^n x^{2n}/4^n(n!)^2$ is the Bessel function of order zero.

Proof. Omitted. □

Folland Exercise 2.32 Suppose $\mu(X) < \infty$. If f and g are complex-valued measurable functions on X, define

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}d\mu.$$

Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e., and $f_n \to f$ with respect to this metric iff $f_n \to f$ in measure.

PROOF. Checking that it is a metric is straightforward.

If $f_n \to f$ w.r.t the metric, then we have $\int |f_n - f| \leq \rho(f,f) \to 0$ which means $f_n \to f$ in L^1 and Proposition 2.29 shows $f_n \to f$ in measure.

Suppose $f_n \to f$ in measure. Then,

$$\rho(f_n, f) \leq \mu(E_n, \epsilon + \int_{E^c_{n,\epsilon}} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \leq \mu(E_m, \epsilon) + \mu(X)\frac{\epsilon}{1+\epsilon} < \mu(X)\left(\epsilon + \frac{\epsilon}{1+\epsilon}\right)$$

since $\frac{t}{1+t}$ is increasing on $[0, \infty)$. □

Folland Exercise 2.33 If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.

PROOF. Consider how $\liminf_{n\to\infty} \int f_n = \lim_{k\to\infty} \inf_{n\geq k} \int f_n$. We can choose a subsequence $\left\{\int f_{n_j}\right\}$ converging to $\liminf_{n\to\infty} \int f_n$. Thus,

$$\lim_{j\to\infty} \int f_{n_j} = \liminf_{n\to\infty} \int f_n \tag{19}$$

Next, since $f_n \to f$ in measure, we know that $f_{n_j} \to f$ in measure and this follows by the definition of convergence in measure. By Theorem 2.30, we choose a subsequence $f_{n_{j_i}} \to f$ s.t. $f_{n_{j_i}} \to f$ almost everywhere. Apply Fatou's Lemma at $(*)$ (since $f_n \geq 0$ our sequences are in L^+) to get

$$\int f = \int \lim_{i\to\infty} f_{n_{j_i}} = \int \liminf_{i\to\infty} f_{n_{j_i}} \overset{(*)}{\leq} \liminf_{i\to\infty} \int f_{n_{j_i}} \tag{20}$$

$$= \liminf_{j\to\infty} \int f_{n_j} \overset{(19)}{=} \liminf_{n\to\infty} \int f_n, \tag{21}$$

where we have (because pointwise limits are unique)

$$\lim_{i\to\infty} f_{n_{j_i}} = f,$$

and

$$\liminf_{i\to\infty} \int f_{n_{j_i}} = \liminf_{j\to\infty} \int f_{n_j}.$$

□

Exercise 2.34 Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure.

a. $\int f = \lim \int f_n$,

b. $f_n \to f$ in L^1.

PROOF.

(a) We wish to apply the Dominated Convergence Theorem. We first pass to a subsequence f_{n_j} such that $\int f_{n_j} \to \lim_{n\to\infty} \int f_n$. Because $f_n \to f$ in measure, we know $f_{n_j} \to f$ in measure as well. By the last part of Theorem 2.30, we can find a subsequence $f_{n_{j_i}} \to f$ a.e.. Since $|f_{n_{j_i}}| \leq g$, we apply the Dominated Convergence Theorem on $f_{n_{j_i}}$ to conclude that $\int f = \lim_{i\to\infty} \int f_{n_{j_i}}$. But then,

$$\int f = \lim_{i\to\infty} \int f_{n_{j_i}} = \lim_{j\to\infty} \int f_{n_j} = \lim_{n\to\infty} \int f_n.$$

(b) We first show that $|f_n - f| \to 0$ in measure. This is clear because

$$\mu(\{x : ||f_n(x) - f(x)| - 0| \geq \epsilon\} = \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$$

as $n \to \infty$ by the hypothesis.

Now, because $|f_n - f| \to 0$ in measure, we may choose a subsequence s.t. $\int |f_{n_j} - f| \to \lim \int |f_n - f|$. Because $|f_{n_j} - f|$ is a subsequence of $|f_n - f|$ which converges in measure to 0, we conclude that $|f_{n_j} - f|$ converges to 0 in measure as well. By Theorem 2.30, we can choose a subsequence $|f_{n_{j_i}} - f|$ which converges to zero a.e.. Apply the Dominated Convergence Theorem with $|f_{n_{j_i}} - f| \leq 2g$ ($2g \in L^1$ since $g \in L^1$ and using the triangle inequality for the bound). By the DCT, we conclude that $\lim_{i\to\infty} \int |f_{n_{j_i}} - f| = \int 0 = 0$. But then, we have

$$0 = \lim_{i\to\infty} \int |f_{n_{j_i}} - f| = \lim_{j\to\infty} \int |f_{n_j} - f| = \lim_{n\to\infty} \int |f_n - f|. \tag{22}$$

The second equality follows since $|f_{n_{j_i}} - f|$ is a subsequence of $|f_{n_j} - f|$ which converges a.e. to the same limit as $|f_{n_j} - f|$. The third equality is by the definition of $|f_{n_j} - f|$.

□

Exercise 2.35 $f_n \to f$ in measure iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon$ for $n \geq N$

PROOF. ($\implies$) Suppose $f_n \to f$ in measure. Then for every $\epsilon > 0$,

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$$

as $n \to \infty$. Then, we let $\epsilon > 0$, we can choose an $N \in \mathbb{N}$ s.t. $n \geq N$ implies

$$|\mu(\{x : |f_n(x) - f(x) \geq \epsilon\})| < \epsilon.$$

($\impliedby$) Let $\epsilon > 0$. Let $\kappa > 0$. Suppose $\kappa \leq \epsilon$. Choose an $N \in \mathbb{N}$ sufficiently large s.t. for all $n \geq N$, we get

$$\mu(\{x : |f_n(x) - f(x)| \geq \kappa\}) < \kappa.$$

Then,

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| \geq \kappa\}) < \kappa$$

since the set being measured on the left is contained in the set on the right and we use monotonicity. Now, if $\epsilon < \kappa$,

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon < \kappa.$$

We conclude that $f_n \to f$ in measure. □

Exercise 2.36 If $\mu(E_n) < \infty$ for $n \in \mathbb{N}$ and $\chi_{E_n} \to f$ in L^1, then f is (a.e. equal to) the characteristic function of a measurable set.

PROOF. Since $\chi_{E_n} \to f$ in L^1, we have a subsequence that converges a.e. $\chi_{E_{n_j}} \to f$ a.e.. But then if E is s.t. $\lim_{j\to\infty} \chi_{E_{n_j}} = \chi_E$, then f is a.e. equal to χ_E. □

Exercise 2.37

PROOF. **a.** Omitted.
b. Omitted.
c. Omitted. □

Exercise 2.38 Suppose $f_n \to f$ in measure and $g_n \to g$ in measure. a. $f_n + g_n \to f + g$ in measure.

b. $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

PROOF.

(a) We have

$$\mu(\{x : |f_n + g_n - (f+g)| \geq \epsilon\}) \leq \mu(\{x : |f_n - f| \geq \frac{\epsilon}{2}\}) + \mu(\{x : |g_n - g| \geq \frac{\epsilon}{2}\}) \to 0$$

as $n \to \infty$ because

$$\{x : |f_n + g_n - f - g| \geq \epsilon\} \subseteq \left\{x : |f_n - f| \geq \frac{\epsilon}{2}\right\} \cup \left\{x : |g_n - g| \geq \frac{\epsilon}{2}\right\}) \tag{23}$$

and the sets on the right have measure approaching zero as $n \to \infty$ by hypothesis. So, $f_n + g_n \to f + g$ in measure.

Let us justify the set containment we used. Suppose

$$y \in \{x : |f_n + g_n - f - g| \geq \epsilon\}.$$

Then, we have

$$|f_n(y) - f(y)| + |g_n(y) - g(y)| \geq |f_n(y) + g_n(y) - f(y) - g(y)| \geq \epsilon.$$

and we must $|f_n(y) - f(y)| \geq \frac{\epsilon}{2}$ or $|g_n(y) - g(y)| \geq \epsilon$. Otherwise, we would have a contradiction. So, we get the set containment in (23).

(b) Suppose $f_n \to f$ and $g_n \to g$ in measure. Let $\epsilon > 0$ and $\kappa > 0$.

Note,

$$|f_n g_n - fg| \leq |f_n||g_n - g| + |g||f_n - f|. \tag{24}$$

Now consider the sets $\{x : |f| > n\}$. These sets are decreasing and $\bigcap_{n=1}^{\infty}\{x : |f| > n\} = \emptyset$. Since the measure space is finite by continuity above we can choose a $M > 0$ sufficiently large s.t. $\mu(\{x : |f| \geq M\}) < \kappa$. Similarly, we choose an $L > 0$ sufficiently large s.t. $\mu(\{x : |g| \geq L\}) < \kappa$. We set $K := \max\{M, L\}$.

We get a set containment

$$\{x : |f_n| \geq K\} \subseteq \left\{x : |f_n - f| \geq \frac{K}{2}\right\} \cup \left\{x : |f| \geq \frac{K}{2}\right\} \tag{25}$$

because if not, we would have a contradiction by the fact that $|f_n(x)| + |f(x) - f_n(x) \geq K$.

Next, we choose K sufficiently large and $N_1 \in \mathbb{N}$ sufficiently large so that for all $n \geq N_1$, we get

$$\mu(\{x : |f_n| \geq K\}) < \kappa. \tag{26}$$

Such a choice is possible since $f_n \to f$ in measure and we can always get $\mu\{(x : |f| \geq \frac{K}{2}\}$ to be very small by taking K to be be very big. Using these two facts, we can bound the measure of (25) above by $\frac{\kappa}{2} + \frac{\kappa}{2}$ and that gives us (26).

By $f_n \to f$ in measure, choose an N_2 s.t. for all $n \geq N_2$,

$$\mu\left(\left\{x : |f_n - f| \geq \frac{\epsilon}{2K}\right\}\right) < \kappa. \tag{27}$$

Since $g_n \to g$ in measure, choose an N_3 s.t. for all $n \geq N_3$, we get

$$\mu\left(\left\{x : |g_n - g| \geq \frac{\epsilon}{2K}\right\}\right) < \kappa. \tag{28}$$

We have,

$$\mu(\{x : |f_n g_n - fg| \geq \epsilon\}) \leq \mu\left(\left\{x : |f_n||g_n - g| \geq \frac{\epsilon}{2}\right\}\right) + \mu\left(\left\{x : |g||f_n - f| \geq \frac{\epsilon}{2}\right\}\right). \tag{29}$$

We obtain the above using monotonicty and the fact that the set on the LHS is contained in the union of the two sets on the RHS. The containment comes from how at least one term on the RHS of (24) must be greater than or equal to $\frac{\epsilon}{2}$. For if neither were greater than or equal to $\frac{\epsilon}{2}$, we would get from (24)

$$\epsilon \leq |f_n g_n - fg| < \epsilon$$

which is a contradiction.

We wish to bound the terms in (29). For all $n \geq N_1$,

$$\mu\left(\left\{x : |f_n||g_n - g| \geq \frac{\epsilon}{2}\right\}\right) \leq \mu\left(\{x : |f_n| \geq K\}\right) + \mu\left(\left\{x : |g_n - g| \geq \frac{\epsilon}{2K}\right\}\right) \leq 2\kappa. \tag{30}$$

We justify this; first we have a set containment

$$\left\{x : |f_n||g_n - g| \geq \frac{\epsilon}{2}\right\} \subseteq \{x : |f_n| \geq K\} \cup \left\{x : |g_n - g| \geq \frac{\epsilon}{2K}\right\}$$

This is true because if not we have an x s.t. $|f_n| < K$ and $|g_n - g| < \frac{\epsilon}{2K}$ which implies $\frac{\epsilon}{2} \leq |f_n||g_n - g| < \frac{\epsilon K}{2K} = \frac{\epsilon}{2}$ and this is absurd. With the desired set containment, we use the monotonicity of measures to get the first inequality of (30). Then we use (26) and (28) for the last inequality.

Also, we have

$$\mu(\{x : |g||f_n - f| \geq \frac{\epsilon}{2}\}) \leq \mu(\{x : |g| \geq K\}) + \mu\left(\left\{x : |f_n - f| \geq \frac{\epsilon}{2K}\right\}\right) \leq 2\kappa \tag{31}$$

This follows from the set containment

$$\left\{x : |g||f_n - f| \geq \frac{\epsilon}{2}\right\} \subseteq \{x : |g| \geq K\} \cup \left\{x : |f_n - f| \geq \frac{\epsilon}{2K}\right\}.$$

The set containment is justified because if not, we would have $\frac{\epsilon}{2} \leq |g||f_n - f| < \frac{K\epsilon}{2K} = \frac{\epsilon}{2}$ which is absurd.

Now, let $N = \max\{N_1, N_2, N_3\}$. Then for all $n \geq N$, we have

$$\mu(\{x : |f_n g_n - fg| \geq \epsilon\} \leq 2\kappa + 2\kappa = 4\kappa. \tag{32}$$

which follows from (29), (30), and (31).

A more efficient proof for the statement above is as follows. Suppose not. Then there exists $\epsilon > 0$ and a subsequence $f_{n_j} g_{n_j}$ s.t.

$$\mu(\{: |f_{n_j}(x) g_{n_j}(x) - f(x)g(x)| > \epsilon\}) > \epsilon.$$

But it must then have a subsequence that converges to fg pointwise a.e. which is a contradiction. The existence of such a subsequence follows from the convergence of $\{f_n\}$ and $\{g_n\}$ in measure and Theorem 2.30.

Now we provide a counterexample for the case where $\mu(X) = \infty$. Take the case where $X = \mathbb{R}$. Consider the functions $f_n(x) = g_n(x) = x + \frac{1}{n}\chi_{(n,n+1)}$. We certainly have that $f_n \to f$ and $g_n \to g$ where $f = g = x$. Next, we check that these functions are convergent in measure. It suffices to check $f_n(x)$. Let $\epsilon > 0$. Then,

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\} = \mu\left(\left\{x : \left|\frac{1}{n}\chi_{(n,n+1)}\right| \geq \epsilon\right\}\right) \to 0$$

as $n \to \infty$ since we can take $N > 0$ large enough so that $\frac{1}{n} < \epsilon$ when $n \geq N$. Now, we check that the product $f_n(x)g_n(x)$ does not converge in measure. Take $\epsilon = 1$. We have

$$\mu(\{x : |f_n(x)g_n(x) - f(x)g(x)| \geq\} = \mu\left(\left\{x : \left|x^2\frac{1}{n}\chi_{(n,n+1)} + 2x\frac{1}{n}\chi_{(n,n+1)} + \frac{1}{n^2}\chi^2_{(n,n+1)}\right| \geq \epsilon\right\}\right),$$

and notice the RHS is always at least equal to 1 the set always contains $(n, n+1)$. Indeed, if $x \in (n, n+1)$, then $|f_n(x)g_n(x) - f(x)g(x)| \geq 1$. So, the measure does not converge to 0 as $n \to \infty$. Thus, $f_n(x)g_n(x)$ does not converge to $f(x)g(x)$ in measure.

□

Folland 2.39 If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e. and in measure.

Proof. $f_n \to f$ *a.e.*: To show that $f_n \to f$ a.e., let $m \in \mathbb{N}$. Then there is an $E_{1/m}$ such that $\mu(E_{1/m}) < \frac{1}{m}$ and $f_n \to f$ uniformly on $E^c_{1/m}$. Then let $E := \bigcap_{m\in\mathbb{N}} E_{1//m}$ and we see that $f_n \to f$ on E^c. Notice that

$$\mu(X) \geq \mu(E^c) = \mu\left(\bigcup_{m\in\mathbb{N}} E^c_{1/m}\right) \geq \mu(E^c_{1/m}) = \mu(X) - \frac{1}{m}.$$

Since this holds for all $m \in \mathbb{N}$, we deduce that $\mu(E^c) = \mu(X)$.

$f_n \to f$ *in measure:* Given $\epsilon > 0$ and $\delta > 0$. There is a set E_δ s.t. $\mu(E_\delta) < \delta$ and $f_n \to f$ uniformly on E^c_δ. Then by definition, for n sufficiently large,

$$\{x : |f_n(x) - f(x)| \geq \epsilon\} \subseteq E_\delta.$$

Therefore, $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \delta$. Since this holds for all $\delta > 0$, we deduce $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0$. □

Folland 2.40 In Egoroff's theorem, the hypothesis " $\mu(X) < \infty$ " can be replaced by " $|f_n| \leq g$ for all n, where $g \in L^1(\mu)$."

Proof. Let $E_n(k) = \bigcup_{m=n}^\infty \{x : |f_m(x) - f(x)| \geq k^{-1}\}$.

In proof of Egorov's Theorem, we needed $\mu(X) < \infty$ to get $\mu(E_1(k)) < \infty$ to use continuity from above. So, it suffices that we show $\mu(E_1(k)) < \infty$ under the new hypothesis. Note

$$E_1(k) = \bigcup_{m=1}^\infty \{x : |f_m(x) - f(x)| \geq k^{-1}\}.$$

Now, we have $|f_m(x) - f(x)| \leq 2g(x)$ using the hypothesis that $|f_n| \leq g$ for all n and hence, $|f| \leq g$. So, for every m

$$\{x : |f_m(x) - f(x)| \geq k^{-1}\} \subseteq \{x : 2g(x) \geq k^{-1}\}.$$

Therefore, $E_1(k) \subseteq \{x : 2g(x) \geq k^{-1}\}$. It suffices to show that $\mu(\{x : 2g(x) \geq k^{-1}\} < \infty$. Suppose not. Then, we get

$$\infty = \frac{1}{2k}\mu(\{x : 2g(x) \geq k^{-1}\} \leq \int_{\{x:2g(x)\geq k^{-1}\}} g(x)d\mu = \int_{\{x:2g(x)\geq k^{-1}\}} |g(x)|d\mu < \infty$$

contradiction.

We can then repeat the proof of Egorov's Theorem. By $\bigcap_{n=1}^{\infty} E_n(k) = \emptyset$ and what we showed, $\mu(E_n(k)) \to 0$ as $n \to \infty$. We may take n_k large enough s.t. $\mu(E_{n_k}(k)) < \epsilon 2^{-k}$ and let $E = \bigcup_{k=1}^{\infty} E_{n_k}(k)$. By subadditivty,

$$\mu(E) = \mu\left(\bigcup_{k=1}^{\infty} E_{n_k}(k)\right) < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

Also, $f_n \to f$ uniformly on $X \setminus E$ since $|f_n(x) - f(x)| < \frac{1}{k}$ for every $x \in X \setminus E$ if we take $n > n_k$.

□

Exercise 2.41 If μ is σ-finite and $f_n \to f$ a.e., there exist measurable $E_1, E_2, \ldots \subset X$ such that $\mu\left((\cup_1^{\infty} E_j)^c\right) = 0$ and $f_n \to f$ uniformly on each E_j.

Proof. Let $X = \bigcup_{j=1}^{\infty} F_j$ where $\mu(F_j) < \infty$. WLOG, we assume that $F_j \subseteq F_{j+1}$ by possibly writing $E_j := \bigcup_{i=1}^{j} F_i$ and using E_j in place of F_j. Applying Egorov's Theorem on F_j with $\epsilon := \frac{1}{j}$, we can find a subset $F_{j,\epsilon}$ s.t. $\mu(F_{j,\epsilon}) < \epsilon$ and $f_n \to f$ uniformly on $F_j \setminus F_{j,\epsilon}$. Then,

$$\begin{aligned}
\mu\left(\left(\bigcup_{j=1}^{\infty} F_j \setminus F_{j,\epsilon}\right)^c\right) &= \mu\left(\bigcap_{j=1}^{\infty} (F_j \setminus F_{j,\epsilon})^c\right) = \mu\left(\bigcap_{j=1}^{\infty} (F_j^c \cup F_{j,\epsilon})\right) \\
&= \mu(F_j^c \cup F_{j,\epsilon}) \leq \mu(F_j^c) + \mu(F_{j,\epsilon}) = \mu(F_j^c) + \frac{1}{j}.
\end{aligned}$$

We have that $\mu(F_j^c) \to 0$ as $j \to \infty$ because F_j is increasing and increases to X. Because the inequality holds for all j, we conclude that

$$\mu\left(\left(\bigcup_{j=1}^{\infty} F_j \setminus F_{j,\epsilon}\right)^c\right) = 0.$$

Since $f_n \to f$ uniformly on each $F_j \setminus F_{j,\epsilon}$, our desired sets are $E_j := F_j \setminus F_{j,\epsilon}$. □

Folland 2.42 Let μ be counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure iff $f_n \to f$ uniformly.

Proof. Assume $f_n \to f$ in measure. That is,

$$\mu(E_{n,\epsilon}) = \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \tag{33}$$

as $n \to \infty$ for all $\epsilon > 0$. Therefore, as μ is the counting measure, there is some N sufficiently large for which $\mu(E_{n,\epsilon}) = 0$ for all $n \geq N$. But this is precisely the definition of uniform continuity.

Assume $f_n \to f$ uniformly. Let $\epsilon > 0$. Choose an $N \in \mathbb{N}$ sufficiently large s.t. $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$. Therefore, $\mu(E_{n,\epsilon}) = 0$ for $n \geq N$. But this is precisely the definition of convergence in measure. □

Folland 2.43 Suppose that $\mu(X) < \infty$ and $f : X \times [0,1] \to \mathbb{C}$ is a function such that $f(\cdot, y)$ is measurable for each $y \in [0,1]$ and $f(x, \cdot)$ is continuous for each $x \in X$.

a. If $0 < \epsilon, \delta < 1$ then $E_{\epsilon,\delta} = \{x : |f(x,y) - f(x,0)| \leq \epsilon \text{ for all } y < \delta\}$ is measurable.

b. For any $\epsilon > 0$ there is a set $E \subset X$ such that $\mu(E) < \epsilon$ and $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E^c as $y \to 0$.

Proof.

(a) Enumerate the set $\mathbb{Q} \cap [0, \delta)$ as $\{r_n\}$. Define sets

$$E_n := \{x : |f(x, r_n) - f(x, 0)| \leq \epsilon\}. \tag{34}$$

Then, E_n is measurable because

$$E_n = |f(\cdot, r_n) - f(\cdot, 0)|^{-1}(\{[0, \epsilon)\}) \tag{35}$$

and $|f(\cdot, r_n) - f(\cdot, 0)|$ is measurable because $f(\cdot, y)$ is measurable for every $y \in [0,1]$. But then,

$$E_{\epsilon,\delta} = \bigcap_{n=1}^{\infty} E_n. \tag{36}$$

(b) For this exercise, we repeat the proof of Egorov's Theorem with modification. Let $\epsilon > 0$. Let $\{y_n\}$ be a sequence converging to 0. Consider the sets

$$E_n(k) := \left\{ x : |f(x, y_n) - f(x, 0)| \geq \frac{1}{k} \right\}. \tag{37}$$

Then,

$$\bigcap_{n=1}^{\infty} E_n(k) = \emptyset \tag{38}$$

because $f(x, \cdot)$ is continuous. Also, $\mu(E_1(k)) < \infty$ because $\mu(X) < \infty$ and therefore, there exists an $N \in \mathbb{N}$ s.t. $\mu(E_N(k)) < \epsilon$. By definition of $E_N(k)$, that means $f(x, y) \to f(x, 0)$ uniformly on $E_N(k)^c$ as desired.

□

Folland 2.44 (Lusin's Theorem) If $f : [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a,b]$ such that $\mu(E^c) < \epsilon$ and $f \mid E$ is continuous. (Use Egoroff's theorem and Theorem 2.26.)

[Replace μ with m]

Proof. To apply Theorem 2.26, we show that $f \in L^1([a,b])$. Consider sets $\{x : |f| \geq N\}$ which decrease to the emptyset because the complement is $\{x : |f| < N\}$ and $m([a,b]) < \infty$. From continuity from above, we deduce the $|f|$ is bounded a.e.. Hence, $\int |f| dm < Mm([a,b])$ where $|f| < M$ a.e.. This shows $f \in L^1([a,b])$.

By Theorem 2.26, choose a sequence of continuous functions $\{f_n\}$ s.t. $\int |f_n - f| d\mu < \frac{1}{n}$ for each n and f_n vanishes outside some bounded interval $I \subseteq [a,b]$. Note that continuous functions are Lebesgue measurable. Since $f_n \to f$ pointwise a.e., we conclude that f is measurable.

Because $m([a,b]) = b - a < \infty$, $f_n \to f$ a.e., and f_n and f are measurable, we use Egorov's Theorem and choose a set $F \subseteq [a,b]$ s.t. $m([a,b] \setminus F) < \frac{\epsilon}{2}$ and $f_n \to f$ uniformly on F.

Because $f_n \to f$ uniformly on F, f is continuous.[2] Then, by Theorem 1.18, we use inner regularity and take a compact subset $E \subseteq F$ s.t. $m(F \setminus E) < \frac{\epsilon}{2}$. In this case,

$$m(E^c) = m([a,b] \setminus E) = m(([a,b] \setminus F) \cup (F \setminus E)) = m(F^c) + m(F \setminus E) < \epsilon + \epsilon = \epsilon.$$

Because $E \subseteq F$, we know that $f|E$ is continuous. So, the theorem has been proven. $\square$

Folland 2.45 If $(X_j, \mathcal{M}_j)$ is a measurable space for $j = 1, 2, 3$, then $\bigotimes_1^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$. Moreover, if μ_j is a σ -finite measure on $(X_j, \mathcal{M}_j)$, then $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$.

Proof. We prove the first statement. We have

$$\begin{aligned}
\bigotimes_{j=1}^{3} \mathcal{M}_j &= \{\pi_i^{-1}(E_i) : E_i \in \mathcal{M}_i, i = 1, 2, 3\} \\
&= \{\tau_j^{-1}(E'_j) : E'_1 \in \mathcal{M}_1 \otimes \mathcal{M}_2, E_2 \in \mathcal{M}_3, \tau_1^{-1}(E'_1) = (\pi_1^{-1}(E_1), \pi_2^{-1}(E_2))\} \\
&= (M_1 \otimes M_2) \otimes M_3.
\end{aligned}$$

We prove the second statement. Suppose $X_i = \cup_{j=1}^{\infty} A_{i,j}$ with $\mu_i(A_{i,j}) < \infty$. By the remarks of the chapter, we have uniqueness. We have

$$\begin{aligned}
\mu_1 \times \mu_2 \times \mu_3(A_1 \times A_2 \times A_3) &= \mu_1(A_1)\mu_2(A_2)\mu_3(A_3) \\
&= (\mu_1 \times \mu_2)(A_1 \times A_2) \times \mu_3(A_3) \\
&= (\mu_1 \times \mu_2) \times \mu_3((A_1 \times A_2) \times A_3))
\end{aligned}$$

since $A_1 \times A_2 \in \mathcal{M}_1 \otimes \mathcal{M}_2$. $\square$

Folland 2.46 Let $X = Y = [0,1], \mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}, \mu =$ Lebesgue measure, and $\nu =$ counting measure. If $D = \{(x,x) : x \in [0,1]\}$ is the diagonal in $X \times Y$, then $\iint \chi_D d\mu d\nu$ $\iint \chi_D d\nu d\mu$, and $\int \chi_D d(\mu \times \nu)$ are all unequal. (To compute $\int \chi_D d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.)

Proof. We have

$$\iint \chi_D d\mu dv = \int \mu(D^y) dv(y) = \int \mu(\{y\}) dv(y) = \int 0 dv = 0$$

and

$$\iint \chi_D d\nu d\mu = \int \nu(D^x) d\mu(x) = \int \nu(\{x\}) d\mu(x) = \int d\mu(x) = 1.$$

Finally, $\int \chi_D d(\mu \times \nu) = \mu \times \nu(D)$. Let $A_j \times B_j$ be a sequence of rectangles covering D . So, $(x,x) \in A_j \times B_j$ implies $x \in A_j, B_j$ so the sets $A_j \cap B_j$ cover $[0,1]$. Since it covers $[0,1]$, we have $\sum_{j=1}^{\infty} m(A_j \cap B_j) \geq 1$ which implies that at least one of the $A_j \cap B_j$ has positive Lebesgue Measure. So, it is has infinite ν measure. Hence, $\mu(A_j)\nu(B_j) = \infty$ since $\nu(B_j) = \infty$. So, $\int \chi_D d(\mu \times \nu) = \infty$.

$\square$

[2]This is a standard undergraduate analysis result.

Folland 2.47 Let $X = Y$ be an uncountable linearly ordered set such that for each $x \in X$, $\{y \in X : y < x\}$ is countable. (Example: the set of countable ordinals.) Let $\mathcal{M} = \mathcal{N}$ be the σ -algebra of countable or co-countable sets, and let $\mu = \nu$ be defined on $\mathcal{M}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is co-countable. Let $E = \{(x, y) \in X \times X : y < x\}$. Then E_x and E^y are measurable for all x, y and $\iint \chi_E d\mu d\nu$ and $\iint \chi_E d\nu d\mu$ exist but are not equal. (If one believes in the continuum hypothesis, one can take $X = [0, 1]$ [with a nonstandard ordering] and thus obtain a set $E \subset [0, 1]^2$ such that E_x is countable and E^y is co-countable [in particular, Borel] for all x, y, but E is not Lebesgue measurable.)

Proof. Let $x \in X$. We show that E_x is measurable. First, $E_x = \{y \in Y : (x, y) \in E\}$ and by hypothesis, this is a countable collection. Thus, E_x is measurable.

Let $y \in Y$. Then, $E^y = \{x \in X : (x, y) \in E\}$ and this collection is co-countable since it is the complement of $(E^y)^c = \{x \in X : (x, y) \notin E\}$. Indeed,

$$(E^y)^c = \{x \in X : (x, y) \notin E\} = \{x \in X : y \geq x\} \tag{39}$$

$$= \{x \in X : x \leq y\} = \{x \in X : x < y\} \cup \{y\} \tag{40}$$

and the final set above is countable because the hypothesis tells us that $\{x \in X : x < y\}$ is countable.

We compute $\int\int \chi_E d\mu d\nu$ which will also show existence. We get

$$\int\int \chi_E d\mu d\nu = \int \left[\int \chi_E d\mu\right] d\nu = \int \mu(E^y) d\nu \tag{41}$$

$$= \int 1 d\nu = 1. \tag{42}$$

We compute $\int\int \chi_E d\nu d\mu$. We get

$$\int\int \chi_E d\nu d\mu = \int \left[\int \chi_E d\nu\right] d\mu = \int \nu(E_x) d\mu \tag{43}$$

$$= \int 0 d\mu = 0. \tag{44}$$

Thus, the two integrals are not equal. □

Folland 2.48 Let $X = Y = \mathbb{N}, \mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N}), \mu = \nu =$ counting measure. Define $f(m, n) = 1$ if $m = n, f(m, n) = -1$ if $m = n + 1$, and $f(m, n) = 0$ otherwise. Then $\int |f| d(\mu \times \nu) = \infty$, and $\iint f d\mu d\nu$ and $\iint f d\nu d\mu$ exist and are unequal.

Proof. We have the following computations

$$\iint f d\mu dv = \sum_n \sum_m f(m, n) = \sum_n (f(1, n) + f(2, n) + \dots) = 0$$

$$\begin{aligned}\iint f dv d\mu &= \sum_m \sum_n f(m, n) = \sum_n (f(m, 1) + f(m, 2) + \dots) \\ &= 1 + \sum_m (f(m, 2) + f(m, 3) + \dots) = 1 + 0 = 1.\end{aligned}$$

So, the two sums are unequal. Note that we interpret the counting measure on $\mathbb{N}$ as summation. On the other hand, we know

$$\int |f| d(\mu \times \nu) = \infty \tag{45}$$

because $|f|$ is constant equal to 1 and $\mu \times \nu(X \times Y) = \infty$ by definition of the product measure. $\square$

Folland 2.49 Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas.

a. If $E \in \mathcal{M} \otimes \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.

b. If f is $\mathcal{L}$ -measurable and $f = 0$ λ -a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x d\nu = \int f^y d\mu = 0$ for a.e. x and y. (Here the completeness of μ and ν is needed.)

Proof. Assume $(X, \mathcal{M}, \mu)$ and $(X, \mathcal{N}, \nu)$ are complete σ-finite measure spaces and $(X \times Y, \mathcal{L}, \lambda)$ is the completion of the product measure space.

We prove the aforementioned lemmas.

(a) Let $E \in \mathcal{M} \otimes \mathcal{N}$ and assume $\mu \times \nu(E) = 0$. Now, $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$. Because of the assumptions and $\mu \times \nu(E) = 0$, we may apply Fubini's Theorem. Then,

$$0 = \mu \times \nu(E) = \int \mu(E^y) d\nu \tag{46}$$

which implies that $\mu(E^y) = 0$ a.e. $y \in Y$. Similarly,

$$0 = \mu \times \nu(E) = \int \nu(E_x) d\mu \tag{47}$$

implies that $\nu(E_x) = 0$ a.e $x \in X$.

(b) Suppose f is λ-measurable a.e. and $f = 0$ λ a.e.. Then, $f \in L^1(\mu \times \nu)$ because

$$\int f d(\mu \times \nu) = \int_N f d(\mu \times \nu) + \int_{N^c} 0 d(\mu \times \nu) = \int_{N^c} 0 d(\mu \times \nu) = 0 \tag{48}$$

where N is the set of measure zero on which $f \neq 0$. Hence, $f \in L^1(\mu \times \nu)$. We apply Fubini's Theorem to conclude that $f_x \in L^1(\nu)$ and $f^y \in L^1(\mu)$ for a.e. $y \in Y$ and a.e. $x \in X$. This proves the first part of (b).

Because $\mu \times \nu(N) = 0$, we conclude that $\nu(N_x) = \mu(N^y) = 0$ a.e. x and y by part (a). By completeness of ν and μ, we conclude that N_x and N^y are measurable. Hence, N_x^c and $(N^y)^c$ are measurable as well.

Using the final part of Fubini's Theorem,

$$0 = \int f \chi_{N^c} d(\mu \times \nu) = \int \int f^y(x) \chi_{N_y^c} d\mu d\nu \tag{49}$$

which implies $\int f^y(x) d\mu = 0$ for a.e. $y \in Y$. Similarly,

$$0 = \int f d\mu(\mu \times \nu) = \int \int f_x(y) \chi_{(N^y)^c} d\mu d\nu \tag{50}$$

which implies $\int f_x(y) d\nu = 0$ for a.e. $x \in X$.

We now prove Theorem 2.49.

Suppose $f \geq 0$. Then f_x is $\mathcal{N}$-measurable and f_y is $\mathcal{M}$-measurable because $f \in L^+(\lambda)$ and Theorem 2.37. Additionally, Theorem 2.37 shows that $\int f_x d\nu \in L^+(X)$ and $\int f_y d\mu \in L^+(Y)$.

Suppose $f \in L^1(\lambda)$. Apply Theorem 2.37b to see that f_x and f^y are integrable a.e. x and y. Also, $\int f_x d\nu$ and $\int f^y d\mu$ are measurable by Theorem 2.37. By Proposition 2.12, choose a g s.t. $f = g$ $(\mu \times \nu)$-a.e. and g is $\mathcal{M} \times \mathcal{N}$-measurable. Let N be the set of measure zero s.t. $f \neq g$. By part (a), we know that $\nu(N_x) = \mu(N_y) = 0$. By Theorem 2.37, we get

$$\int f d\lambda = \int g d\lambda \qquad \text{(Definition)} \tag{51}$$

$$= \int_{N^c} g d\lambda + \int_N g d\lambda \qquad \text{(Definition)} \tag{52}$$

$$\stackrel{(*)}{=} \int_{N^c} g d\lambda \tag{53}$$

$$= \int \int g \chi_{N^c} d\nu d\mu \qquad \text{(Fubini-Tonelli Theorem)} \tag{54}$$

$$= \int \int f(x,y) d\nu d\mu (\text{Definition of } N^c) \tag{55}$$

$$\int f d\lambda = \int g d\lambda \qquad \text{(Definition)} \tag{56}$$

$$= \int_{N^c} g d\lambda + \int_N g d\lambda \qquad \text{(Definition)} \tag{57}$$

$$\stackrel{(*)}{=} \int_{N^c} g d\lambda \tag{58}$$

$$= \int \int g \chi_{N^c} d\mu d\nu \qquad \text{(Fubini-Tonelli Theorem)} \tag{59}$$

$$= \int \int f(x,y) d\mu d\nu. \qquad (\text{Definition of } N^c) \tag{60}$$

We must justify the steps at $\stackrel{(*)}{=}$. Because we assumed g is $\mathcal{M} \otimes \mathcal{N}$-measurable, we know that g is $\mathcal{L}$-measurable and by definition, $g\chi_N = 0$ λ-a.e.. By part (b), this implies $\int g_x d\nu = \int g^y d\mu = 0$ for a.e. x and y. But that means, by the Fubini-Tonelli Theorem,

$$\int_N g d\lambda = \int \int g \chi_{N_x} d\mu d\nu = \int 0 d\nu = 0.$$

This justifies both $\stackrel{(*)}{=}$. Thus,

$$\int f d\lambda = \int \int f(x,y) d\mu d\nu = \int \int f(x,y) d\nu d\mu. \tag{61}$$

$\square$

Folland 2.50 Suppose $(X, \mathcal{M}, \mu)$ is a σ -finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x,y) \in X \times [0,\infty] : y \leq f(x)\}.$$

Then G_f is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$ -measurable and $\mu \times m\,(G_f) = \int f d\mu$; the same is also true if the inequality $y \leq f(x)$ in the definition of G_f is replaced by $y < f(x)$. (To show measurability of G_f, note that the map $(x, y) \mapsto f(x) - y$ is the composition of $(x, y) \mapsto (f(x), y)$ and $(z, y) \mapsto z - y$.) This is the definitive statement of the familiar theorem from calculus, "the integral of a function is the area under its graph."

[You can replace $y \leq f(x)$ by $y < f(x)$ if you want]

PROOF. We will assume $y < f(x)$ in the definition of G_f.

To see that G_f is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$-measurable, we will use the hint. Define

$$\psi : X \times [0, \infty] \to [0, \infty]^2$$

by $(x, y) \mapsto (f(x), y)$, and

$$\theta : [0, \infty]^2 \to [0, \infty]$$

by $(z, y) \mapsto z - y$. Then, define

$$\phi : X \times [0, \infty] \to [0, \infty]$$

by $\phi := \theta \circ \psi$. Then, $G_f = \phi^{-1}((0, \infty])$ by the definition of G_f and the construction of ϕ. It suffices to show that ϕ is $(\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}})$-measurable. Since $\phi = \theta \circ \psi$, it suffices to show θ and ψ are measurable.

First, θ is $(\mathcal{B}_{\mathbb{R}^2}, \mathcal{B}_{\mathbb{R}})$-measurable because it is continuous. Second, ψ is $(\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}^2})$-measurable because $f(x)$ is $\mathcal{M}$-measurable and the map on the second coordinate is the identity.

Thus, for $E \in \mathcal{B}_{\mathbb{R}}$, we have

$$\phi^{-1} = \psi^{-1}(\theta^{-1}(E)) \tag{62}$$

where $\theta^{-1}(E)$ is $\mathcal{B}_{\mathbb{R}^2}$-measurable and hence, $\psi^{-1}(\theta^{-1}(E))$ is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$-measurable.

Now that G_f is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$-measurable, we can prove the second statement. By Theorem 2.36,

$$\mu \times m(G_f) = \int m((G_f)_x) d\mu = \int f(x) d\mu.$$

The last equality follows from how

$$(G_f)_x = \{y \in Y : y < f(x)\}$$

which means $m((G_f)_x) = m((0, f(x)) = f(x)$. □

Folland 2.51 Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be arbitrary measure spaces (not necessarily σ finite).

a. If $f : X \to \mathbb{C}$ is $\mathcal{M}$ -measurable, $g : Y \to \mathbb{C}$ is $\mathcal{N}$ -measurable, and $h(x, y) = f(x)g(y)$, then h is $\mathcal{M} \otimes \mathcal{N}$ -measurable.

b. If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h d(\mu \times \nu) = \left[\int f d\mu\right] \left[\int g d\nu\right]$.

PROOF. WLOG, we may take our functions f, g to have codomain $[0, \infty)$ because we can use the result and prove it for real and imaginary parts as well as for positive and negative parts.

(a) Let $h(x, y) = f(x)g(y)$. We show that $h(x, y)$ is $\mathcal{M} \otimes \mathcal{N}$ measurable. For $a \in (0, \infty)$,

$$h^{-1}((a, \infty)) = \left[f^{-1}((a, \infty) \times g^{-1}\left(\left(\frac{1}{a}, \infty\right)\right)\right] \cup \left[g^{-1}((a, \infty)) \times f^{-1}\left(\left(\frac{1}{a}, \infty\right)\right)\right] \tag{63}$$

which is certainly $\mathcal{M} \otimes \mathcal{N}$-measurable because f and g are $\mathcal{M}$-measurable and $\mathcal{N}$-measurable respectively.

For $a = 0$,

$$h^{-1}((a,\infty)) = f^{-1}((a,\infty)) \times g^{-1}((a,\infty)) \tag{64}$$

which is $\mathcal{M} \otimes \mathcal{N}$-measurable since f, g are $\mathcal{M}$ and $\mathcal{N}$ measurable respectively.

(b) Suppose $f \in L^1(\mu)$ and $g \in L^1(\nu)$. By Theorem 2.10, choose increasing sequences of simple functions ϕ_n and ψ_n which increase to f and g respectively and s.t. $0 \leq \phi_n \leq f$ and $0 \leq \psi_n \leq g$.

Set $\phi_n = \sum_{j=1}^m a_j \chi_{E_j}$ and $\psi_n = \sum_{i=1}^l b_i \chi_{F_i}$.

Then,

$$\int |\phi_n \psi_n| d(\mu \times \nu) = \int \phi_n(x)\psi_n(y) d(\mu \times \nu) \tag{65}$$

$$= \int \sum_{j=1}^m a_j \chi_{E_j} \sum_{i=1}^l b_i \chi_{F_i} d(\mu \times \nu) \tag{66}$$

$$= \sum_{j=1}^m \sum_{i=1}^l \int a_j b_i \chi_{E_j \times F_i} d(\mu \times \nu) \qquad \text{Theorem 2.15} \tag{67}$$

$$= \sum_{j=1}^m \sum_{i=1}^l a_j b_i \mu(E_j)\nu(F_i) \qquad \text{Def. of Product} \tag{68}$$

$$= \left(\sum_{j=1}^m a_j \mu(E_j)\right)\left(\sum_{i=1}^l b_i \nu(F_i)\right) \qquad \text{Rewrite Sum} \tag{69}$$

$$= \int \phi_n d\mu \int \psi_n d\nu. \tag{70}$$

Thus, $\int |\phi_n \psi_n| d(\mu \times \nu) = \int \phi_n d\mu \int \psi_n d\nu$. Applying the Monotone Convergence Theorem, we conclude that $\int h d(\mu \times \nu) = \int f d\mu \int g d\nu$. Because f and g are L^1 and we showed the equality, we conclude that $h \in L^1(\mu \times \nu)$. Indeed,

$$\int |h| d(\mu \times \nu) = \int |f| d\mu \int |g| d\nu < \infty \cdot \infty = \infty.$$

$\square$

Folland Exercise 2.52 The Fubini-Tonelli theorem is valid when $(X, \mathcal{M}, \mu)$ is an arbitrary measure space and Y is a countable set, $\mathcal{N} = \mathcal{P}(Y)$, and ν is counting measure on Y.(Cf. Theorems 2.15 and 2.25.)

Proof. Just as in the proof of the Fubini-Tonelli Theorem, we will prove Fubini's part and then derive Tonelli immediately.

Let $f \in L^+(X \times Y)$ and set $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$. Because ν is the counting measure, we have $g(x) = \sum_{y \in Y} f_x(y)$ and so, $g(x) \in L^+$ due to f being measurable and g being a sum of measurable functions. Indeed, f_x is measurable by Proposition 2.34. Also, $h(y)$ is measurable because all subsets of Y are in the σ-algebra and hence, the inverse image of any measurable set under h is measurable.

Next, we show that we can interchange the order of integration. To do this, we first show $\int f d(\mu \times \nu) = \int \int f(x,y) d\mu d\nu$. It suffices to show $\int f d(\mu \times \nu) = \int h(y) d\nu$. By Theorem

2.10a, choose a sequence of simple functions $0 \leq \phi_n \leq f$ increasing to f. Such a sequence defines a sequence $h_n(y)$ increasing to $h(y)$. Then,

$$\int h d\nu = \lim \int h_n d\nu = \lim \int f_n d(\mu \times \nu) = \int f d(\mu \times \nu) \tag{71}$$

by the Monotone Convergence Theorem.

We have

$$\int f d(\mu \times \nu) = \int \int f(x,y) d\nu d\mu = \int \sum_{y \in Y} f(x,y) d\mu \tag{72}$$

$$= \sum_{y \in Y} \int f(x,y) d\mu = \int \int f(x,y) d\mu d\nu \tag{73}$$

where we interchanged the sum and the integral by Theorem 2.15.

With the above results, Tonelli's Theorem follows by considering the positive and negative parts of the real and imaginary parts of f. The equality above also allows us to conclude that if $\int |f| d(\mu \times \nu) < \infty$, then we must have $\int |h(y)| d\nu < \infty$ for a.e. y and $\int |g(x)| d\mu < \infty$ for a.e. x. This complete's the Tonelli part of the theorem. $\square$

Folland Exercise 2.53 Fill in the details of the proof of Theorem 2.41.

Proof. Following the approach in the proof of Theorem 2.41 is relatively straightforward. A more general approach is to generalize Lusin's Theorem to $\mathbb{R}^n$ for $n \in \mathbb{N}$ and use that result. See [2]. $\square$

Folland Exercise 2.54 How much of Theorem 2.44 remains valid if T is not invertible?

Proof. Only some facts remain true. In the sequel, when we write T as an $n \times n$ matrix, we are viewing $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $Tx = y$ where $x \in \mathbb{R}^n$ is a column vector.

(a) We will show that the first statement is in general false: "if f is Lebesgue measurable and T is a linear transformation, then $f \circ T$ is Lebesgue measurable".

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation that projects onto the first coordinate. That is, $(x, y) \mapsto (x, 0)$.

Next, let N be a Lebesgue nonmeasurable set (which exists by constructions in Chapter 1). Define $f : \mathbb{R}^2 \to [0, \infty)$ by

$$f(x,y) := \begin{cases} \frac{y}{1+|y|} & \text{if } y \neq 0 \\ \chi_{N \times \{0\}} & \text{if } y = 0 \end{cases} \tag{74}$$

Then, $f(x, y)$ is Lebesgue measurable since $N \times \{0\} \subseteq \mathbb{R}^2$ is Lebesgue measurable (as it is measure zero and by completeness) and $\frac{y}{1+|y|}$ is Lebesgue measurable.

Then, $f^{-1}(\{1\}) = N \times \{0\}$ and $T^{-1}(N \times \{0\}) = N \times \mathbb{R}$. Hence, $T^{-1}(f^{-1}(\{1\})) = N \times \mathbb{R}$. But $N \times \mathbb{R}$ is not Lebesgue measurable since N is not Lebesgue measurable.

For the second statement: "if $f \geq 0$ or $f \in L^1(\mathbb{R})$, then $\int f(x)dx = |\det T| \int f \circ T(x) dx$".

Let $f(x) = |x|$ and so $f \geq 0$. Let $T = (0)$ be the zero matrix. Then, $\int_{\mathbb{R}} f(x)dx = \infty$. However, $|\det T| \int f \circ T(x)|dx = 0$ because $|\det T| = 0$.

Let $f(x) = \frac{1}{1+x^2}$ and so, $f \in L^1(\mathbb{R})$. Let $T = (0)$ be the zero matrix. Then, $|\det T| \int |f \circ T(x) dx = 0$ however, $\int_{\mathbb{R}} f(x)dx \neq 0$.

(b) We show the first statement is true in general: "If $E \in \mathcal{L}^n$, then $T(E) \in \mathcal{L}^n$".

Let T be a linear transformation and E an Lebesgue measurable set. We show $T(E)$ is Lebesgue measurable too. Let $E = F \cup N$ where F is an F_σ-set and N is a null-measure set. Then, $T(E)$ is F_σ as well because linear transformations are continuous. So, it suffices to show that $T(N)$ has measure zero. Linear transformations are Lipschitz and so,

$$|T(u) - T(v)| \leq c|u - v|$$

for $u, v \in \mathbb{R}^n$ and c depends only on T. So, $m(T(I)) \leq cm(I)$ for any I a product of intervals. Let $\epsilon > 0$. Then, cover $N \subseteq \cup_{k=1}^{\infty} I_k$ by I_k's which are products of intervals. We may assume $m(\cup_{k=1}^{\infty} I_k) < \epsilon$ is of small measure because N is measure zero. Obviously, the $T(I_k)$'s cover $T(N)$ because N is covered by the I_k's. By properties of measures,

$$m(T(N)) \leq \sum_{k=1}^{\infty} m(T(I_k)) \leq \sum_{k=1}^{\infty} cm(I_k) < c\epsilon. \tag{75}$$

Letting $\epsilon \to 0$, we conclude that $m(T(N)) = 0$ as desired.

We show the second statement is true in general: "if E is Lebesgue measurable, then $m(T(E)) = |\det T| m(E)$".

Let T be a noninvertible matrix i.e. $|\det(T) = 0|$. Then, $|\det T| m(E) = 0$ so we need to show $m(T(E)) = 0$. Because T is noninvertible, we have $T(E) \subset T(\mathbb{R}^n) \subset S$ where S is some vector subspace of $\mathbb{R}^n$ with dimension less than $\mathbb{R}^n$. However, $m(S) = 0$ because S is isomorphic to $\mathbb{R}^m$ where $m < n$ and so, $m(S) = m(\mathbb{R}^m) = m(\mathbb{R}^m \times \{0\} \cdots \times \{0\}) = 0$ where $\mathbb{R}^m \times \{0\} \ldots \{0\} \subset \mathbb{R}^n$. By completeness, we deduce that $T(E)$ is a subset of a null set and hence, $T(E)$ is measurable. Therefore $m(T(E)) = 0$ and so, $m(T(E)) = |\det T| m(E)$ as desired.

□

Folland 2.56 If f is Lebesgue integrable on $(0, a)$ and $g(x) = \int_x^a t^{-1} f(t) dt$, then g is integrable on $(0, a)$ and $\int_0^a g(x) dx = \int_0^a f(x) dx$.

Proof. Apply the Fubini part of the Fubini-Tonelli Theorem and use $f \in L^1((0, a))$ to get

$$\int_0^a |g(x)| dx \leq \int_0^a \int_x^a t^{-1} |f(t)| dt dx = \int_0^a \int_0^t t^{-1} |f(t)| dx dt = \int_0^a |f(t)| dt = \int_0^a |f(x)| dx < \infty \tag{76}$$

which shows that $g(x)$ is Lebesgue integrable. Note that to get the second equality, we changed the order of integration using Fubini's Theorem with $t^{-1}|f(t)| \in L^+([0, \infty])$.

Now, to prove the equality we just use the Tonelli part of the Fubini-Tonelli Theorem. Indeed, by a similar changing of the order of integration.,

$$\int_0^a g(x) dx = \int_0^a \int_x^a t^{-1} f(t) dt dx = \int_0^a \int_0^t t^{-1} f(t) dx dt = \int_0^a f(t) dt = \int_0^a f(x) dx \tag{77}$$

□

Folland 2.58 Show that $\int_0^\infty e^{-sx} x^{-1} \sin^2 x dx = \frac{1}{4} \log(1 + 4s^{-2})$ for $s > 0$ by integrating $e^{-sx} \sin 2xy$ with respect to x and y.

PROOF. Consider the double integral

$$\int_0^\infty \int_0^1 e^{-sx} \sin(2xy) dy dx.$$

We will justify changing the order of integration. Indeed,

$$\begin{aligned}
\int_0^\infty \int_0^1 |e^{-sx} \sin(2xy)| dy dx &\le \int_0^\infty \int_0^1 e^{-sx}(2xy) dy dx && (\text{use } |\sin(x)| \le x) \\
&= \int_0^\infty x e^{-sx} dx && (\text{Riemann Integration}) \\
&= \frac{1}{s^2} < \infty && (\text{Riemann Integration})
\end{aligned}$$

for $s > 0$. Thus the hypothesis of the Fubini-Tonelli Theorem are fulfilled and we can change the order of integration. Now,

$$\int_0^\infty \int_0^1 e^{-sx} \sin(2xy) dy dx = \int_0^\infty \frac{e^{-sx}}{2x}(1 - \cos(2x)) dx \qquad (\text{Integrate}) \tag{78}$$

$$= \int_0^\infty \frac{e^{-sx}}{2x}(2\sin^2 x) dx \qquad (\text{Trig. Identity}) \tag{79}$$

$$= \int_0^\infty \frac{e^{-sx} \sin^2 x}{x} dx. \qquad (\text{Simplify}) \tag{80}$$

We now evaluate $\int_0^1 \int_0^\infty e^{-sx} \sin(2xy) dx dy$. Let us set $I := \int_0^\infty e^{-sx} \sin(2xy) dx dy$. We will use integration by parts twice because the integrand is Riemann integrable by the Lebesgue Criterion for Riemann integration. Indeed, $e^{-sx} \sin(2xy)$ is nowhere discontinuous.

We have

$$\int_0^\infty e^{-sx} \sin(2xy) dx = \left(-\frac{e^{-sx}}{s} \sin(2xy) \right)\Big|_0^\infty - \int_0^\infty -\frac{e^{-sx}}{s} 2y \cos(2xy) dx \tag{81}$$

$$= \int_0^\infty \frac{e^{-sx}}{s} 2y \cos(2xy) dx \tag{82}$$

$$= \left(-\frac{e^{-sx}}{s^2} 2y \cos(2xy) \right)\Big|_0^\infty - \int_0^\infty + \frac{e^{-sx}}{s^2} 4y^2 \sin(2xy) dx \tag{83}$$

$$= \frac{2y}{s^2} - \frac{4y^2}{s^2} \int_0^\infty e^{-sx} \sin(2xy) dx. \tag{84}$$

The first equality follows by integration by parts, the second by evaluating the term and simplifying, the third by a second integration by parts, and the last equality follows by simplifying the terms.

Recall the definition of I and notice that the above shows

$$I = \frac{2y}{s^2} - \frac{4y^2}{s^2} I \tag{85}$$

and therefore,

$$I = \frac{\frac{2y}{s^2}}{\left(1 + \frac{4y^2}{s^2}\right)}. \tag{86}$$

Thus, $I = \frac{2y}{s^2+4y^2}$ after rewriting the above. Now, we determine $\int_0^1 I dy$,

$$\int_0^1 I dy = \int_0^1 \frac{2y}{s^2+4y^2} dy = \frac{1}{4}\log(s^2+4y)\Big|_0^1 = \frac{1}{4}(s^2+4) - \frac{1}{4}\log(s^2) = \frac{1}{4}\log(1+4s^{-2}). \tag{87}$$

Equating the two integrals in (78) and (87) by the Fubini-Tonelli Theorem, we conclude that

$$\int_0^\infty \frac{e^{-sx}\sin^2 x}{x} dx = \frac{1}{4}\log(1+4s^{-2}).$$

□

Folland 2.59 Let $f(x) = x^{-1}\sin x$

a. Show that $\int_0^\infty |f(x)|dx = \infty$.

b. Show that $\lim_{b\to\infty}\int_0^b f(x)dx = \frac{1}{2}\pi$ by integrating $e^{-xy}\sin x$ with respect to x and y. (In view of part (a), some care is needed in passing to the limit as $b \to \infty$.)

PROOF. (a) To show $\int_0^\infty |f(x)|dx = \infty$, we estimate the integral from below and show that $\int_0^\infty |f(x)|dx \geq \infty$ using the estimate.

Let $F := \left\{x \in [0,\infty] : |\sin(x)| \geq \frac{1}{2}\right\}$. By definition of $\sin(x)$, we know that $F = \bigcup_{n=1}^\infty \left[\frac{\pi}{6}+\pi n, \frac{5}{6}\pi + \pi n\right]$. Also, by definition of $\sin(x)$, we know $F \neq [0,\infty]$. So, we can bound the integral from below $\int_0^\infty f(x)dx \geq \int_F f(x)dx \geq \int_F \frac{1}{2x}dx$.

It would unwise to compute $\int_F \frac{1}{2x}dx$ because the integral $\int_{\left[\frac{\pi}{6}+\pi n, \frac{5}{6}\pi+\pi n\right]} \frac{1}{2x}dx$ does not yield a result for which the series obviously converges. So, we do one more estimate.

On $\left[\frac{\pi}{6}+\pi n, \frac{5}{6}\pi+\pi n\right]$, we get $\frac{1}{x} \geq \frac{1}{\frac{5}{6}\pi+\pi n} = \frac{6}{\pi(5+6n)} \geq \frac{1}{\pi(n+1)}$. Therefore, we get

$$\int_{\left[\frac{\pi}{6}+\pi n, \frac{5}{6}\pi+\pi n\right]} \frac{1}{2x}dx \geq \int_{\left[\frac{\pi}{6}+\pi n, \frac{5}{6}\pi+\pi n\right]} \frac{1}{2}\frac{1}{\pi(n+1)}dx \tag{88}$$

$$= \frac{5+6n}{12\,(n+1)} - \frac{1+6n}{12\,(n+1)} \tag{89}$$

$$= \frac{1}{3\,(n+1)}. \tag{90}$$

Therefore,

$$\int_F \frac{1}{2x}dx \geq= \sum_{n=0}^\infty \int_{\left[\frac{\pi}{6}+\pi n, \frac{5}{6}\pi+\pi n\right]} \frac{1}{2x}dx \tag{91}$$

$$\geq \sum_{n=0}^\infty \frac{1}{3(n+1)} \qquad \text{(From Above)} \tag{92}$$

$$= \frac{1}{3}\sum_{n=0}^\infty \frac{1}{n+1} \tag{93}$$

and the series on the RHS diverges because that is just the harmonic series. Therefore,

$$\int_0^\infty |f(x)|dx = \infty. \tag{94}$$

(b) Let $g(x,y) = e^{-xy}\sin x$. Consider how

$$\int_0^b \int_0^\infty |g| dy dx = \int_0^b x^{-1}|\sin x| dx \tag{95}$$

because

$$\int_0^\infty |g| dy = -e^{-xy}\frac{1}{x}|\sin(x)|\Big|_0^\infty = 0 - \left(-\frac{|\sin(x)|}{x}\right) = \frac{|\sin(x)|}{x}. \tag{96}$$

We show that the RHS of (95) is finite. We have

$$\int_0^b x^{-1}|\sin x| dx \leq \int_0^1 x^{-1}|\sin x| dx + \int_1^b x^{-1}|\sin x| dx \leq 1 + \ln|b| < \infty$$

by using $|\sin x| \leq 1$ to bound the second integral above by $\ln|b|$ and the fact that $x^{-1}|\sin x| \leq 1$ on $(0,1)$ to bound the first integral.

So, $f \in L^1$ on $[a,b] \times [0,\infty)$. Applying Fubini's Theorem to change the order of integration, we have

$$\int_0^b \int_0^\infty g dy dx = \int_0^\infty \int_0^b g dx dy. \tag{97}$$

The antiderivative is, by calculus, $\int e^{-xy}\sin x dx = -e^{-xy}\frac{y\sin x+\cos x}{y^2+1}$. So,

$$\int_0^\infty \int_0^b g dx dy. = \int_0^\infty \frac{1}{y^2+1} dy - \int_0^\infty \frac{e^{-by} y\sin b + \cos b}{y^2+1} dy.$$

It suffices to show that $\int_0^\infty \frac{e^{-by} y \sin b + \cos b}{y^2+1} dy \to 0$ as $b \to \infty$ since $\int_0^\infty \frac{1}{y^2+1} = \frac{\pi}{2}$.

We have

$$\int_0^\infty \left|\frac{e^{-by}(y\sin b + \cos b)}{y^2+1}\right| dy \leq \int_0^\infty e^{-by}\left|\frac{y+1}{y^2+1}\right| dy \leq \int_0^\infty e^{-by} dy = \frac{1}{b}.$$

Letting $b \to \infty$, the RHS goes to zero.

We conclude that $\lim_{b\to\infty}\int_0^\infty \int_0^b g dx dy = \frac{\pi}{2}$.

This with (97) completes the proof.

□

3. Signed Measures and Differentiation

Folland Exercise 3.1 Prove Proposition 3.1.

PROOF. Repeat the proof of Theorem 1.8 with appropriate changes. The only nontrivial checks one has to make is that if one side attains the value of $\pm\infty$, then the other side must always attain $\pm\infty$ as well. □

Folland Exercise 3.2 If ν is a signed measure, E is ν-null iff $|\nu|(E) = 0$. Also, if ν and μ are signed measures, $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

PROOF. If E is ν-null, then $\nu^+(E) = \nu^-(E) = 0$ which means $|\nu|(E) = 0$. The other direction is obvious since ν^+ and ν^- are positive measures.

We have $\nu \perp \mu$ iff there exist disjoint measurable E, F s.t. $E \cup F = X$, $\nu(F) = 0$ and $\mu(E) = 0$ iff with the same E, F we have $|\nu|(E) = 0$ and $\mu(E) = 0$ uff $|\nu| \perp \mu$ which occurs iff $\nu^+(E) = \nu^-(E) = 0$ so this is iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. $\square$

Folland Exercise 3.3 Let ν be a signed measure on $(X, \mathcal{M})$.

a. $L^1(\nu) = L^1(|\nu|)$.

b. If $f \in L^1(\nu)$, $\left|\int f d\nu\right| \leq \int |f| d|\nu|$.

c. If $E \in \mathcal{M}$, $|\nu|(E) = \sup\left\{\left|\int_E f d\nu\right| : |f| \leq 1\right\}$.

PROOF. **(a):** Since $\nu \leq |\nu|$, $L^1(|\nu|) \subseteq L^1(\nu)$ is obvious. If $f \in L^1(\nu)$, we know that $f \in L^1(\nu^+) \cap L^1(\nu^-)$ which means $\int |f| d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^- < \infty$ so $f \in L^1(|\nu|)$.

(b): Suppose $f \in L^1(\nu)$. We have,

$$\left|\int f d\nu\right| = \left|\int f d\nu^+ - \int f d\nu^-\right| \qquad (f \in L^1(\nu)) \tag{98}$$

$$\leq \left|\int f d\nu^+\right| + \left|\int f d\nu^-\right| \qquad \text{(Triangle Inequality)} \tag{99}$$

$$\leq \int |f| d\nu^+ + \int |f| d\nu^- \qquad \text{(Proposition 2.22)} \tag{100}$$

$$= \int |f| d|\nu|. \qquad \text{(Total Variation)} \tag{101}$$

(c): Let $E \in \mathcal{M}$ and $|f| \leq 1$. It is clear that if $f \notin L^1(\nu)$, then we have $|\nu|(E) = \infty$ implies that $|\nu|(E) = \int_E \frac{d\nu}{d|\nu|} d\nu = \infty$ which means the the supremum on the right is infinite as well. We have, for $f \in L^1(\nu)$,

$$\left|\int_E f d\nu\right| \leq \left|\int_E f d\nu^+\right| + \left|\int_E f d\nu^-\right| \tag{102}$$

$$\leq \int_E |f| d|\nu| \leq \int_E 1 d|\nu| = |\nu|(E). \tag{103}$$

Therefore,

$$\sup\left\{\left|\int_E f d\nu\right| : |f| \leq 1\right\} \leq |\nu|(E). \tag{104}$$

We now prove the other direction. Choose a Hahn decomposition P, N for ν. Then,

$$|\nu|(E) = |\nu^+(E) + \nu^-(E)| = |\nu(E \cap P) + \nu(E \cap N)| \tag{105}$$

$$= \left|\int_E \chi_P d\nu + \int_E \chi_N d\nu\right| \tag{106}$$

$$= \left|\int_E \chi_P + \chi_N d\nu\right|. \tag{107}$$

Notice that since P, N are disjoint, $|\chi_P + \chi_N| \leq 1$. But that means

$$|\nu|(E) \leq \left|\int_E \chi_P + \chi_N d\nu\right| \in \left\{\left|\int_E f d\nu\right| : |f| \leq 1\right\} \tag{108}$$

and therefore,

$$\nu(E) \leq \sup\left\{\left|\int_E f d\nu\right| : |f| \leq 1\right\} \tag{109}$$

as desired.

□

Folland Exercise 3.4 If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$

Proof. Suppose not and that $\nu^+ > \lambda$ or $\nu^- > \mu$. Suppose $\nu^+ > \lambda$ occurs. Then

$$\lambda - \mu = \nu^+ - \nu^- > \lambda - \nu^- \qquad \Longrightarrow \qquad \nu^- > \mu.$$

A similar argument shows that if $\nu^- > \mu$ occurs, then $\nu^+ > \lambda$ must also occur. But then,

$$\nu^+ - \mu < \lambda - \mu = \nu^+ - \nu^- < \nu^+ - \mu$$

which is a contradiction. □

Folland 3.5 If ν_1, ν_2 are signed measures that both omit the value $+\infty$ or $-\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Use Exercise 4.)

Proof. By definition,

$$\begin{aligned} |\nu_1 + \nu_2| &= (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \\ |\nu_1| &= \nu_1^+ + \nu_1^- \\ |\nu_2| &= \nu_2^+ + \nu_2^-. \end{aligned}$$

Let $\lambda = \nu_1^+ + \nu_2^+$ and $\mu = \nu_1^- + \nu_2^-$. Then $\nu_1 + \nu_2 = \lambda - \mu$. By Exercise 3.4, $\nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$ and $\nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$. Thus,

$$|\nu_1 + \nu_2| = (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \tag{110}$$

$$\leq \nu_1^+ + \nu_2^+ + \nu_1^- + \nu_2^- \tag{111}$$

$$\leq |\nu_1| + |\nu_2|. \tag{112}$$

We note that we used the hypothesis of omitting both $+\infty$ and $-\infty$ to write the above inequalities. For if not, then we would □

Folland 3.6 Suppose $\nu(E) = \int_E f d\mu$ where μ is a positive measure and f is an extended μ -integrable function. Describe the Hahn decompositions of ν and the positive, negative, and total variations of ν in terms of f and μ.

Proof. Suppose $\nu(E) = \int_E f d\mu$ for μ a positive measure and f an extended μ-integrable function.

A possible Hahn Decomposition is given by

$$P = \{x \in E : f(x) \geq 0\} \qquad \text{and} \qquad N = \{x \in E : f(x) < 0\}.$$

Thus, $\int_P f(x) d\mu \geq 0$ and $\int_N f(x) d\mu < 0$. Also, by the remarks on p. 86 of Folland, we know P and N are positive and negative sets respectively. Also, $P \cup N = X$ and $P \cap N = \emptyset$. The other possible Hahn Decompositions are P', N' for which $P \Delta P'$ is μ-null and $N \Delta N'$ is μ-null.

Next,

$$\nu(E)=\int_E f d\mu=\int_{E\cap P} f(x)d\mu-\int_{E\cap N} f(x)d\mu \tag{113}$$

where only one of the integrals is infinite.

The positive variation is given by

$$\nu^+(E)=\nu(E\cap P)=\int_{E\cap P} f d\mu=\int_{E\cap P}|f|d\mu. \tag{114}$$

and the negative variation is given by

$$\nu^-(E)=-\nu(E\cap N)=-\int_{E\cap N} f d\mu=\int_{E\cap N}|f|d\mu. \tag{115}$$

The total variation is then

$$|\nu|(E)=\nu^+(E)+\nu^-(E)=\int_{E\cap P}|f|d\mu+\int_{E\cap N}|f|d\mu=\int_E|f|d\mu. \tag{116}$$

Notice that in describing the positive, negative, and total variation, we could have used another Hahn Decomposition P', N' and the values would have been unchanged. Indeed,

$$\int_{E\cap P} f d\mu=\int_{E\cap(P\cap P')} f d\mu+\int_{E\cap P\setminus P'} f d\mu \qquad (P\setminus P' \text{ is } \mu-\text{null}) \tag{117}$$

$$=\int_{E\cap(P\cap P')} f d\mu+\int_{E\cap(P'\setminus P)} f d\mu \qquad (P'\setminus P \text{ is } \mu-\text{null}) \tag{118}$$

$$=\int_{E\cap P'} f d\mu \tag{119}$$

and by similar reasoning, $\int_{E\cap N} f d\mu=\int_{E\cap N'} f d\nu$. Because ν^+ and ν^- does not depend on the choice of Hahn Decomposition, $|\nu|$ also does not depend on choice of Hahn Decomposition.

□

Folland 3.7 Suppose that ν is a signed measure on $(X,\mathcal{M})$ and $E\in\mathcal{M}$.

a. $\nu^+(E)=\sup\{\nu(F): E\in\mathcal{M}, F\subset E\}$ and $\nu^-(E)=-\inf\{\nu(F): F\in\mathcal{M}, F\subset E\}$.

b. $|\nu|(E)=\sup\left\{\sum_1^n|\nu(E_j)|: n\in\mathbb{N}, E_1,\ldots,E_n \text{ are disjoint, and } \bigcup_1^n E_j=E\right\}$.

Proof. **a.** We do the proof of the first statement and the second is similar. Let P be the positive set in the Hahn-Jordan Decomposition of ν. Then, $\nu^+(E)=\nu(P\cap E)$ and since $P\cap E\subseteq E$, we know ν^+ is at most the supremum. For the other direction, let $\epsilon>0$ and there exists an $F\subseteq E$ s.t.

$$\sup\{\nu(F): E\in\mathcal{M}, F\subset E\}-\epsilon\le\nu(F)$$

and because $\nu(F)=\nu^+(F)-\nu^-(F)\le\nu^+(F)\le\nu^+(E)$, letting $\epsilon\to 0$, we have the other inequality.

b. For one direction, if P, N is as in Theorem 3.3 and Theorem 3.4, then $|\nu(E)|=\nu(E\cap P)+\nu(E\cap N)=|\nu(E\cap P)|+|\nu(E\cap N)|$ and the RHS is a candidate for the supremum so $\nu(E)$ is bounded above by the supremum.

Let $\epsilon > 0$ and $C := \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \dots, E_n$ are disjoint, and $\bigcup_1^n E_j = E\}$. Then there exist $E_1, \dots, E_n$ disjoint s.t. $\bigcup_1^n E_j = E$ and

$$C-\epsilon \leq \sum_1^n |\nu(E_j)| \leq \sum_1^n |\nu^+(E_j)|+|\nu^-(E_j)| = \sum_1^n \nu^+(E_j)+\nu^-(E_j) = \sum_1^n \nu(E_j) = \nu\left(\bigcup_1^n E_j\right) \leq \nu(E).$$

Since $\epsilon > 0$ was arbitrary, we get $C \leq \nu(E)$ and this establishes equality. □

Folland 3.8 $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Proof. If $\nu \ll \mu$, then $|\nu| \ll \mu$ is clear since $\nu \ll |\nu|$.
If $|\nu| \ll \mu$, then because $|\nu|(E) = 0$ iff $\nu^+(E) = \nu^-(E) = 0$, we get $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.
If $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, then $\nu^+ - \nu^- \ll \mu$ which implies $\nu \ll \mu$ by Proposition 3.11. □

Folland 3.9 Suppose $\{\nu_j\}$ is a sequence of positive measures. If $\nu_j \perp \mu$ for all j, then $\sum_1^\infty \nu_j \perp \mu$; and if $\nu_j \ll \mu$ for all j, then $\sum_1^\infty \nu_j \ll \mu$.

Proof. Let $\delta := \sum_{j=1}^\infty \nu_j$. Assume $\nu_j \perp \mu$ for each j. Then, there exist sets E_j, F_j for which $E_j \cup F_j = X$ and $\mu(E_j) = \nu(F_j) = 0$. Let $E = \bigcup_{j=1}^\infty E_j$ and $F = \bigcap_{j=1}^\infty F_j$. Therefore, $\delta(F) = 0$ because

$$\delta(F) = \sum_{j=1}^\infty \nu_j(F) \leq \sum_{j=1}^\infty \nu_j(F_j) = \sum_{j=1}^\infty 0 = 0 \tag{120}$$

where we use monotonicity to get $\nu_j(F) \leq \nu_j(F_j)$ for the inequality. Also,

$$\mu(E) = \mu\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty \mu(E_j) = \sum_{j=1}^\infty 0 = 0. \tag{121}$$

Now suppose $\nu_j \ll \mu$ for all j. Suppose $\mu(E) = 0$. Then, $\nu_j(E) = 0$ for all j. Therefore, $\delta(E) = 0$. By definition, $\sum_{j=1}^\infty \nu_j \ll \mu$. □

Folland Exercise 3.10 Theorem 3.5 may fail when ν is not finite. (Consider $d\nu(x) = dx/x$ and $d\mu(x) = dx$ on $(0,1)$, or ν = counting measure and $\mu(E) = \sum_{n\in E} 2^{-n}$ on $\mathbb{N}$.)

Proof. Let ν be the counting measure and μ the sum as stated. Then $\mu \ll \nu$ is clear. Take $\epsilon = \frac{1}{2}$ and let $\delta > 0$. Let n be large enough s.t. $\frac{1}{2^n} < \delta$ and set $E := \{n\}$. Then $\mu(E) < \delta$, but we have $|\nu(E)| = 1 \geq \epsilon$. □

Folland Exercise 3.11 Let μ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha\in A} \subset L^1(\mu)$ is called uniformly integrable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\left|\int_E f_\alpha d\mu\right| < \epsilon$ for all $\alpha \in A$ whenever $\mu(E) < \delta$.

a. Any finite subset of $L^1(\mu)$ is uniformly integrable.

b. If $\{f_n\}$ is a sequence in $L^1(\mu)$ that converges in the L^1 metric to $f \in L^1(\mu)$, then $\{f_n\}$ is uniformly integrable.

Proof. **a.** If we have a finite subset, then we just choose the minimum of all the δ_i we obtain from Corollary 3.6.

b. Choose N large enough s.t. $n \geq N$ implies $\int |f_n - f| d\mu < \frac{epsilon}{2}$. Then choose δ_i for each of $f, f_1, f_2, \dots, f_{N-1}$. Then, we have $|\int f_n d\mu| \leq |\int f d\mu| + \int |f_n - f| d\mu < \epsilon$ if we choose δ_0 for f so that $|\int f d\mu| < \frac{\epsilon}{2}$ when integrating over a set of $< \delta_0$ measure. Then take $\delta := \min\{\delta_0, \delta_1, \dots, \delta_{N-1}\}$. □

Folland Exercise 3.12 For $j = 1, 2$, let μ_j, ν_j be σ -finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$ Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2).$$

Proof. For $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$, notice that

$$\mu_1 \times \mu_2(E) = \int \mu_1(E^{x_2}) d\mu_2 = 0$$

by the Fubini-Tonelli Theorem. Then, $\mu_1(E^{x_2}) = 0$ for μ_2-a.e.. Because $\nu_1 \ll \mu_1$, we deduce $\nu_1(E^{x_2}) = 0$ for ν_2-a.e.. So,

$$\nu_1 \times \nu_2(E) = \int \nu_1(E^{x_2}) d\nu_2 = 0.$$

For the next part, the trick is to use the Fubini-Tonelli theorem to relate the product measure to the measures separately.

Consider $\frac{d\nu_1}{d\mu_1}$ and $\frac{d\nu_2}{d\mu_2}$ and they are unique by the Radon-Nikodym Theorem. From the definition of the product measure and Proposition 1.3, it suffices to reduce to the case where $E = E_1 \times E_2$ where $E_1 \in \mathcal{M}_1$ and $E_2 \in \mathcal{M}_2$,

$$(\nu_1 \times \nu_2)(E) = \nu_1(E_1)\nu_2(E_2) = \int_{E_1} \frac{d\nu_1}{d\mu_1} d\mu_1 \int_{E_2} \frac{d\nu_2}{d\mu_2} d\mu_2.$$

Rewrite the RHS, as

$$\int_{E_1} \int_{E_2} \frac{d\nu_1}{d\mu_1} \frac{d\nu_2}{d\mu_2} d\mu_1 d\mu_2$$

since the two integrals do not depend on each other.

Applying the Fubini-Tonelli theorem shows that the above is equal to

$$\int_E \frac{d\nu_1}{d\mu_1} \frac{d\nu_2}{d\mu_2} d(\mu_1 \times \mu_2).$$

In which case, the uniqueness of the Radon-Nikodym Theorem gives the equality

$$\frac{d\nu_1}{d\mu_1} \frac{d\nu_2}{d\mu_2} = \frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)} \tag{122}$$

□

Folland 3.13 Let $X = [0, 1], \mathcal{M} = \mathcal{B}_{[0,1]}, m =$ Lebesgue measure, and $\mu =$ counting measure on $\mathcal{M}$.

a. $m \ll \mu$ but $dm \neq f d\mu$ for any f.

b. μ has no Lebesgue decomposition with respect to m.

Proof.

(a) Suppose $\mu(E) = 0$. Since μ is the counting measure, this shows $E = \emptyset$. But then $m(E) = 0$ and therefore, $m \ll \mu$.

Suppose the Radon-Nikodym derivative exists. Now, $m(E) = \int_E f d\mu = \sum_{x \in E} f(x)$ since μ is the counting measure. Now,

$$0 = m(\{x\}) = \sum_{x \in \{x\}} f(x) = f(x). \tag{123}$$

for any $x \in X$. Therefore, f must be the constant zero function. However,

$$1 = m(X) = \sum_{x \in X} f(x) = 0. \tag{124}$$

is a contradiction. So, there does not exist a function f for which $dm = f d\mu$.

(b) We claim μ has no Lebesgue Decomposition w.r.t m. Suppose it did and we have λ and ρ s.t. $\mu = \lambda + \rho$, $\lambda \perp m$, and $\rho \ll m$.

Then, for all $\{x\} \subseteq [0,1]$, we have $m(\{x\}) = 0$ and since $\rho \ll m$, $\rho(\{x\}) = 0$. Therefore,

$$1 = \mu(\{x\}) = \lambda(\{x\}) + \rho(\{x\}) = \lambda(\{x\}) \tag{125}$$

.

Because $\lambda(E) = 1$ for any singleton set E, we deduce that $\lambda(F) \neq 0$ for any nonempty $F \subseteq [0,1[]$. But since $\lambda \perp m$, and $\lambda(F) \neq 0$ for all $F \subseteq [0,1]$, we are forced to have $m(F) = 0$ for all $F \subseteq [0,1]$. But this contradicts the definition of m (for example, $m([0,0.5]) = 0.5 \neq 0$).

□

Folland 3.14 If ν is an arbitrary signed measure and μ is a σ -finite measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$, there exists an extended μ -integrable function $f : X \to [-\infty, \infty]$ such that $d\nu = f d\mu$.

Hints:

a. It suffices to assume that μ is finite and ν is positive.

b. With these assumptions, there exists $E \in \mathcal{M}$ that is σ -finite for ν such that $\mu(E) \geq \mu(F)$ for all sets F that are σ -finite for ν.

c. The Radon-Nikodym theorem applies on E. If $F \cap E = \varnothing$, then either $\nu(F) = \mu(F) = 0$ or $\mu(F) > 0$ and $|\nu(F)| = \infty$.

Proof. We follow the hints and break the proof into parts.

(a) We justify assuming μ is finite and ν is positive.

Suppose the result held true when ν is positive. Then, if ν is a general signed measure, we write $\nu = \nu^+ - \nu^-$ and choose extended μ-integrable function s.t. $d\nu^+ = f d\mu$ and $d\nu^- = g d\mu$. Therefore,

$$d\nu = d(\nu^+ - \nu^-) = d\nu^+ - d\nu^- = f d\mu - g\mu = (f - g) d\mu. \tag{126}$$

Now because ν assumes at most one of $\pm\infty$, we deduce that only one of $\int_X f d\mu$ and $\int_X g d\mu$ assumes value ∞. Therefore, $f - g$ is extended μ-integrable and it is the function we wanted.

Now, we justify assuming μ is finite. Suppose the result held true when μ is finite. We show the result for when μ is σ-finite. Let $X = \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) < \infty$. WLOG, we may assume that X is a disjoint union of E_n's by possibly replacing it with $\cup_{n=1}^{\infty} F_n$ where $F_n := E_n \setminus \bigcup_{k=1}^{n-1} E_k$. Now, apply the result to find f_n s.t $d\nu = f_n d\mu$ on E_n. That is, $\nu(G) = \int_G f_n d\mu$ for every measurable $G \subseteq E_n$. Define $f := \sum_{n=1}^{\infty} f_n$. To define f_n on X, set $f_n(x) = 0$ for all $x \in X \setminus E_n$. Then, if $E \subseteq X$, we get

$$\nu(E) = \nu\left(\bigcup_{n=1}^{\infty}(E \cap E_n)\right) \tag{127}$$

$$= \sum_{n=1}^{\infty} \nu(E \cap E_n) \tag{128}$$

$$= \sum_{n=1}^{\infty} \int_{E \cap E_n} f_n d\mu \tag{129}$$

$$= \int_E \sum_{n=1}^{\infty} f_n d\mu = \int_E f d\mu \tag{130}$$

as desired. Now we check that f is extended μ-integrable. On E_n,

$$d\nu = f_n^+ d\mu - f_n^- d\mu = (f_n^+ - f_n^-)d\mu \tag{131}$$

where $f_n^+ - f_n^- = f_n$. Because of the way we constructed f, we have $f^+ = \sum_{n=1}^{\infty} f_n^+$, $f^- = \sum_{n=1}^{\infty} f_n^-$, and $f = f^+ - f^-$. Furthermore, $d\nu = f^+ d\mu - f^- d\mu$. Because ν is a signed measure, only one of $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ assumes the value ∞. Therefore, f is extended μ-integrable.

(b) Assume μ finite and ν positive for the rest of this proof. Suppose not and that for all $E \in \mathcal{M}$ σ-finite for ν, there exists an F σ-finite for ν s.t. $\mu(E) < \mu(F)$. Then, we get a chain of sets

$$\mu(\emptyset) < \mu(F_1) < \mu(F_2) < \mu(F_3) < \dots \tag{132}$$

where each F_i is σ-finite for ν. We claim that this chain does not terminate i.e. $\mu(F_i) \to \infty$ as $i \to \infty$. Suppose not and there is a $C > 0$ chose to be the infimum value for which $\mu(F_i) \leq C$ for all i. Then let $F := \bigcup_{n=1}^{\infty} F_n$ and since each F_i is σ-finite for ν, F is also σ-finite for μ. Also, by choice of C and finiteness of μ, $\mu(F) = C$. But by hypothesis, there is some G which is σ-finite for ν s.t. $\mu(F) < \mu(G)$. And so, $C < \mu(G)$ which contradicts the choice of C.

(c) Now we apply the Radon-Nikodym Theorem on E. This gives a ν-extended integrable function $f : E \to \overline{\mathbb{R}}$ s.t. $d\nu = f d\mu$ and any two such functions are equal a.e.. Now we extend to all of X. Suppose $F \subset X$ and $F \cap E = \emptyset$. If $\mu(F) = 0$, absolute continuity of ν w.r.t μ gives $\nu(F) = 0$. So, suppose $\mu(F) > 0$.

Since $\mu(F) > 0$, we claim that $|\nu(F)| < \infty$ cannot occur. Indeed, if it did, then that means $E \cup F$ is σ-finite w.r.t ν and $\mu(E \cup F) = \mu(E) + \mu(F)$. But this contradicts the maximality of E.

Now we extend the domain of ν when we define $d\nu = f d\mu$. If B is measurable, then we get

$$\nu(B) = \nu(B \cap E) + \nu(B \setminus E) = \int_{B \cap E} f d\mu + \nu(B \setminus E) = \int_{B \cap E} f d\mu. \tag{133}$$

by having set $f = \infty$ on $B \setminus E$ and assuming $\nu(B \setminus E) = 0$ for the case where $\mu(B \setminus E) = \nu(B \setminus E) = 0$. In the other case, the equality is immediate. Since B was an arbitrary measurable set, we take $f = \infty$ on $X \setminus E$ and $f = 0$ and by the above, this is our desired function. Note that f is obviously extended μ-integrable by ν positive.

□

Folland 3.16 Suppose that μ, ν are measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$. If $f = d\nu/d\lambda$, then $0 \leq f < 1\mu$ -a.e. and $d\nu/d\mu = f/(1-f)$.

Proof. We note that since $\nu \ll \mu$, we must have $\lambda \ll \mu$ (since λ is the sum of μ and ν so $\mu(E) = 0$ implies $\nu(E) = 0$ and so, $\lambda(E) = 0$). Certainly, we have

$$\frac{d\lambda}{d\mu} = \frac{d\mu}{d\mu} + \frac{d\nu}{d\mu} = 1 + \frac{d\nu}{d\mu} \tag{134}$$

by considering how

$$\int_E \frac{d\lambda}{d\mu} d\mu = \lambda(E) = \mu(E) + \nu(E) = \int_E 1 + \frac{d\nu}{d\mu} d\mu. \tag{135}$$

Therefore, because $\frac{d\nu}{d\lambda} = f$ and the chain rule in the text, we can write

$$\frac{d\lambda}{d\mu} = 1 + \frac{d\nu}{d\mu} = 1 + \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu} = 1 + f\frac{d\lambda}{d\mu} \tag{136}$$

Therefore, $1 = (1-f)\frac{d\lambda}{d\mu}$.

It suffices to show that the sets $E_0 := \{x : f(x) < 0\}$ and $E_1 := \{x : f(x) \geq 0\}$ are μ-null. For E_0, we have

$$0 \leq \nu(E_0) = \int_{E_0} f d\lambda \leq 0. \tag{137}$$

Therefore, $\nu(E_0) = 0$. Furthermore, $\lambda(E_0) = 0$ because $f \neq \lambda$ a.e. in the above. Because $\lambda(E_0) - \nu(E_0) = \mu(E_0)$, we conclude that $\mu(E_0) = 0$ as desired.

Now for the set E_1. On E_1, we have

$$0 \leq \mu(E_1) = \lambda(E_1) - \nu(E_1) = \int_{E_1} 1 - f d\lambda \leq 0 \tag{138}$$

assuming that both λ and ν are finite to write the equality. In the general case, we use the fact that for us to even write $f = d\nu/d\lambda$, we had to assume λ was σ-finite. We could write $E_1 = \bigcup_{i=1}^{\infty} F_i \cap E_1$ where $\bigcup_{i=1}^{\infty} F_i = X$ in which $\lambda(F_i) < \infty$. But then $\mu(E_1) \leq \sum_{i=1}^{\infty} \mu(E_1 \cap F_i)$. Now the above argument showed that $\mu(E_1 \cap F_i) = 0$ for each i and therefore, $\mu(E_1) = 0$. So, we conclude that $f < 1$ μ-a.e..

Let $f = d\nu/d\lambda$. Then, $\int_E f d\lambda = \nu(E)$ for $E \in \mathcal{M}$. Because $\nu \ll \mu$, $\nu \ll \lambda$, and $\mu \ll \lambda$, we have $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}$. By hypothesis, $\frac{d\nu}{d\lambda} = f$ and so, $f = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}$. It suffices to show $\frac{d\mu}{d\lambda} = 1 - f$.

We have

$$\mu(E) = \lambda(E) - \nu(E) \tag{139}$$
$$= \int_E 1 d\lambda - \nu(E) \tag{140}$$
$$= \int_E 1 d\lambda - \int_E f d\lambda \qquad (f = d\nu/d\lambda) \tag{141}$$
$$= \int_E (1-f) d\lambda. \tag{142}$$

Therefore, $\frac{d\mu}{d\lambda}$ as desired and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$. ☐

Folland 3.17 Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\mathcal{N}$ a sub-σ -algebra of $\mathcal{M}$, and $\nu = \mu \mid \mathcal{N}$. If $f \in L^1(\mu)$, there exists $g \in L^1(\nu)$ (thus g is $\mathcal{N}$ -measurable) such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathcal{N}$; if g' is another such function then $g = g'\nu$ -a.e. (In probability theory, g is called the conditional expectation of f on $\mathcal{N}$.)

PROOF. Since μ is finite, the restriction ν is also finite. Define $d\rho = f d\mu$ on $\mathcal{N}$ and notice that $\rho \ll \nu$ because if $\nu(E) = 0$, then $\nu(E) = \mu(E)$ implies that $\int_E f d\mu = 0$ and hence, $\rho(E) = 0$. By Theorem 3.8, there exists a $g \in L^1(\nu)$ s.t. $\rho(E) = \int_E g d\nu$ when $E \in \mathcal{N}$. But by definition of $\rho(E)$, we get $\int_E f d\mu = \int_E g d\nu$. Furthermore, Theorem 3.8 asserts that any g satisfying the condition is unique ν-a.e.. ☐

Folland 3.18 Prove Proposition 3.13c.

PROOF. A function f is in $L^1(\nu)$ iff its real and negative parts are in $L^1(\nu_r)$ and $L^1(\nu_i)$ which is iff $f \in L^1(|\nu|)$ so the first statement is done.

For the second statement, take $\mu = |\nu_r| + |\nu_i|$ and find an $g \in L^1(\mu)$ s.t. $d\nu = g d\mu$. Then, $d|\nu| = |g| d\mu$. If $f \in L^1(\nu)$, then Proposition 3.9 (we refer to the generalization which Folland hints at on p. 93) tells us

$$\left| \int f d\nu \right| = \left| \int f g d\mu \right| \leq \int |fg| d\mu = \int |f||g| d\mu = \int |f| d|\nu|$$

☐

Folland 3.19 If ν, μ are complex measures and λ is a positive measure, then $\nu \perp \mu$ iff $|\nu| \perp |\mu|$, and $\nu \ll \lambda$ iff $|\nu| \ll \lambda$.

PROOF. We see that $\nu \perp \mu$ iff there exists E, F disjoint s.t. $E \cup F = X$ with E,F being μ, ν-null respectively iff the same sets hold but are $|\mu|, |\nu|$-null respectively. The second iff follows from Proposition 3.13a using Exercise 3.2.

For the second statement, $|\nu| \ll \lambda \implies \nu \ll \lambda$ is immediate. On the other hand, $\nu \ll \lambda$ iff $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$ iff $|\nu| \ll \lambda$ by decomposing f into real and imaginary parts where $d\nu = f d\mu$ for μ a positive measure. ☐

Folland 3.20 If ν is a complex measure on $(X, \mathcal{M})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

PROOF. Let $f : X \to \mathbb{C}$ be measurable with $|f| = 1$ on X and $d\nu = f d|\nu|$. Then we have $0 = |\nu|(X) - \nu(X) = \int (1-f) d|\nu|$, so $f = 1$ a.e. because $\operatorname{Re}(1-f) \geq 0$. This implies $d\nu = d|\nu|$ and so $\nu = |\nu|$. ☐

Folland Exercise 3.21 Let ν be a complex measure on $(X,\mathcal{M})$. If $E\in\mathcal{M}$, define

$$\begin{aligned}\mu_1(E)&=\sup\left\{\textstyle\sum_1^n|\nu(E_j)|:n\in\mathbb{N},E_1,\ldots,E_n\text{ disjoint, }E=\bigcup_1^n E_j\right\},\\ \mu_2(E)&=\sup\left\{\textstyle\sum_1^\infty|\nu(E_j)|:E_1,E_2,\ldots\text{ disjoint, }E=\bigcup_1^\infty E_j\right\},\\ \mu_3(E)&=\sup\left\{\left|\textstyle\int_E fd\nu\right|:|f|\le 1\right\}\end{aligned}.$$

Then $\mu_1=\mu_2=\mu_3=|\nu|$. (First show that $\mu_1\le\mu_2\le\mu_3$. To see that $\mu_3=|\nu|$, let $f=d\nu/d|\nu|$ and apply Proposition 3.13. To see that $\mu_3\le\mu_1$, approximate f by simple functions.)

Proof. Note that there is a typo in Folland. The definition of μ_3 should have a ν im place of the μ.

For simplicity, define

$$T_1:=\left\{\sum_1^n|\nu(E_j)|:n\in\mathbb{N},E_1,\ldots,E_n\text{ disjoint, }E=\bigcup_1^n E_j\right\}\tag{143}$$

$$T_2:=\left\{\sum_1^\infty|\nu(E_j)|:E_1,E_2,\ldots\text{ disjoint, }E=\bigcup_1^\infty E_j\right\}\tag{144}$$

$$T_3:=\left\{\left|\int_E fd\nu\right|:|f|\le 1\right\}.\tag{145}$$

By the definition of μ_1,μ_2, and μ_3, we immediately have $\mu_1(E)\le\mu_2(E)\le\mu_3(E)$ for all $E\in\mathcal{M}$ and hence, $\mu_1\le\mu_2\le\mu_3$. In more detail, we show $\mu_1\le\mu_2$ and $\mu_2\le\mu_3$.

For $\mu_1\le\mu_2$, let E be measurable. We get,

$$\sum_1^n|\nu(E_j)|\le\sum_1^\infty|\nu(F_j)|\tag{146}$$

by setting $F_j=E_j$ for $1\le j\le n$ and $F_j=\emptyset$ when $j>n$. Taking the supremum over both sides all possible $\{E_j\}$ disjoint covering E, we get $\mu_1(E)\le\mu_2(E)$.

For $\mu_2\le\mu_3$, let E be measurable and $E=\bigcup_1^\infty E_j$ be a disjoint union. Because $|\nu|\ll\nu$, choose an f s.t. $fd|\nu|=d\nu$. Therefore,

$$\sum_1^\infty|\nu(E_j)|=\left|\sum_1^\infty|\nu(E_j)|\right|\qquad\text{(positive)}\tag{147}$$

$$=\left|\sum_1^\infty\left|\int_{E_j}fd|\nu|\right|\right|\qquad(\nu=fd|\nu|)\tag{148}$$

$$\le\left|\sum_1^\infty\int_{E_j}\frac{d\nu}{d|\nu|}d|\nu|\right|\qquad(f=d\nu/d|\nu|)\tag{149}$$

$$=\left|\sum_1^\infty\int_{E_j}d\nu\right|\tag{150}$$

$$=\left|\int_E d\nu\right|\tag{151}$$

$$\le\mu_3(E)\qquad\text{(Definition of }\mu_3)\tag{152}$$

Now we move on to show $\mu_3 = \mu_1$ which completes the proof since all of inequalities $\mu_1 \leq \mu_2 \leq \mu_3$ become equalities.

First, we show $\mu_3 = |\nu|$. Because $\nu \ll |\nu|$, there exists an f equal to $\frac{d\nu}{d|\nu|}$. By Proposition 3.13, $|f| = 1$. Assuming $f \in L^1(\nu)$ (because the case where $f \notin L^1(\nu)$ gives equality immediately),

$$\left|\int_E f d\nu\right| \leq \int_E |f| d|\nu| \leq \int_E 1 d|\nu| = |\nu|(E).$$

Taking the supremum over all functions g s.t. $|g| \leq 1$ on the LHS, we get $\mu_3(E) \leq |\nu|(E)$ for all $E \in \mathcal{M}$. Therefore, $\mu_3 \leq |\nu|$.

For other inequality, let $f := \overline{\frac{d\nu}{d|\nu|}}$. By Proposition 3.13, $\overline{f} = \frac{d\nu}{d|\nu|}$ has absolute value equal to 1 and $f\overline{f} = |f|^2 = 1$. Therefore, for any E measurable,

$$\begin{aligned}
|\nu|(E) &= \left|\int_E d|\nu|\right| && \text{(using } |\nu| \text{ positive)}\\
&= \left|\int_E \overline{f} f d|\nu|\right| \\
&= \left|\int_E \overline{\frac{d\nu}{d|\nu|}}\frac{d\nu}{d|\nu|} d|\nu|\right| && \text{(definition of } f \text{ \& } \overline{f}\text{)}\\
&= \left|\int_E \overline{\frac{d\nu}{d|\nu|}} d\nu\right| && \text{(Proposition 3.9)}\\
&= \left|\int_E \overline{\frac{d\nu}{d|\nu|}}\right| \\
&\leq \mu_3(E) && \text{(use } \left|\overline{f}\right| \leq 1\text{)}
\end{aligned}$$

Now that $\mu_3 = |\nu|$ is established, we show $\mu_3 \leq \mu_1$ by approximating f by simple functions. Choose simple functions ϕ_n s.t. $\left|\int_X f d\nu\right| - \frac{1}{n} < \left|\int_X \phi_n d\nu\right|$ and $|\phi_n| \leq |f|$. Then,

$$\left|\int_E f d\nu\right| - \frac{1}{n} < \left|\int_E \phi_n d\nu\right| \tag{153}$$

$$= \left|\int_E \sum_{j=1}^N a_j \chi_{E_j} d\nu\right| \tag{154}$$

$$= \left|\sum_{j=1}^N a_j \nu(E_j)\right| \tag{155}$$

$$\leq \sum_{j=1}^N |a_j||\nu(E_j)| \tag{156}$$

$$= \sum_{j=1}^N |\nu(E_j)| \tag{157}$$

for $E \in \mathcal{M}$ and where $E_j \subseteq E$ so that $E = \bigcup_{j=1}^N E_j$. Taking the supremum over all simple functions ϕ_n on the RHS and letting $n \to \infty$ on the RHS, we get $\left|\int_E f d\nu\right| \leq \mu_1(E)$. But by definition of f, the LHS is greater than $\int_E f d|\nu|$ and hence, greater than $|\nu|(E)$. Hence, $|\nu|(E) \leq \mu_1(E)$ which means $\mu_3(E) \leq \mu_1(E)$. Since E was arbitrary, $\mu_3 \leq \mu_1$ as desired.

We now have $\nu_1 \leq \nu_2 \leq \nu_3 = |\nu| \leq \nu_1$ so we have equality throughout. $\square$

Folland Exercise 3.22 If $f \in L^1(\mathbb{R}^n), f \neq 0$, there exist $C, R > 0$ such that $Hf(x) \geq C|x|^{-n}$ for $|x| > R$. Hence $m(\{x : Hf(x) > \alpha\}) \geq C'/\alpha$ when α is small, so the estimate in the maximal theorem is essentially sharp.

Proof. Because $f \neq 0$ a.e. and, there is a nonnull set on which $|f(x)| > 0$ we can choose an $R > 0$ sufficiently large so that

$$\int_{B(R,0)} |f(y)|dy > D > 0 \tag{158}$$

for some D. Now suppose $|x| > R$. Then,

$$\begin{aligned}
Hf(x) &= \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)|dy && \text{(Definition)} && (159)\\
&\geq \frac{1}{m(B(R+|x|,x))} \int_{B(R+|x|,x)} |f(y)|dy && \left(\text{Definition of } \sup_{r>0}\right) && (160)\\
&= \frac{1}{(R+|x|)^n m(B(1,x))} \int_{B(R+|x|,x)} |f(y)|dy && (*) && (161)\\
&\geq \frac{1}{(R+|x|)^n m(B(1,x))} \int_{B(R,0)} |f(y)|dy && (B(R,0) \subseteq B(R+|x|,x)) && (162)\\
&\geq \frac{1}{(2|x|)^n m(B(1,x))} \int_{B(R,0)} |f(y)|dy && (|x| > R) && (163)\\
&> \frac{1}{(2|x|)^n m(B(1,x))} D && \text{(Apply (158) above)} && (164)\\
&\geq \frac{C}{|x|^n}. && \left(C := \frac{D}{2^n m(B(1,x))}\right) && (165)
\end{aligned}$$

We justify the step above labeled (*). At that step, we used the fact that $m(B(S,0)) = S^n m(B(1,0))$ for $S > 0$. This is a corollary of Theorem 2.44b. The ball $B(1,0)$ is certainly measurable in $\mathbb{R}^n$ (with the Borel-measure). To transform $B(1,0)$ to $B(S,0)$, we use the linear transformation defined by SI where I is the identity matrix. Then Theorem 2.44b says that $m(B(S,0)) = \det(SI)m(B(1,0)) = S^n m(B(1,0))$ as desired.

We now show the second statement. To get the bound, we will show that the set $\{x : Hf(x) > \alpha\}$ has a subset whose measure is bounded below by $\frac{C'}{\alpha}$. We will also determine C' (which must not depend on α since it is constant).

First, we determine all x for which $Hf(x) \geq \frac{C}{|x|^n} > \alpha$ and $|x| > R$. Now, $\frac{C}{|x|^n} > \alpha$ is equivalent to requiring $\frac{C}{\alpha} > |x|^n$. Furthermore,

$$R < |x| < \sqrt[n]{\frac{C}{\alpha}}. \tag{166}$$

Therefore, we have

$$K := \left\{ x : R < |x| < \sqrt[n]{\frac{C}{\alpha}} \right\} \subseteq \{x : Hf(x) > \alpha\}. \tag{167}$$

We now determine $m(K)$. We will use invariance properties of the Lebesgue measures for the computation. Notice that $K = B\left(\sqrt[n]{\frac{C}{\alpha}}, 0\right) - B(R, 0)$ is just an anuulus centered at the origin.

Therefore,

$$m(K) = m\left(B\left(\sqrt[n]{\frac{C}{\alpha}}, 0\right) - B(R,0)\right) \qquad \text{(Definition)} \tag{168}$$

$$= m\left(B\left(\sqrt[n]{\frac{C}{\alpha}}, 0\right)\right) - m(B(R,0)) \qquad \text{(Properties of measure)} \tag{169}$$

$$= \frac{C}{\alpha} m(B(1,0)) - R^n m(B(1,0)) \qquad \text{(Invariance of L. Measure)} \tag{170}$$

$$= \frac{(C - \alpha R^n) m(B(1,0))}{\alpha}. \qquad \text{(Simplify)} \tag{171}$$

$$= \frac{C'}{\alpha} \qquad (C' := (C - \alpha R^n) m(B(1,0))\ ) \tag{172}$$

If we do not want C' to depend on α, we could have required α to be small enough i.e. $R^n < \frac{1}{\alpha}$. Then in the above,

$$\frac{(C - \alpha R^n) m(B(1,0))}{\alpha} \geq \frac{(C-1) m(B(1,0))}{\alpha} \tag{173}$$

and we take $C' = (C-1)m(B(1,0))$.

By monotonicity and (167), we conclude

$$\frac{C'}{\alpha} \leq m(\{x : Hf(x) > \alpha\}) \tag{174}$$

as desired.

□

Folland Exercise 3.23 A useful variant of the Hardy-Littlewood maximal function is

$$H^* f(x) = \sup\left\{ \frac{1}{m(B)} \int_B |f(y)| dy : B \text{ is a ball and } x \in B \right\}$$

Show that $Hf \leq H^* f \leq 2^n Hf$.

Proof. The fact that $Hf \leq H^* f$ is obvious because of the fact that the set in the definition of Hf is contained in the set defining Hf^*. That is, any ball centered at x is certainly a ball containing x.

Let $B \ni x$ be a ball with radius r, then $B \subseteq B(2r, x)$ and hence,

$$\frac{1}{m(B)} \int_B |f(y)| dy \leq \frac{m(B(2r,x))}{m(B)} \frac{1}{m(B(2r,x))} \int_B |f(y)| dy \leq \frac{m(B(2r,x))}{m(B)} \frac{1}{m(B(2r,x))} \int_{B(2r,x)} |f(y)| dy$$
$$= 2^n \frac{1}{m(B(2r,x))} \int_{B(2r,x)} |f(y)| dy \leq Hf(x).$$

□

Folland Exercise 3.24 If $f \in L^1_{\text{loc}}$ and f is continuous at x, then x is in the Lebesgue set of f.

Proof. Since f is continuous at x, choose a $\delta > 0$ s.t. $|y-x| < \delta$ implies $|f(y)-f(x)| < \epsilon$. Let $r > 0$ be s.t. $r < \delta$. Therefore,

$$\frac{1}{m(B(r,x))}\int_{B(r,x)} |f(y)-f(x)|dy \leq \frac{1}{m(B(r,x))}\int_{B(r,x)} \epsilon dy = \epsilon$$

and since $\epsilon > 0$ was arbitrary, $\lim_{r\to 0} \frac{1}{m(B(r,x))}\int_{B(r,x)} |f(y)-f(x)|dy = 0$. This means $x \in L_f$. □

Proof. Assume f is continuous at x_0. First off, $x_0 \in L_f$ iff $\lim_{r\to 0} A_r f(x_0) = f(x_0)$ by definition. But since $A_r f(x)$ is jointly continuous in both variables, we know

$$f(x_0) = \lim_{x_n\to x_0} f(x_n) = \lim_{x_n\to x_0}\lim_{r\to 0} A_r f(x_n) = \lim_{r\to 0}\lim_{x_n\to x_0} A_r f(x_n) = \lim_{r\to 0} A_f(x_0)$$

where we take a sequence of points $x_n \to x$ with $x_n \in L_f$ since L_f is a set of full measure in $\mathbb{R}^m$. □

Folland Exercise 3.25 If E is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r\to 0}\frac{m(E\cap B(r,x))}{m(B(r,x))}$$

whenever the limit exists. a. Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

b. Find examples of E and x such that $D_E(x)$ is a given number $\alpha \in (0,1)$, or such that $D_E(x)$ does not exist.

Proof. (a) Let $f := \chi_E$. Then f is certainly in L^1_{loc} and we can write

$$\frac{m(E\cap B(r,x))}{m(B(r,x))} = \frac{1}{m(B(r,x))}\int_{B(r,x)} f(y)dy = A_r f(x). \tag{175}$$

Therefore, $D_E(x) = \lim_{r\to 0} A_r f(x)$. Then we apply Theorem 3.18 to deduce that $\lim_{r\to 0} A_r f(x) = f(x) = \chi_E(x)$ for a.e. $x \in \mathbb{R}^n$. In particular, this means that $D_E(x) = 1$ a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

(b) Consider the case of $n = 2$. We let E be the region defined by

$$E := \{x \in \mathbb{R}^2 | x = re^{i\theta},\ r \geq 0,\ 0 \leq \theta < \alpha(2\pi)\} \tag{176}$$

where we identified $\mathbb{R}^2$ with $\mathbb{C}$ to simplify writing in terms of polar coordinates.

In this case, $m(E\cap B(r,0)) = \alpha m(B(r,0))$ for any $\alpha \in (0,1)$. Therefore, $D_E(0) = \alpha$ because the terms in the limit are always α.

We note that it is not important that $\alpha \in (0,1)$. If $\alpha = 0$, then there is nothing to show. If $\alpha = 1$, then one takes E to be the whole space. Issues only arise if $\alpha \notin [0,1]$.

Now we give an example where $D_E(x)$ does not exist. The idea is to make the the values bounce between certain values by making our E very messy. We make this rigorous. Define

$$E := \bigcup_{n=0}^{\infty} \left[\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}} \right]. \tag{177}$$

To show that the limit does not exist, it suffices to show that there are subsequences r_k and r_j of radii of $B(r_k, 0)$ in which the ratios of the measure are always different. We do this exactly as follows.

Consider the case where r_k is equal to $\frac{1}{2^{2k}}$. In the case of $k = 1$, what we have is that

$$m(E \cap B(r_1, 0)) = m(E) = \sum_{n=0}^{\infty} \frac{1}{2^{2n}} - \frac{1}{2^{2n+1}} = \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{4^n} \tag{178}$$

$$= \frac{1}{2} \frac{1}{1 - \frac{1}{4}} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}. \tag{179}$$

More generally, for r_k, we get

$$m(E \cap B(r_k, 0)) = \sum_{n=k}^{\infty} \frac{1}{2^{2n}} - \frac{1}{2^{2n+1}} = \frac{1}{2} \sum_{n=k}^{\infty} \frac{1}{4^n} = \frac{1}{2(4^k)} \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{6 \cdot 4^{k-1}}. \tag{180}$$

But note that $B(r_k, 0)$ has measure equal to $2r_k = \frac{1}{2^{2k-1}}$. Therefore,

$$\frac{m(E \cap B(r_k, 0)}{m(B(r_k, 0))} = \frac{\frac{1}{6 \cdot 4^{k-1}}}{\frac{1}{2^{2k-1}}} = \frac{2^{2k-1}}{6 \cdot 4^{k-1}} = \frac{2^{2k-1}}{6 \cdot 2^{2k-2}} = \frac{1}{3} \tag{181}$$

Taking the limit as $k \to \infty$, the RHS goes to $\frac{1}{4}$.

Now consider the subsequence given by $r_j = \frac{1}{2^{2j+1}}$. In this case, what we get is that

$$m(E \cap B(r_j, 0)) = \sum_{n=j}^{\infty} \frac{1}{2^n} - \frac{1}{2^{2n+1}} = \sum_{n=j}^{\infty} \frac{1}{2^{2n+1}} \tag{182}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{2(n+j)+1}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^{2(n+j)}} = \frac{1}{2} \cdot \frac{1}{4^j} \cdot \sum_{n=0}^{\infty} \frac{1}{4^n} \tag{183}$$

$$= \frac{1}{2 \cdot 4^j} \frac{1}{1 - \frac{1}{4}} = \frac{1}{2 \cdot 4^j \cdot \frac{3}{4}} = \frac{4}{6 \cdot 4^j} = \frac{1}{6 \cdot 4^{j-1}}. \tag{184}$$

But notice that $B(r_j, 0)$ has measure $2r_j = \frac{2}{2^{2j+1}} = \frac{1}{2^{2j}}$. Therefore,

$$\frac{m(E \cap B(r_j, 0))}{m(B(r_j, 0))} = \frac{\frac{1}{6 \cdot 4^{j-1}}}{\frac{1}{2^{2j}}} = \frac{2}{3}. \tag{185}$$

Now we take a new sequence $\{r_m\}$ in which when m is even, it equals the least r_k that has not shown up in the sequence and when m is odd, it equals the least r_j that has not shown up in the sequence. For $D_E(0)$ to exist in this case, these two subsequences should converge to the same limit, but they do not. So, $D_E(0)$ does not exist.

□

Folland 3.26 If λ and μ are positive, mutually singular Borel measures on $\mathbb{R}^n$ and $\lambda + \mu$ is regular, then so are λ and μ.

PROOF. By mutual singularity, let $L \cup M = X$ s.t. L is μ-null and M is λ-null. Suppose K were a compact set. Then,

$$\infty > (\lambda + \mu)(K) = \lambda(K \cap L) + \mu(K \cap M) = \lambda(K) + \mu(K). \tag{186}$$

Since λ and μ are positive, we must have $\lambda(K), \mu(K) < \infty$ for the above inequality to hold.

To show outer regularity, we WLOG just show λ is outer regular. Outer regularity means

$$\lambda(E) := \inf\{\lambda(U) \mid E \subseteq U \text{ open}\} \tag{187}$$

and this is equivalent to showing that for all $\epsilon > 0$, there exists an open set U containing E s.t. $\lambda(U) \leq \lambda(E) + \epsilon$.

Given $\epsilon > 0$. By outer regularity of $\lambda + \mu$, we there is an open $U \supseteq E$ s.t.

$$(\lambda + \mu)(U) \leq (\lambda + \mu)(E) + \epsilon. \tag{188}$$

But then we have

$$\begin{aligned}
\lambda(U) &= \lambda(U) + \mu(U) - \mu(U) \\
&= \lambda(U \cap L) + \mu(U \cap M) - \mu(U \cap M) \\
&= (\lambda + \mu)(U) - \mu(U \cap M) \\
&\leq (\lambda + \mu)(E) - \mu(U \cap M) + \epsilon \\
&= \lambda(E \cap L) + \mu(E \cap M) - \mu(U \cap M) + \epsilon \\
&= \lambda(E \cap L) + \epsilon \\
&= \lambda(E) + \epsilon.
\end{aligned}$$

Thus, (187) holds and we have that λ is regular. As noted, the same argument applies to μ and so they are both regular. □

Folland 3.28 If $F \in NBV$, let $G(x) = |\mu_F|((-\infty, x])$. Prove that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$ via the following steps.

a. From the definition of $T_F, T_F \leq G$

b. $|\mu_F(E)| \leq \mu_{T_F}(E)$ when E is an interval, and hence when E is a Borel set.

c. $|\mu_F| \leq \mu_{T_F}$, and hence $G \leq T_F$. (Use Exercise 21.)

PROOF. We follow the steps.

(a) We have,

$$T_F(x) = \sup\left\{\sum_1^n |F(x_j) - F(x_{j-1})| : -\infty < x_0 < \cdots < x_n = x\right\} \tag{189}$$

and

$$\begin{aligned}
\sum_1^n |F(x_j) - F(x_{j-1})| &= \sum_1^n |\mu_F((x_{j-1}, x_j]| && (190) \\
&\leq |\mu_F|((-\infty, x]) && (191) \\
&= G(x). && (192)
\end{aligned}$$

Taking the supremum over all possible partitions gives us the result

(b) If E is an interval $(a,b]$, then the result is immediate by Lemma 3.26 (and in class, Lemma 3.26 was shown to imply that $|F(b)-F(a)| \leq T_F(b) - T_F(a)$ for $a < b$).

Next, we generalize to the Borel σ-algebra. The collection of half open intervals forms an algebra by Proposition 1.15. The σ-algebra generated by the half open intervals is the Borel σ-algebra. By the Monotone Class Lemma, the monotone class generated by the half open intervals coincide with the Borel σ-algebra. So, we just show that the result holds true for countable unions of increasing half-open intervals and countable intersections of decreasing half-open intervals. Suppose $(a,b_j]$ were an increasing union of half-open intervals s.t. $b_j \to b$ then $E := \bigcup_{j=1}^{\infty}(a,b_j]$ gives

$$|\mu_F(E)| = \lim_{j\to\infty} |\mu_F((a,b_j]| \leq \lim_{j\to\infty} \mu_{T_F}((a,b_j]) = \mu_{T_F}(E) \tag{193}$$

as desired. Similarly, if $(a_j,b_j]$ is a decreasing sequence of half-open intervals and $E := \bigcap_{j=1}^{\infty}(a_j,b_j]$ then continuity from below (as $(a_1,b_1]$ is finite)

$$|\mu_F(E)| = \lim_{j\to\infty} |\mu_F((a_j,b_j]| \leq \lim_{j\to\infty} \mu_{T_F}((a,b_j]) = \mu_{T_F}(E). \tag{194}$$

Thus, the result is true for the monotone class generated by the half-open intervals. Therefore, it is true for the Borel σ-algebra by the Monotone Class Lemma.

(c) By Exercise 3.21, we deduce that $|\mu_F| = (\mu_F)_1$ as defined in the exercise. Given an Borel measurable E. By the definition of $(\mu_F)_1$ and part (b) we get

$$|\mu_F|(E) = \sup\left\{\sum_1^n |\mu_F(E_j)| : n\in\mathbb{N}, E_1,\dots,E_n \text{ disjoint}, E = \bigcup_1^n E_j\right\} \tag{195}$$

$$\leq \sup\left\{\sum_1^n \mu_{T_F}(E_j) : n\in\mathbb{N}, E_1,\dots,E_n \text{ disjoint}, E = \bigcup_1^n E_j\right\} \tag{196}$$

$$\leq \mu_{T_F}(E). \tag{197}$$

Since E was arbitrary, we are done with the first part.

To see that $G \leq T_F$, we use the preceding parts. We have

$$G(x) = |\mu_F|((-\infty,x]) \overset{(b)}{\leq} \mu_{T_F}((-\infty,x]) = T_F(x) - T(-\infty) \overset{Lemma\ 3.28}{=} T_F(x).$$

$\square$

Folland Exercise 3.29 If $F \in NBV$ is real-valued, then $\mu_F^+ = \mu_P$ and $\mu_F^- = \mu_N$ where P and N are the positive and negative variations of F. (Use Exercise 28.)

Proof. It suffices to show $\mu_F^+ = \mu_P$ since that implies $\mu_F^- = |\mu_F| - \mu_P$. We have

$$\mu_P = \frac{1}{2}(\mu_{T_F} + \mu_F) = \frac{1}{2}(\mu_F^+ + \mu_F^- + \mu_F^+ - \mu_F^-) = \frac{1}{2}(2\mu_F^+) = \mu_F^+$$

as desired.

$\square$

Folland Exercise 3.30 Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is $\mathbb{Q}$.

Proof. [3] Take $F(x) := \sum_{k=1}^{\infty} \frac{1}{2^k}\chi_{[q_k,\infty)}$ where $\{q_k\}$ is an enumeration of $\mathbb{Q}$. **[Incomplete]**

$\square$

[3]The following proof is motivated form the post here.

Folland 3.31 Let $F(x) = x^2 \sin(x^{-1})$ and $G(x) = x^2 \sin(x^{-2})$ for $x \neq 0$, and $F(0) = G(0) = 0$.

a. F and G are differentiable everywhere (including $x = 0$).

b. $F \in BV([-1,1])$, but $G \notin BV([-1,1])$

PROOF. (a) WLOG, it suffices to check differentiability at the point $x = 0$. The two functions given are clearly differentiable elsewhere (by undergraduate analysis).

We claim $F(x)$ is differentiable at $x = 0$. We have, where $x = 0$,

$$\frac{|F(x+h) - F(x)|}{|h|} = \frac{|F(h)|}{|h|} = \frac{\left|h^2 \sin\left(\frac{1}{h}\right)\right|}{|h|} = \left|h \sin\left(\frac{1}{h}\right)\right|. \tag{198}$$

Taking the limit as $h \to 0$, we get

$$\lim_{h\to 0}\left|h \sin\left(\frac{1}{h}\right)\right| = \lim_{h\to\infty}\left|\frac{1}{h}\right| |\sin(h)| = 0. \tag{199}$$

where the last equality follows because $|\sin(h)|$ is bounded.

Similarly, we know

$$\lim_{h\to 0}\frac{|G(x+h) - G(x)|}{|h|} = \lim_{h\to 0}\frac{|G(h)|}{|h|} = \lim_{h\to 0}\left|h \sin\left(\frac{1}{h^2}\right)\right| \tag{200}$$

$$= \lim_{h\to\infty}\left|\frac{1}{h}\sin(h^2)\right| = 0 \tag{201}$$

where the last equality follows because $|\sin(h^2)|$ is bounded.

(b) To see that $F(x)$ is of bounded variation, use part (a) to differentiate $F'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$. Therefore,

$$|F'(x)| \leq |2x||\sin\left(\frac{1}{x}\right)| + |\cos\left(\frac{1}{x}\right)| \leq 2\cdot 1\cdot 1 + 1 = 3$$

and $|F'(x)| \leq 3$ on $[-1,1]$. By the result on p. 102, F' is bounded on $[-1,1]$ and a.e. differentiable means $F \in BV([-1,1])$ and we are done.

To see $G \notin BV([-1,1])$, we show that $T_G(1) = \infty$.

Define $x_n := \frac{1}{\sqrt{\frac{\pi}{2}+\pi n}}$. Then we get a sequence $1 > x_1 > x_2 > x_3 > \cdots \geq 0$. Also,

$$|G(x_n) - G(x_{n-1})| = \left|\frac{1}{\frac{\pi}{2}+\pi n} - \frac{1}{\frac{\pi}{2}+\pi(n-1)}\right| = \left|-\frac{4n}{\pi(2n+1)(2n-1)}\right|. \tag{202}$$

But since $\sum_{n=1}^{k}|G(x_n) - G(x_{n-1})| \leq T_G(b)$, we have that

$$\lim_{k\to\infty}\sum_{n=1}^{k}|G(x_n) - G(x_{n-1})| = \sum_{n=1}^{k}\left|-\frac{4n}{\pi(2n+1)(2n-1)}\right| \tag{203}$$

$$= \sum_{n=1}^{\infty}\frac{4n}{\pi(4n^2-1)} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n-\frac{1}{n}} \tag{204}$$

$$\leq T_G(1) \tag{205}$$

where the series on the LHS of the inequality diverges. Hence, $T_G(1) = \infty$ and $G \notin BV([-1,1])$.

$\square$

Folland Exercise 3.32 If $F_1, F_2, \dots, F \in NBV$ and $F_j \to F$ pointwise, then $T_F \leq \liminf T_{F_j}$.

PROOF. [4] Let $x \in \mathbb{R}$. Take a partition $-\infty < x_1 < x_2 < \cdots < x_m =: x$. Then, observing the definition of T_{F_j},

$$\sum_{n=1}^{m} |F(x_n) - F(x_{n-1})| = \lim_{j\to\infty} \sum_{j=1}^{m} |F_j(x_n) - F_j(x_{n-1})| \leq \liminf_{j\to\infty} T_{F_j}$$

and taking the supremum over all partitions gives the desired result. □

Folland Exercise 3.33

PROOF. By Theorem 3.23, we know $F' = G'$ a.e. when $G = F(x+)$ so $\int_a^b F'(t)dt = \int_a^b G(t)dt$. Then, since G is increasing,

$$\int_a^b G(t)dt \leq \int_a^b \frac{d\mu_G}{dm} dm \leq \mu_G((a,b)) \leq G(b-) - G(a) \leq F(b) - F(a).$$

□

Folland 3.35 If F and G are absolutely continuous on $[a,b]$, then so is FG, and

$$\int_a^b (FG' + GF')(x)dx = F(b)G(b) - F(a)G(a)$$

PROOF. First, F and G are continuous because absolute continuity implies continuity. Because they are continuous on a compact set $[a,b]$, the are bounded. Let L be the maximum of the bounds on F and G.

Given $\epsilon > 0$. Choose a $\delta > 0$ and $\eta > 0$ for the definition of absolute continuity of F and G with $\frac{\epsilon}{2L+1}$ i.e. if $\{(a_j, b_j)\}$ is finite set of disjoint intervals in $[a,b]$, then

$$\sum_1^N (b_j - a_j) < \delta \implies \sum_1^N |F(b_j) - F(a_j)| < \frac{\epsilon}{2L+1} \tag{206}$$

and similarly for η with G. Now we take $\xi := \min\{\delta, \eta\}$. Then if $\sum_1^N (b_j - a_j) < \xi$, we get

$$\begin{aligned}
\sum_1^N |F(b_j)G(b_j) - F(a_j)G(b_j)| &\leq \sum_1^N |F(b_j)G(b_j) - F(a_j)G(b_j)| + \sum_1^N |F(a_j)G(b_j) - F(a_j)G(a_j)| \\
&= \sum_1^N |G(b_j)||F(b_j) - F(a_j)| + \sum_1^N |F(a_j)||G(b_j) - G(a_j)| \\
&\leq L\sum_1^N |F(b_j) - F(a_j)| + L\sum_1^N |G(b_j) - G(a_j)| \\
&\leq L\left(\frac{\epsilon}{2L+1} + \frac{\epsilon}{2L+1}\right) < \epsilon
\end{aligned}$$

[4] I am unsure if the following proof works. Apply Fatou's Lemma,

$$T_F(x) = \mu_{T_F}((-\infty, x]) = \int_{-\infty}^{x} d\mu_{T_F} = \int_{-\infty}^{x} dT_F = \int_{-\infty}^{1} \lim_{j\to\infty} dT_{F_j} \leq \liminf_{j\to\infty} \int_{-\infty}^{1} dT_{F_j} = \liminf T_{F_j}(x).$$

and so, FG is absolutely continuous on $[a, b]$.

We note that if F and G are differentiable a.e., then the product FG is also differentiable a.e.. since the points of nondifferetiability are precisely where either F or G is not differentiable. Also, because $F', G' \in L^1$, we deduce that $(FG)'$ is also in L^1. Also,

$$\int_a^b (FG' + GF')(x)dx = \int_a^b (FG)'(x)dx = F(b)G(b) - F(a)G(a) \tag{207}$$

where we use the product rule for the first equality and the criterion of Theorem 3.35c. □

Folland 3.36 Let G be a continuous increasing function on $[a, b]$ and let $G(a) = c, G(b) = d$.

a. If $E \subset [c, d]$ is a Borel set, then $m(E) = \mu_G\left(G^{-1}(E)\right)$. (First consider the case where E is an interval.)

b. If f is a Borel measurable and integrable function on $[c, d]$, then $\int_c^d f(y)dy = \int_a^b f(G(x))dG(x)$. In particular, $\int_c^d f(y)dy = \int_a^b f(G(x))G'(x)dx$ if G is absolutely continuous.

c. The validity of (b) may fail if G is merely right continuous rather than continuous.

Proof. (a) As per the hint, consider the case $E := (c', d') \subset [c, d]$ and $G^{-1}(E) = (a', b')$ (since G is continuous *and* increasing, there is such an interval (a', b') and it is contained in $[a, b]$). Then $m(E) = d' - c'$ and

$$\mu_G(G^{-1}(E)) = \mu_G((a', b')) = G(b') - G(a') = d' - c'$$

which is what we wanted. Consider the case of a general open subset $E \subset [c, d]$. We can write E as a countable disjoint union of open intervals in $[c, d]$ i.e. $E = \bigcup_{j=1}^\infty I_j$. Then,

$$m(E) = m\left(\bigcup_{j=1}^\infty I_j\right) = \sum_{j=1}^\infty m(I_j) = \sum_{j=1}^\infty \mu_G(G^{-1}(I_j)) \tag{208}$$

$$= \mu_G\left(\bigcup_{j=1}^\infty G^{-1}(I_j)\right) = \mu_G\left(G^{-1}\left(\bigcup_{j=1}^\infty I_j\right)\right) \tag{209}$$

$$= \mu_G(G^{-1}(E)) \tag{210}$$

which is exactly we we desired. Then for the general case of a Borel set, we argue as follows. The Borel σ-algebra is generated by open subsets and hence, every Borel set can be written as the union of open and closed sets. If we write a Borel set as the countable union of open and closed sets, we can write it as a disjoint union as per techniques in Chapter 1. Then, we can write it as a countable union of open intervals and complements of open intervals. Then by countable additivity of measures and the fact that inverse images commutes with complements and unions, we get $m(E) = \mu_G(G^{-1}(E))$ for general Borel sets.

(b) The result for the case of characteristic functions over Borel sets is immediate from (a). Indeed, if $E \subseteq [c, d]$ were Borel, then

$$\int_c^d \chi_E dy = m(E) = \mu_G(G^{-1}(E)) = \int_a^b \chi_{G^{-1}(E)} d\mu_G = \int_a^b \chi_E \circ G(x)dG(x).$$

Now if we were to consider a simple function f (whose characteristic functions are over Borel sets), then the linearity of the integral gives the result for simple functions immediately. Now that we know the result for simple functions, suppose we are given a general f that is Borel measurable *and* and integrable on $[c,d]$. By Theorem 2.10b, choose a sequence of simple functions ϕ_n s.t. $|\phi_n|$ increases to $|f|$ on $[c,d]$. Since we know the result for simple functions, we have

$$\int_c^d \phi_n(y)dy = \int_a^b \phi_n(G(x))dG(x). \tag{211}$$

and taking limits, we get

$$\int_c^d f(y)dy \overset{*}{=} \int_a^b f(G(x))dG(x). \tag{212}$$

We must justify interchanging the integral with the limit at $*$. We know $|\phi_n| \leq f$ for all $n \in \mathbb{N}$. But f is integrable on $[c,d]$ and we just apply the DCT to get interchange of the limit and integral.

Now for the statement about absolutely continuous G, we just need to check that $dG(x) = G'(x)dx$ on $[a,b]$. Note that $dG(x) = d\mu_G$ and Theorem 3.35 gives us what we need. We always have

$$\mu_G((a,b)) = G(b) - G(a) = \int_a^b G'(t)dt. \tag{213}$$

The extension to general Borel sets on $\mathbb{R}$ follows by Theorem 1.16 and the above.

(c) For a counterexample in the case of only right continuity, we take $c = 0, d = 1$ so that $[c,d] = [0,1]$. Define

$$G(x) := \begin{cases} 1 & x \geq \frac{1}{2} \\ 0 & x < \frac{1}{2} \end{cases} \tag{214}$$

and take $[a,b] = [0,1] = G^{-1}([c,d])$. Also, $G(x)$ is right continuous. Define

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{215}$$

and $f(x)$ is certainly Borel measurable and integrable on $[0,1]$. Notice that

$$f \circ G(x) := \begin{cases} 1 & x < \frac{1}{2} \\ 0 & x \geq \frac{1}{2} \end{cases} \tag{216}$$

Next,

$$\int_0^1 f(y)dy = 0 \tag{217}$$

because $f(y)$ is a.e. zero. Meanwhile,

$$\begin{aligned} \int_0^1 f(G(x))dG(x) &= \int_0^{\frac{1}{2}} 1dG(x) && (f(G(x)) \text{ zero on } (1/2,1)) \qquad (218) \\ &= G\left(\frac{1}{2}\right) - G(0) && \qquad (219) \\ &= 1. && (\text{Def of } G) \qquad (220) \end{aligned}$$

$\square$

Folland 3.37 Suppose $F : \mathbb{R} \to \mathbb{C}$. There is a constant M such that $|F(x)-F(y)| \leq M|x-y|$ for all $x, y \in \mathbb{R}$ (that is, F is Lipschitz continuous) iff F is absolutely continuous and $|F'| \leq M$ a.e.

PROOF. ($\implies$) Given $\epsilon > 0$. Assume $\sum_{j=1}^N |b_j - a_j| < \frac{\epsilon}{M}$. Let $\delta := \frac{\epsilon}{M+1}$. Then if $\sum_{j=1}^N |b_j - a_j| < \frac{\epsilon}{M+1}$ for some partition $x_1 < x_2 < \cdots < x_n$, we have

$$\sum_{j=1}^N |F(b_j) - F(a_j)| \leq \sum_{j=1}^N M|b_j - a_j| = M\sum_{j=1}^N |b_j - a_j| < \frac{M\epsilon}{M+1} < \epsilon \tag{221}$$

which shows absolute continuity. To see that $F'(x) \leq M$ a.e., we note that $F'(x)$ exists for a.e. $x \in [n, n+1]$ where $n \in \mathbb{Z}$ (since it is absolutely continuous on each $[n, n+1]$, just use Theorem 3.35). Therefore, $F'(x)$ exists a.e. on $\mathbb{R}$. Suppose the derivative exists at x_0 and consider $x_0 \in [a, b]$ for a, b chosen s.t $x_0 \in (a, b)$. Then $F'(x_0)$ exists since F is of bounded variation on $[a, b]$. Then $|F(x_0 + h) - F(x_0)| \leq M|x_0 + h - x_0|$ for h sufficiently small s.t. $x_0 + h \in [a, b]$. Therefore,

$$\frac{|F(x_0 + h) - F(x_0)|}{|h|} \leq M. \tag{222}$$

Taking the limit as $h \to 0$ gives us $|F'(x_0)| \leq M$ as desired.

($\impliedby$) By Theorem 3.35, absolute continuity gives $F(y) - F(x) = \int_x^y F'(t)dt$ a.e. and $F'(t) \in L^1$. But then,

$$|F(y) - F(x)| = \left|\int_x^y F'(t)dt\right| \leq \int_x^y |F'(t)|dt \leq \int_x^y M dt = |y - x|M \tag{223}$$

where we assumed $y > x$. But y and x were arbitrary so we have our desired conclusion. ☐

Folland Exercise 3.39 If $\{F_j\}$ is a sequence of nonnegative increasing functions on $[a, b]$ such that $F(x) = \sum_1^\infty F_j(x) < \infty$ for all $x \in [a, b]$, then $F'(x) = \sum_1^\infty F_j'(x)$ for a.e. $x \in [a, b]$. (It suffices to assume $F_j \in NBV$. Consider the measures μ_{F_j}.

PROOF. Since we are working on $[a, b]$, the F_j may be assumed to be NBV by the following reductions. First, F_j is bounded and increasing on $[a, b]$ (provided we redefine at endpoints as needed) and this means $F \in BV$ by Example 3.25a. Since F is in BV, the $T_F(-\infty) = 0$. To get right continuity, redefine F as $F(x+)$.

Now, $F(x) = \int_{-\infty}^x F'(t)dt$ and then,

$$F(x) = \sum_1^\infty F_j(x) = \sum_1^\infty \int_{-\infty}^x F_j'(t)dt = \int_{-\infty}^x \sum_1^\infty F_j'(t)dt$$

and by Corollary 3.33, we dedude that $\sum_1^\infty F_j'(t) = F'(x)$ a.e. $x \in [a, b]$.

☐

Folland 3.42 A function $F : (a, b) \to \mathbb{R}(-\infty \leq a < b \leq \infty)$ is called convex if

$$F(\lambda s + (1-\lambda)t) \leq \lambda F(s) + (1-\lambda)F(t)$$

for all $s, t \in (a, b)$ and $\lambda \in (0, 1)$. (Geometrically, this says that the graph of F over the interval from s to t lies underneath the line segment joining $(s, F(s))$ to $(t, F(t))$.)

a. F is convex iff for all $s, t, s', t' \in (a,b)$ such that $s \leq s' < t'$ and $s < t \leq t'$

$$\frac{F(t)-F(s)}{t-s} \leq \frac{F\left(t'\right)-F\left(s'\right)}{t'-s'}$$

b. F is convex iff F is absolutely continuous on every compact subinterval of (a,b) and F' is increasing (on the set where it is defined).

c. If F is convex and $t_0 \in (a,b)$, there exists $\beta \in \mathbb{R}$ such that $F(t) - F\left(t_0\right) \geq \beta\left(t - t_0\right)$ for all $t \in (a,b)$

d. (Jensen's Inequality) If $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) = 1$, $g : X \to (a,b)$ is in $L^1(\mu)$, and F is convex on (a,b), then

$$F\left(\int g d\mu\right) \leq \int F \circ g d\mu$$

(Let $t_0 = \int g d\mu$ and $t = g(x)$ in (c), and integrate.)

PROOF. (a) ($\Longrightarrow$): We do a two step process by showing the following chain of inequalities

$$\frac{F(t)-F(s)}{t-s} \overset{(i)}{\leq} \frac{F(t')-F(s)}{t'-s} \overset{(ii)}{\leq} \frac{F(t')-F(s')}{t'-s'} \tag{224}$$

where we need to use $s < t < t'$ at (i) and $s < s' < t'$ at (ii).

We justify (i). First, $s < t < t'$ implies $0 < \frac{t-s}{t'-s} < 1$. If we take $\lambda = \frac{t-s}{t'-s}$, we get that $t = \lambda t' + (1-\lambda)s$ and by convexity,

$$F(t) \leq \lambda F(t') + (1-\lambda)F(s). \tag{225}$$

Subtracting both sides by $F(s)$ and multiplying through by $\frac{1}{t-s}$ gives

$$\frac{F(t)-F(s)}{t-s} \leq \frac{F(t')-F(s)}{t'-s}. \tag{226}$$

We justify (ii), notice that because $s < s' < t'$, we get $0 < t' - s' < t' - s$ and this implies $0 < \frac{t'-s'}{t'-s} < 1$. Take $\lambda = \frac{t'-s'}{t'-s}$ and notice that we can write $s' = \lambda s + (1 - \lambda t')$. Using convexity, we get

$$F(s') \leq (1-\lambda)F(t') + \lambda F(s) \tag{227}$$

and we can rewrite the above as

$$\lambda(F(t') - F(s)) \leq F(t') - F(s'). \tag{228}$$

Writing out λ and multiplying through by $t' - s$ and dividing by $s' - s$,

$$\frac{F(t')-F(s)}{t'-s} \leq \frac{F(t')-F(s')}{t'-s'}. \tag{229}$$

Now all that is left is to deal with the cases where $s = s'$, $t = t'$, and when $s = s'$ and $t = t'$.

If $s = s'$, then (224) gives

$$\frac{F(t)-F(s)}{t-s} \leq \frac{F(t')-F(s)}{t'-s} = \frac{F(t')-F(s')}{t'-s'}. \tag{230}$$

Similarly, if $t = t'$, (224) gives,

$$\frac{F(t)-F(s)}{t-s} \leq \frac{F(t)-F(s')}{t-s'} = \frac{F(t')-F(s')}{t'-s'}. \tag{231}$$

If both $s = s'$ and $t = t'$ occur, then there is nothing to even show.

($\Longleftarrow$): Given $\lambda \in (0,1)$ and s, t. WLOG, assume $s < t$. Set $\theta := \lambda s + (1-\lambda)t$. Then $s \leq \theta \leq t$ and we can use the given inequality to get

$$\frac{F(\theta)-F(s)}{(1-\lambda)(t-s)} = \frac{F(\theta)-F(s)}{\theta - s} \leq \frac{F(t)-F(s)}{t-s}. \tag{232}$$

Then, we can rewrite the above as

$$\frac{F(\theta)-F(s)}{(1-\lambda)} \leq F(t) - F(s). \tag{233}$$

But the above inequality can used to get the following chain of equivalences (the first step is just obtained by multiplying the above by $(1-\lambda)$):

$$F(\theta) \leq (F(t)-F(s))(1-\lambda) + F(s) \tag{234}$$

$$\Longleftrightarrow F(\theta) \leq (1-\lambda)F(t) - F(s) + F(s)\lambda + F(s) \tag{235}$$

$$\Longleftrightarrow F(\theta) \leq (1-\lambda)F(t) + F(s)\lambda \tag{236}$$

$$\Longleftrightarrow F(\lambda s + (1-\lambda)t) \leq \lambda F(s) + (1-\lambda)F(t). \tag{237}$$

But this last inequality is precisely the definition of convexity and we are done.

(b) ($\Longrightarrow$): Assume $F(x)$ is convex. Let $[c,d]$ be a compact subinterval of (a,b). Then the above condition allows us to find a β s.t. $\frac{F(t)-F(s)}{t-s} \leq \beta$ for all $t, s \in [c,d]$ i.e. we take the supremum over all possible quotients on the RHS of the inequality in (a) with $t', s' \in [c,d]$. Given $\epsilon > 0$, take $\delta := \frac{\epsilon}{|\beta|+1}$. Then, if (a_j, b_j) is a partition of the compact subinterval, then

$$\sum_{j=1}^{n} b_j - a_j < \delta \quad \Longrightarrow \quad \sum_{j=1}^{n} |F(b_j) - F(a_j)| \leq \sum_{j=1}^{n} (b_j - a_j)\beta < \delta\beta < \epsilon. \tag{238}$$

To see that F' is increasing where it is defined, suppose $t \leq t'$ and these are points where F' is defined. We choose s s.t. $s < t \leq t'$ and $s' = t' - (t-s)$ which implies $s \leq s' < t'$ (and we choose s sufficiently close to s' so that $s \in [c,d]$). Then (a) gives

$$\frac{F(t)-F(s)}{t-s} \leq \frac{F(t')-F(s')}{t'-s'}. \tag{239}$$

Letting $s \to t$, we get $s' \to t'$ and by the above inequality, $F'(t) \leq F'(t')$ as desired.

($\Longleftarrow$): Let $\theta := \lambda s + (1-\lambda)t$ and WLOG, assume $s > t$. Define $G(x) := \lambda x + (1-\lambda)t$. We want to use Exercise 3.36 to show convexity directly from the definition and use $G(x)$ to do this. Conveniently, note that $G'(x) = \lambda$. Because F is absolutely continuous, Theorem 3.35 gives

$$F(G(s)) - F(G(t)) = \int_{G(t)}^{G(s)} F'(z)dz. \tag{240}$$

Then by Exercise 3.36,

$$\int_{G(t)}^{G(s)} F'(z)dz = \int_t^s F'(G(z)) \cdot \lambda dz = \lambda \int_t^s F'(G(z))dz \leq \lambda \int_t^s F'(z)dz \tag{241}$$

$$= \lambda(F(s) - F(t)) = \lambda F(s) - \lambda F(t) \tag{242}$$

where to get the second to last inequality, we used the fact that $G(z) \leq z$ for $z \in (t, s)$ and $F'(z)$ is increasing.

Also by Exercise 3.36,

$$\int_{G(t)}^{G(s)} F'(z)dz = F(G(s)) - F(G(t)). \tag{243}$$

Putting together the last two inequalities / equalities,

$$F(G(s)) - F(G(t)) \leq \lambda F(s) - \lambda F(t). \tag{244}$$

Notice that $F(G(s)) = F(\lambda s + (1-\lambda)t)$ and $F(G(t)) = F(t)$. Then the above can be rewritten and we have a chain of equivalent inequalities:

$$F(\lambda s + (1-\lambda)t) - F(t) \leq \lambda F(t) - \lambda F(s) \tag{245}$$

$$F(\lambda s + (1-\lambda)t) \leq F(t) + \lambda F(s) - \lambda F(t) \tag{246}$$

$$F(\lambda s + (1-\lambda)t) \leq \lambda F(s) + (1-\lambda)F(t) \tag{247}$$

which is what we wanted for convexity.

(c) Given t_0. Now if t were s.t. $a < t < t_0$, we get

$$\beta \leq \frac{F(t) - F(t_0)}{t - t_0} \tag{248}$$

where we take t to be t' in (a) and t_0 to be s' in (a) and β to be the supremum over all possible quotients on the LHS of the inequality in (a). Now suppose u were s.t. $t < t_0 < u < b$. Then by the inequality we showed in part (a), we get

$$\beta \leq \frac{F(t) - F(t_0)}{t - t_0} \leq \frac{F(u) - F(t_0)}{u - t_0}. \tag{249}$$

Therefore, for any $t \in (a, b)$, we have

$$\beta \leq \frac{F(t) - F(t_0)}{t - t_0} \tag{250}$$

as desired.

(d) We follow the hint and set $t_0 = \int g d\mu$ and $t = g(x)$. The (c) says that

$$F(t) \geq F(t_0) + \beta(t - t_0). \tag{251}$$

Integrate over X in terms of x to get

$$\int_X F(t) \geq \int_X F(t_0) + \int_X \beta(t - t_0). \tag{252}$$

It is important to keep in mind what is constant in this situation. Then t_0 is constant w.r.t x while $t = g(x)$ means it is being integrated and we cannot treat it as a constant. Therefore,

$$\int_X F(t) \geq \int_X F(t_0)d\mu + \int_X \beta t - \int_X \beta t_0 \tag{253}$$

$$= \int_X F(t_0)d\mu + \int_X \beta g(x)dx - \int_X \beta t_0 \tag{254}$$

$$= F(t_0) + \int_X \beta g(x)dx - \int_X \beta \left(\int_X g d\mu\right) dx \tag{255}$$

$$= F(t_0) + \int_X \beta g(x)dx - \beta \int_X g d\mu \tag{256}$$

$$= F(t_0) = F\left(\int_X g(x)\right) dx \tag{257}$$

where we used $\mu(X) = 1$ as needed. This gives us the desired inequality.

□

4. Point Set Topology

Folland Exercise 4.1 If $\text{card}(X) \geq 2$, there is a topology on X that is T_0 but not T_1.

PROOF. Choose $x \in X$. Then define a topology whose open sets are $\emptyset, X, X \setminus \{x\}$ and this is T_0 by choosing $y \neq x$ and observing the axiom. □

Folland Exercise 4.2 If X is an infinite set, the cofinite topology on X is T_1 but not T_2, and is first countable iff X is countable.

PROOF. □

Folland Exercise 4.3 Every metric space is normal. (If A, B are closed sets in the metric space (X, ρ), consider the sets of points x where $\rho(x, A) < \rho(x, B)$ or $\rho(x, A) > \rho(x, B)$.)

PROOF. □

Folland Exercise 4.4 Let $X = \mathbb{R}$, and let $\mathcal{T}$ be the family of all subsets of $\mathbb{R}$ of the form $U \cup (V \cap \mathbb{Q})$ where U, V are open in the usual sense. Then $\mathcal{T}$ is a topology that is Hausdorff but not regular. (In view of Exercise 3, this shows that a topology stronger than a normal topology need not be normal or even regular.)

PROOF. □

Folland Exercise 4.5 Every separable metric space is second countable.

PROOF. First off, choose a countable dense subset $\{x_n\}$. Then, take an open neighborhood $U_n \ni x_n$ for each n and the collection consisting of all finite intersections of the U_n forms a countable basis. □

Folland Exercise 4.6 Let $\mathcal{E} = \{(a, b] : -\infty < a < b < \infty\}$

a. $\mathcal{E}$ is a base for a topology $\mathcal{T}$ on $\mathbb{R}$ in which the members of $\mathcal{E}$ are both open and closed.

b. $\mathcal{T}$ is first countable but not second countable. (If $x \in \mathbb{R}$, every neighborhood base at x contains a set whose supremum is x.)

c. $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to $\mathcal{T}$. (Thus the converse of Proposition 4.5 is false.)

PROOF. (a) Apply Proposition 4.3 to simplify showing $\mathcal{E}$ is a base for a topology. First, if $x \in \mathbb{R}$, then x lies in $(x-1, x+1] \in \mathcal{E}$. For the second condition, suppose $(a_1, b_1], (a_2, b_2] \in \mathcal{E}$ and $x \in (a_1, b_1] \cap (a_2, b_2]$. But $(a_1, b_1] \cap (a_2, b_2] = (c, d]$ for some c, d because half-open intervals form an algebra. Now choosing $\epsilon > 0$ sufficiently small, $x \in (c - \epsilon, d]$ and $(c - \epsilon, d] \subseteq (a_1, b_1] \cap (a_2, b_2]$ as desired.

Certainly elements of $\mathcal{E}$ are open. Notice that $(a, b]^c = (-\infty, a] \cup (b, \infty)$. Notice $(b, \infty) = \bigcup_{n=1}^{\infty} (b, n]$ and $(-\infty, a] = \bigcup_{n=1}^{\infty} (-n, a]$ so these are open sets. Hence, $(a, b]^c$ is open and therefore, $(a, b]$ is closed.

(b) Let $x \in \mathbb{R}$. Then the collection

$$\mathcal{N}_x := \{(x - \frac{1}{n}, x] : n \in \mathbb{N}\} \tag{258}$$

forms a neighborhood base about x. Indeed, x lies in every element of $\mathcal{N}_x$ and if $U \in \mathcal{T}$ s.t. $x \in U$, we can always take n sufficiently large so that $(x - \frac{1}{n}, x] \subset U$.

Next, we claim that the space is not second countable. Let $\mathcal{B}$ be a base. Then there is a neighborhood base about each $x \in \mathbb{R}$. Given an x and a neighborhood base $\mathcal{N}_x$. Then $\mathcal{N}_x$ must contain a set U whose maximum is x because the open set $(c, x] \in \mathcal{E}$ contains x for $c < x$ and by definition of a base, there is a $U \in \mathcal{N}_x$ s.t. $x \in U$ and $U \subset (c, x]$. This occurs for each $x \in \mathbb{R}$ and hence, the base must contain uncountably many such sets.

(c) We claim $\overline{\mathbb{Q}} = \mathbb{R}$ w.r.t this topology. Suppose $r \in \mathbb{R} \setminus \mathbb{Q}$. It suffices by Proposition 4.1. to show that r is an accumulation point of $\mathbb{Q}$. Let $U \ni x$ be a neighborhood of x. WLOG, assume U is open by replacing U by U^o. Since $x \in U$, there exists a $(a, b] \in \mathcal{E}$ s.t. $x \in (a, b]$. But then $\mathbb{Q} \cap (U \setminus \{x\}) \supseteq \mathbb{Q} \cap ((a, b] \setminus \{x\}) \neq \emptyset$ by the density of $\mathbb{Q}$ in the usual topology.

The converse of Proposition 4.5 is then false because $\mathbb{Q}$ is a countable dense subset but $(\mathbb{R}, \mathcal{T})$ is not a second countable topological space.

□

Folland Exercise 4.7 If X is a topological space, a point $x \in X$ is called a cluster point of the sequence $\{x_j\}$ if for every neighborhood U of $x, x_j \in U$ for infinitely many j. If X is first countable, x is a cluster point of $\{x_j\}$ iff some subsequence of $\{x_j\}$ converges to x.

PROOF. ($\implies$) : If x is a cluster point of $\{x_j\}$, then just use the first count ability to pick a countable neighborhood base about x. Per the remarks on p. 116, we can form a countable neighborhood base $\{U_n\}_{n \in \mathbb{N}}$ satisfying $U_1 \supset U_2 \supset U_3 \supset \dots$. Then take the subsequence $\{x_{j_n}\}$ to be a sequence of points satisfying $x_{j_n} \in U_n$.

Let U be a neighborhood of x. Then there is an $N \in \mathbb{N}$ s.t. $U_N \subset U$ and for all $n \geq N$, we have $x_{j_n} \in U_n \subset U_N \subset U$. Hence, $\{x_{j_n}\} \to x$.

($\impliedby$) : If we have a subsequence which converges to x, then for any neighborhood U_1 of x, there is an $x_{n_1} \in U_1$. We need to show that U_1 contains infinitely many elements of the sequence $\{x_j\}$. By first countability, there is a countable neighborhood base about x and so, we can choose a descending chain $U_1 \supset U_2 \supset \dots$ and since $x_j \to x$, choose elements $x_{n_i} \in U_i$. But $U_1 \supset U_i$ for all i and so, there are infinitely many elements of the sequence contained in U_1 as desired. □

Folland Exercise 4.8 If X is an infinite set with the cofinite topology and $\{x_j\}$ is a sequence of distinct points in X, then $x_j \to x$ for every $x \in X$.

Proof. This follows essentially from definition. □

Folland Exercise 4.10 A topological space X is called disconnected if there exist nonempty open sets U, V such that $U \cap V = \varnothing$ and $U \cup V = X$; otherwise X is connected. When we speak of connected or disconnected subsets of X, we refer to the relative topology on them.

a. X is connected iff $\varnothing$ and X are the only subsets of X that are both open and closed.

b. If $\{E_\alpha\}_{\alpha \in A}$ is a collection of connected subsets of X such that $\bigcap_{\alpha \in A} E_\alpha \neq \varnothing$, then $\bigcup_{\alpha \in A} E_\alpha$ is connected.

c. If $A \subset X$ is connected, then $\bar{A}$ is connected.

d. Every point $x \in X$ is contained in a unique maximal connected subset of X, and this subset is closed. (It is called the connected component of x.)

Proof.

a. ($\Longrightarrow$): Assume X is connected and let U be a set that is both open and closed but $U \neq \emptyset$ and $U \neq X$. Then $U \cup U^c = X$ is a union of two nonempty disjoint open sets which contradicts X connected.

($\Longleftarrow$): Assume only $\emptyset$ and X are open and closed subsets of X. Suppose $X = U \cup V$ for disjoint nonempty open sets U and V. Notice that

$$U^c = X \setminus U = (U \cup V) \setminus U = (U \cup V) \cap U^c = V \cap U^c = V$$

where the last equality follows by $U^c \subseteq V$. Then $U^c = V$ is open so $U^c = \emptyset$ or $U^c = X$. Therefore, X is connected.

b. Let $x \in \bigcap_{\alpha \in A} E_\alpha$. Suppose U, V are disjoint open sets s.t. $\bigcup_{\alpha \in A} E_\alpha = U \cup V$. WLOG, assume $x \in U$. Then $E_\alpha \subseteq U$ for all $\alpha \in A$ since if not, there is an α for which $E_\alpha \cap V$ and $E_\alpha \cap U$ are nonempty disjoint open sets whose union is E_α which is a contradiction. Thus, $U = \bigcup_{\alpha \in A} E_\alpha$ which means $\bigcup_{\alpha \in A} E_\alpha$ is connected.

c. Let $A \subset X$ be connected but assume not and that $\overline{A} = U \cup V$ for nonempty disjoint open sets U, V. Then $(A \cap U) \cup (A \cap V) = A$ where $A \cap U$ and $A \cap V$ are nonempty disjoint open sets in the relative topology contradicting A connected. We need to check that $A \cap U = \emptyset$ or $A \cap V = \emptyset$ do not occur. If $A \cap U = \emptyset$, then $V = \partial \overline{A}$ which is not open.

d. Let $C := \bigcup_{E \in \mathcal{E}} E$ where $\mathcal{E} := \{E \subseteq X \mid E \text{ is connected } \& \ x \in E\}$. We claim that C is the connected component. To see that C is connected, notice that $\bigcap_{E \in \mathcal{E}} E \neq \emptyset$. Hence, part b shows that C, being the union, is also connected. To see that C is maximally connected, suppose C' were a connected set containing x. Then $C' \in \mathcal{E}$ and hence, $C' \subseteq C$. Since C' was arbitrary, C is a maximal connected set of x. Finally, C is closed because part c showed that $\overline{C}$ would also be connected which means $\overline{C} \subseteq C$ by maximality of C. Therefore, $\overline{C} \subseteq C \subseteq \overline{C}$ and so, $\overline{C} = C$ implying that C is closed.

□

Folland Exercise 4.11 If $E_1, \ldots, E_n$ are subsets of a topological space, the closure of $\bigcup_1^n E_j$ is $\bigcup_1^n \bar{E}_j$.

Proof. This is a standard undergraduate topology exercises. □

Folland Exercise 4.12 If X is a topological space, U is open in X, and A is dense in X, then $\bar{U} = \overline{U \cap A}$.

Proof. Essentially from definition. □

Folland 4.16 Let X be a topological space, Y a Hausdorff space, and f, g continuous maps from X to Y.

a. $\{x : f(x) = g(x)\}$ is closed.

b. If $f = g$ on a dense subset of X, then $f = g$ on all of X.

PROOF. (a) Let W be the set given. It suffices to show W^c is open. Let $x \in W^c$. By Y Hausdorff, choose U, V open disjoint sets containing $f(x)$ and $g(x)$ respectively. Since f, g are continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are open and so $f^{-1}(U) \cap g^{-1}(V)$ is open. In particular $x \in f^{-1}(U) \cap g^{-1}(V) \subseteq W^c$ because if $y \in f^{-1}(U) \cap g^{-1}(V)$, then $f(y) \neq g(y)$ since $U \cap V = \emptyset$. Since $x \in W^c$ was arbitrary, W^c is open and the set is closed.

(b) Suppose $f = g$ on a dense subset $D \subseteq X$. Then, $D \subseteq \{x \mid f(x) = g(x)\}$ and so, $X = \overline{D} \subseteq \overline{\{x \mid f(x) = g(x)\}} \subseteq X$ implying $\overline{\{x \mid f(x) = g(x)\}} = X$. But the set was already closed by (a) which means $\{x \mid f(x) = g(x)\} = X$.

□

Folland Exercise 4.17 If X is a set, $\mathcal{F}$ a collection of real-valued functions on X, and $\mathcal{T}$ the weak topology generated by $\mathcal{F}$, then $\mathcal{T}$ is Hausdorff iff for every $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{F}$ with $f(x) \neq f(y)$

PROOF. ($\implies$): Fix $x \neq y$. Since $\mathcal{T}$ is Hausdorff, there are disjoint open sets $U \ni x$ and $V \ni y$. By definition of the topology, $U = \bigcup_{\alpha \in A} f_\alpha^{-1}(U_\alpha)$ for U_α open sets of $\mathbb{R}$ and $f_\alpha \in \mathcal{F}$. Since $x \in U$, there is an $f_\beta^{-1}(U_\beta) \ni x$ which is disjoint from V. Therefore, $f_\beta(x) \neq f_\beta(y)$ for if $f_\beta(x) = f_\beta(y)$, that means $y \in f_\beta^{-1}(U_\alpha)$ which is absurd from $U \cap V = \emptyset$.

($\impliedby$): Fix an f as stated in the problem. Since $\mathbb{R}$ is Hausdorff, there are U, V disjoint open sets s.t. $f(x) \in U$ and $f(y) \in V$. Then, $f^{-1}(U)$ and $f^{-1}(V)$ are our desired disjoint open sets containing x and y respectively. Hence, $\mathcal{T}$ is Hausdorff. □

Folland Exercise 4.20 If A is a countable set and X_α is a first (resp. second) countable space for each $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is first (resp. second) countable.

PROOF. If each X_α is first countable, then every point has a countable neighborhood base. Let $x \in X := \prod_{\alpha \in A} X_\alpha$. For each $x_\alpha \in X_\alpha$, choose a neighborhood base $\mathcal{N}_\alpha$. We claim the collection $\mathcal{S}$ of finite intersections $\bigcap_{\alpha \in I, U_\alpha \in \mathcal{N}_\alpha} \pi_\alpha^{-1}(U_\alpha)$ forms the countable neighborhood base about x.

First, the collection is countable. For every cardinality $|I| \in \mathbb{N}$, the collection $\mathcal{S}_{|I|}$ of sets of the form $\bigcap_{\alpha \in I, U_\alpha \in \mathcal{N}_\alpha} \pi_\alpha^{-1}(U_\alpha)$ is countable since each $\mathcal{N}_\alpha$ is a countable collection and A is a countable set. Then $\mathcal{S}$ is just the union of $\mathcal{S}_{|I|}$ over all $|I| \in \mathbb{N}$ which is a countable union of countable sets. Hence, $\mathcal{S}$ is countable.

Second, $\mathcal{S}$ forms a neighborhood base of x. If $U \in \mathcal{S}$, then $x \in U$ since $x \in \pi_\alpha^{-1}(U_\alpha)$ for all $U_\alpha \in \mathcal{N}_\alpha$. Suppose $V \ni x$ were an arbitrary open set of X. By Proposition 4.2, $V = \bigcup_\beta \pi_\beta^{-1}(V_\beta)$ where $V_\beta \subseteq X_\beta$ is open. Then $x \in \pi_\gamma^{-1}(V_\gamma)$ for some γ which means $x_\gamma \in V_\gamma$. Since $\mathcal{N}_\gamma$ is a neighborhood base for $x_\gamma \in X_\gamma$, choose $U_\gamma \in \mathcal{N}_\gamma$ s.t $U_\gamma \subseteq V_\gamma$. Then, $x \in \pi_\gamma^{-1}(U_\gamma) \in \mathcal{S}$ which fulfills the second condition of being a neighborhood base.

Suppose each X_α is second countable and let $\mathcal{E}_\alpha$ be a countable base for topology on X_α. Let

$$\mathcal{E} := \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{E}_\alpha, \alpha \in A\} = \bigcup_{\alpha\in A}\{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{E}_\alpha\}.$$

Since A is countable, the RHS is a countable union of countable sets which means $\mathcal{E}$ is countable. We use Proposition 4.2 to show $\mathcal{E}$ is a base for the topology. From p. 120, the product topology on X is the weak topology generated by the coordinate maps. By Proposition 4.2, every nonempty open subset of X is of the form

$$U := \bigcup_{\beta\in B\subseteq A} \pi_\beta^{-1}(U_\beta)$$

By Proposition 4.2, for each α, we can write $U_\alpha := \bigcup_{\gamma\in C, E_{\alpha,\gamma}\in\mathcal{E}_\alpha} E_{\alpha,\gamma}$ which means

$$U := \bigcup_{\beta\in B\subseteq A}\ \bigcup_{\gamma\in C, E_{\beta,\gamma}\in\mathcal{E}_\beta} \pi_\beta^{-1}(E_{\beta,\gamma})$$

as desired: the collection $\mathcal{E}$ is a basis for the product topology.

To summarize, open sets of X are unions of sets of the form $\pi_\beta^{-1}(U_\beta)$, but each U_β is a union of sets of the form $E_{\beta,\gamma} \in \mathcal{E}_\beta \subseteq \mathcal{E}$ and since inverse images commute with unions, open sets of X are unions of sets of the form $\pi_\beta^{-1}(E_{\beta,\gamma})$. Proposition 4.2 shows that $\mathcal{E}$ forms a basis which we showed was countable. At last, X is second countable. $\square$

Folland Exercise 4.25 If $(X, \mathcal{T})$ is completely regular, then $\mathcal{T}$ is the weak topology generated by $C(X)$.

Proof. First, let $\mathcal{S}$ be the weak topology generated by $C(X)$ and we show $\mathcal{S} \subseteq \mathcal{T}$. Note that when we are writing $f \in C(X)$, we are really referring to $f : (X, \mathcal{T}) \to \mathbb{C}$ being continuous w.r.t this topology. By definition, a basis element of $\mathcal{S}$ is of the form $f^{-1}(U)$ for some $U \subseteq \mathbb{C}$ open. But clearly, $f^{-1}(U)$ has to be open w.r.t $\mathcal{T}$ and therefore, $f^{-1}(U) \in \mathcal{T}$. Since the base of $\mathcal{S}$ is in $\mathcal{T}$, $\mathcal{S} \subseteq \mathcal{T}$.

Next to show that $\mathcal{T} \subseteq \mathcal{S}$, if suffices, by Proposition 4.2, to show that for all open $U \in \mathcal{T}$, the set U can be written as the union of sets of the form $f_\alpha^{-1}(U_\alpha)$ for $U_\alpha \subseteq \mathbb{C}$ open and $f_\alpha \in C(X)$. Assume U is nonempty since that is a trivial case. Certainly $U^c \subseteq X$ is a closed set. Then for $x \in U$, there is an $f_x \in C(X)$ (technically $f_x \in C(X, [0, 1])$, but $[0, 1] \subseteq \mathbb{C}$) s.t. $f_x(x) = 1$ and $f_x|_{U^c} = 0$. Consider the open sets $f_x^{-1}(\{0\}^c)$ ranging over $x \in U$. We claim $U = \bigcup_{x\in U} f_x^{-1}(\{0\}^c)$. First off, $f_x^{-1}(\{0\}^c) \subseteq U$ because $U^c \cap f_x^{-1}(\{0\}^c) = \emptyset$. Next, $U \subseteq \bigcup_{x\in U} f_x^{-1}(\{0\}^c)$ because for any $x \in U$, we obviously have $x \in f_x^{-1}(\{0\}^c)$. Hence, $U = \bigcup_{x\in U} f_x^{-1}(\{0\}^c)$. Thus, $U \in \mathcal{S}$. $\square$

Folland 4.26 Let X and Y be topological spaces.

a. If X is connected (see Exercise 10) and $f \in C(X, Y)$, then $f(X)$ is connected.

b. X is called arcwise connected if for all $x_0, x_1 \in X$ there exists $f \in C([0, 1], X)$ with $f(0) = x_0$ and $f(1) = x_1$. Every arcwise connected space is connected.

c. Let $X = \{(s, t) \in \mathbb{R}^2 : t = \sin(s^{-1})\} \cup \{(0, 0)\}$, with the relative topology induced from $\mathbb{R}^2$. Then X is connected but not arcwise connected.

Proof. a. We prove the contrapositive. Suppose $f(X)$ were not connected and $f(X) = U \cup V$ for disjoint nonempty open sets U and V. Then $f^{-1}(U) \cup f^{-1}(V) = X$ is a union of disjoint nonempty open sets which means X is not connected.

b. Let X be arcwise connected. Suppose not and $X = U \cup V$ for disjoint nonempty open sets U and V. Choose $x_0 \in U$ and $x_1 \in V$ alongside and $f \in C([0,1], X)$ s.t. $f(0) = x_0$ and $f(1) = x_1$. By part a, $f([0,1])$ is connected. However, one can write $f([0,1]) = (f([0,1]) \cap U) \cup (f([0,1] \cap V)$ which is a union of disjoint nonempty open sets. Contradiction.

c. This is famously known as the topologist's sine curve.

First, $X \setminus \{(0,0)\}$ is connected because we will show $X \setminus \{(0,0)\} = \{(s,t) \in \mathbb{R}^2 \mid t = \sin(s^{-1})\}$ is arcwise connected and apply part a.

Suppose $z_0, z_1 \in X \setminus \{(0,0\}$ are distinct points. WLOG, let $z_0 = (s_0, \sin(s_0^{-1}))$ and $z_1 = (s_1, \sin(s_1^{-1}))$ and assume $s_0 < s_1$. Define $f \in C(X \setminus \{(0,0)\}, [0,1])$ by

$$f(t) = (s_0(1-t) + s_1 t, \sin((s_0(1-t) + s_1 t)^{-1})).$$

The fact that f is a continuous function can follow from an ϵ-δ argument. Instead of an ϵ-δ argument, let us apply Proposition 4.11 where f is viewed as a function from $[0,1]$ to $\mathbb{R}^2$. The projection $\pi_1 \circ f := s_0(1-t) + s_1 t$ is continuous, since it is a linear function in t, and the projection $\pi_2 \circ f = \sin((s_0(1-t)+s_1 t)^{-1}))$ is continuous since $s_0(1-t) + s_1 t$ is never equal to zero on $[0,1]$ and $\sin(x^{-1}))$ is continuous for $x \in \mathbb{R} \setminus \{0\}$. Therefore, $f(t)$ is a continuous function. Hence, it is a continuous path connected z_0 and z_1.

Next, we show $(0,0) \in \operatorname{acc}(X \setminus \{(0,0)\})$. For every open ball $B_r((0,0))$ where $r > 0$, we have $B_r((0,0)) \cap (X \setminus \{(0,0)\} \neq \emptyset$ since we can choose $s \in (0,1]$ sufficiently small s.t. $(s, f(s)) \in B_r((0,0))$. This means $(0,0) \in \operatorname{acc}(X \setminus \{(0,0)\}$ and $\overline{X \setminus \{(0,0)\}} = X$. From part c of Folland 4.10, X is also connected.

To show that X is not arcwise connected, consider points $(0,0)$ and $(1, \sin(1))$ of X. A path $p : [0,1] \to X$ that connects the two points has the property $p(0) = (0,0)$ and $p(1) = (1, \sin(1))$ and p continuous. However, p cannot be continuous for the following reason. At $(0,0)$, consider a ball $B_\epsilon((0,0))$ of radius $\epsilon := \frac{1}{2}$. For p to be continuous, there must be some $\delta > 0$ s.t. for all $x \in [0, \delta)$, we have $p(x) \in B_\epsilon((0,0))$. However, for any $\delta > 0$, we can consider $x := \frac{1}{\frac{\pi}{2} + 2n\pi}$ for $n \in \mathbb{N}$ large enough s.t. $x \in [0, \delta)$. Then, $p(x) \notin B_\epsilon((0,0))$.

$\square$

Folland Exercise 4.27 If X_α is connected for each $\alpha \in A$ (see Exercise 10), then $X = \prod_{\alpha \in A} X_\alpha$ is connected. (Fix $x \in X$ and let Y be the connected component of x in X. Show that Y includes $\{y \in X : \pi_\alpha(y) = \pi_\alpha(x)$ for all but finitely many $\alpha\}$ and that the latter set is dense in X. Use Exercises 10 and 18.)

Proof. Fix $x \in X$ and, from part d of Folland 4.10, Y, a connected component of x in X and Y is closed. Let $Z := \{y \in X : \pi_\alpha(y) = \pi_\alpha(x) \text{ for all but finitely many } \alpha\}$. To show $Z \subseteq Y$, let $y \in Z$. By definition of Z, suppose $\pi_{\gamma_i}(y) \neq \pi_{\gamma_i}(x)$ for $1 \leq i \leq n$ and $\gamma_i \in A$.

We make use of the following fact: if a is in the same connected component as b, denoted $a \sim b$, and $b \sim c$, then $a \sim c$. Indeed, $a \sim b$ means there is a connected set $U \ni a, b$ and $b \sim c$ means there is a connected set $V \ni b, c$. But $U \cap V \ni b$ and $U \cup V$ is connected by Exercise 4.10. But $a, c \in U \cup V$ and $a \sim c$ by definition of a connected component.

Construct $Y_1 := X_{\gamma_1} \times \prod_{\beta \in A \setminus \{\gamma_1\}} \{\pi_\beta(x)\}$. Exercise 4.18 shows that Y_1 is homeomorphic[5] to X_{γ_1} and is also connected. Notice that $x \in Y_1$ and $y_1 \in Y_1$ where y_1 is defined by $\pi_{\gamma_1}(y_1) := \pi_{\gamma_1}(y)$ and $\pi_\alpha(y_1) := \pi_\alpha(x)$ for all $\alpha \neq \gamma_1$. Therefore, x is in the same connected component as y_1. Now construct $Y_2 := X_{\gamma_2} \times \{\pi_{\gamma_1}(y_1)\} \times \prod_{\beta \in A \setminus \{\gamma_1, \gamma_2\}} \{\pi_\beta(x)\}$ which is also connected by Exercise 4.18 and Exercise 4.10. Notice that $y_1 \in Y_2$ and $y_2 \in Y_2$ where y_2 is defined by $\pi_{\gamma_2}(y_2) = \pi_{\gamma_2}(y)$ and $\pi_\alpha(y_2) = \pi_\alpha(y)$ for all $\alpha \neq \gamma_2$. Therefore y_1 and y_2 are in the same connected component. Repeating this method, we can construct points $y_1, y_2, \ldots, y_n$ so that y_i and y_{i+1} lie in the same connected component and hence, y_1 and y_n lie in the same connected component. Notice that x and y_1 lie in the same connected component and so, x and y_n are in the same component. These conclusions from follow the fact we proved above. Finally, notice that inductively, y_n is actually just the point y because the coordinates of y differed from x in only those γ_i coordinates for $1 \leq i \leq n$ and the process defined y_{i+1} by changing the γ_{i+1} coordinate of y_i to $\pi_{\gamma_{i+1}}(x)$. Hence, x and y are in the same connected component. Therefore, $y \in Y$ as desired.

Next, we show Z is dense in X: for all $w \in X$, and all open neighborhood $U \ni w$, we have $U \cap Z \neq \emptyset$. We reduce to showing this for the base of the product topology since U is a union of elements in the base. By the remarks on p. 120, WLOG we may assume $U := \prod_{\alpha \in A} U_\alpha$ where only finitely many $U_\alpha \neq X_\alpha$. Since $w \in U$, $\pi_\alpha(w) \in U_\alpha$ for all $\alpha \in A$. We define a $z \in Z \cap U$ which shows $Z \cap U \neq \emptyset$. For the infinitely many $U_\alpha = X_\alpha$, set $\pi_\alpha(z) = \pi_\alpha(x)$. For the finitely many $U_\alpha \neq X_\alpha$, set $\pi_\alpha(z) = \pi_\alpha(w)$. In this way, $z \in Z$, since infinitely many $\pi_\alpha(z) = \pi_\alpha(x)$ and $z \in U$, since $\pi_\alpha(z) \in U_\alpha$ for all α. Therefore, Z is dense in X.

Since Z is dense in X, $Z \subseteq Y$, Y is closed, and Y is connected, we have $X = \overline{Z} \subseteq Y \subseteq X$ and hence, $X = Y$ which means X is connected.

$\square$

Folland Exercise 4.30 If A is a directed set, a subset B of A is called cofinal in A if for each $\alpha \in A$ there exists $\beta \in B$ such that $\beta \gtrsim \alpha$.

a. If B is cofinal in A and $\langle x_\alpha \rangle_{\alpha \in A}$ is a net, the inclusion map $B \to A$ makes $\langle x_\beta \rangle_{\beta \in B}$ a subnet of $\langle x_\alpha \rangle_{\alpha \in A}$.

b. If $\langle x_\alpha \rangle_{\alpha \in A}$ is a net in a topological space, then $\langle x_\alpha \rangle$ converges to x iff for every cofinal $B \subset A$ there is a cofinal $C \subset B$ such that $\langle x_\gamma \rangle_{\gamma \in C}$ converges to x.

Proof. (a) First, we claim B is a directed set. Clearly, $\beta \gtrsim \beta$ for all $\beta \in B$. If $\alpha \gtrsim \beta$ and $\beta \gtrsim \gamma$ are elements of B, clearly $\alpha \gtrsim \gamma$ since it is inherited from A. If $\alpha, \beta \in B$, then there is a $\gamma \in A$ s.t. $\gamma \gtrsim \alpha$ and $\gamma \gtrsim \beta$. By cofinality, choose a $\gamma_1 \in B$ s.t. $\gamma_1 \gtrsim \gamma$ and so, $\gamma_1 \gtrsim \alpha$ and $\gamma_1 \gtrsim \beta$ by transitivity.

Let $\phi : B \hookrightarrow A$ be the inclusion map. We define x_β as x_{α_β} where $\alpha_\beta = \beta$ using the inclusion map. Suppose $\alpha_0 \in A$ is given. By cofinality, we can find a $\beta_0 \in B$ s.t. $\beta_0 \gtrsim \alpha_0$. Now, whenever $\beta \gtrsim \beta_0$, we have that $\alpha_\beta = \beta \gtrsim \beta_0 \gtrsim \alpha_0$. This shows that $\langle x_\beta \rangle_{\beta \in B}$ is a subnet.

(b) ($\implies$): Assume $\langle x_\alpha \rangle \to x$. Let $B \subset A$ be cofinal in A. Clearly, B is cofinal in itself: if $\beta \in B$, just choose β so that $\beta \gtrsim \beta$.

We show $\langle x_\beta \rangle_{\beta \in B} \to x$ i.e. B is our desired cofinal C. Let $U \ni x$ be an open set. Then choose $\alpha_0 \in A$ s.t. for all $\alpha \gtrsim \alpha_0$, $x_\alpha \in U$. By B cofinal in A, choose $\beta_0 \in B$

[5]We are applying Exercise 4.18 with X_{γ_1} and viewing the point $\prod_{\beta \in A \setminus \{\gamma_i\}} \{\pi_\beta(x)\}$ as an element of the topological space $\prod_{\beta \in A \setminus \{\gamma_i\}} X_\beta$.

s.t. $\beta_0 \gtrsim \alpha_0$. Then, for all $\alpha_\beta = \beta \gtrsim \beta_0$, we have $\beta \gtrsim \alpha_0$ and hence, $x_\beta \in U$. This means $\langle x_\beta \rangle_{\beta \in B} \to x$.

($\Longleftarrow$): Suppose not and $\exists U \ni x$ open s.t. $\forall \alpha_0 \in A$, $\exists \alpha \gtrsim \alpha_0$ s.t. $x_\alpha \notin U$. Define $B := \{\beta \in A \mid x_\beta \notin U\}$. Then B is clearly cofinal by our first assumption: if $\alpha \in A$, we can find $\beta \gtrsim \alpha$ s.t. $x_\beta \notin U$ and hence, $\beta \in B$. By the hypothesis, there exists $C \subset B$ which is cofinal in B for which $\langle x_\gamma \rangle_{\gamma \in C} \to x$. Then there is a $\gamma_0 \in C$ s.t. for all $\gamma \in C$ and $\gamma \gtrsim \gamma_0$, one has $x_\gamma \in U$. By cofinality of C in B, if we view γ_0 as an element of B, there exists a $\gamma \in C$ s.t. $\gamma \gtrsim \gamma_0$. Hence, $x_\gamma \in U$. However, $\gamma \in C \subseteq B$ means that $\gamma \in B$ and hence, $x_\gamma \notin U$. Contradiction.

□

Folland Exercise 4.32 A topological space X is Hausdorff iff every net in X converges to at most one point. (If X is not Hausdorff, let x and y be distinct points with no disjoint neighborhoods, and consider the directed set $\mathcal{N}_x \times \mathcal{N}_y$ where $\mathcal{N}_x, \mathcal{N}_y$ are the families of neighborhoods of x, y.)

Proof.
($\Longrightarrow$) Suppose X were Hausdorff. Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net. We claim it converges to at most one point. Suppose it converges to two points, y and z. But that means for every neighborhood U_y of y, $\langle x_\alpha \rangle$ is eventually in U_y and the same holds true for U_z. By Hausdorff, we can take U_y and U_z neighborhoods of y and z that are disjoint. Then there is an α_0 and α_1 s.t. for all $\alpha \gtrsim \alpha_0$, $x_\alpha \in U_y$ and for all $\alpha \gtrsim \alpha_1$, $x_\alpha \in U_z$. Since A is a directed set, choose an $\alpha_3 \in A$ s.t. $\alpha_3 \gtrsim \alpha_0$ and $\alpha_3 \gtrsim \alpha_1$. Then $x_{\alpha_3} \in U_y$ and $x_{\alpha_3} \in U_z$. However, $U_y \cap U_z = \emptyset$. Contradiction.
($\Longleftarrow$) Suppose every net converges to at most one point. Let $y \neq z$ be distinct points in X. Suppose not and that there are no open sets the satisfy the Hausdorff condition. Then for all open sets U, V containing y and z respectively, $U \cap V \neq \emptyset$. Then choose $x_{(U,V)} \in U \cap V$. This defines a net $\langle x_{(U,V)} \rangle$ whose directed set is $\mathcal{N}_y \times \mathcal{N}_z$ and we order by $(U', V') \leq (U, V)$ iff $U' \supseteq U$ and $V' \supseteq V$.

We claim this net converges to both y and z. Let N_y be a neighborhood of y. Then for every $(U, V) \geq (N_y, X)$, we have $x_{(U,V)} \in U \cap V \subseteq N_y \cap X \subseteq N_y$; that is, $\langle x_{(U,V)} \rangle$ is eventually in N_y. Hence, $\langle x_{(U,V)} \rangle \to y$. Let N_z be a neighborhood of z. Then for $(U, V) \geq (X, N_z)$, we have $x_{(U,V)} \in U \cap V \subseteq N_z \cap X \subseteq N_z$; that is, $\langle x_{(U,V)} \rangle$ is eventually in N_z. Hence, $x_{(U,V)} \to z$ as well.

Since nets converge to at most one point, $y = z$. Contradiction. □

Folland Exercise 4.34 If X has the weak topology generated by a family $\mathcal{F}$ of functions, then $\langle x_\alpha \rangle$ converges to $x \in X$ iff $\langle f(x_\alpha) \rangle$ converges to $f(x)$ for all $f \in \mathcal{F}$. (In particular, if $X = \prod_{i \in I} X_i$, then $x_\alpha \to x$ iff $\pi_i(x_\alpha) \to \pi_i(x)$ for all $i \in I$.).

Proof. ($\Longrightarrow$): By definition of the weak topology, each $f \in \mathcal{F}$ is continuous. Proposition 4.19 says that $f \in \mathcal{F}$ is continuous iff for every net $\langle x_\alpha \rangle$ converging to x, one has $\langle f(x_\alpha) \rangle \to f(x)$. So, $\langle f(x_\alpha) \rangle \to f(x)$ for all $f \in \mathcal{F}$.

($\Longleftarrow$): Suppose $\langle f(x_\alpha) \rangle \to f(x)$ for every $f \in \mathcal{F}$. Let $U \ni x$ be open. By definition of the base of the weak topology, there is a neighborhood base about x. Then there is an open set $W := \bigcap_{i=1}^n f_i^{-1}(V_i)$ s.t. V_i open, $f \in \mathcal{F}$, $x \in W$ and $W \subseteq U$. For each i, since $f^{-1}(V_i)$ is a neighborhood containing x, choose an α_i s.t. for all $\alpha \gtrsim \alpha_i$, one has $x_\alpha \in f_i^{-1}(V_i)$. Since A is a directed set, there is an α_0 s.t. $\alpha_0 \gtrsim \alpha_i$ for $1 \leq i \leq n$. Then, for all $\alpha \gtrsim \alpha_0$, one has

$\alpha \gtrsim \alpha_i$ for all $1 \leq i \leq n$ and hence, $x_\alpha \in f_i^{-1}(V_i)$ for each i. Therefore, $x_\alpha \in W \subseteq U$ and since U was arbitrary, $\langle x_\alpha \rangle \to x$.

Extra: The extra statement regarding the Cartesian product follows by taking the family $\mathcal{F}$ to be the family of projection maps $\pi_i : X \to X_i$. $\square$

Folland Exercise 4.35 Let X be a set and $\mathcal{A}$ the collection of all finite subsets of X, directed by inclusion. Let $f : X \to \mathbb{R}$ be an arbitrary function, and for $A \in \mathcal{A}$, let $z_A = \sum_{x \in A} f(x)$ Then the net $\langle z_A \rangle$ converges in $\mathbb{R}$ iff $\{x : f(x) \neq 0\}$ is a countable set $\{x_n\}_{n \in \mathbb{N}}$ and $\sum_1^\infty |f(x_n)| < \infty$, in which case $z_A \to \sum_1^\infty f(x_n)$. (Cf. Proposition 0.20.[6])

Proof. ($\implies$): Suppose $\langle z_A \rangle \to z$ to some z.

First, $S := \{x : f(x) \neq 0\}$ is a countable set. Suppose not and that it were uncountable. By Proposition 0.20, $\sum_{x \in S} |f(x)| = \infty$. Then,

$$\sum_{x \in S^+} f(x) - \sum_{x \in S^-} f(x) = \sum_{x \in S} |f(x)| = \infty \qquad \text{where } S^+ := \{x : f(x) > 0\} \,\&\, S^- := \{x : f(x) < 0\}.$$

At least one of the sets on the LHS is uncountable and the corresponding sum diverges. So WLOG, assume $\sum_{x \in S^+} f(x) = \infty$ where $|S^+| > |\mathbb{N}|$. Consider the open set $(z-1, z+1) \subset \mathbb{R}$ about z. Convergence means there is an $A_0 \in \mathcal{A}$ s.t. for all $A \supseteq A_0$, $z_A \in (z-1, z+1)$. Consider $S^+ \cup A_0$ and since $S^+ \cup A_0 \supseteq A_0$, we have $z_{S^+ \cup A_0} \in (z-1, z+1)$. However, since A_0 is a finite subset, $\sum_{x \in S^+ \setminus A_0} f(x)$ diverges and

$$|z_{S^+ \cup A_0} - z| = \left| \sum_{x \in S^+ \setminus A_0} f(x) + \sum_{x \in A_0} f(x) - z \right| \geq \left| \left| \sum_{x \in S^+ \setminus A_0} f(x) \right| - \left| z - \sum_{x \in A_0} f(x) \right| \right| \geq |\infty - 1| = \infty$$

Then, $z_{S^+ \cup A_0} \notin (z-1, z+1)$ and this is a contradiction. If $|S^-| > |\mathbb{N}|$ and $-\sum_{x \in S^-} f(x) = \infty$ instead, we could consider $z_{S^- \cup A_0}$ and have $|z - z_{S^- \cup A_0}| = \infty$ and so, $z_{S^- \cup A_0} \notin (z-1, z+1)$ which is also a contradiction.

Second, we show $\sum_1^\infty |f(x_n)| < \infty$ for an enumeration $\{x_n\}$ of S. Suppose not and that $\sum_1^\infty |f(x_n)| = \infty$. We can once again consider the sets S^+ and S^- as above and at least one of $\sum_{x \in S^+} f(x)$ or $-\sum_{x \in S^-} f(x)$ are infinite. WLOG assume $\sum_{x \in S^+} f(x) = \infty$, and then consider $S^+ \cup A_0$ once more. By the same reasoning, $|z_{S^+ \cup A_0} - z| = \infty$ which is a contradictions since this means $z_{S^+ \cup A_0} \notin (z-1, z+1)$. Again, if $-\sum_{x \in S^-} f(x)$ is infinite, we would have $|z - z_{S^- \cup A_0}| = \infty$ which means $z_{S^- \cup A_0} \notin (z-1, z+1)$.

The final statement $z_A \to \sum_1^\infty f(x_n)$ follows from the fact that $\sum_1^\infty |f(x_n)| < \infty$ and hence, any enumeration of S does not affect the value of the limit $\sum_1^\infty f(x_n)$.

($\impliedby$): Assume $S = \{x_n\}_{n \in \mathbb{N}}$ is countable and $\sum_1^\infty |f(x_n)| < \infty$. To show that $\langle z_A \rangle$ converges, we show $\langle z_A \rangle \to z := \sum_1^\infty f(x_n)$. First,

$$\left| \sum_1^\infty f(x_n) \right| \leq \sum_1^\infty |f(x_n)| < \infty \qquad \implies \qquad \sum_1^\infty f(x_n) \in \mathbb{R}.$$

Let $U \ni z$ be an open neighborhood of z. Since a base of the topology on $\mathbb{R}$ consists of open intervals, assume $z \in (z - \epsilon, z + \epsilon) \subseteq U$ for an $\epsilon > 0$. Next, since $\sum_1^\infty |f(x_n)| < \infty$, choose

[6] "Given $f : X \to [0, \infty]$, let $A := \{x : f(x) > 0\}$. If A is uncountable, then $\sum_{x \in X} f(x) = \infty$. If A is countably infinite, then $\sum_{x \in X} f(x) = \sum_1^\infty f(g(n))$ where $g : \mathbb{N} \to A$ is a bijection and the sum on the right is an ordinary infinite series.

an $N \in \mathbb{N}$ s.t. for all $m \geq N$, we have $\sum_m^\infty |f(x_n)| < \epsilon$. Let $A_0 := \{x_1, x_2, \dots, x_N\}$. Suppose $A \supseteq A_0$. Then,

$$\left|z - \sum_{x \in A} f(x)\right| = \left|\sum_1^\infty f(x_n) - \sum_{x \in A \cap S} f(x)\right| = \left|\sum_{x_n \in S \setminus A} f(x_n)\right| \leq \sum_{x_n \in S \setminus A} |f(x_n)| \leq \sum_{N+1}^\infty |f(x_n)| < \epsilon$$

which means $\sum_{x \in A} f(x) \in (z - \epsilon, z + \epsilon) \subseteq U$. Therefore, $\langle z_A \rangle \to z$ as desired. □

Folland Exercise 4.38 Suppose that $(X, \mathcal{T})$ is a compact Hausdorff space and $\mathcal{T}'$ is another topology on X. If $\mathcal{T}'$ is strictly stronger than $\mathcal{T}$, then $(X, \mathcal{T}')$ is Hausdorff but not compact. If $\mathcal{T}'$ is strictly weaker than $\mathcal{T}$, then $(X, \mathcal{T}')$ is compact but not Hausdorff.

Proof. Let $1_X : (X, \mathcal{T}) \to (X, \mathcal{T}')$ be the identity map defined by $x \mapsto x$. Let $1'_X : (X, \mathcal{T}') \to (X, \mathcal{T})$ be the identity map with topologies switched around.

First Statement: Suppose $\mathcal{T}'$ is strictly stronger than $\mathcal{T}$ i.e. $\mathcal{T} \subsetneq \mathcal{T}'$. Then $1'_X : (X, \mathcal{T}') \to (X, \mathcal{T})$ is a continuous map.

First, $\mathcal{T}$ is Hausdorff because if $x \neq y$, then we can choose open disjoint sets $U, V \in \mathcal{T}$ which contain x, y respectively and then consider $1'^{-1}_X(U)$ and $1'^{-1}_X(V)$ which are our desired disjoint open sets in the $\mathcal{T}'$ topology containing x and y respectively.

Second, suppose not and that $\mathcal{T}'$ were compact.

Explicit Proof: Suppose $U \in \mathcal{T}' \setminus \mathcal{T}$. Set $Z := U^c$ which is closed w.r.t $\mathcal{T}'$. Since $(X, \mathcal{T}')$ is compact, Proposition 4.22 says Z is also compact. By Proposition 4.26, $1'_X(Z)$ is compact in $(X, \mathcal{T})$. But $(X, \mathcal{T})$ is Hausdorff so $1'_X(Z)$ is closed w.r.t. $(X, \mathcal{T})$. Thus, $(1'_X(Z))^c = 1'_X(U)$ is open w.r.t. $(X, \mathcal{T})$. But then $1'_X(U) = U$ open in $(X, \mathcal{T})$ means $U \in \mathcal{T}$ which is a contradiction.

Theorem Proof: If $\mathcal{T}'$ were compact, $1'_X$ is a bijective continuous map from a compact space to a Hausdorff space. Hence, it is a homeomorphism by Proposition 4.28. A homeomorphism preserves the topologies so $\mathcal{T} = \mathcal{T}'$ which contradicts $\mathcal{T} \subsetneq \mathcal{T}'$.

Second Statement: Suppose $\mathcal{T}'$ were strictly weaker than $\mathcal{T}$ i.e. $\mathcal{T}' \subsetneq \mathcal{T}$. Then $1_X : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is a continuous map.

First, we show $(X, \mathcal{T}')$ is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X w.r.t $\mathcal{T}'$. Then $1_X^{-1}(U_\alpha)$ is open w.r.t $\mathcal{T}$ and by compactness of $\mathcal{T}$, $\{1_X^{-1}(U_\alpha)\}_{\alpha \in A}$ has a finite subcover $\{1_X^{-1}(U_{\alpha_i})\}_{i=1}^n$. Then that means $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\{U_\alpha\}_{\alpha \in A}$ because $1_X^{-1}(U_{\alpha_i}) = U_{\alpha_i}$. This shows compactness.

Second, suppose not and that $\mathcal{T}'$ were still Hausdorff.

Explicit Proof: Let $U \in \mathcal{T} \setminus \mathcal{T}'$. Set $Z := U^c$. Then Z is closed w.r.t. $\mathcal{T}$ and by Proposition 4.22, it is compact. By Proposition 4.26, $1_X(Z)$ is compact in $\mathcal{T}'$. Since $\mathcal{T}'$ is assumed to be Hausdorff, then $1_X(Z)$ is closed in $\mathcal{T}'$. Thus, $(1_X(Z))^c = 1_X(Z^c) = 1_X(U) = U$ is open in $\mathcal{T}'$ and so, $U \in \mathcal{T}'$. However, $U \in \mathcal{T} \setminus \mathcal{T}'$ so this is a contradiction.

Theorem Proof: If $\mathcal{T}'$ were Hausdorff, 1_X is a bijective continuous map from a compact space to a Hausdorff space. Hence, it is a homeomorphism by Proposition 4.28. A homeomorphism preserves the topologies so $\mathcal{T} = \mathcal{T}'$ which contradicts $\mathcal{T}' \subsetneq \mathcal{T}$. □

Folland Exercise 4.39 Every sequentially compact space is countably compact.

Proof. Suppose X were sequentially compact but not countably compact and $\{U_k\}_{k \in \mathbb{N}}$ were a countable open cover with no finite subcover. Then the union $U_1 \cup \cdots \cup U_n$ is never a cover for X when $n \in \mathbb{N}$ and so, $X \setminus \bigcup_{k=1}^n U_k \neq \emptyset$. For each $n \in \mathbb{N}$, choose an

$x_n \in X \setminus (\cup_{k=1}^n U_k)$. This defines a sequence $\langle x_n \rangle$ in X with the property that $x_n \notin U_i$ for $i \leq n$.

By sequential compactness, there is a convergent subsequence say $\langle x_{n_k} \rangle \to x$. Since $\{U_k\}_{k\in\mathbb{N}}$ is a covering of X, we know that $x \in U_L$ for some $L \in \mathbb{N}$. Then by convergence of the subsequence, there is an n_l s.t. for all $n_k \geq n_l$, $x_{n_k} \in U_L$. Set $N := \max\{L, n_l\}$. For all $n_k \geq N$, we have $x_{n_k} \in U_L$ and $x_{n_k} \notin U_L$ by construction which is a contradiction. □

Folland Exercise 4.40 If X is countably compact, then every sequence in X has a cluster point. If X is also first countable, then X is sequentially compact.

Proof. We prove the first and second statement separately.

(a) Define $E_n := \{x_m \mid m \geq n\}$. Let $B \subseteq \mathbb{N}$ be a finite subset and $b := \max B$. Then, $\bigcap_{n\in B} \overline{E_n} \neq \emptyset$ since $x_b \in \bigcap_{n\in B} \overline{E_n}$. Since B was an arbitrary finite subset of $\mathbb{N}$, this shows that $\{\overline{E_n}\}_{n\in\mathbb{N}}$ has the finite intersection property. We claim $\bigcap_{n\in\mathbb{N}} \overline{E_n} \neq \emptyset$.

If not and $\bigcap_{n\in\mathbb{N}} \overline{E_n} = \emptyset$, we have a countable open cover $\bigcup_{n\in\mathbb{N}} X \setminus \overline{E_n} = X$ by taking the complement. By countable compactness, there is a finite subcover $\bigcup_{n\in B} X \setminus \overline{E_n} = X$. But then, $\bigcap_{n\in B} \overline{E_n} = \emptyset$ which contradicts the finite intersection property we established.

Since $\bigcap_{n\in\mathbb{N}} \overline{E_n} \neq \emptyset$, choose an $x \in \bigcap_{n\in\mathbb{N}} \overline{E_n}$. We claim x is a cluster point for $\langle x_n \rangle$. Let $U \ni x$ be an arbitrary open neighborhood. By choice of x and definition of $\overline{E_n}$, we have $U \cap E_n \neq \emptyset$ for all $n \in \mathbb{N}$. Since this occurs, for any $n \in \mathbb{N}$, we can choose an $l \geq n$ s.t. $x_l \in U \cap E_n$ by definition of E_n. Hence, $x_l \in U$ and $\langle x_n \rangle$ is frequently in U. Because U was arbitrary, x is a cluster point of $\langle x_n \rangle$.

(b) Suppose X were first countable as well. Given a sequence $\langle x_n \rangle$. By Exercise 4.7, it has a cluster point $x \in X$ and x has a countable neighborhood basis $\{V_n\}_{n\in\mathbb{N}}$ as well. WLOG, we choose a countable neighborhood basis s.t. $V_1 \supset V_2 \supset \ldots$ and we construct a convergent subsequence converging to x as follows. Take x_{n_1} where $n_1 \geq 1$ to be a point that lies in V_1 which exists by definition of a cluster point. Then take x_{n_2} to be a point that lies in V_2 s.t. $n_2 > n_1$ which also exists by definition of a cluster point. Inductively, we construct a subsequence $\{x_{n_k}\}$ and by construction this is a subsequence that converges to x: for any neighborhood $U \ni x$, choose a $V_N \subseteq U$ and then for all $n_j \geq n_N$, one has $x_{n_j} \in V_j \subset V_N \subseteq U$.

□

Folland Exercise 4.43 For $x \in [0, 1)$, let $\sum_1^\infty a_n(x)2^{-n}$ $(a_n(x) = 0 \text{ or } 1)$ be the base- 2 decimal expansion of x. (If x is a dyadic rational, choose the expansion such that $a_n(x) = 0$ for n large.) Then the sequence $\langle a_n \rangle$ in $\{0, 1\}^{[0,1)}$ has no pointwise convergent subsequence. (Hence $\{0, 1\}^{[0,1)}$, with the product topology arising from the discrete topology on {0,1} , is not sequentially compact. It is, however, compact, as we shall show in §4.6.)

Proof. A pointwise convergence subsequence of $\langle a_n \rangle$ is a subsequence $\langle a_{n_k} \rangle$ s.t. for every $x \in [0, 1)$, $\langle a_{n_k}(x) \rangle \to \langle a(x) \rangle$ for some $a(x)$. However, for any subsequence $\langle a_{n_k} \rangle$, one can construct a point $x \in [0, 1)$ of the form $\sum_1^\infty a_n(x)2^{-n}$ where $a_{n_k}(x) = 1$ for k even, $a_{n_k}(x) = 0$ for k odd, and for all other values of n not included among the n_k's, set them equal to zero. Notice that our point is not a dyadic rational because for any $m \in \mathbb{N}$, there is an $n_k \geq m$ where k is even and so, $a_{n_k}(x) \neq 0$. In which case, the subsequence $\langle a_{n_k}(x) \rangle$ does not converge to an $a(x)$ at x.

In particular, $\langle a_{n_k}(x)\rangle$ does not converge because it would need to converge to one point by Exercise 4.32, and for any small enough interval about the limit $a(x)$, the sequence is not eventually in the interval since we can always find a large enough n_k s.t. $a_{n_k}(x) = 0$ or $a_{n_k}(x) = 1$. $\square$

Folland Exercise 4.47 Prove Proposition 4.36. Also, show that if X is Hausdorff but not locally compact, Proposition 4.36 remains valid except that X^* is not Hausdorff.

Proof. $(X^*, \mathcal{T})$ **is a topological space:**

(1) First, $\emptyset \in \mathcal{T}$ because $\emptyset$ is an open subset of X. Also, $X^* \in \mathcal{T}$ because $\infty \in X^*$ and $(X^*)^c = \emptyset$ is a compact subset of X.
(2) Suppose $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ is some arbitrary collection. Then, $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ because we can handle the two cases that occur. If $\infty \notin U_\alpha$ for all $\alpha \in A$, then $U_\alpha \subseteq X$ is open and $\bigcup_{\alpha \in A} U_\alpha$ is open in X. If $\infty \in U_\beta$ for some β, then $U_\beta^c \subseteq X$ is compact. Then, $\bigcap_{\alpha \in A} U_\alpha^c$ is compact because it is an intersection closed sets depending and so it is closed in U_β^c which means it is a closed subset of a compact set which means it is compact.
(3) Let $U_i \in \mathcal{T}$ and consider $\bigcap_{i=1}^n U_i$. If $\infty \notin U_i$ for all i, then $\bigcap_{i=1}^n U_i$ is already open in X. If $\infty \in \bigcap_{i=1}^n U_i$, then $\bigcup_{i=1}^n U_i^c$ is a finite union of compact sets which is again compact. Hence, $\bigcap_{i=1}^n U_i$ is open.

X^* **is compact:** Suppose $\{U_i\}_{i \in I}$ were a covering of X^*. Then each U_i is either open in X or $\infty \in U_i$ with U_i^c compact. Since the U_i cover X^*, we have $\infty \in U_j$ for some j. Hence, U_j^c is compact and $\{U_i\}_{i \in I \setminus \{j\}}$ covers U_j^c. Hence, there is a finite subcover $\{U_{i_k}\}_{k=1}^n$ of U_j^c. Therefore, $\{U_{i_k}\}_{k=1}^n \cup \{U_j\}$ forms a finite subcover for X^*.
X^* **is Hausdorff:** The verification is obvious if we choose $x, y \in X$ distinct points since X is Hausdorff. So, consider $x \in X$ and ∞ and we show that they can be separated by open sets. Since X is locally compact, there is a compact set K s.t. $x \in K \subseteq X$. By Proposition 4.31, there exists a precompact open V s.t. $K \subset V \subset \overline{V} \subset X$. Then $x \in V$ and $\infty \in X^* \setminus \overline{V} =: U$. Certainly, U is open because $\infty \in U$ and $X^* \setminus U = \overline{V}$ is compact in X. However, $V \cap U = V \cap (X^* \setminus \overline{V}) = \emptyset$ which shows that U, V are the disjoint open neighborhoods of x and ∞ we desired.
$i : X \to X^*$ **is an embedding:** First, $i : X \to X^*$ is injective because it is just the identity map on X and $X \subset X^*$. Next, $i : X \to i(X)$ is a homeomorphism because if $i(X)$ has the relative topology as a subset of X^*, then a subset $U \subset i(X)$ is open iff U is open in X because $\infty \notin U$. Indeed, i and i^{-1} are naturally continuous because $i(U)$ is open when U is open and $i^{-1}(V)$ is open when V is open. The identity map is clearly a bijection between X and $i(X)$.
The iff statement in the proposition:

($\Longrightarrow$): If we continuously extend f, then $f(\infty) = c$ for some c. Now consider $(f - c)$ which is also continuous[7]. For all $\epsilon > 0$, $|f - c|^{-1}([0, \epsilon))$ needs to be open since $|f - c|$ is also continuous. Notice that $\infty \in |f - c|^{-1}([0, \epsilon))$ and hence, the complement $|f - c|^{-1}([\epsilon, \infty))$ needs to be compact. But that precisely means that $f - c \in C_0(X)$. Hence, $f = (f - c) + c$ with $f - c \in C_0(X)$ and we are done.

($\Longleftarrow$): If $f = g + c$ where $g \in C_0(X)$ and $f(\infty) = c$, we must show that f is continuous on X^*. WLOG, we may assume $c = 0$ because if we show continuity of $f - c = g$ on X^*, then

[7] See following footnote.

adding by a constant does not affect continuity[8]. By considering real and imaginary parts, assume f is a real-valued[9]. We show $f^{-1}((a,b))$ is open for any open interval $(a,b) \subset \mathbb{R}$ since open intervals form a base for the topology on $\mathbb{R}$. There are two cases to consider.

If $0 \notin (a,b)$, then $f^{-1}((a,b)) \subset X$. Since $f = g$ and $g \in C_0(X)$, we know f is continuous on X and therefore, $f^{-1}((a,b))$ is open in X.

If $0 \in (a,b)$, then $\infty \in f^{-1}((a,b)) \subseteq X^*$ which is open if we can show $f^{-1}((-\infty,a] \cup [b,\infty))$ is compact. We have

$$f^{-1}((-\infty,a] \cup [b,\infty)) = f^{-1}((-\infty,a]) \cup f^{-1}([b,\infty)). \tag{259}$$

Since $0 \notin (-\infty,a]$ and so $\infty \notin f^{-1}((-\infty,a]) \subseteq X$, we know $f^{-1}((-\infty,a])$ is a closed subset of X and $a < 0$. Similarly, $0 \notin [b,\infty)$ and so $\infty \notin f^{-1}([b,\infty)$ which means $f^{-1}([b,\infty))$ is a closed subset of X and $b > 0$. Because $f = g \in C_0(X)$,

$$|f|^{-1}([|a|,\infty)) \qquad \& \qquad |f|^{-1}([|b|,\infty)) \text{ are compact in } X.$$

But $f^{-1}((-\infty,a]) \subseteq |f|^{-1}([|a|,\infty))$ and $f^{-1}([b,\infty)) \subseteq |f|^{-1}([|b|,\infty)$ are now closed subsets of compact sets and are therefore compact. Since (259) shows $f^{-1}((-\infty,a] \cup [b,\infty))$ is a union of compact sets, it is also compact. This was what we wanted.

Proposition 4.36 mostly true if X^* not locally compact: We just verify that our proof did not rely on X^* being locally compact. In showing that X^* is compact, notice we did not use that hypothesis. Similarly for showing that $i : X \to X^*$ is an embedding. For the iff statement, we did not use the hypothesis in proving ($\Longrightarrow$) nor did we use it in proving ($\Longleftarrow$). The only part of the proof where we used the locally compact hypothesis was in showing X^* is Hausdorff.

X^* is not Hausdorff if X^* not locally compact: Towards a contradiction, assume X is Hausdorff and not locally compact. Then there is a point $x \in X$ that has no compact neighborhood and one can find disjoint open neighborhoods U, V of $x \in X$ and $\infty \in X^*$ respectively. Since $\infty \in V$, we know $X^* \setminus V$ is compact in X and contains $U \ni x$. Therefore, V^c is a compact neighborhood of x. Contradiction. ☐

Folland Exercise 4.55 Every open set in a second countable LCH space is σ -compact.

Proof. Let U be an open set and assume it is nonempty. Let $\mathcal{B}$ be a countable basis. Let $x \in U$ and by Proposition 4.30, there exists a compact neighborhood N_x s.t. $x \in N_x \subset U$. By Proposition 4.31, there exists a precompact open set V_x s.t. $N_x \subset V_x \subset \overline{V_x}$. Since the basis is countable, $V_x := \bigcup_{n\in\mathbb{N}} B_{n,x}$ for $B_{n,x} \in \mathcal{B}$. Furthermore, $\overline{B_{n,x}} \subseteq \overline{V_x}$ are closed subsets of $\overline{V_x}$ which is a compact set and therefore, $\overline{B_{n,x}}$ is compact. Since our point x was arbitrary, we can repeat this process to write

$$U := \bigcup_{x\in U} \bigcup_{n\in\mathbb{N}, B_{n,x}\in\mathcal{B}} \overline{B_{n,x}}.$$

Since $B_{n,x} \in \mathcal{B}$ and $\mathcal{B}$ is countable, the union above is a countable union. Therefore, U is a countable union of compact sets. Hence, U is σ-compact. ☐

Folland Exercise 4.56 Define $\Phi : [0,\infty] \to [0,1]$ by $\Phi(t) = t/(t+1)$ for $t \in [0,\infty)$ and $\Phi(\infty) = 1$

[8] Constant functions from one topological space to another are always continuous.

[9] Recall $f(x,y)$ is continuous iff $\pi_i \circ f$ is continuous for $i = 1,2$ projection maps on first and second coordinates. This is a consequence of Proposition 4.11.

a. Φ is strictly increasing and $\Phi(t+s) \leq \Phi(t) + \Phi(s)$.

b. If (Y, ρ) is a metric space, then $\Phi \circ \rho$ is a bounded metric on Y that defines the same topology as ρ.

c. If X is a topological space, the function $\rho(f,g) = \Phi\left(\sup_{x\in X} |f(x) - g(x)|\right)$ is a metric on $\mathbb{C}^X$ whose associated topology is the topology of uniform convergence.

d. If X is a σ -compact LCH space and $\{U_n\}_1^\infty$ is as in Proposition 4.39, the function

$$\rho(f,g) = \sum_1^\infty 2^{-n}\Phi\left(\sup_{x\in \bar{U}_n} |f(x) - g(x)|\right)$$

is a metric on $\mathbb{C}^X$ whose associated topology is the topology of uniform convergence on compact sets.

Proof. **Part a.** Suppose $s < t$. Note that if $t = \infty$, then the result is trivial since $\frac{s}{s+1} \leq 1$ always. So assuming $t < \infty$,

$$\frac{s}{s+1} < \frac{t}{t+1} \iff st + s < ts + t \iff s < t.$$

For the second statement, the case of $t = \infty$ or $s = \infty$ is obvious and we have the following equivalences where the last statement is true since $t, s \in [0, \infty)$:

$$\Phi(t+s) \leq \Phi(t) + \Phi(s) \iff \frac{t+s}{t+s+1} \leq \frac{t}{t+1} + \frac{s}{s+1} \iff$$
$$t^2s + 2ts + ts^2 + s^2 + s + t^2 + t \leq 2t^2s + 2ts^2 + 4ts + t^2 + t + s^2 + s \iff 0 \leq t^2s + 2ts + ts^2.$$

Part b. Note that $\Phi^{-1} : [0,1] \to [0,\infty]$ exists and is defined by $\Phi^{-1}(s) = \frac{s}{1-s}$. First, $\Phi \circ \rho$ is bounded because $\rho : Y \times Y \to [0,\infty)$ and $\Phi|_{[0,\infty)} \leq 1$. Next, the metric axioms are verified:

(1) $\Phi \circ \rho(x,y) = 0$ iff $\rho(x,y) = 0$ iff $x = y$,
(2) $\Phi \circ \rho(x,y) = \frac{\rho(y,x)}{\rho(y,x)+1} = \Phi \circ \rho(x,y)$,
(3) and $\Phi \circ \rho(x,z) \leq \Phi(\rho(x,y) + \rho(y,z)) \leq \Phi \circ \rho(x,y) + \Phi \circ \rho(y,z)$ using part a. twice.

Let $\mathcal{Q}$ be the topology generated by ρ and $\mathcal{Q}'$ be the topology generated by $\Phi \circ \rho$.

Containment $\mathcal{Q} \subseteq \mathcal{Q}'$: Suppose $U \in \mathcal{Q}$ were nonempty. Let $x \in U$ and we show there is a ball about x contained in U w.r.t the metric $\Phi \circ \rho$. There is an $r > 0$ s.t.

$$B_r^\rho(x) := \{y \in Y : \rho(y,x) < r\} \subseteq U.$$

Let $s := \frac{r}{r+1} = \Phi(r)$. Then,

$$B_s^{\Phi\circ\rho}(x) := \{y \in Y : \Phi \circ \rho(y,x) < s\} = \{y \in Y : \rho(y,x) < r\} \subset U$$

which shows $U \in \mathcal{Q}'$.

Containment $\mathcal{Q}' \subseteq \mathcal{Q}$: Suppose $V \in \mathcal{Q}'$ were nonempty. Let $x \in V$. Then, there is an $r > 0$ s.t. $B_r^{\Phi\circ\rho}(x) \subseteq V$. If $r \geq 1$, we can shrink r so that $r < 1$ since we still have $B_r^{\Phi\circ\rho}(x) \subseteq V$. Let $s := \frac{r}{1-r}$. Then $V \in \mathcal{Q}$ because

$$B_s^\rho(x) := \{y \in Y : \rho(y,x) < s\} = \{y \in Y : \Phi \circ \rho(y,x) < r\} \subseteq B_r^{\Phi\circ\rho}(x) \subseteq V.$$

and $x \in V$ was arbitrary.

Part c. First, note that we cannot apply part b. since the supremum does not define a metric on $\mathbb{C}^X$ ($\sup_{x\in X} |f - g|$ may not be finite). So, we do everything directly from the definition. For ease of notation, we fix $\Psi(f,g) := \Phi(\sup_{x\in X} |f(x) - g(x)|) = \Phi(\sup |f - g|)$ and omit the $x \in X$ subscript when it is clear.

(1) $\Psi(f,g)=0$ iff $\sup|f-g|=0$ iff $f(x)-g(x)=0$ for all $x\in X$ iff $f=g$.
(2) $\Psi(f,g)=\Psi(g,f)$ is immediate.
(3) $\Phi(\sup|f-h|)\le\Phi(\sup|f-g|+\sup|g-h|)\le\Phi(\sup|f-g|)+\Phi(\sup|g-h|)$ by application of part a.. Note that if $\sup|f-h|=\infty$, then the inequality is immediate since that forces either $\sup|f-g|=\infty$ or $\sup|g-h|=\infty$ for any $g\in\mathbb{C}^X$.

Let $\mathcal{T}$ be the topology generated by Ψ on $\mathbb{C}^X$. Let $\mathcal{T}'$ be the topology of uniform convergence on p. 133.
Containment $\mathcal{T}\subseteq\mathcal{T}'$: Let $U\in\mathcal{T}$ and assume $U\neq\emptyset$ since that case is vacuous. Then for all $f\in U$, there exists an $r>0$ s.t.

$$B_r^\Psi(f):=\{g\in\mathbb{C}^X:\Phi(\sup|f-g|)<r\}\subseteq U.$$

Choose $n\in\mathbb{N}$ s.t. $\Phi(\frac{1}{n})<r$. Then,

$$\left\{g\in\mathbb{C}^X:\sup|f-g|<\frac{1}{n}\right\}\subseteq\{g\in\mathbb{C}^X:\Phi(\sup|f-g|)<r\}\subseteq U.$$

So, U is a union of sets of the form on the LHS. The sets of the form on the LHS generate the topology $\mathcal{T}'$ and so, $U\in\mathcal{T}'$.
Containment $\mathcal{T}'\subseteq\mathcal{T}$: Let $V\in\mathcal{T}'$ (again, assume nonempty). Then V is a union of finite intersections of sets of the form $\{g\in\mathbb{C}^X:\sup|f-g|<n^{-1}\}$. To show $V\in\mathcal{T}$, let $f\in V$ be arbitrary and we show there is a ball about f contained in V. By definition of the topology of uniform convergence, there is a basis element that contains f s.t.

$$f\in\bigcap_{i\in 1}^{m}\{g\in\mathbb{C}^X:\sup|g-f_i|<n_i^{-1}\}\subset V.$$

Set $M:=\min_{1\le i\le m}\Phi(n_i^{-1})$. Then,

$$f\in B_M^\Psi(f)=\{g\in\mathbb{C}^X:\Phi(\sup|g-f|)<M\}\subseteq\{g\in\mathbb{C}^X:\sup|g-f|<n_i^{-1}\}\quad\forall 1\le i\le m.$$

Hence, $B_M^\Psi(f)\subseteq\bigcap_{i\in1}^m\{g\in\mathbb{C}^X:\sup|g-f_i|<n_i^{-1}\}\subseteq V$ and we conclude that $V\in\mathcal{T}$.
Part d. First, we check that $\rho(\cdot,\cdot)$ is a metric.

(1) $\rho(f,g)=0$ iff $\sum_1^\infty 2^{-n}\Phi(\sup_{\overline{U_n}}|f-g|)=0$ iff $\Phi(\sup_{\overline{U_n}}|f-g|)=0$ for all n iff $\sup_{x\in X}|f(x)-g(x)|=0$ iff $f(x)=g(x)$ for all $x\in X$ iff $f=g$.
(2) The symmetry $\rho(f,g)=\rho(g,f)$ is immediate.
(3) $\rho(f,h)\le\sum_1^\infty 2^{-n}\Phi(\sup_{\overline{U_n}}|f-g|)+\sum_1^\infty 2^{-n}\Phi(\sup_{\overline{U_n}}|g-h|)=\rho(f,g)+\rho(g,h)$ is immediate by part c. or part a. as needed.

Let $\mathcal{S}$ be the topology generated by ρ on $\mathbb{C}^X$. Let $\mathcal{S}'$ be the topology of uniform convergence on compact sets. We show the two topologies contain each other.
Containment $\mathcal{S}\subseteq\mathcal{S}'$: Let $U\in\mathcal{S}$ (again, assume nonempty). Let $f\in U$. Then $B_r^\rho(f)\subseteq U$ for some $r>0$. WLOG, we may shrink $r>0$ so that $1>r>0$. Then, consider the set

$$f\in H_f:=\left\{g\in\mathbb{C}^X:\sup_{x\in\overline{U_1}}|g(x)-f(x)|<\frac{1}{m}\right\}\qquad\text{where }\frac{1}{m}<\frac{r}{1-r}$$

Then,

$$\left\{g\in\mathbb{C}^X:\sup_{x\in\overline{U_1}}|g(x)-f(x)|<m^{-1}\right\}\subseteq B_r^\rho(f)\subseteq U$$

because if g is in the LHS set, we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n}\Phi\left(\sup_{x\in\overline{U_n}} |g(x) - f(x)|\right) < \sum_{n=1}^{\infty} \frac{1}{2^n} r = r.$$

Notice that $H_f \in \mathcal{S}'$ since $\overline{U_1}$ is compact. Since $f \in U$ was arbitrary, that and there always exists a ball $B_r^\rho(f) \subseteq U$, we deduce that $U := \bigcup_{f\in U} H_f$ i.e. U is a union of sets that are in $\mathcal{S}'$. Hence, $U \in \mathcal{S}'$.

Containment $\mathcal{S}' \subseteq \mathcal{S}$: Suppose $V \in \mathcal{S}'$ (assumed nonempty). By definition of the topology, V is a union of finite intersection sets of the form given on p. 133. We show $V \in \mathcal{S}$ by showing that sets of the form $\{g \in \mathbb{C}^X : \sup_{x\in K} |g - f| < n^{-1}\}$ for K compact and $n \in \mathbb{N}$ are in $\mathcal{S}$. The reason we can do is this because sets of these form generate $\mathcal{S}'$ and that means sets in $\mathcal{S}'$ are unions of finite intersections such sets. Once $\mathcal{S}$ contains these sets, since $\mathcal{S}$ is a topology, it will necessarily contain unions of finite intersections of such sets. WLOG let $V := \{g \in \mathbb{C}^X : \sup_{x\in K} |g - f| < n^{-1}\}$.

Let $h \in V$ and we must show there is an $r > 0$ s.t. $B_r^\rho(h) \subset V$. First, choose an $m \in \mathbb{N}$ s.t. $K \subset U_m \subset \overline{U_m}$.

Then choose $\alpha > 0$ small enough s.t. $\alpha 2^m < \Phi(n^{-1} - \sup_{x\in K} |h - f|)$ and $0 < \alpha 2^m < 1$.

Then, we show $B_\alpha^\rho(h) \subset V$. Suppose $g \in B_\alpha^\rho(h)$. Then,

$$\rho(g, h) = \sum_{1}^{\infty} 2^{-n}\Phi(\sup_{x\in\overline{U_n}} |g - h|) < \alpha$$

which means

$$2^{-m}\Phi(\sup_{x\in\overline{U_m}} |g - h|) < \alpha$$

and so,

$$\Phi(\sup_{x\in\overline{U_m}} |g - h|) < \alpha 2^m < \Phi(n^{-1} - \sup_{x\in K} |h - f|) \qquad \Longrightarrow \qquad \sup_{\overline{U_m}} |g - h| < n^{-1} - \sup_{x\in K} |h - f|.$$

Then, since $K \subseteq \overline{U_m}$,

$$\sup_{x\in K} |g - f| \le \sup_{x\in K} |g - h| + \sup_{x\in K} |h - f| < n^{-1} - \sup_{x\in K} |h - f| + \sup_{x\in K} |h - f| = n^{-1}$$

and so, $g \in V$. □

Folland Exercise 5.5

5. Elements of Functional Analysis

Folland Exercise 5.1 If X is a normed vector space over $K(= \mathbb{R}$ or $\mathbb{C})$, then addition and scalar multiplication are continuous from $X \times X$ and $K \times X$ to X. Moreover, the norm is continuous from X to $[0, \infty)$; in fact, $|\|x\| - \|y\|| \le \|x - y\|$.

Proof. **Addition and scalar multiplication are linear maps.** We check this to apply Proposition 5.2 and reduce the problem. Note that K is a normed vector space over K with $|k|$ for $k \in K$ as its norm. The checks are straightforward.

(a) Define $\phi : X \times X \to X$ by $(x, x') \mapsto x + x'$. Then $X \times X$ is a normed vector space with the product norm. Checking linearity $\forall k, l \in K$ and $\forall x, x', y, y' \in X$:

$$\phi(k(x, x') + l(y, y')) = kx + ly + kx' + ly' = k\phi((x, x')) + l\phi(y, y').$$

(b) Define $\psi : K \times X \to X$ by $(k, x) \mapsto kx$. Then $K \times X$ is a normed vector space with the product norm. Checking linearity, $\forall \alpha, \beta, k, k' \in K$ and $\forall x, x' \in X$:

$$\psi(\alpha(k, x) + \beta(k', x')) = \alpha kx + \beta k'x' = \alpha\psi((k, x)) + \beta\psi((k', x')).$$

Addition is continuous. Let $\epsilon > 0$ and $\delta := \frac{\epsilon}{2}$. For all $(x, x') \in X$, if $\|(x, x')\| < \delta$, then we have $\max(\|x\|, \|x'\|) < \frac{\epsilon}{2}$ and so,

$$\|\phi(x, x')\| = \|x + x'\| \leq \|x\| + \|x'\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Scalar multiplication is continuous. Let $\epsilon > 0$. Let $\delta := \sqrt{\epsilon}$. For all $(k, x) \in K \times X$, if $\|(k, x)\| < \delta$, we have $\max(|k|, \|x\|) < \sqrt{\epsilon}$ and so,

$$\|\psi(k, x)\| = \|kx\| = |k|\|x\| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon.$$

Norm is continuous. Let $\|\cdot\| : X \to [0, \infty)$. Let $y \in X$ and $\epsilon > 0$. Let $\delta := \epsilon$. Then, for all $x \in X$ s.t. $\|x - y\| < \delta$, one has

$$|\|x\| - \|y\|| \leq \|x - y\| < \delta = \epsilon.$$

We used the inequality $|\|x\| - \|y\|| \leq \|x - y\|$ in a previous HW, but we justify the inequality here. One has

$$\begin{aligned} \|x\| = \|x - y + y\| \leq \|x - y\| + \|y\| \qquad &\Longrightarrow \qquad \|x\| - \|y\| \leq \|x - y\| \\ \|y\| = \|y - x + x\| \leq \|y - x\| + \|x\| \qquad &\Longrightarrow \qquad \|y\| - \|x\| \leq \|x - y\| \end{aligned}$$

which together implies $|\|x\| - \|y\|| \leq \|x - y\|$. □

Folland Exercise 5.6 Suppose that X is a finite-dimensional vector space. Let $e_1, \ldots, e_n$ be a basis for X, and define $\|\sum_1^n a_j e_j\|_1 = \sum_1^n |a_j|$

a. $\|\cdot\|_1$ is a norm on X.

b. The map $(a_1, \ldots, a_n) \mapsto \sum_1^n a_j e_j$ is continuous from K^n with the usual Euclidean topology to X with the topology defined by $\|\cdot\|_1$.

c. $\{x \in X : \|x\|_1 = 1\}$ is compact in the topology defined by $\|\cdot\|_1$.

d. All norms on X are equivalent. (Compare any norm to $\|\cdot\|_1$.)

PROOF. **Part a.** We verify the axioms of a norm. Fix $v = \sum_1^n a_j e_j$ and $w = \sum b_j e_j$.

(1) First, $\|v\|_1 = 0$ iff $\sum_1^n |a_j| = 0$ iff $|a_j| = 0$ for each j iff $v = 0$.
(2) Second, $\|\lambda v\|_1 = \sum_1^n |\lambda||a_j| = |\lambda|\|v\|_1$.
(3) Third, $\|v + w\|_1 = \sum_1^n |a_j + b_j| \leq \sum_1^n |a_j| + \sum_1^n |b_j| = \|v\|_1 + \|w\|_1$.

Part b. By Proposition 5.2, it suffices to check continuity at zero because part a. showed X is a normed vector space and K^n is a normed vector space with usual Euclidean norm while the given map is clearly linear since

$$k(a_1, \ldots, a_n) + l(b_1, \ldots, b_n) \mapsto k \sum_1^n a_j e_j + l \sum_1^n b_j e_j.$$

Let $\epsilon > 0$. Let $\delta := \frac{\epsilon}{\sqrt{n}}$. Then if $\|(a_1, \ldots, a_n)\| = \sqrt{\sum_1^n a_j^2} < \delta$, by applying the Cauchy-Schwarz Inequality, it follows that

$$\left\| \sum_1^n a_j e_j \right\|_1 = \sum_1^n |a_j| = \sum_1^n \sqrt{a_j^2} \cdot 1 \leq \sqrt{\sum_1^n a_j^2} \sqrt{\sum_1^n 1} = \sqrt{\sum_1^n a_j^2} \sqrt{n} < \delta\sqrt{n} = \epsilon.$$

Part c. Let $B := \{x \in X : \|x\|_1 = 1\}$ and $A := \{(a_1, \ldots, a_n) \in K^n : \sum_1^n |a_j| = 1\}$. The set A is bounded in K^n (it is contained insider the ball $B_{100}(0)$ with Euclidean metric) and it is closed because $A = f^{-1}(\{1\})$ where $f : K^n \to [0, \infty)$ is defined by $f((a_1, \ldots, a_n)) = \sum_{i=1}^n |a_j|$ which is continuous since this is a norm on K^n and Exercise 5.1 shows that norms are continuous. Therefore, A is a compact subset of K^n (recalling that Folland writes K to be either $\mathbb{R}$ or $\mathbb{C}$ and we just use the Heine-Borel property here). The image of A under the map in part b. is the set B. Since the map in part b. is continuous, B is the continuous image of a compact set and hence, B is compact.

Part d. Since $\|\cdot\| : X \to [0, \infty)$ is continuous, the continuous image of $B := \{x \in X : \|x\|_1 = 1\}$ (which is compact by part c.) is also compact and so, there is a minimum L and a maximum M. That is, for all $v \in B$, we have $L \leq \|v\| \leq M$. Note that $L \neq 0$ because $L = 0$ iff $\|x\| = 0$ for some $x \in B$ iff $x = 0$ iff $\|x\|_1 = 0$ which is impossible. From these bounds, for any $v \in B$, since $\|v\|_1 = 1$, it follows that

$$L\|v\|_1 \leq \|v\| \leq M\|v\|_1.$$

Then, for any $x \in X$,

$$\begin{aligned} \|x\| &= \|x\|_1 \left\| \frac{x}{\|x\|_1} \right\| \leq \|x\|_1 M \left\| \frac{x}{\|x\|_1} \right\|_1 = M\|x\|_1 \\ \text{and} \qquad \|x\| &= \|x_1\| \left\| \frac{x}{\|x\|_1} \right\| \geq \|x\|_1 L \left\| \frac{x}{\|x\|_1} \right\|_1 = L\|x\|_1. \end{aligned}$$

Therefore, for any $x \in X$, we have $L\|x\|_1 \leq \|x\| \leq M\|x\|_1$ and this means $\|\cdot\|$ is equivalent to $\|\cdot\|_1$. Since $\|\cdot\|$ was arbitrary, every norm on X is equivalent to $\|\cdot\|_1$ and because being equivalent as norms is an equivalence relation, this shows every norm is equivalent on X.

□

Folland Exercise 5.7 Let X be a Banach space.

a. If $T \in L(X, X)$ and $\|I - T\| < 1$ where I is the identity operator, then T is invertible; in fact, the series $\sum_0^\infty (I - T)^n$ converges in $L(X, X)$ to T^{-1}.

b. If $T \in L(X, X)$ is invertible and $\|S - T\| < \|T^{-1}\|^{-1}$, then S is invertible. Thus the set of invertible operators is open in $L(X, X)$.

Proof. By Theorem 5.1, any absolutely convergent series in X converges in X. Proposition 5.4 shows that $L(X, X)$ is complete. Note that if $S, T \in L(X, X)$, then $ST = S \circ T$ by abuse of notation. By convention, for all $T \in L(X, X)$, it is defined that $T^0 = I$.

Part a. Since $T, I \in L(X, X)$, we know $I - T \in L(X, X)$ and in particular, $(I - T)^n \in L(X, X)$ for any $n \in \mathbb{N}_0$. We claim $T^{-1} := \sum_{n=0}^\infty (I - T)^n$. By the hypothesis, $\|I - T\| < 1$ and so,

$$\sum_{n=0}^\infty \|(I - T)^n\| \leq \sum_{n=0}^\infty \|I - T\|^n < \infty. \tag{260}$$

By Theorem 5.1, $T^{-1} = \sum_{n=0}^{\infty}(I-T)^n$ is defined and an element of $L(X,X)$ which means it is bounded. Next, we verify T^{-1} is actually the inverse of T:

$$\begin{aligned} TT^{-1} &= (-(I-T)+I)\sum_{n=0}^{\infty}(I-T)^n = \sum_{n=0}^{\infty} -(I-T)(I-T)^n + \sum_{n=0}^{\infty}(I-T)^n \\ &= \sum_{n=0}^{\infty}(I-T)^n - \sum_{n=0}^{\infty}(I-T)^{n+1} = I. \end{aligned}$$

and

$$T^{-1}T = \left(\sum_{n=0}^{\infty}(I-T)^n\right)(I-(I-T)) = \sum_{n=0}^{\infty}(I-T)^n - \sum_{n=0}^{\infty}(I-T)^{n+1} = I.$$

Since T has a two-sided inverse, T is bijective. Because we showed T is bijective and $T^{-1} \in L(X,X)$, we deduce that T is invertible.

Part b. From the assumptions, we know that

$$\|I - ST^{-1}\| = \|TT^{-1} - ST^{-1}| \le \|T-S\|\|T^{-1}\| = \|T^{-1}\|\|S-T\| < 1.$$

Then $\sum_{n=0}^{\infty}(I-ST^{-1})^n$ converges in $L(X,X)$ since

$$\sum_{n=0}^{\infty}\|(I-ST^{-1})^n\| \le \sum_{n=0}^{\infty}(\|T^{-1}\|S-T\|)^n < \infty$$

and so it is an absolutely convergent series. Completeness of $L(X,X)$ shows $\sum_{n=0}^{\infty}(I - ST^{-1})^n \in L(X,X)$. Next, we show that the inverse of S is defined by $S^{-1} := T^{-1}\sum_{n=0}^{\infty}(I - ST^{-1})^n$. Since $T^{-1} \in L(X,X)$, the RHS is defined and $S^{-1} \in L(X,X)$. Now we check that S^{-1} is actually a two-sided inverse for S. Indeed,

$$\begin{aligned} SS^{-1} &= S\left(T^{-1}\sum_{n=0}^{\infty}(I-ST^{-1})^n\right) = (T-(T-S))\left(T^{-1}\sum_{n=0}^{\infty}(I-ST^{-1})^n\right) \\ &= \sum_{n=0}^{\infty}(I-ST^{-1})^n - \sum_{n=0}^{\infty}(I-ST^{-1})^{n+1} = I. \end{aligned}$$

For the next computation, we first show $T^{-1}(I-ST^{-1})^n = (I-T^{-1}S)^nT^{-1}$ by induction on n. The base case $n=0$ is obvious since $T^{-1}I = IT^{-1}$. Suppose the result holds up to some $n \in \mathbb{N}$. Then,

$$\begin{aligned} T^{-1}(I-ST^{-1})^{n+1} &= T^{-1}(I-ST^{-1})^n(I-ST^{-1}) = (I-T^{-1}S)^nT^{-1}(I-ST^{-1}) \\ &= (I-T^{-1}S)^n(I-T^{-1}S)T^{-1} = (I-ST^{-1})^{n+1}T^{-1} \end{aligned}$$

where the we used the inductive hypothesis for the second equality. This auxiliary result is used in the third equality of the computation and the second to last equality follows by $T^{-1}(T-S) = I - T^{-1}S$:

$$\begin{aligned} S^{-1}S &= \left(T^{-1}\sum_{n=0}^{\infty}(I-ST^{-1})^n\right)S = \left(\sum_{n=0}^{\infty}(I-T^{-1}S)^nT^{-1}\right)S = \left(\sum_{n=0}^{\infty}(I-T^{-1}S)^nT^{-1}\right)(T-(T-S)) \\ &= \sum_{n=0}^{\infty}(I-T^{-1}S)^n - \sum_{n=0}^{\infty}(I-T^{-1}S)^nT^{-1}(T-S) = \sum_{n=0}^{\infty}(I-T^{-1}S)^n - \sum_{n=0}^{\infty}(I-T^{-1}S)^{n+1} = I. \end{aligned}$$

Since we exhibited a two-sided inverse, S is a bijective map. Because S is bijective and S^{-1} is bounded, S is invertible.

If $\mathcal{I}$ is the set of invertible linear operators, then for any $T \in \mathcal{I}$, the ball $B_{\|T^{-1}\|^{-1}}(T)$ about T is entirely contained in $\mathcal{I}$ by what we showed above. Hence, $\mathcal{I}$ is open w.r.t the operator norm on $L(X, X)$. $\square$

Folland Exercise 5.8 Let $(X, \mathcal{M})$ be a measurable space, and let $M(X)$ be the space of complex measures on $(X, \mathcal{M})$. Then $\|\mu\| = |\mu|(X)$ is a norm on $M(X)$ that makes $M(X)$ into a Banach space. (Use Theorem 5.1.)

Proof. First, we verify the axioms of a norm.

(1) If $\|\mu\| = 0$, then $|\mu|(X) = 0$ and this means μ is the zero measure.
(2) $\|\alpha\mu\| = |\alpha\mu|(X) = |\alpha||\mu|(X) = |\alpha|\|\mu\|$
(3) $\|\mu+\nu\| = |\mu+\nu|(X) \leq |\mu|(X) + |\nu|(X) \leq \|\mu\| + \|\nu\|$ by applying Proposition 3.14.

Next, the vector space axioms are essentially immediate because for $\alpha, \beta \in \mathbb{C}$ and $\mu, \nu \in M(X)$, it follows $\alpha\mu + \beta\nu \in M(X)$. Indeed, $\alpha\mu + \beta\nu(\emptyset) = 0$ and

$$\alpha\mu + \beta\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \alpha\sum_{j=1}^{\infty}\mu(E_j) + \beta\sum_{j=1}^{\infty}\nu(E_j) = \sum_{j=1}^{\infty}\alpha\mu + \beta\nu(E_j)$$

for $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and the series in the middle obviously converge absolutely and hence, the series on the RHS converges absolutely.

We have thus shown that $M(X)$ is a normed vector space and to show completeness, we employ Theorem 5.1. Suppose $\sum_{n=1}^{\infty}\|\mu_n\| < \infty$. Then $\sum_{n=1}^{\infty}\mu_n$ is a well-defined since it converges i.e. for any $F \in \mathcal{M}$, we have, by application of Proposition 3.13,

$$\sum_{n=1}^{\infty}|\mu_n(F)| \leq \sum_{n=1}^{\infty}|\mu_n|(F) \leq \sum_{n=1}^{\infty}|\mu_n|(X) = \sum_{n=1}^{\infty}\|\mu_n\| < \infty.$$

Next, we verify that this is actually a complex measure.

(1) First, $\left(\sum_1^{\infty}\mu_n\right)(\emptyset) = \sum_1^{\infty}\mu_n(\emptyset) = 0$.
(2) Let $\{E_j\}_{j=1}^{\infty}$ be a disjoint sequence of measurable sets. Let $E := \bigcup_{j=1}^{\infty} E_j$. Then $\mu_n(E) = \mu_n\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty}\mu_n(E_j)$ and the sum converges absolutely by definition of a complex measure. Therefore,

$$\sum_{n=1}^{\infty}\mu_n(E) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\mu_n(E_j) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\mu_n(E_j) = \sum_{j=1}^{\infty}\left(\sum_{n=1}^{\infty}\mu_n(E_j)\right)$$

where changing the order of integration is justified by applying the Fubini-Tonelli Theorem with the counting measure, using the fact that $\sum_{j=1}^{\infty}\mu_n(E_j)$ converges absolutely, and $\sum_{n=1}^{\infty}\|\mu_n\|<\infty$. Notice that the RHS converges absolutely because

$$\begin{aligned}\sum_{j=1}^{\infty}\left|\left(\sum_{n=1}^{\infty}\mu_n(E_j)\right)\right| &\leq \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}|\mu_n(E_j)| \leq \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}|\mu_n|(E_j) && \text{(second inequality by Propositio}\\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}|\mu_n|(E_j) \leq \sum_{j=1}^{\infty}|\mu_n|(E) && \text{(apply Fubini-Tonelli and } |\mu_n| \text{ is a complex me}\\ &\leq \sum_{j=1}^{\infty}|\mu_n|(X) = \sum_{j=1}^{\infty}\|\mu_n\| < \infty && \text{(from monotonicity and hypot}\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty}\mu_n \in M(X)$. □

Folland Exercise 5.9 Let $C^k([0,1])$ be the space of functions on $[0,1]$ possessing continuous derivatives up to order k on $[0,1]$, including one-sided derivatives at the endpoints.

a. If $f\in C([0,1])$, then $f\in C^k([0,1])$ iff f is k times continuously differentiable on $(0,1)$ and $\lim_{x\searrow 0} f^{(j)}(x)$ and $\lim_{x\nearrow 1} f^{(j)}(x)$ exist for $j\leq k$. (The mean value theorem is useful.)

b. $\|f\|=\sum_0^k \left\|f^{(j)}\right\|_u$ is a norm on $C^k([0,1])$ that makes $C^k([0,1])$ into a Banach space. (Use induction on k. The essential point is that if $\{f_n\}\subset C^1([0,1]), f_n\to f$ uniformly, and $f_n'\to g$ uniformly, then $f\in C^1([0,1])$ and $f'=g$. The easy way to prove this is to show that $f(x)-f(0)=\int_0^x g(t)dt$.)

Folland Exercise 5.10 Let $L_k^1([0,1])$ be the space of all $f\in C^{k-1}([0,1])$ such that $f^{(k-1)}$ is absolutely continuous on $[0,1]$ (and hence $f^{(k)}$ exists a.e. and is in $L^1([0,1])$). Then $\|f\|=\sum_0^k\int_0^1\left|f^{(j)}(x)\right|dx$ is a norm on $L_k^1([0,1])$ that makes $L_k^1([0,1])$ into a Banach space. (See Exercise 9 and its hint.)

Proof. □

Folland Exercise 5.11 If $0<\alpha\leq 1$, let $\Lambda_\alpha([0,1])$ be the space of Hölder continuous functions of exponent α on $[0,1]$. That is, $f\in\Lambda_\alpha([0,1])$ iff $\|f\|_{\Lambda_\alpha}<\infty$, where

$$\|f\|_{\Lambda_\alpha}=|f(0)|+\sup_{x,y\in[0,1],x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

a. $\|\cdot\|_{\Lambda_\alpha}$ is a norm that makes $\Lambda_\alpha([0,1])$ into a Banach space.

b. Let $\lambda_\alpha([0,1])$ be the set of all $f\in\Lambda_\alpha([0,1])$ such that $\frac{|f(x)-f(y)|}{|x-y|^\alpha}\to 0$ as $x\to y$, for all $y\in[0,1]$. If $\alpha<1, \lambda_\alpha([0,1])$ is an infinite-dimensional closed subspace of $\Lambda_\alpha([0,1])$. If $\alpha=1, \lambda_\alpha([0,1])$ contains only constant functions.

Proof. **[Incomplete]** a. Verifying that this is a norm is relatively straightforward.

b. The second statement is clear when $\alpha=1$ because if not the function is not constant, then there is a point in which the limit would not converge to zero which is absurd. For the statement about $\alpha<1$ verifying that it is a closed subspace is not difficult. However, the infinite dimensional part is not entirely certain for me. □

Folland Exercise 5.12 Let X be a normed vector space and $\mathcal{M}$ a proper closed subspace of x.

a. $\|x+\mathcal{M}\|=\inf\{\|x+y\|:y\in\mathcal{M}\}$ is a norm on $X/\mathcal{M}$

b. For any $\epsilon>0$ there exists $x\in X$ such that $\|x\|=1$ and $\|x+\mathcal{M}\|\geq 1-\epsilon$.

c. The projection map $\pi(x)=x+\mathcal{M}$ from X to $X/\mathcal{M}$ has norm 1 .

d. If X is complete, so is $X/\mathcal{M}$. (Use Theorem 5.1.)
e. [Incomplete]

Proof. **Part a.** First, we check that this is actually well-defined. Suppose $x - x' \in \mathcal{M}$. Then, $-x' + x \in \mathcal{M}$ by $\mathcal{M}$ subspace of X and so,

$$\|x + \mathcal{M}\| = \|x + (-x + x') + \mathcal{M}\| = \|x' + \mathcal{M}\|.$$

Next, we verify the three axioms of a norm.

i. If $\|x + \mathcal{M}\| = 0$, then there is a sequence of $y_n \in \mathcal{M}$ s.t. $\|x + y_n\| \to 0$. Then the sequence y_n is converges to $-x$ because $\|y_n - (-x)\| \to 0$ as $n \to \infty$ and since $\mathcal{M}$ is a closed subspace, $-x \in \mathcal{M}$. Since $\mathcal{M}$ is a subspace, $x \in \mathcal{M}$. Hence, $x + \mathcal{M} = \mathcal{M}$ as desired.

If $x + \mathcal{M} = \mathcal{M}$, then $x \in \mathcal{M}$ which means $\|x + \mathcal{M}\| = 0$ since $-x \in \mathcal{M}$ by the subspace axioms and hence, $\|x + \mathcal{M}\| \leq \|x - x\| = 0$.

ii. Suppose $\alpha \in \mathbb{C}$. Then,

$$|\alpha|\|x+\mathcal{M}\| = |\alpha| \inf\{\|x+y\| : y \in \mathcal{M}\} = \inf\{\|\alpha x+\alpha y\| : y \in \mathcal{M}\} = \inf\{\|\alpha x+y\| : y \in \mathcal{M}\} = \|\alpha x+\mathcal{M}\|.$$

iii. Given $x + \mathcal{M}, y + \mathcal{M} \in X/\mathcal{M}$. Since we can always write $z = \frac{1}{2}z + \frac{1}{2}z$ and $\mathcal{M}$ is a subspace,

$$\begin{aligned}\|x + y + \mathcal{M}\| &= \inf\{\|x + y + z\| : z \in \mathcal{M}\} \leq \inf\{\|x + z_1\| : z_1 \in \mathcal{M}\} + \inf\{\|y + z_2\| : z_2 \in \mathcal{M}\} \\ &= \|x + \mathcal{M}\| + \|y + \mathcal{M}\|.\end{aligned}$$

Part b. Let $\epsilon > 0$. If $\epsilon > 1$, then the result is obvious since any unit vector would work. So assume $0 < \epsilon < 1$. Since $\mathcal{M}$ is a proper subspace, let $x \in X \setminus \mathcal{M}$ and consider $x + \mathcal{M}$. By definition of $\|x + \mathcal{M}\|$, we can find a $y \in \mathcal{M}$ s.t.

$$\|x + \mathcal{M}\| \geq \|x + y\|(1 - \epsilon). \tag{261}$$

But notice that $\|x + \mathcal{M}\| = \|x + y + \mathcal{M}\|$ and so if we take $z := \frac{x+y}{\|x+y\|}$, we get

$$\begin{aligned}\|z + \mathcal{M}\| &= \inf\left\{\left\|\frac{x + y}{\|x + y\|} + w\right\| : w \in \mathcal{M}\right\} = \inf\left\{\left\|\frac{x}{\|x + y\|} + w\right\| : w \in \mathcal{M}\right\} = \left\|\frac{x}{\|x + y\|} + \mathcal{M}\right\| \\ &= \frac{1}{\|x + y\|}\|x + \mathcal{M}\| \geq \frac{1}{\|x + y\|}\|x + y\|(1 - \epsilon) = (1 - \epsilon)\end{aligned}$$

where we used (261) to get the inequality. Note that the second equality above follows from $\frac{y}{\|x+y\|} \in \mathcal{M}$. Now, this means z is our desired element since it has norm one by our definition. The only issue that can occur is if $\|x + y\| = 0$ but this cannot occur since $x \in X \setminus \mathcal{M}$ and $y \in \mathcal{M}$.

Part c. From (b), for all $\epsilon > 0$, there is a $y \in X$ s.t. $\|y\| = 1$ and $\|y + \mathcal{M}\| \geq 1 - \epsilon$ and so,

$$\|\pi\| = \sup\{\|\pi(x)\| : \|x\| = 1\} = \sup\{\|x + \mathcal{M}\| : \|x\| = 1\} \geq \|y + \mathcal{M}\| \geq 1 - \epsilon$$

Since this is true for all $\epsilon > 0$, $\|\pi\| \geq 1$. It suffices to show $\|\pi\| \leq 1$. Indeed, since $\|x + \mathcal{M}\| \leq \|x\|$ (because $0 \in \mathcal{M}$),

$$\|\pi\| = \inf\{C \geq 0 : \|x + \mathcal{M}\| \leq C\|x\|\} \leq 1. \tag{262}$$

Altogether, $\|\pi\| = 1$.

Part d. To show that $X/\mathcal{M}$ is complete, we use Theorem 5.1. Let $F := \sum_{k=1}^{\infty} x_k + \mathcal{M}$ be an absolutely convergence series i.e. $\sum_{k=1}^{\infty} \|x_k + \mathcal{M}\| < \infty$. By definition of $\|x_k + \mathcal{M}\|$, for every $\epsilon > 0$, there exists a $y_k \in \mathcal{M}$ s.t.

$$\|x_k + y_k\| \leq \|x_k + \mathcal{M}\| + \epsilon.$$

Taking $\epsilon = \frac{1}{2^k}$ where $k \in \mathbb{N}$, then for all $k \in \mathbb{N}$, there exists a $y_k \in \mathcal{M}$ s.t. $\|x_k + y_k\| \leq \|x_k + \mathcal{M}\| + \frac{1}{2^k}$. Therefore,

$$\sum_{k=1}^{\infty} \|x_k + y_k\| \leq \sum_{k=1}^{\infty} \|x_k + \mathcal{M}\| + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$$

which means $\sum_{k=1}^{\infty} x_k + y_k$ is an absolutely convergent series and hence, converges in X. Notice that $\{y_k\}$ is a sequence in $\mathcal{M}$ and since $\mathcal{M}$ is closed, $\sum_{k=1}^{\infty} y_k$ converges in $\mathcal{M}$. Therefore, if $z = \lim_{n\to\infty} \sum_{k=1}^{n} x_k + y_k$ we have $z + \mathcal{M} = \sum_{k=1}^{\infty} x_k + \mathcal{M}$. Altogether, this shows that

$$\sum_{k=1}^{\infty} x_k + \mathcal{M} = \sum_{k=1}^{\infty} x_k + \sum_{k=1}^{\infty} y_k + \mathcal{M} = \lim_{n\to\infty} \sum_{k=1}^{n} \pi(x_k + y_k) = \lim_{n\to\infty} \pi\left(\sum_{k=1}^{n} x_k + y_k\right) = \pi\left(\sum_{k=1}^{\infty} x_k + y_k\right) \in X/\mathcal{M}$$

Hence, $\sum_{k=1}^{\infty} x_k + \mathcal{M}$ converges in $X/\mathcal{M}$.

□

[10]**Folland Exercise 5.18** Let X be a normed vector space.

a. If $\mathcal{M}$ is a closed subspace and $x \in X \backslash \mathcal{M}$ then $\mathcal{M} + \mathbb{C}x$ is closed. (Use Theorem 5.8a.)

b. Every finite-dimensional subspace of X is closed.

PROOF. **a.** By Theorem 5.8a, there exists an $f \in X^*$ s.t. $f(x) \neq 0$ and $f|_{\mathcal{M}} = 0$. Multiplying f by the constant $\frac{1}{f(x)}$, we may assume $f(x) = 1$.

To show $\mathcal{M} + \mathbb{C}x$ is closed, suppose there were a sequence $z_n := m_n + \lambda_n x \in \mathcal{M} + \mathbb{C}x$ and $z_n \to z$ converges.

Consider the sequence $\{z_n - f(z_n)x\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$. The reason this sequence is in $\mathcal{M}$ is because $f(z_n) = f(m_n) + \lambda_n f(x) = \lambda_n$ which means

$$z_n - f(z_n)x = z_n - \lambda_n x = m_n \in \mathcal{M}.$$

Then, by closedness of $\mathcal{M}$, this means

$$\lim_{n\to\infty} z_n - f(z_n)x = z - \lim_{n\to\infty} \lambda_n f(x)x = z - \lim_{n\to\infty} \lambda_n x \in \mathcal{M}.$$

So, there is an $m \in \mathcal{M}$ s.t. $m = z - \lim_{n\to\infty} \lambda_n x$. Hence, $z = m - \lim_{n\to\infty} \lambda_n x$. Therefore, $z \in \mathcal{M} + \mathbb{C}x$ as desired.

b. We induct on the dimension of the subspace. For the base case, if the dimension is zero, we just have the zero subspace and it is closed under any norm. Assume the result is true for all subspaces of dimension $\leq n$. Let $\mathcal{M}$ be a subspace of dimension $n+1$. Choose a basis $\{x_i\}_{i=1}^{n+1}$ for $\mathcal{M}$. Then $\mathbb{C}x_1 + \cdots + \mathbb{C}x_n$ is a closed subspace by the inductive hypothesis. Clearly $x_{n+1} \notin \mathbb{C}x_1 + \cdots + \mathbb{C}x_n$ by linear independence and so, by part a., $(\mathbb{C}x_1 + \cdots + \mathbb{C}x_n) + \mathbb{C}x_{n+1} = \mathcal{M}$ is closed.

[10]For this Homework, I mainly collaborated and discussed with Scotty Tilton. I also received a hint from James Wong on Exercise 5.18. Supplemental resources were the course text, extra reading from "Real Analysis" by Stein and Shakarchi, "Real Analysis" by Royden, and "Real and Complex Analysis" by Walter Rudin.

□

Folland Exercise 5.19 Let X be an infinite-dimensional normed vector space.

a. There is a sequence $\{x_j\}$ in X such that $\|x_j\| = 1$ for all j and $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$. (Construct x_j inductively, using Exercises 12 b and 18.)

b. X is not locally compact.

PROOF. **a.** First, choose an element $x_1 \in X$ s.t. $\|x_1\| = 1$. Then $\mathcal{M}_1 := \mathbb{C}x_1$ is a closed subspace by Exercise 18b. Let $\epsilon = \frac{1}{2}$ and by Exercise 12b, we can find an $x_2 \in X$ s.t. $\|x_2\| = 1$ and

$$\|x_2 - x_1\| \geq \|x_2 + \mathcal{M}_1\| \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

By Exercise 18, the subspace $\mathcal{M}_2 := \mathbb{C}x_1 + \mathbb{C}x_2$ is closed since it is finite dimensional. Indeed, $\|x_2 + \mathcal{M}_1\| \geq \frac{1}{2}$ means that $x_2 \notin \mathcal{M}_1$ and therefore, x_1 and x_2 are linearly independent. This means $\mathcal{M}_2 = \mathbb{C}x_1 \oplus \mathbb{C}x_2$. By Exercise 12b, we can find an $x_3 \in X$ s.t. $\|x_3 + \mathcal{M}_2\| \geq \frac{1}{2}$. Since $x_1, x_2 \in \mathcal{M}_2$, this implies

$$\begin{aligned} &\|x_3 - x_1\| \geq \|x_3 + \mathcal{M}_2\| \geq \frac{1}{2} \\ \text{and} \qquad &\|x_3 - x_2\| \geq \|x_3 + \mathcal{M}_2\| \geq \frac{1}{2}. \end{aligned}$$

Then, repeat the process with $\mathcal{M}_3 := \mathcal{M}_2 + \mathbb{C}x_3 = \mathcal{M}_2 \oplus \mathbb{C}x_3$ since $x_3 \notin \mathcal{M}_2$ means $\{x_1, x_2, x_3\}$ forms a linearly independent spanning set of $\mathcal{M}_3$.

Now if $\mathcal{M}_n$ is given in this process, then $\mathcal{M}_n = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$ and we can find an $x_{n+1} \in X$ s.t. $\|x_{n+1}\| = 1$ and $\|x_{n+1} + \mathcal{M}_n\| \geq \frac{1}{2}$ which means $\|x_{n+1} - x_j\| \geq \frac{1}{2}$ for all $1 \leq j \leq n$.

Inductively, we obtain a sequence of points $\|x_j - x_k\| \geq \frac{1}{2}$ whenever $j > k$. By positive homogeneity of the norm, $\|x_j - x_k\| = \|x_k - x_j\| \geq \frac{1}{2}$ whenever $k < j$ as well. Hence, $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$.

b. To show that X is not locally compact, we show that there is a point in X that has no compact neighborhood. Because X is a normed vector space and the maps $x \mapsto rx$ and $x \mapsto x + y$ are homeomorphisms for all scalars r and $y \in X$, it suffices to show there is no compact neighborhood about the origin containing the unit ball.

Suppose not and N were a compact neighborhood at 0. By Folland's definition, this means $0 \in N^\circ$ so there a is ball $B_r(0)$ of some radius $r > 0$ s.t. $0 \in B_r(0) \subseteq N$. By scaling the space (using a homeomorphism $x \mapsto \frac{1}{2r}x$), assume that N contains the closed unit ball $\overline{B}$. From part a., we can find a sequence $\{x_j\}_{j=1}^\infty$ in $\overline{B} \subseteq N$ s.t. $\|x_j - x_k\| \geq \frac{1}{2}$ for all $j \neq k$. This latter condition implies that $\{x_j\}_{j=1}^\infty$ has no convergent subsequence. Since, N is compact and closed subsets of compact sets are compact, $\overline{B}$ is compact. By Theorem 4.29, since $\overline{B}$ is compact, any sequence in $\overline{B}$ must have a convergent subsequence. Contradiction.

□

Folland Exercise 5.20 If $\mathcal{M}$ is a finite-dimensional subspace of a normed vector space X, there is a closed subspace $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = X$.

PROOF. The subspace $\mathcal{M}$ is closed by Exercise 5.18b and has basis $\{e_i\}_{i=1}^n$. Each element of $x \in \mathcal{M}$ is represented by $P(x) = \alpha_1(x)e_1 + \cdots + \alpha_n(x)e_n$ and each α_i extends to a bounded linear functional on $\mathcal{X}$. The space $\mathcal{N}$ is then the intersection of the null spaces of these bounded linear functionals. Then any $x \in X$ has form $x = Px + (x - Px)$ and $Px \in \mathcal{M}$ while $x - Px \in \mathcal{N}$ (applying P would give $Px - P^2x = Px - Px = 0$ and similarly for any extension of α_i). $\square$

Folland Exercise 5.21 If X and y are normed vector spaces, define $\alpha : X^* \times Y^* \to (X \times Y)^*$ by $\alpha(f,g)(x,y) = f(x) + g(y)$. Then α is an isomorphism which is isometric if we use the norm $\|(x,y)\| = \max(\|x\|, \|y\|)$ on $X \times Y$, the corresponding operator norm on $(X \times Y)^*$, and the norm $\|(f,g)\| = \|f\| + \|g\|$ on $X^* \times Y^*$.

Proof. It is clearly isomorphic since the map is linearly and one defines the inverse map in the obvious way (over each component). Checking the isometry follows by using the norm on $X \times Y$ to deduce that if $\|x\| = \|y\| = 1$, then $\|(x,y)\| = 1$. Indeed, we would get

$$\|\alpha(f,g)\|_{op} = \sup\{|f(x)+g(y)| : \|(x,y)\| = 1\} = \sup\{|f(x)|+|g(y)| : \|x\| = \|y\| = 1\} = \|f\|+\|g\| = \|(f,g)\|$$

The middle inequality is nontrivial but $\leq$ is obvious and $\geq$ follows by scaling x by $\operatorname{sgn}((x))\operatorname{sgn}((y))$ which does not affect the norm of x nor the absolute value of $f(x)$. $\square$

Folland Exercise 5.22 Suppose that X and Y are normed vector spaces and $T \in L(X,Y)$.

a. Define $T^\dagger : Y^* \to X^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger \in L(Y^*, X^*)$ and $\left\|T^\dagger\right\| = \|T\|$. $T^\dagger$ is called the adjoint or transpose of T.

b. Applying the construction in (a) twice, one obtains $T^{\dagger\dagger} \in L(X^{**}, Y^{**})$. If X and Y are identified with their natural images $\widehat{X}$ and $\widehat{Y}$ in X^{**} and Y^{**}, then $T^{\dagger\dagger} \mid X = T$

c. $T^\dagger$ is injective iff the range of T is dense in Y.

d. If the range of $T^\dagger$ is dense in X^*, then T is injective; the converse is true if X is reflexive.

Proof. **a.** First, we check linearity, for all $f, g \in Y^*$ and $a, b \in K$, we have

$$T^\dagger(af + bg) = (af + bg) \circ T = af \circ T + bg \circ T = aT^\dagger f + bT^\dagger.$$

Next, since for all $x \in X$, we can write

$$\|fTx\| \leq \|fT\|\|x\| \leq \|T\|\|f\|\|x\|,$$

it follows

$$\|T^\dagger f\| = \|f \circ T\| \leq \|f\|\|T\| < \infty.$$

This means $\|T^\dagger\| \leq \|T\|$ and also, $T^\dagger \in L(Y^*, X^*)$.

Now we show the other inequality. Let $x \in X$. If $Tx = 0$, then the inequality $\|Tx\| \leq C\|x\|$ is trivial since we can just take $C = \|T^\dagger\|$. So suppose we are given $Tx \neq 0$. Then, choose an $f \in Y^*$ s.t. $\|f\| = 1$ and $f(Tx) = \|Tx\|$ by Theorem 5.8b. Then,

$$\|Tx\| = |f(Tx)| = \|f(Tx)\| = \|(f \circ T)x\| \leq \|f \circ T\|\|x\| = \|T^\dagger f\|\|x\| \leq \|T^\dagger\|\|f\|\|x\| = \|T^\dagger\|\|x\|$$

where the last equality follows by $\|f\| = 1$. So, for all $x \in X$, we know $\|Tx\| \leq \|T^\dagger\|\|x\|$. By definition of $\|T\|$, this means $\|T\| \leq \|T^\dagger\|$. At last, $\|T^\dagger\| = \|T\|$ as desired.

b. Embed X and Y into X^{**} and Y^{**} by identifying X with $\widehat{X}$ and Y with $\widehat{Y}$. By applying the construction twice, this means $T^{\dagger\dagger} = (T^\dagger)^\dagger$. Then showing $T^{\dagger\dagger}|_X = T$ entails showing $T^{\dagger\dagger}(\widehat{x}) = \widehat{Tx}$ for all $x \in X$. Indeed, for all $x \in X$ and all $g \in Y^*$, we have

$$[T^{\dagger\dagger}(\widehat{x})](g) = (\widehat{x}T^\dagger)(g) = \widehat{x}(\big(T^\dagger(g)\big)) = \widehat{x}(g(T)) = g(T(x)) = g(Tx) = \widehat{Tx}(g) \qquad \Longrightarrow \qquad T^{\dagger\dagger}(\widehat{x}) = \widehat{Tx}.$$

This shows $T^{\dagger\dagger}|_X = T$.

c. ($\implies$): Suppose $T^\dagger$ is injective which means, since $T^\dagger$ is linear, if $T^\dagger f = 0$ for some $f \in Y^*$, then $f = 0$. Towards, a contradiction, assume T is not dense in Y and therefore, there exists $y \in Y \setminus \overline{T(X)}$. By Theorem 5.8a, there is an $f \in Y^*$ s.t. $f(y) \neq 0$, $f|_{\overline{T(X)}} = 0$. Therefore, $T^\dagger f = f \circ T = 0$ by definition of f. However, this contradicts injectivity of $T^\dagger$ since f is not the zero linear functional.

($\impliedby$): Suppose the range of $T^\dagger$ is dense in Y. Because $T^\dagger$ is a linear map, to show injectivity, it suffices to show $T^\dagger f = 0$ if, then $f = 0$.

Let $f \in Y^*$ s.t. $T^\dagger f = f \circ T = 0$. Let g denote the zero linear functional. Then, for all x,

$$f(Tx) = f \circ T(x) = 0 = g \circ T(x) = g(Tx) \qquad \implies \qquad f = g \text{ on } T(X).$$

Since $\mathbb{C}$ is Hausdorff, $T(X)$ is dense in Y, and $f, g \in Y^*$ are continuous, applying Exercise 4.16, we deduce that $f = g$ on Y.

d. ($\implies$) To show injectivity, it suffices to show that if $Tx = 0$, then $x = 0$. Suppose $Tx = 0$. Then $\widehat{x} : X^* \to \mathbb{C}$ is defined by $f \mapsto f(x)$ and so, $\widehat{x}$ is defined on $T^\dagger(Y^*) \subseteq X^*$. For all $f \circ T \in T^\dagger(Y^*)$, we have

$$f \circ T(x) = f(0) = f \circ T(0).$$

Therefore, $\widehat{x} = \widehat{0}$ on $T^\dagger(Y^*)$. But this means $\widehat{x} = \widehat{0}$ on the dense set $T^\dagger(Y^*) \subseteq X^*$ and since $\mathbb{C}$ is Hausdorff, we can apply Proposition 4.16 to conclude $\widehat{x} = \widehat{0}$ on X^*.

Next, we show this means $x = 0$. Because linear functionals separate points of X by Proposition 5.8c, if $x \neq 0$, there exists an $f \in X^*$ s.t. $f(x) \neq f(0)$ and this means $\widehat{x}f \neq \widehat{0}f$ which is a contradiction.

($\impliedby$) Assume further that X is reflexive i.e. $\widehat{X} = X^{**}$. Towards a contradiction, suppose $T^\dagger(Y^*)$ is not dense in X^*. So, there is an $f \in X^* \setminus \overline{T^\dagger(Y^*)}$. By Theorem 5.8a, there exists $\phi \in X^{**}$ s.t. $\phi f \neq 0$ and $\phi|_{\overline{T^\dagger(Y^*)}} = 0$. Since X is reflexive, there exists an $x \in X$ s.t. $\widehat{x} = \phi$ and we write $\widehat{x}$ for ϕ from here on.

Because T is injective, we know $Tx' = 0$ iff $x' = 0$. Certainly, $x \neq 0$ because $\widehat{x}f = \phi f \neq 0$ (and by definition, $\widehat{x}f = f(x)$). Therefore, $Tx \neq 0$. By Theorem 5.8c, there exists an $h \in Y^*$ s.t. $h(Tx) \neq h(0)$. Then $T^\dagger h \in T^\dagger(Y^*) \subseteq X^*$ and since $\widehat{x}|_{\overline{T^\dagger(Y^*)}} = 0$, it follows $\widehat{x}(T^\dagger h) = \widehat{x}(hT) = h(Tx) = 0$. However, by definition of h,

$$\widehat{x}(h \circ T) = (h \circ T)(x) = h(Tx) \neq h(0) = 0$$

which is a contradiction. ☐

Folland Exercise 5.25 If X is a Banach space and X^* is separable, then X is separable. (Let $\{f_n\}_1^\infty$ be a countable dense subset of X^*. For each n choose $x_n \in X$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Then the linear combinations of $\{x_n\}_1^\infty$ are dense in X.) Note: Separability of X does not imply separability of X^*.

Proof. [11] If $\{f_n\}_1^\infty \subseteq X^*$ is a countable dense subset, then for each n, we can choose $x_n \in X$ s.t. $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. We are able to make such a choice because by definition,

$$\|f_n\| = \sup\{\|f_n x\| : x \in X, \|x\| = 1\}.$$

Let $\mathcal{S}$ be the set of finite linear combinations of $\{x_n\}_1^\infty$ whose coefficients are in $\mathbb{Q}$ or $\mathbb{Q} + i\mathbb{Q}$ depending on if our ground field K is $\mathbb{R}$ or $\mathbb{C}$. This ensures that $\mathcal{S}$ is countable.

[11] The hypothesis that X is Banach does not look like it is needed, we do not need it for Theorem 5.8 nor do we need it to find the sequence $f_{n_k} \to f$ in X^*.

Indeed, let $\mathcal{S}_m$ be the set of finite linear combinations with m terms. Since the scalars are in a countable set, $\mathcal{S}_m$ is countable. Then, $\mathcal{S} = \bigcup_{m=0}^{\infty} \mathcal{S}_m$ is a countable union of countable sets and hence, $\mathcal{S}$ is countable.

Towards a contradiction, assume $\mathcal{S}$ is not dense. Then $\overline{\mathcal{S}}$ is a proper closed subspace of X. So, there is an $x \in X \setminus \overline{\mathcal{S}}$ and by Theorem 5.8a, there exists an $f \in X^*$ s.t. $f(x) \neq 0$, $f|_{\overline{\mathcal{S}}} = 0$, $\|f\| = 1$, and $f(x) = \inf_{y \in \overline{\mathcal{S}}} \|x - y\|$.

Since $f \in X^*$ and $\{f_n\}_1^{\infty}$ is dense in X^*, choose a sequence $f_{n_k} \to f$ in X^*. Then, by construction of f,

$$\frac{1}{2}\|f_{n_k}\| \leq |f_{n_k}(x_{n_k})| = |f_{n_k}(x_{n_k}) - f(x_{n_k})| \leq \|f_{n_k} - f\|\|x_{n_k}\| = \|f_{n_k} - f\|.$$

The first step is from our choice of x_n, the second step is by $f(x_{n_k}) = 0$ since $x_{n_k} \in \overline{\mathcal{S}}$, the third step is because $f_{n_k} - f \in X^*$ and the absolute value is the norm on K, and the final step is from $\|x_{n_k}\| = 1$. Let $k \to \infty$. Then the above inequality has $\|f_{n_k} - f\| \to 0$ as $k \to \infty$ which means $\|f_{n_k}\| \to 0$. Therefore, $f_{n_k} \to 0$ in X^* where 0 denotes the zero linear functional. However, we required $f_{n_k} \to f$ and $\|f\| = 1 \neq \|0\|$ which is a contradiction. $\square$

Folland Exercise 5.29 Let $Y = L^1(\mu)$ where μ is counting measure on $\mathbb{N}$, and let $X = \{f \in Y : \sum_1^{\infty} n|f(n)| < \infty\}$, equipped with the L^1 norm.

a. X is a proper dense subspace of Y; hence X is not complete.

b. Define $T : X \to Y$ by $Tf(n) = nf(n)$. Then T is closed but not bounded.

c. Let $S = T^{-1}$. Then $S : Y \to X$ is bounded and surjective but not open.

PROOF.

a. Let $f(n) = \frac{1}{n^2}$. Then, $\|f(n)\|_1 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ so $f \in Y$. However, $\sum_1^{\infty} n|f(n)| = \sum_1^{\infty} \frac{1}{n} = \infty$ which means $f \notin X$. This shows $X \subsetneq Y$.

Next, X is a subspace since for all $\alpha, \beta \in \mathbb{C}$ and for all $f, g \in X$, we have $\alpha f + \beta g \in X$. Indeed,

$$\sum_1^{\infty} n|\alpha f(n) + \beta g(n)| \leq \alpha \sum_1^{\infty} n|f(n)| + \beta \sum_1^{\infty} n|g(n)| < \infty.$$

To show that X dense in Y, let $f \in Y$ be arbitrary. Then $\sum_1^{\infty} |f(n)| < \infty$. Given $\epsilon > 0$. Choose an $N \in \mathbb{N}$ s.t. $\sum_N^{\infty} |f(n)| < \frac{\epsilon}{2}$. Then choose a $g(m)$ s.t. $|f(m) - g(m)| < \frac{\epsilon}{2^{m+1}}$ for $1 \leq m \leq N$ and then set $g(n) = 0$ for all $n \geq N + 1$. Therefore,

$$\|f - g\|_1 = \sum_1^{\infty} |f(n) - g(n)| < \sum_{m=1}^{N} \frac{\epsilon}{2^{m+1}} + \sum_{m=N+1}^{\infty} |f(m)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

On the other hand, we have

$$\sum_1^{\infty} n|g(n)| = \sum_1^{N} n|g(n)| < \infty$$

which means $g \in X$.

Conclusion: X is not complete so the hypothesis of the Closed Graph Theorem and Open Mapping Theorem do not hold in this situation.

b. To show T is unbounded, it suffices to show for all $M \in \mathbb{N}$, there exists an $f \in X$ s.t. $\|T(f)\| > M\|f\|$. Given $M \in \mathbb{N}$. Define f by setting $f(n) = 0$ for $1 \leq n \leq M$ and for $f(n) = \frac{1}{n^3}$ for $n \geq M+1$. Then,

$$\|T(f)\| = \sum_{1}^{\infty} |nf(n)| = \sum_{M+1}^{\infty} nf(n) = \sum_{M+1}^{\infty} \frac{1}{n^2} > \sum_{M+1}^{\infty} \frac{M}{n^3} = M\|f\|.$$

Next, we show that T is a closed operator[12]. Suppose $f_m \to f$ and $Tf_m \to g$ for some $g \in Y$. We want to show $Tf = g$. Because $f_m \to f$, we know

$$\|f_m - f\| = \sum_{n=1}^{\infty} |f_m(n) - f(n)| \to 0 \text{ as } m \to \infty.$$

So, for all n, we have $|f_m(n) - f(n)| \to 0$ as $m \to \infty$. Hence, $f_m \to f$. Because $Tf_m \to g$, this means

$$\|Tf_m - g\| = \sum_{n=1}^{\infty} |nf_m(n) - g(n)| \to 0 \text{ as } m \to \infty.$$

This implies for all n, $|nf_m(n) - g(n)| \to 0$ as $m \to \infty$. So, for all n, $g(n) = nf(n)$ and this means $Tf = g$ as desired.

Conclusion: By showing that T is closed and not bounded, we have shown that the hypothesis that X be Banach in the Close Graph Theorem is necessary.

c. First, S is surjective because if $f \in X$, then $nf \in Y$ because $\sum_1^\infty n|f(n)| < \infty$ and $S(nf) = f$. The equality $S(nf) = f$ is due to what we show next.

Second, we claim $S : Y \to X$ is defined by mapping $f(n) \in Y$ to $\frac{1}{n}f(n) \in X$. In order to ensure that we can actually write $S = T^{-1}$, we verify that T is bijective. It suffices to show that S is a two sided inverse of T when defined in this way. Indeed,

i. Given $f(n) \in X$, we have $ST(f(n)) = S(nf(n)) = \frac{1}{n}nf(n) = f(n)$ which shows $STf = f$ for all $f \in X$. Hence, $ST = \mathrm{id}_X$.
ii. Given $g(n) \in Y$. Then $TS(g(n)) = T\left(\frac{1}{n}g(n)\right) = g(n)$ which shows $f \in X$ for all $TSg = g$. Then, $TS = \mathrm{id}_Y$.

This shows our definition of S is correct and equals T^{-1} which is well-defined.

Next, S is bounded. Given any $f \in Y$, we have

$$\|Sf\| = \sum_{1}^{\infty} \frac{1}{n}|f(n)| \leq \sum_{1}^{\infty} |f(n)| = \|f\|.$$

Therefore, $\|S\| \leq 1$ and this guarantees that S is bounded.

To show S is not an open map, we use the remarks on p. 162. The map S is open iff $S(B_1)$ contains a ball centered at 0 in X where B_1 is the unit ball about 0 in Y. Therefore, we show that there is no ball centered at 0 contained in $S(B_1)$.

So suppose $r > 0$ and we want $B_r(0) \not\subseteq S(B_1)$ in X. Define f by $f(n) = \frac{1}{n^2}\frac{6}{\pi^2}\frac{r}{2}$.Then,

$$\|f\| = \sum_{1}^{\infty} \frac{1}{n^2}\frac{6}{\pi^2}\frac{r}{2} = \frac{\pi^2}{6}\frac{6}{\pi^2}\frac{r}{2} = \frac{r}{2}$$

[12]Recall T is closed if for any sequence $x_n \to x$ s.t. $Tx_n \to y$ for some y, this implies $y = Tx$.

which means $f \in B_r(0)$. But $f \notin S(B_1)$ because if it were, there exists a $g \in B_1$ s.t. $f = \frac{1}{n}g$ and $\|g\| < 1$. This means $g(n) = \frac{1}{n}\frac{6}{\pi^2}\frac{r}{2}$ for all $n \in \mathbb{N}$. However,

$$\|g\| = \sum_1^\infty \frac{1}{n}\frac{6}{\pi^2}\frac{r}{2} = \infty \not< 1.$$

Conclusion: By showing that S is bounded and surjective, but not open, we have shown that the Open Mapping Theorem fails if the completeness condition on the domain is dropped. □

Folland Exercise 5.30 Let $\mathcal{Y} = C([0,1])$ and $X = C^1([0,1])$, both equipped with the uniform norm. a. X is not complete.

b. The map $(d/dx) : X \to Y$ is closed (see Exercise 9) but not bounded.

PROOF. **a.** For noncompleteness, take a sequence of polynomials (which are in $C^1([0,1])$) that approximate $|x - \frac{1}{2}|$ uniformly (apply the Stone-Weierstrass Theorem, see Section 4.7 pp. 138-139). However, the limit is $|x - \frac{1}{2}|$ which is not in $C^1([0,1])$.
b. Checking that the map is closed is easy because if $f_n \to f$ in X and $\frac{d}{dx}f_n \to g$ for some g, then obviously $g = \frac{d}{dx}f$ (differentiation behaves well with limits in this case). To see that it is not bounded, consider the sequence of functions x^n which gives $\|\frac{d}{dx}x^n\|_u = n$ despite $\|x^n\|_u = 1$. □

Folland Exercise 5.32 Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the vector space X such that $\|\cdot\|_1 \leq \|\cdot\|_2$. If X is complete with respect to both norms, then the norms are equivalent.

PROOF. Let $X_1 := (X, \|\cdot\|_1)$ and $X_2 := (X, \|\cdot\|_2)$ denote the respective Banach space structures. Define $T : X_2 \hookrightarrow X_1$ to be the identity map on X. Then T is bounded because for all $x \in X_1$, the hypothesis gives $\|Tx\|_1 = \|x\|_1 \leq \|x\|_2$ which means $\|T\| \leq 1$. Also, T is bijective because it is just the identity map on X. By the Open Mapping Theorem (in particular, Corollary 5.11), T is an isomorphism of normed vector spaces and in particular, $T^{-1} \in L(Y, X)$. Hence, there is a constant $C > 0$ s.t. $\|x\|_2 = \|T^{-1}x\|_2 \leq C\|x\|_1$ for all $x \in X_1$. Therefore,

$$\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \qquad \forall x \in X.$$

This means that the norms are equivalent. □

Folland Exercise 5.37 Let X and Y be Banach spaces. If $T : X \to Y$ is a linear map such that $f \circ T \in X^*$ for every $f \in \mathcal{Y}^*$, then T is bounded.

PROOF. By the Closed Graph Theorem, it suffices to show that T is a closed linear map. Suppose $x_n \to x$ in X and $Tx_n \to y$ in Y. We must show that $y = Tx$.

Suppose not and $y \neq Tx$. By Theorem 5.8c, there exists a linear functional $f \in Y^*$ s.t. $f(y) \neq f(Tx)$. However, $f \circ T \in X^*$ is bounded and so it is continuous which means, $f \circ T(x_n) \to f \circ T(x)$ and therefore, $f \circ T(x_n) \not\to f(y)$. However, we assumed $T(x_n) \to y$ and since $f \in Y^*$ is bounded, it is continuous and this means we must have $f \circ T(x_n) \to f(y)$. In other words,

$$\lim_{n\to\infty}(f \circ T)(x_n) = (f \circ T)(\lim_{n\to\infty} x_n) = f(Tx) \neq f(y) = f(\lim_{n\to\infty} Tx_n) = \lim_{n\to\infty}(f \circ T)(x_n).$$

Contradiction. □

PROOF USING UNIFORM BOUNDEDNESS PRINCIPLE. Linear functionals $f \circ T$ with $\|f\| \leq 1$ fulfill the hypothesis of the Uniform Boundedness Principle (X is Banach and $\sup_{\|f\|\leq 1} \|f \circ T(x)\| < \infty$ by $f \circ T$ always bounded). Therefore, there is a C s.t. $\|f(Tx)\| \leq C\|x\|$ for all $x \in X$ and $f \in Y^*$ s.t. $\|f\| \leq 1$. For any x, ther exists $f \in Y^*$ s.t. $\|f\| = 1$ ad $f(Tx) = \|Tx\|$. Therefore, $\|f(Tx)\| = \|Tx\| \leq C\|x\|$ which means $\|T\| \leq C$. $\square$

Folland Exercise 5.41 Let X be a vector space of countably infinite dimension (that is, every element is a finite linear combination of members of a countably infinite linearly independent set). There is no norm on X with respect to which X is complete. (Given a norm on X, apply Exercise 18 b and the Baire category theorem.)

PROOF. Towards a contradiction, let $\|\cdot\|$ be a norm on X in which $(X, \|\cdot\|)$ is complete. Fix a basis $\{e_i\}_{i\in\mathbb{N}}$ of X. Then for all $N \in \mathbb{N}$, the subspace S_N spanned by $\{e_i\}_{i=1}^N$ is a closed subspace by Exercise 5.18b. Then, $X := \bigcup_{n\in\mathbb{N}} S_n$ and by the Baire Category Theorem, as X is complete, there is an $N \in \mathbb{N}$ s.t. S_N is not nowhere dense. So, there is an $r > 0$ and $x \in S_N$ s.t. $\overline{B_r(x)} \subseteq S_N$. Now suppose $x := \sum_1^N a_i e_i \in S_N$. Define $y := \sum_1^N a_i e_i + \delta e_{N+1}$ where $\delta := \frac{r}{\|e_{N+1}\|+1}$. Then,

$$\|y - x\| = \|\delta e_{N=1}\| = |\delta|\|e_{N+1}\| < r \qquad \Longrightarrow \qquad y \in \overline{B_r(x)} \qquad \Longrightarrow \qquad y \in S_N.$$

However, $y - x = \delta e_{N+1} \notin S_N$ which contradicts S_N being a subspace. Contradiction. $\square$

[13]**Folland Exercise 5.43** Prove Proposition 5.16. (For part (b), proceed as in Exercise 56 d in §4.5.).

5.16 Proposition. Let X be a vector space equipped with the topology defined by a family $\{p_\alpha\}_{\alpha\in A}$ of seminorms.

a. X is Hausdorff iff for each $x \neq 0$ there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$.

b. If X is Hausdorff and A is countable, then X is metrizable with a translationinvariant metric (i.e., $\rho(x, y) = \rho(x + z, y + z)$ for all $x, y, z \in X$)

PROOF. **Proposition 5.16 (a)** ($\Longrightarrow$) Suppose $x \neq 0$. Since X is Hausdorff, there exists disjoint open sets U, V containing x and 0 respectively. By definition of the topology, there is a basis element $x \in \bigcap_{\beta\in B\subseteq A} p^{-1}(W_\alpha) \subseteq U$ where the W_α are open. Then, $x \in p_\alpha^{-1}(W_\alpha)$ and $0 \notin p_\alpha^{-1}(W_\alpha)$ which means $p(x) \neq p(0)$.

($\Longleftarrow$) Let $x, y \in X$ be distinct points and by translating, X using the homeomorphism $X \to X$ defined by $z \mapsto z - y$, we may assume $y = 0$ and $x \neq 0$. Then, there is an α s.t. $p_\alpha(x) \neq 0 = p_\alpha(y)$. Since K is Hausdorff, there are disjoint open sets $W \ni p_\alpha(x)$ and $V \ni 0$ of $[0, \infty)$. Then, $p_\alpha^{-1}(W) \ni x$ and $p_\alpha^{-1}(V) \ni 0$ and these sets are disjoint by construction which means these are our desired sets for the definition of Hausdorff.

Proposition 5.16 (b) Enumerate A as $\mathbb{N}$ using a bijection. Define $\rho : X \times X \to [0, \infty)$ by

$$\rho(x, y) := \sum_1^\infty 2^{-n}\Phi(p_n(y - x)).$$

First, this map is well-defined by because $\Phi \leq 1$ so the series on the RHS always converges and $p_n(y - x) \geq 0$ means $\Phi(p_n(y - x))$ is well-defined. Next, the metric axioms and the translation invariance,

[13]For this Homework, I mainly collaborated and discussed with Jeb Runnoe and Scotty Tilton. Supplemental resources were the course text, extra reading from "Real Analysis" by Royden and "Real and Complex Analysis" by Walter Rudin.

(1) We have $\rho(x, y) = 0$ iff $\Phi(p_n(y-x)) = 0$ for all n iff $p_n(y-x) = 0$ for all n iff $x = y$. The last iff is due to the part a. because if $x \neq y$, then $y - x \neq 0$ which means there is a $p_n(y-x) \neq 0$.
(2) We have $\rho(y, x) = \rho(x, y)$ because $\{p_n\}$ is a family of seminorms.
(3) We have $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ because of the results in Exercise 4.56 (in particular, Exercise 4.56a and Exercise 4.56d).
(4) We have $\rho(x+z, y+z) = \sum_1^\infty 2^{-n}\Phi(p_n((y+z) - (x+z)) = \rho(x, y)$.

With these verifications completed, we proceed to show that the metric coincides with the topology defined by $\{p_n\}$. Recall the observations we made about Φ^{-1} in that exercise. Let $\mathcal{S}$ and $\mathcal{S}'$ denote the topology defined by the metric and the family of seminorms $\{p_n\}$ respectively.

Show $\mathcal{S} \subseteq \mathcal{S}'$: Given $U \in \mathcal{S}$ nonempty. Let $z \in U$ be a point and $B_r^\rho(z) \subseteq \mathcal{S}$ for some $r > 0$. Choose an N sufficiently large s.t. $\sum_{N+1}^\infty \frac{1}{2^n} < r$. Then, choose $\epsilon > 0$ s.t. $0 < \epsilon < r - \sum_{N+1}^\infty \frac{1}{2^n}$. Then, consider the set in the basis of $\mathcal{S}'$

$$z \in V := \bigcap_{i=1}^N U_{z,i,\delta}$$

and we chose δ s.t. if $p_n(w_z) < \delta$, then $\Phi(p_n(w-z)) < \epsilon$. We claim $V \subseteq U$. Let $w \in V$. Then,

$$\rho(w, z) = \sum_1^\infty \frac{1}{2^n}\Phi(p_n(w-z)) < \sum_1^N \frac{1}{2^n}\Phi(p_n(w-z)) + \sum_{N+1}^\infty \frac{1}{2^n} < \epsilon + \sum_{N+1}^\infty \frac{1}{2^n} < r.$$

That is, $w \in B_r^\rho(z) \subset U$ as desired.

Show $\mathcal{S}' \subseteq \mathcal{S}$: Suppose $U \in \mathcal{S}'$ is nonempty. Since there is a basis for $\mathcal{S}'$ by Theorem 5.14, it suffices to assume $U := \bigcap_{i=1}^N U_{x_i,n_i,\epsilon_i}$. Even better, since the sets $U_{x\alpha\epsilon}$ generate the topology of $\mathcal{S}$ by Theorem 5.14, it suffices to assume $U = U_{z,m,\epsilon}$ for some $z \in X$, $m \in A$, and $\epsilon > 0$. We claim $U \in \mathcal{S}$ and since $\mathcal{S}$ is just the metric topology, we just show points of U contain balls about them inside U.

Let $z \in U$ and we want to find an $r > 0$ s.t. $B_r^\rho(z) \subseteq U$. Choose $r > 0$ sufficiently small so that $r < 1$ and

$$\Phi^{-1}(2^m r) < \epsilon.$$

In this case, we deduce that

$$\frac{1}{2^m}\Phi(p_m(w-z)) < r \qquad \Longrightarrow \qquad p_m(w-z) < \epsilon.$$

So for any $w \in B_r^\rho(z)$, we know

$$\rho(w, z) = \sum_1^\infty \frac{1}{2^n}\Phi(p_n(w-z)) \qquad \Longrightarrow \qquad \frac{1}{2^m}\Phi(p_m(w-z)) < r \qquad \Longrightarrow \qquad p_m(w-z) < \epsilon$$

which means $w \in U$ as desired. $\square$

Folland Exercise 5.45 The space $C^\infty(\mathbb{R})$ of all infinitely differentiable functions on $\mathbb{R}$ has a Fréchet space topology with respect to which $f_n \to f$ iff $f_n^{(k)} \to f^{(k)}$ uniformly on compact sets for all $k \geq 0$.

PROOF. *Proof of First Claim:* Define a countable family of seminorms by

$$p_{m,k}(f) = \sup_{x\in[-m,m]} |f^{(k)}(x)| \qquad m \in \mathbb{N}, k \in \mathbb{N}_0.$$

If $f \neq 0$, then we can choose m sufficiently large and $k = 0$ to get $p_{m,1}(f) \neq 0$. Then, the preceding exercise shows that the topology induced by the seminorms is Hausdorff. It is also clear that the family of seminorms we just defined is a countable family and by the preceding exercise, $C^\infty(\mathbb{R})$ is metrizable with a translation invariant metric ρ.

The last part is to check that this yields a complete metric space (since the comments on p. 167 explains that being complete would just mean being complete in the metric sense for this case).

Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $C^\infty(\mathbb{R})$. Theorem 5.14. Because of how we defined the metric in the preceding exercise, we know that means $\{f_n\}$ is a Cauchy sequence in $C^\infty([-m,m])$ with metric defined by $p_{m,k}$ and where we restrict f_n to $[-m,m]$. But then $C^\infty([-m,m])$ is complete by the remarks on p. 168. So, $f_n|_{[-m,m]} \to f$ for some $f \in C^\infty([-m,m])$.

Suppose $f_n|_{[-m,m]} \to f$ for some $f \in C^\infty([-m,m])$ and $f_n|_{[-l,l]} \to g$ for some $g \in C^\infty([-l,l])$ where $l > m$. Then by definition of the metric, for all $m \in \mathbb{N}$, we get $g|_{[-m,m]} = f$ and $g^{(k)}|_{[-m,m]} = f^{(k)}$ for all $k \geq 0$. Because this holds for all $m \in \mathbb{N}$, we deduce that $f_n \to f$ in $C^\infty(\mathbb{R})$.

Proof of Second Claim: Suppose $f_n \to f$ w.r.t the topology. Then Theorem 5.14 shows that means $p_{m,k}(f_n - f) \to 0$ for all m and k. But every compact set is contained in some interval $[-m,m]$ where m is chose sufficiently large and therefore, $p_{m,k}(f_n - f) \to 0$ implies $f_n^{(k)} \to f^{(k)}$ uniformly on compact sets for all $k \geq 0$.

Conversely, if $f_n^{(k)} \to f^{(k)}$ uniformly on all compact sets for all $k \geq 0$, then that means $f_n^{(k)} \to f^{(k)}$ uniformly on all sets of the form $[-m,m]$ for $m \in \mathbb{N}$. Therefore, $p_{m,k}(f_n - f) \to 0$ for all m and k. Then by Theorem 5.14, that means $f_n \to f$ in the topology generated by the seminorms. □

Folland Exercise 5.47 Suppose that X and Y are Banach spaces.

a. If $\{T_n\}_1^\infty \subset L(X,Y)$ and $T_n \to T$ weakly (or strongly), then $\sup_n \|T_n\| < \infty$.

b. Every weakly convergent sequence in X, and every weak*-convergent sequence in X^*, is bounded (with respect to the norm).

PROOF. **a.** We apply the Uniform Boundedness Principle. By definition, $T_n \to T$ weakly if for all $x \in X$, we have $T_n x \to Tx$ in the weak topology of Y. That is, $(f \circ T_n)x \to (f \circ T)x$ for all $f \in Y^*$. This means $\sup_n \|T_n x\| < \infty$ for all n and therefore, $\sup_n \|T_n\| < \infty$.

Since the strong topology is stronger than the weak topology, $T_n \to T$ strongly means $T_n \to T$ weakly and therefore $\sup_n \|T_n\| < \infty$.

b. *First Statement:* Let $x_n \to x$ be weakly convergent in X. Then for every $f \in X^*$, we have $f(x_n) \to f(x)$. Consider $\widehat{x_n} \in X^{**}$ and then,

$$f(x_n) \to f(x) \qquad \Longrightarrow \qquad \widehat{x_n} f \to \widehat{x} f$$

for every $f \in X^*$. Since $\sup_n \|\widehat{x_n} f\| < \infty$ for every $f \in X^*$, by the Uniform Boundedness Principle, we have $\sup_n \|x\| = \sup_n \|\widehat{x_n}\| < \infty$ since $\|\widehat{x_n}\| = \|x\|$.

Second Statement: Assume $f_\alpha \to f$ weakly*. Then $f_\alpha(x) \to f(x)$ for all $x \in X$. Therefore, $\sup_\alpha \|f_\alpha(x)\| < \infty$. By the Uniform Boundedness Principle, $\sup_\alpha \|f_\alpha\| < \infty$. □

Folland Exercise 5.56 If E is a subset of a Hilbert space $\mathcal{H}$, $\left(E^{\perp}\right)^{\perp}$ is the smallest closed subspace of $\mathcal{H}$ containing E.

PROOF. [14] We are essentially asked to show $(E^{\perp})^{\perp} = \overline{E}$.

By definition, $(E^{\perp})^{\perp} \oplus E^{\perp} = H$ and $E^{\perp}$ is closed by the remarks on p. 173. Clearly, $E \perp E^{\perp}$ so that means $E \subseteq (E^{\perp})^{\perp}$. This shows $(E^{\perp})^{\perp} \supseteq E$ and $(E^{\perp})^{\perp}$ is closed so we just need to show it is the smallest closed subspace that contains E.

Suppose not and $M \subsetneq (E^{\perp})^{\perp}$ were a proper closed subspace and contained E. The hypothesis implies $((E^{\perp})^{\perp})^{\perp} \subsetneq M^{\perp}$ and this gives $(E^{\perp}) \subsetneq M^{\perp}$. So there is an $m \in M^{\perp}$ s.t. $m \notin E^{\perp}$. That means $(m, m') = 0$ for all $m' \in M$, but $(m, n) \neq 0$ for some $n \in E$. However, $M \supseteq E$ which is a contradiction. □

Folland Exercise 5.57 Suppose that $\mathcal{H}$ is a Hilbert space and $T \in L(\mathcal{H}, \mathcal{H})$

a. There is a unique $T^* \in L(\mathcal{H}, \mathcal{H})$, called the adjoint of T, such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$. (Cf. Exercise 22. We have $T^* = V^{-1}T^{\dagger}V$ where V is the conjugate-linear isomorphism from $\mathcal{H}$ to $\mathcal{H}^*$ in Theorem 5.25, $(Vy)(x) = \langle x, y\rangle$.)

b. $\|T^*\| = \|T\|, \|T^*T\| = \|T\|^2, (aS + bT)^* = \bar{a}S^* + \bar{b}T^*, (ST)^* = T^*S^*$, and $T^{**} = T$

c. Let $\mathcal{R}$ and $\mathcal{N}$ denote range and nullspace; then $\mathcal{R}(T)^{\perp} = \mathcal{N}(T^*)$ and $\mathcal{N}(T)^{\perp} = \overline{\mathcal{R}(T^*)}$.

d. T is unitary iff T is invertible and $T^{-1} = T^*$.

PROOF. **(a).** *Uniqueness:* Suppose there were $S, R \in L(H, H)$ s.t. $\langle x, Sy\rangle = \langle Tx, y\rangle = \langle x, Ry\rangle$ for all $x, y \in H$. Therefore,

$$\overline{\langle Sy, x\rangle} - \overline{\langle Ry, x\rangle} = \overline{\langle (S-R)y, x\rangle} = 0$$

for all $x, y \in H$. Fixing y, we can let x vary through elements of an orthonormal basis of H. Therefore, $(S - R)y = 0$. Hence, $Sy = Ry$ for all $y \in H$.

Existence: Given a $y \in H$, define the map $\Phi_y : H \to K$ by $x \mapsto \langle Tx, y\rangle$. This is a bounded linear functional because

$$\Phi_y(ax + bz) = \langle T(ax + bz), y\rangle = a\langle Tx, y\rangle + b\langle Tx, y\rangle$$

shows it is linear and

$$|\langle Tx, y\rangle| \leq \|Tx\|\|y\| \leq \|T\|\|x\|\|y\|.$$

By Theorem 5.25, there is a $w \in H$ s.t. $\Phi_y(x) = \langle x, w\rangle$ for all $x \in H$. Set $T^*y = w$. Since w is unique for any given y, this we obtain $T^* : H \to H$.

We check linearity. Let a, b be scalars and $x, y \in H$ and we want to show that $T^*(ax + by) = aT^*x + bT^*y$. By existence of orthonormal basis and completeness, it suffices to show that for all $z \in H$, we have,

$$\langle z, T^*(ax + by) - aT^*x - bT^*y\rangle = 0.$$

Indeed, using adjointness and the conjugate bilinearity of the inner product in the second coordinate,

$$\langle z, T^*(ax + by) - aT^*x - bT^*y\rangle = \langle z, T^*(ax + by)\rangle - \langle z, aT^*x + bT^*y\rangle = \langle Tz, ax + by\rangle - \overline{a}\langle Tz, x\rangle - \overline{b}\langle Tz,$$

[14]Yunyi suggested that it would be better to explain $((E^{\perp})^{\perp})^{\perp} = E^{\perp}$ which I used implicitly here. Indeed, this follows from the fact that $(E^{\perp})^{\perp}$ was shown to be closed and that just means the $^{\perp}$ of it is going to be $E^{\perp} = \overline{E}^{\perp}$.

We check that T^* is bounded. We have

$$\|T^*x\| = \sqrt{\langle TT^*x, x\rangle} \leq \sqrt{\|T\|\|x\|\|T^*x\|} \qquad \Longrightarrow \qquad \|T^*x\|^2 \leq \|T\|\|x\|\|T^*x\| \qquad \Longrightarrow \qquad \|T^*x\| \leq \|$$

(b). *Showing* $\|T^*\| = \|T\|$*:* First, for all $x \in H$,

$$\|T^*x\| = \sqrt{\langle T^*x, T^*x\rangle} = \sqrt{\langle TT^*x, y\rangle} \qquad \Longrightarrow \qquad \|T^*x\|^2 = \langle TT^*x, y\rangle \leq \|TT^*x\|\|x\| \leq \|T\|\|T^*x\|\|x\|$$

by an application of the Schwarz inequality. Therefore, $\|T^*\| \leq \|T\|$. For the other inequality, repeat what we just did except switch the the T^* with T and vice-versa.

Showing $\|T^*T\| = \|T\|^2$*:* First,

$$\|T^*Tx\| \leq \|T\|\|T^*\|\|x\| \qquad \Longrightarrow \qquad \|T^*T\| \leq \|T\|\|T^*\| = \|T\|^2$$

using the preceding part. It remains to show the other inequality. We have

$$\|Tx\|^2 = |\langle Tx, Tx\rangle| = |\langle T^*Tx, x\rangle| \leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|^2.$$

Taking the supremum over all $\|x\| = 1$, we have $\|T\|^2 \leq \|T^*T\|$.

Showing linearity of **:* By uniqueness, it suffices to show that $\langle (aS+bT)x, y\rangle = \langle x, (\overline{a}S^* + \overline{b}T^*y\rangle$ for all $x, y \in H$. Let $x, y \in H$ be arbitrary. Then,

$$\langle (aS+bT)x, y\rangle = a\langle Sx, y\rangle + b\langle Tx, y\rangle = a\langle x, S^*y\rangle + b\langle x, T^*y\rangle = \langle x, (\overline{a}S^* + \overline{b}T^*)y\rangle$$

where the second equality follows by conjugate linearity of the inner product in the second coordinate.

Showing $(ST)^* = T^*S^*$*:* For all $x, y \in H$, we have

$$\langle STx, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle \qquad \Longrightarrow \qquad (ST)^* = T^*S^*$$

by uniqueness of $(ST)^*$.

Showing $T^{**} = T$*:* For all $x, y \in H$,

$$\langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle \qquad \Longrightarrow \qquad T^{**} = T$$

by uniqueness of T^*.

(c). *Showing* $R(T)^\perp = N(T^*)$*:* We have

$$\begin{aligned} R(T)^\perp &= \{z \in H : \langle z, Tx\rangle = 0, \ \forall x \in H\} \\ &= \{z \in H : \langle T^*z, x\rangle = 0, \ \forall x \in H\} \\ &= \{z \in H : T^*z = 0\} \\ &= N(T^*). \end{aligned}$$

Showing $N(T)^\perp = \overline{R(T^*)}$*:* We reduce to showing that $\overline{N(T)} = \overline{(R(T^*))}^\perp$ because $(M^\perp)^\perp = \overline{M}$.

$$\begin{aligned} \overline{N(T)} &= \overline{\{z \in H : Tz = 0\}} \\ &= \overline{\{z \in H : \langle Tz, w\rangle = 0, \ \forall w \in H\}} \\ &= \overline{\{z \in H : \langle z, T^*w\rangle = 0, \ \forall w \in H\}} \\ &= \overline{(R(T^*))}^\perp \end{aligned}$$

where the second equality follows because the inner product is continuous in both coordinates.

(d). ($\Longrightarrow$) Since T is unitary, we know T^{-1} is invertible and $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all x, y which means T is defined on all of H and so is its inverse (from Exercise 5.55, we know T is surjective as well). So,

$$\langle Tx, y\rangle = \langle Tx, TT^{-1}y\rangle = \langle x, T^{-1}y\rangle$$

By uniqueness of the adjoint, this shows $T^* = T^{-1}$.

($\Longleftarrow$) Since T is invertible, T^{-1} is well-defined and bounded. Since $T^{-1} = T^*$, for any $x, y \in H$,

$$\langle Tx, Ty\rangle = \langle x, T^*Ty\rangle = \langle x, T^{-1}Ty\rangle = \langle x, y\rangle \qquad \Longrightarrow \qquad T \text{ is unitary.}$$

□

Folland Exercise 5.58 Let $\mathcal{M}$ be a closed subspace of the Hilbert space $\mathcal{H}$, and for $x \in \mathcal{H}$ let Px be the element of $\mathcal{M}$ such that $x - Px \in \mathcal{M}^\perp$ as in Theorem 5.24.

a. $P \in L(\mathcal{H}, \mathcal{H})$, and in the notation of Exercise 57 we have $P^* = P, P^2 = P\ \mathcal{R}(P) = \mathcal{M}$, and $\mathcal{N}(P) = \mathcal{M}^\perp$. P is called the orthogonal projection onto $\mathcal{M}$

b. Conversely, suppose that $P \in L(\mathcal{H}, \mathcal{H})$ satisfies $P^2 = P^* = P$. Then $\mathcal{R}(P)$ is closed and P is the orthogonal projection onto $\mathcal{R}(P)$.

c. If $\{u_\alpha\}$ is an orthonormal basis for $\mathcal{M}$, then $Px = \sum \langle x, u_\alpha\rangle\, u_\alpha$

Proof. **a.** *Showing $P \in L(H, H)$:* Define $P : H \to H$ by $x \mapsto Px$ where $x - Px \in \mathcal{M}^\perp$. This map is well-defined by Theorem 5.24. Next, checking linearity: let $a, b \in \mathbb{C}$ and $x, y \in H$ and then,

$$ax + by - (aP(x) + bP(y)) = ax + by - aP(x) - bP(x) = a(x - Px) + b(y - Py) \in \mathcal{M}^\perp$$

since $\mathcal{M}^\perp$ is a subspace and by well-definedness of P, we deduce $P(ax+by) = aP(x)+bP(y)$. Next, we show that P is bounded. We have the following identity:

$$\begin{aligned}\langle Px, Px\rangle &= \langle x, Px\rangle - \langle x - Px, Px\rangle = \langle x, Px\rangle \\ &= \langle x, x\rangle - \langle x, x - Px\rangle \\ &= \langle x, x\rangle - [\langle x - Px, x - Px\rangle + \langle Px, x - Px\rangle] \\ &= \langle x, x\rangle - \langle x - Px, x - Px\rangle.\end{aligned}$$

Indeed, for any $x \in H$, we have

$$\|Px\| = \sqrt{\langle Px, Px\rangle} = \sqrt{\langle x, x\rangle - \langle x - Px, x - Px\rangle} \le \sqrt{\langle x, x\rangle} = \|x\|.$$

which shows $\|P\| \le 1$.

Showing $P^* = P$: For any $x, y \in H$, we have

$$\begin{aligned}\langle Px, y\rangle &= \langle x, y\rangle - \langle x - Px, y\rangle = \langle x, y\rangle - \langle x - Px, y - Py + Py\rangle \\ &= \langle x, y\rangle - \langle x - Px, y - Py\rangle - \langle x - Px, Py\rangle \\ &= \langle x, y\rangle - \langle x, y - Py\rangle = \langle x, y\rangle - \langle x, y - Py\rangle = \langle x, Py\rangle.\end{aligned}$$

Showing $P^2 = P$: For any x, we know $Px - P^2x \in \mathcal{M}^\perp$ and therefore, $P(x - Px) \in \mathcal{M}^\perp$ which implies

$$x - Px - (P(x - Px)) \in \mathcal{M}^\perp.$$

Since $x - Px$ is already in $\mathcal{M}^\perp$, this means $P(x - Px) = 0$ so $Px = P^2x$.

Showing $R(P) = M$*:* Clearly $R(P) \subseteq M$ so we prove the other inequality. Let $x \in \mathcal{M}$. Notice that

$$(2x - Px) - x = x - Px \in \mathcal{M}^\perp$$

and by uniqueness, this means $P(2x - Px) = x$. Since $2x - Px \in H$, this means that $x \in R(P)$.

Showing $N(P) = M^\perp$*:* Clearly, if $x \in N(P)$, then $x \in M^\perp$ because $\mathcal{M}^\perp \ni x - Px = x$. So suppose $x \in \mathcal{M}^\perp$. Then, $x - Px \in \mathcal{M}^\perp$ means $Px = 0$ by uniqueness of Px.

b. Let $\{x_n\} = \{Py_n\} \subseteq R(P)$ be a convergent sequence and $x_n \to x$. Then by continuity of P,

$$Py_n = P^2 y_n = P(Py_n) \to Px$$

which means $Py_n \to Px \in R(P)$ and we deduce that the limit of $x_n = Py_n$ is an element of $R(P)$.

By part a., to show that P is the orthogonal projection onto $R(P)$, we need to show that $N(P) = R(P)^\perp$. From Exercise 5.57c, we know

$$N(P) = N(P^*) = R(P)^\perp.$$

c. Let $\{u_\alpha\}$ be an orthonormal basis of M. We know $Px \in \mathcal{M}$ and by Theorem 5.27c,

$$Px = \sum \langle Px, u_\alpha \rangle u_\alpha = \sum (\langle x, u_\alpha \rangle + \langle Px - x, u_\alpha \rangle) u_\alpha = \sum \langle x, u_\alpha \rangle u_\alpha$$

since $\langle Px - x, u_\alpha \rangle = 0$ from the fact that $x - Px \in \mathcal{M}^\perp$ and $u_\alpha \in M$.

$\square$

Folland Exercise 5.59 Every closed convex set K in a Hilbert space has a unique element of minimal norm. (If $0 \in K$, the result is trivial; otherwise, adapt the proof of Theorem 5.24.)

Proof. If $0 \in K$, then it is clearly the element of minimal norm. Furthermore, it is unique because any other element with norm zero must be zero by definition.

Suppose $0 \notin K$ and let $\delta := \inf_{y \in K} \|y\|$. Choose a sequence $\{y_n\} \subseteq K$ s.t. $\|y_n\| \to \delta$. Then, by convexity, $\frac{1}{2}y_n + \frac{1}{2}y_m \in K$ for any $m, n \in \mathbb{N}$. By the Parallelogram law, we deduce

$$\left\| \frac{1}{2}(y_n - y_m) \right\|^2 + \left\| \frac{1}{2}(y_n + y_m) \right\|^2 = \frac{\|y_n\|^2}{2} + \frac{\|y_m\|^2}{2}.$$

This means

$$\|y_n - y_m\| = -4 \left\| \frac{1}{2}(y_n + y_m) \right\|^2 + 2\|y_n\|^2 + 2\|y_m\|^2 = 2\|y_n\|^2 + 2\|y_m\|^2 - 4 \left\| \frac{1}{2}(y_n + y_m) \right\|^2 \leq 2\|y_n\|^2 + 2\|y_m\|^2 - 4$$

Letting $n, m \to \infty$, the RHS converges to $2\delta^2 + 2\delta^2 - 4\delta^2 = 0$. This shows $\{y_n\}$ is a Cauchy sequence and therefore, $y_n \to y$ for some y and in this case, y converges to an element of norm equal to δ. Indeed,

$$\|y\| = \| \lim_{n \to \infty} y_n \| = \lim_{n \to \infty} \|y_n\| = \delta$$

by continuity of the norm.

For uniqueness, suppose $\|x\| = \|y\|$ are of minimal norm. Then,

$$0 < \| \frac{1}{2}x + \frac{1}{2}y \|^2 < \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 = \|x\|^2$$

taking square roots,

$$0 < \|\frac{1}{2}x + \frac{1}{2}y\| < \|x\|$$

which is a contradiction since equality in the triangle inequality for norm occurs iff the two elements are linearly independent. $\square$

Folland Exercise 5.61 Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ -finite measure spaces such that $L^2(\mu)$ and $L^2(\nu)$ are separable. If $\{f_m\}$ and $\{g_n\}$ are orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$ and $h_{mn}(x,y) = f_m(x)g_n(y)$, then $\{h_{mn}\}$ is an orthonormal basis for $L^2(\mu \times \nu)$.

Proof. We verify that $\{h_{mn}\}$ forms an orthonormal set.

First, $\|h_{mn}\|_{L^2(\mu\times\nu)} = 1$ since $\{f_m\}$ and $\{g_n\}$ are orthonormal bases of the respective spaces and

$$\begin{aligned}\|h_{mn}\|^2_{L^2(\mu\times\nu)} &= \int_{X\times Y} |f_n(x)g_m(y)|^2 d(\mu\times\nu) = \int_Y\int_X |f_n(x)|^2|g_n(y)|^2 d\mu(x)d\nu(y) = \int_X |f_n(x)|^2 d\mu(x)\int_Y |g_m(\\ &= \|f\|^2_{L^2(\mu)}\|g\|^2_{L^2(\nu)} = 1\end{aligned}$$

where we applied the the Tonelli part of the Fubini-Tonelli Theorem (Theorem 2.37a). The hypothesis Tonelli's Theorem are fulfilled because $|f_n(x)g_m(y)|^2 \in L^+(\mu\times\nu)$. This computation also confirms that $\{h_{mn}\} \subseteq L^2(\mu\times\nu)$.

Next, assume $(m,n) \neq (m', n')$ and WLOG assume $m \neq m'$. Then,

$$\begin{aligned}\langle h_{mn}, h_{m'n'}\rangle &= \int_{X\times Y} h_{mn}\overline{h_{m'n'}}d(\mu\times\nu) = \int_{X\times Y} f_m(x)\overline{f_{m'}(x)}g_n(y)\overline{g_{n'}(y)}d(\mu\times\nu)\\ &= \int_X f_m(x)\overline{f_{m'}(x)}d\mu\int_Y g_n(y)\overline{g_{n'}(y)}d\nu = \langle f_m f_{m'}\rangle\langle g_n, g_{n'}\rangle = 0\langle g_n, g_{n'}\rangle = 0\end{aligned}$$

where we applied the Fubini-Tonelli Theorem for the third equality. Next, to show it is an orthonormal basis, we use Theorem 5.27. Suppose $F(x,y) \in L^2(\mu\times\nu)$ s.t. $\langle F, h_{mn}\rangle = 0$. That is,

$$\begin{aligned}\langle F, h_{mn}\rangle &= \int_{X\times Y} F(x,y)\overline{h_{mn}(x,y)}d(\mu\times\nu)\\ &= \int_{X\times Y} F(x,y)\overline{f_m(x)g_n(y)}d(\mu\times\nu)\\ &= \int_Y\int_X F(x,y)\overline{f_m(x)}d\mu(x)\overline{g_n(y)}d\nu \qquad \text{(Fubini-Tonelli Theorem)}\\ &= \int_Y \langle F, f_m\rangle\overline{g_n(y)}d\nu = \langle\langle F, f_m\rangle, g_n\rangle = 0.\end{aligned}$$

Since, for any m, this equality holds for all n, we deduce that $\langle F, f_m\rangle = 0$ for all m (viewed as a function in y). Fixing y, we know that $\langle F, f_m\rangle = 0$ for all m. Hence, $F = 0$ a.e. when fixing the value of x. But notice that our application of Fubini-Tonelli and our deductions afterwards could be done with $g_n(y)$ and $f_m(x)$ switched and x and y switched so that means $F = 0$ a.e. when we fix a value for y. Together, $F = 0$ a.e. on $X \times Y$. $\square$

Folland Exercise 5.63 Let $\mathcal{H}$ be an infinite-dimensional Hilbert space.

a. Every orthonormal sequence in $\mathcal{H}$ converges weakly to 0.

b. The unit sphere $S = \{x : \|x\| = 1\}$ is weakly dense in the unit ball $B = \{x : \|x\| \leq 1\}$. (In fact, every $x \in B$ is the weak limit of a sequence in S.)

PROOF. **a.** Let $\{x_n\}$ be an orthonormal sequence in H. To show $x_n \to 0$ weakly, we must show for all $f \in H^*$, $f(x_n) \to f(0) = 0$. Fix an $f \in H^*$. By the Riesz-Representation Theorem, we know $f(\cdot) = \langle \cdot, y\rangle$ for some $y \in H$. Bessel's inequality,

$$\sum_{n=1}^{\infty} |\langle x_n, y\rangle|^2 \leq \|y\|^2 < \infty$$

which means $\lim_n |\langle x_n, y\rangle|^2 = 0$. Therefore, $\lim_n \langle x_n, y\rangle = 0$ as desired.

b. Let $b \in B$. We show that there exists a sequence in S that converges weakly to b. Since $\{b\}^\perp$ is a closed subspace of H, it is also a Hilbert space. Therefore, it has an orthonormal basis and we can extract from it an orthonormal sequence $\{x_n\}$ (since H is infinite dimensional, $\{b\}^\perp$ is infinite dimensional, then consider infinite countable dimension subspace and choose an orthonormal basis for it by the Gram-Schmidt method). Define,

$$b_n = \theta_n x_n + b.$$

We want θ_n to be a constant s.t. $\|b_n\| = 1$. We can exploit the fact that $\{x_n\} \subseteq \{b\}^\perp$ to get that,

$$1 = \|b_n\| = \sqrt{\langle \theta_n x_n, \theta_n x_n\rangle + \langle \theta_n x_n, b\rangle + \langle b, \theta_n x_n\rangle + \langle b, b\rangle} = \sqrt{|\theta_n|^2 + |b|^2}.$$

Therefore,

$$|\theta_n| = \sqrt{1 - |b|^2}.$$

Since $\|b\| \leq 1$, the above choice for $|\theta_n|$ is possible (we do not get a complex value). So, we set $\theta_n = \sqrt{1 - |b|^2}$ for all n. Next, we show $b_n \to b$ weakly. We employ the Riesz-Representation Theorem once more. Let $z \in H$. Then,

$$\langle b_n, z\rangle = \langle \theta_n x_n, z\rangle + \langle b, z\rangle = \theta_n \langle x_n, z\rangle + \langle b, z\rangle \to \langle b, z\rangle$$

because x_n is an orthonormal sequence and we apply part a. □

Folland Exercise 5.48 Suppose that X is a Banach space.

a. The norm-closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ is also weakly closed. (Use Theorem 5.8d.)

b. If $E \subset X$ is bounded (with respect to the norm), so is its weak closure.

c. If $F \subset X^*$ is bounded (with respect to the norm), so is its weak* closure.

d. Every weak*-Cauchy sequence in X^* converges. (Use Exercise 38.)

PROOF. **a.** Suppose $x_\alpha \to x$ weakly. Assume $x_\alpha \in B$ for all $\alpha \in A$ in the indexing set. We must show $x \in B$. The condition $f(x_\alpha) \to f(x)$ for all $f \in X$ is equivalent to $\widehat{x_\alpha}(f) \to \widehat{x}(f)$ for all $f \in X^*$. By Theorem 5.8d, we know $\|\widehat{x_\alpha}\| = \|x_\alpha\|$. Therefore, $\|\widehat{x}_\alpha\| \to \|\widehat{x}\|$ because $\|\cdot\| : X \to \mathbb{C}$ is also a bounded linear functional. But $\langle \|\widehat{x_\alpha}\|\rangle$ is a net in the closed unit ball $\overline{B_1(0)}$ of $\mathbb{C}$ and so is the limit $\|\widehat{x}\| \in \overline{B_1(0)}$. Hence, $\|x\| = \|\widehat{x}\| \leq 1$.

b. Suppose $E \subset X$ is bounded and $R > 0$ was large enough so that $E \subseteq B_R(0) := \{x \in X : \|x\| \leq R\}$. By part a., if we let $\overline{E}$ denote the weak closure of E and since the norm-closed unit ball is weakly closed, $\overline{E} \subseteq \overline{B} = B$ and this means $\overline{E}$ is also bounded.

To deduce $B_R(0)$ was also weakly closed, we made use of the fact that scalar multiplication is a homeomorphism $(X, w) \to (X, w)$ with respect to the weak topology. Indeed, this is an immediate consequence of the fact that the base for the weak topology is given by sets

$$U_{V_i, f_i} := \bigcap_{i=1}^{n} f_i^{-1}(V_i) \qquad V_i \subseteq \mathbb{C} \text{ open}, f_i \in X^*$$

and scalar multiplication by $R > 0$ gives $R(U_{V_i,f_i}) = \bigcap_{i=1}^n f_i^{-1}(RV_i)$ where $RA := \{Ra : a \in A\}$ when A is a set.

c. If $F \subset X^*$ is bounded w.r.t norm, let $R > 0$ be large enough so that $F \subseteq B_R(0) := \{f \in X^* : \|f\| \leq R\}$. Then $B_R(0)$ is compact in the weak* topology by Alaoglu's theorem and since the weak* topology is Hausdorff, compact sets are closed which means $B_R(0)$ is closed (the weak* topology is Hausdorff because of Exercise 4.17 and Theorem 5.8c). Therefore, if $\overline{F}$ denotes the weak closure of F, then $\overline{F} \subseteq \overline{B_R(0)} = B_R(0)$ is also bounded.

Again, to deduce $B_R(0)$ is compact, we made use of the fact that scalar multiplication $(X^*, w^*) \to (X^*, w^*)$ is a homeomorphism. The fact that this is also a homeomorphism follows by the same reasoning as above (the base for the topology is preserved).

d. Let $\{f_n\}$ be a weak* Cauchy sequence inside X^*. That means $f_n - f_m \to 0$ and therefore, $(f_n - f_m)(x) \to 0$ for all $x \in X$ (and the directed set here is $(n, m) \in \mathbb{N} \times \mathbb{N}$). So for all $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$ and by completeness, $\{f_n(x)\}$ converges. Let $f(x) := \lim_{n\to\infty} f_n(x)$ for $x \in X$. By Exercise 5.38, we deduce that $f \in L(X, \mathbb{C})$ when defined in this way.

We verify that $f_n \to f$ w.r.t the weak* topology. Indeed, $f_n \to f$ iff for all $x \in X$, we have $f_n(x) \to f(x)$ and indeed, this is immediate by the definition of f. □

Folland Exercise 5.51 A vector subspace of a normed vector space X is norm-closed iff it is weakly closed. (However, a norm-closed subspace of X^* need not be weak* -closed unless X is reflexive; see Exercise 52 d.)

PROOF. ($\Longrightarrow$): *Approach (1)*[15] Assume Y is a norm closed vector subspace of X. By Theorem 5.8, for every $x \in X \setminus M$, there exists $f_x \in M^*$ s.t. $f_x(x) \neq 0$ and $f|_M = 0$. Since the weak topology is generated by bounded linear functionals in X^*, we know that $f_x : (X, w) \to \mathbb{C}$ is continuous where (X, w) denotes X with the weak topology. Therefore, $f_x^{-1}(\{0\})$ is a closed subset of (X, w) because $\{0\} \subseteq \mathbb{C}$ is closed. Then,

$$M = \bigcap_{x \in X \setminus M} f_x^{-1}(\{0\})$$

which is immediate because the LHS is clearly contained in the RHS by construction and the RHS is contained in the LHS because the intersection runs over all those $x \in X \setminus M$.

However, the arbitrary intersection of closed sets is necessarily closed. So, this shows M is a closed subset of (X, w).

Approach (2) Assume Y is norm closed. To show Y is weakly closed, let $\langle y_\alpha \rangle$ be a net in Y. Suppose $y_\alpha \to y$ w.r.t the weak topology and we must show $y \in Y$. We know $f(y_\alpha) \to f(y)$ for all $f \in Y^*$. By Theorem 5.8, choose $f \in X^*$ s.t. $f|_Y = 0$ and $f(y) = \delta = \inf_{z \in Y} \|y - z\|$. Then if $y \notin Y$, that means

$$\|f(y_\alpha) - f(y)\| = \|f(y)\| = f(y) \neq 0$$

which is a contradiction.

($\Longleftarrow$): Assume Y is weakly closed. Suppose $y_n \to y$ is a sequence in Y and $\|y_n - y\| \to 0$. We claim $y \in Y$ which shows Y is norm closed. For any bounded linear function $f \in Y^*$,

$$\|f(y_n - y)\| \leq \|f\| \|y_n - y\| \to 0$$

which implies $y_n \to y$ weakly. By weakly closed, $y \in Y$. □

[15] This approach is a consequence of discussion with Jeb Runnoe.

Folland Exercise 5.53 Suppose that X is a Banach space and $\{T_n\}, \{S_n\}$ are sequences in $L(X, X)$ such that $T_n \to T$ strongly and $S_n \to S$ strongly.

a. If $\{x_n\} \subset X$ and $\|x_n - x\| \to 0$, then $\|T_n x_n - Tx\| \to 0$. (Use Exercise 47a.)

b. $T_n S_n \to TS$ strongly.

PROOF. **a.** Let $\epsilon > 0$. By Exercise 5.47a and $T_n \to T$ strongly, we know $C := \sup_n \|T_n\| < \infty$. Since $T_n \to T$ strongly, we also know $T_n x \to Tx$ in the norm topology of Y and therefore, we choose $N \in \mathbb{N}$ s.t $\|T_n x - Tx\| < \epsilon$ for $n \geq N$. Since $\|x_n - x\| \to 0$, choose $M \in \mathbb{N}$ so that $\|x_n - x\| < \epsilon$ for all $n \geq M$. Set $L := \max\{N, M\}$. Then, we deduce that for all $n \geq N$,

$$\begin{aligned}\|T_n x_n - Tx\| &= \|T_n x_n - T_n x + T_n x - Tx\| \leq \|T_n(x_n - x)\| + \|T_n x - Tx)\| \leq \|T_n\|\|x_n - x\| + \|(T_n - T)x| \\ &\leq C\|x_n - x\| + \|(T_n - T)x\| < C\epsilon + \epsilon\end{aligned}$$

Since the RHS is arbitrary, $\|T_n x_n - Tx\| \to 0$ as desired.

b. First, $T_n S_n \to TS$ strongly iff $\|(T_n S_n - TS)(x)\|_Y \to 0$ as $n \to \infty$ for all $x \in X$. Because $S_n \to S$ strongly, for any $x \in X$, we have $\|S_n x - Sx\| \to 0$. From a., we deduce

$$\|T_n(S_n x) - T(Sx)\| \to 0.$$

□

6. L^p Spaces

Folland Exercise 6.1 When does equality hold in Minkowski's inequality? (The answer is different for $p = 1$ and for $1 < p < \infty$. What about $p = \infty$)

PROOF. We deal with the case $p = 1$, $1 < p < \infty$, and $p = \infty$ separately.

(1) For $p = 1$, we need $\int |f + g| = \int |f| + \int |g|$ to occur. But $\int |f| + \int |g| = \int |f| + |g|$ so that means we must have $|f(x) + g(x)| = |f(x)| + |g(x)|$ for a.e. x. Hence, equality holds iff the usual triangle inequality is an equality a.e.. Recall that the usual triangle inequality for complex numbers holds iff $f = 0$ or $g = 0$ or $f = \kappa g$ for some $\kappa \geq 0$.

(2) Suppose $1 < p < \infty$. We needed equality at two points. First, we needed equality at $|f+g|^p \leq (|f|+|g|)|f+g|^{p-1}$ which we need a.e. since we integrated afterwards. So, we need $|f+g| = |f|+|g|$ a.e.. Since f, g are complex valued functions, equality occurs iff f and g have the same argument a.e.. Therefore, $|f+g| = |f|+|g|$ a.e. occurs iff f is a positive multiple of g a.e. or $f = 0$ a.e. or $g = 0$ a.e.

We also need equality to occur in our application of Hölder's inequality. Our first application was $\int |f||f+g|^{p-1} \leq \|f\|_p \||f+g|^{p-1}\|_q$. Equality occurs iff there exists constants α_1, β_1 with both not equal to zero s.t.

$$\alpha_1 |f|^p = \beta_1 |f+g|^{q(p-1)} = \beta_1 |f+g|^p \text{ a.e..} \tag{263}$$

Note that $q(p-1) = p$ since $\frac{1}{p} + \frac{1}{q} = 1$. Our second application was $\int |g||f+g|^{p-1} \leq \|g\|_p \||f+g|^{p-1}\|_q$. Equality occurs iff there exists constants α_2, β_2 with both not equal to zero s.t.

$$\alpha_2 |g|^p = \beta_2 |f+g|^{q(p-1)} = \beta_2 |f+g|^p \text{ a.e..} \tag{264}$$

But then

$$\alpha_1 \beta_2 |f|^p = \beta_1 \beta_2 |f+g|^p = \beta_1 \alpha_2 |g|^p.$$

Taking pth roots and recalling the discussion in the first paragraph, equality in Minkowski's inequality for $1 < p < \infty$ occurs iff $f = 0$ a.e. or $g = 0$ a.e. or f is a positive multiple of g a.e. and there exists constants α, β s.t. $\alpha|f| = \beta|g|$ a.e.. Notice that the condition that there exists such constants α, β follows from the fact that f is a positive multiple of g: set $f = \kappa g$ for $\kappa > 0$, then take $\alpha = 1$ and $\beta = \kappa$ to get $\alpha|f| = |\kappa g| = \beta|g|$. Thus, equality in Minkowi's inequality occurs iff $f = 0$ a.e. or $g = 0$ a.e. or f is a positive multiple of g a.e..

(3) Let $p = \infty$. There is no characterization for equality similar to that for the case $1 \le p < \infty$. Take for example $f = \chi_{(-1,0)}$ and $g = \chi_{(-1,1)}$. In this case, $\|f+g\|_\infty = 2$ and $\|f\|_\infty = \|g\|_\infty = 1$ so equality for Minkowksi's inequality occurs. However, there are no constants α and β s.t. $\alpha|f| = \beta|g|$ except $\alpha = \beta = 0$.

We give a characterization by using Theorem 6.8 and the definition. Note that essential supremum $\|f\|_\infty$ says that $|f| \le \|f\|_\infty$ a.e.. Hence, $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$ from the usual triangle inequality. From the definition, equality occurs iff there is a positive measure set E on which $|f+g| = |f| + |g|$, $|f+g| \le a$, and a is the infimum such value over all possible positive measure sets E.

$\square$

Improving Understanding of $p = \infty$ equality:

The homework solution had: "For $p = \infty$, equality holds iff $\forall \epsilon > 0$, $\exists \theta \in [0, 2\pi]$ s.t.

$$\mu(\{x \in X \mid |f(x) - e^{i\theta}\|f\|_\infty| \le \epsilon \ \& \ |g(x) - e^{i\theta}\|g\|_\infty| \le \epsilon\}) > 0."$$

Proof. ($\Longleftarrow$): Assume the condition holds. Then, for a.e. x in that set of positive measure

$$|||f\|_\infty + \|g\|_\infty| - (|f(x)+g(x)|)| \le |f(x)+g(x) - e^{i\theta}(\|f\|_\infty + \|g\|_\infty)| \le |f(x) - e^{i\theta}\|f\|_\infty| + |g(x) - e^{i\theta}\|g\|_\infty| \le 2\epsilon$$

the first inequality follows from reverse triangle, second from usual triangle, and last from x in the set. Then, for a.e. x in the set

$$\|f\|_\infty + \|g\|_\infty \le |f(x) + g(x)| + 2\epsilon \le \|f+g\|_\infty + 2\epsilon.$$

But since this holds for all $\epsilon > 0$, $\|f\|_\infty + \|g\|_\infty \le \|f+g\|_\infty$ and equality holds. Note: the reason why that set had to have positive measure is because we need to it to deduce $|f(x) + g(x)| \le \|f+g\|_\infty$.

($\Longrightarrow$): Assume that the condition fails so fix an $\epsilon > 0$ and for all $\theta \in [0, 2\pi]$ the set had measure zero. That means, for every $\theta \in [0, 2\pi] \cap \mathbb{Q}$,

$$|f(x) - e^{i\theta}\|f\|_\infty| > \epsilon \qquad \text{or} \qquad |g(x) - e^{i\theta}\|g\|_\infty| > \epsilon$$

for almost every x. WLOG assume the first holds for an x. Then $|f(x) - e^{i\theta}\|f\|_\infty| > \epsilon \implies \|f\|_\infty - \epsilon > |f(x)|$ a.e..

In particular, if $f(x) = r_x e^{i\theta_x}$, then

$$|f(x) - e^{i\theta_x}\|f\|_\infty| > \epsilon \quad \implies \quad |r_x - \|f\|_\infty| > \epsilon \quad \implies \quad \|f\|_\infty - \epsilon > r_x = |f(x)| \ a.e..$$

But this means

$$|f(x)+g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty - \epsilon + \|g\|_\infty \le \|f\|_\infty + \|g\|_\infty - \epsilon.$$

for almost every x. But that means $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty - \epsilon$ and so

$$\|f+g\|_\infty < \|f\|_\infty + \|g\|_\infty.$$

Question from OH, "why write $|f(x) - e^{i\theta}\|f\|_\infty| \le \epsilon$ instead of $|f| \ge \|f\|_\infty - \epsilon$?"

The condition won't work. The ($\Longrightarrow$) direction fails! If we claimed equality when for all $\epsilon > 0$

$$\mu(\{x \in \mathbb{R} \mid |f| \geq \|f\|_\infty - \epsilon \ \ \& \ \ |g| \geq \|g\|_\infty - \epsilon\}) > 0.$$

What is the issue here? Consider $f := \chi_{(0,1)}$ and $g := -\chi_{(0,1)}$. Then

$$\mu(\{x \in X \mid |f| \geq 1 - \epsilon \quad \& \quad |g| \geq 1 - \epsilon\}) > 0$$

but equality doesn't occur in Minkowski's.

Extra Question: Why not require $|f(x) - \|f\|_\infty| < \epsilon$ and $|g(x) - \|g\|_\infty| < \epsilon$ on some positive measure set?

Answer: Then ($\Longleftarrow$) fails. Because we can get a counterexample; $f = (1+i)\chi_{(0,1)}$ and $g = (2+2i)\chi_{(0,1)}$. In this case, the condition is not fulfilled $\|f\|_\infty = \sqrt{2}$ and $\|g\|_\infty = 2\sqrt{2}$. For example, take $\epsilon = 1$ and the set where both inequalities is true is not positive measure;

$$|f(x) - \|f\|_\infty| > 1 \qquad |g(x) - \|g\|_\infty| > 1$$

on $(0,1)$ (the first is $\sim$1.08 and second is $\sim$2.1).

However, equality in Minkowksi's *does* occur

$$\|f + g\|_\infty = 3\sqrt{2} = \|f\|_\infty + \|g\|_\infty.$$

Key Note from OH: The two statements are not equivalent ("if $\forall \theta$, a.e. x" $\Longleftrightarrow$ "a.e. x, $\forall \theta$) Let us write out the conditions.

$\forall \theta \in S$, $\mu(A_\theta^c) = 0$ versus $\mu(A_\theta^c) = 0$, $\bigcap_{\theta \in S} A_\theta = A$ and $\mu(A^c) = 0$ are not equivalent unless S is a countable set.

Example where the above equivalence fails is as follows. Consider $f_y : \mathbb{R} \to \mathbb{R}$ and $f_y(x) = x - y$. Clearly, $\forall y \in \mathbb{R}$, $f_y(x) \neq 0$ for a.e. x. However, we do not have for a.e. x, $\forall y \in \mathbb{R}$, $f_y(x) \neq 0$. Because we can just pick $y_x = x$ and then, $f_{y_x}(x) = 0$. $\square$

Folland Exercise 6.2 Prove Theorem 6.8 .

PROOF. We break the proof into parts.

(a) Let f, g be measurable functions. We may restrict our attention to the set where $f \neq 0$ since the result is vacuous if $f = 0$. Let $E_a := \{x : |g(x)| > a\}$ and $E_a^c := \{x : |g(x)| \leq a\}$. Consider the values $a \geq 0$ s.t. $\mu(E_a) = 0$. Then,

$$\int |fg| = \int |f||g| \leq \int_{E_a} |f||g| + \int_{E_a^c} a|f| = \int_{E_a} |f||g| + a \int_{E_a^c} |f| = a \int_{E_a^c} |f| = a \int_X |f|. \tag{265}$$

Notice that we have equality at the end since $\mu(E_a^c) = \mu(X)$ due to $\mu(E_a) = 0$. Taking the infimum over all such $a \geq 0$, we get

$$\|fg\|_1 \leq \|g\|_\infty \|f\|_1 \tag{266}$$

as desired.

We had an inequality when deducing $\int_{E_a^c} |f||g| \leq a \int_{E_a^c} |f|$ and this was the only time we had an inequality. Notice that we get equality iff $|g| = a$ a.e.. But we took the infimum over all such $a \geq 0$ and hence, since we restricted attention to when $f \neq 0$, equality occurs iff $|g| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$.

(b) We show that $\|\cdot\|_\infty$ is a norm.

(i) Suppose $\|f\|_\infty = 0$. Then that means $\{x : |f(x)| > 0\}$ has measure zero. So, $|f| = 0$ a.e. which means $f = 0$ a.e.. Suppose $f = 0$ a.e.. Then $\{x : |f| > 0\}$ has measure zero and $\|f\|_\infty = 0$.

(ii) Let $\alpha \in \mathbb{C}$. If $\alpha = 0$, then it is clear that $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$ so suppose $\alpha \neq 0$. Then

$$\|\alpha f\|_\infty = \inf\{a \geq 0 : |\alpha f(x)| > a\} = \inf\{a \geq 0 : |f(x)| > \frac{a}{|\alpha|}\} = |\alpha|\|f\|_\infty. \tag{267}$$

(iii) We proved this in Exercise 6.1, so we repeat the proof. By definition, $\|f\|_\infty$ is the infimum such value for which $|f| \leq \|f\|_\infty$ a.e. and since $|f+g| \leq |f| + |g|$, one has $\|f+g\|_\infty \leq |f| + |g|$ a.e.. Hence, $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

(c) ($\Longrightarrow$): Assume $\|f_n - f\|_\infty \to 0$. If $f_n - f = 0$ a.e. for all n, then there is nothing to show. Consider sets

$$E_n := \{x : |f_n(x) - f(x)| > a_n\} \qquad \Longrightarrow \qquad E_n^c := \{x : |f_n(x) - f(x) \leq a_n\} \tag{268}$$

where $a_n \geq 0$ for all n, $a_n \to 0$. Because $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, We may require $\mu(E_n) = 0$ as for all n by taking a_n to be the infimum such value in the definition of $\|\cdot\|_\infty$. Let $E^c := \bigcup_{n=1}^\infty E_n$ which has measure zero by subadditivity. Then $E := \bigcap_{n=1}^\infty E_n^c$ which is a set that $f_n \to f$ uniformly on.

($\Longleftarrow$): Let $\epsilon > 0$. Choose an $N \in \mathbb{N}$ sufficiently large s.t. for all $n \geq N$, we have

$$E_\epsilon := \{x : |f_n(x) - f(x)| > \epsilon\} \subseteq E^c. \tag{269}$$

By monotonicity, $\mu(E_\epsilon) = 0$. Therefore, $\|f_n - f\|_\infty \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, this shows that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

(d) We first verify that $L^\infty(\mu)$ is a vector space. Indeed, if $f, g \in L^\infty(\mu)$ and α, β are scalars, then

$$\|\alpha f + \beta g\|_\infty \leq |\alpha|\|f\|_\infty + |\beta|\|g\|_\infty < \infty \qquad \Longrightarrow \qquad \alpha f + \beta g \in L^\infty(\mu). \tag{270}$$

We employ Theorem 5.1 to show completeness because (b) and the above shows us that L^∞ is a normed vector space. Suppose $\{f_k\} \subseteq L^\infty$ gives an absolutely convergent series $\sum_{k=1}^\infty \|f_k\|_\infty = B < \infty$. Let $F := \sum_{k=1}^\infty f_k$. Then, $\|F\|_\infty \leq \sum_{k=1}^\infty \|f_k\|_\infty = B$ by (a) which shows us that $F \in L^\infty$. Furthermore, since for a.e. x, we have $|F(x)| \leq \sum_{k=1}^\infty |f_k| \leq \sum_{k=1}^\infty \|f_k\|_\infty < B$ a.e. and so $F(x)$ is defined a.e.. All that is left is to show that $\sum_{k=1}^m f_k \to F$ as $m \to \infty$. We have

$$\left\| F - \sum_{k=1}^m f_k \right\|_\infty = \left\| \sum_{k=m+1}^\infty f_k \right\|_\infty \leq \sum_{k=m+1}^\infty \|f_k\|_\infty = \sum_{k=1}^\infty \|f_k\|_\infty \leq B < \infty$$

so letting $m \to \infty$, the series on the $\sum_{k=m+1}^\infty \|f_k\|_\infty$ goes to zero. Hence, the LHS goes to zero as $m \to \infty$ which shows that $\sum_{k=1}^m f_k \to F$ as desired.

(e) First, notice that simple functions are in L^∞. Indeed, suppose $\phi := \sum_{i=1}^n a_i \chi_{E_i}$ were simple where we assume WLOG that $\mu(E_i) > 0$. Then

$$\|\phi\|_\infty = \sup_{1 \leq i \leq n} |a_i| < \infty. \tag{271}$$

Now suppose $f \in L^\infty$ were arbitrary. By Theorem 2.10, choose a sequence of simple functions f_n s.t. $|f_n|$ increases to $|f|$ and $f_n \to f$ uniformly on any set in which f is bounded. In particular, $f_n \to f$ uniformly on the set,

$$E := X \setminus \{x \in X : |f| > \|f\|_\infty\}. \tag{272}$$

But the set E^c is measurable and of μ-measure zero by definition of $\|\cdot\|_\infty$. By (c), this means for all $\epsilon > 0$, we have $|f_n - f| < \epsilon$ on E, and hence $\|f_n - f\|_\infty < \epsilon$ on E. Therefore, $\|f_n - f\|_\infty \to 0$.

□

Folland Exercise 6.3 If $1 \le p < r \le \infty, L^p \cap L^r$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$ and if $p < q < r$, the inclusion map $L^p \cap L^r \to L^q$ is continuous.

Proof. Since L^p and L^r are Banach spaces, and in particular vector spaces, the intersection $L^p \cap L^r$ is also a vector space. To see that $\|\cdot\|$ is a norm, we show the three axioms.

(i) First, $\|f\| = 0$ iff $\|f\|_p = \|f\|_r = 0$ iff $f = 0$ a.e..
(ii) Second, for any scalar $\lambda \in \mathbb{C}$, $\|\lambda f\| = \|\lambda f\|_p + \|\lambda f\|_r = |\lambda|\|f\|$.
(iii) Third, if $f, g \in L^\cap L^r$, then $\|f+g\| = \|f+g\|_p + \|f+g\|_r \le \|f\|_p + \|g\|_p + \|f\|_r + \|f\|_r \le \|f\| + \|g\|$.

Using Theorem 5.1: For showing completeness, we employ Theorem 5.1 of the text. Suppose $\{f_k\} \subset L^p \cap L^r$ is an absolutely convergent series i.e. $\sum_{k=1}^\infty \|f_k\| = B < \infty$. Let $G_n := \sum_{k=1}^n |f_k|$ and $G := \sum_{k=1}^\infty |f_k|$. Then by Minkowski's inequality,

$$\|G_n\| = \|G_n\|_p + \|G_n\|_r \le \sum_{k=1}^n \|G_n\|_p + \sum_{k=1}^n \|G_n\|_r \le \sum_{k=1}^n \|G\| \le B \tag{273}$$

for all $n \in \mathbb{N}$. By MCT,

$$\max\left\{\int G^p, \int G^r\right\} = \max\left\{\lim \int G_n^p, \lim \int G_n^r\right\} \le B < \infty \tag{274}$$

and so $G \in L^p \cap L^r$. Hence, $G(x) < \infty$ a.e. and $F := \sum_{k=1}^\infty f_k$ is defined a.e.. Since $|F| \le G$, $F \in L^p \cap L^r$, $|F - \sum_{k=1}^n f_k|^r \le (2G)^r \in L^1$, and $|F - \sum_{k=1}^n f_k|^p \le (2G)^p \in L^1$. By DCT,

$$\left\|F - \sum_{k=1}^n f_k\right\|_p^p = \int \left|F - \sum_{k=1}^n f_k\right|^p \to 0 \qquad \& \qquad \left\|F - \sum_{k=1}^n f_k\right\|_r^r = \int \left|F - \sum_{k=1}^n f_k\right|^r \to 0 \tag{275}$$

as $n \to \infty$ and so, which means

$$\left\|F - \sum_{k=1}^n f_k\right\| = \left\|F - \sum_{k=1}^n f_k\right\|_p + \left\|F - \sum_{k=1}^n f_k\right\|_r \to 0 \qquad \text{as } n \to \infty \tag{276}$$

i.e. $\sum_{k=1}^\infty f_k$ converges in the $L^p \cap L^r$-norm.

In case we cannot use Theorem 5.1: Let $\{f_n\}$ be a Cauchy sequence in $L^p \cap L^r$. By definition, $\{f_n\}$ is also Cauchy w.r.t. the p-norm and the q-norm. By completeness of L^p and L^r, assume $f_n \to g$ in L^p and $f_n \to h$ in L^r. By Exercise 6.9, convergence in the p-norm and the q-norm shows that $f_n \to g$ and $f_n \to h$ in measure. By Theorem 2.30, since $f_n \to h$ in measure and $f_n \to g$ in measure, $g = h$ a.e.. Therefore, $\{f_n\}$ converges to g in $L^p \cap L^r$.

Inclusion map is continuous: Suppose $p < q < r$ and $i : L^p \cap L^r \hookrightarrow L^q$ is the inclusion map. Let $\epsilon > 0$. Let $\delta := \epsilon$. Let $g \in L^p \cap L^r$. Suppose $f \in L^p \cap L^r$ s.t. $\|f - g\| < \delta$. By Proposition 6.10, for the λ in the proposition

(277)
$$\|i(f) - i(g)\|_q = \|f - g\|_q \leq \|f - g\|_p^\lambda \|f - g\|_q^{1-\lambda} \leq \|f - g\|^\lambda \|f - g\|^{1-\lambda} \leq \|f - g\| < \delta = \epsilon$$

which shows continuity.

□

Folland Exercise 6.4 If $1 \leq p < r \leq \infty, L^p + L^r$ is a Banach space with norm $\|f\| = \inf\{\|g\|_p + \|h\|_r : f = g + h\}$, and if $p < q < r$, the inclusion map $L^q \to L^p + L^r$ is continuous.

Proof. First, we verify that $\|\cdot\|$ is a norm.

i. Suppose $\|f\| = 0$. Then there is a sequence of g_n and h_n s.t. $f = g_n + h_n$ and $\|g_n\|_p, \|h_n\|_r \to 0$. But that means $g_n \to 0$ and $h_n \to 0$ since L^p and L^r are complete. Therefore, $f = 0$.
Suppose $f = 0$. Then $\|f\| = 0$ since $0 \in L^p$ and $0 \in L^r$.

ii. Suppose α is a constant. Then,
$$\|\alpha f\| = \inf\{\|\alpha g\|_p + \|h\|_r : f = g + h\} = |\alpha|\|f\|.$$

iii. For the triangle inequality,
$$\|f + f'\| \leq \inf\{\|g\|_p + \|h\|_r : f = g + h\| + \inf\{\|g'\|_p + \|h'\|_r : f' = g' + h'\} = \|f\| + \|f'\|.$$

Now, we show that $L^p + L^r$ is a complete Banach space w.r.t the given norm. We use Theorem 5.1 to do this. Suppose $\sum_1^\infty \|f_n\| < \infty$. By definition of $\|\cdot\|$, we can find $g_n \in L^p$ and $h_n \in L^r$ s.t. $\|g_n\|_p + \|h_n\|_q < \|f_n\| + \frac{\epsilon}{2^n}$ and $f_n = g_n + h_n$. Then,

$$\sum_1^\infty \|g_n\|_p \leq \sum_1^\infty \|f\| + \epsilon. \tag{278}$$

Since L^p is complete, $\sum_1^\infty g_n$ is defined i.e. it converges. Similarly, $\sum_1^\infty h_n$ is defined due to L^r complete. But then $\sum_1^\infty f_n = \sum_1^\infty g_n + \sum_1^\infty h_n$.

We now show that the map $L^q \hookrightarrow L^p + L^r$ is continuous. Let $\epsilon > 0$. Now, let $E := \{x : |f(x)| > 1\}$ and $F := E^c$. Then we can decompose any $f \in L^q$ as $f := f\chi_E + f\chi_F$ and Proposition 6.9 showed that $f\chi_E$ and $f\chi_F$ are in L^p and L^r respectively. We also have $|f|^p \leq |f\chi_E|^q$ and $|f|^r \leq |f\chi_F|^r$. Integrating and taking pth roots, we deduce that $\|f\|_p \leq \|f\|_q^{\frac{q}{p}}$ and $\|f\|_p \leq \|f\|_r^{\frac{q}{r}}$. Let $\epsilon > 0$ be arbitrary. Let $\delta := \max\{\epsilon^{\frac{p}{q}}, \epsilon^{\frac{r}{q}}\}$.

Since the inclusion map is linear, we just need to show continuity at 0. Suppose $\|f\|_q \leq \delta$. Then,

$$\|f\| \leq \|f\chi_E\|_p + \|f\chi_{E^c}\|_r \leq \|f\chi_E\|_q^{q/p} + \|f\|_q^{q/r} < 2\epsilon. \tag{279}$$

□

Folland Exercise 6.5 Suppose $0 < p < q < \infty$. Then $L^p \not\subset L^q$ iff X contains sets of arbitrarily small positive measure, and $L^q \not\subset L^p$ iff X contains sets of arbitrarily large finite measure. (For the "if" implication: In the first case there is a disjoint sequence $\{E_n\}$ with $0 < \mu(E_n) < 2^{-n}$, and in the second case there is a disjoint sequence $\{E_n\}$ with $1 \leq \mu(E_n) < \infty$. Consider $f = \sum a_n \chi_{E_n}$ for suitable constants a_n.) What about the case $q = \infty$?

Folland Exercise 6.6 Suppose $0 < p_0 < p_1 \leq \infty$. Find examples of functions f on $(0, \infty)$ (with Lebesgue measure), such that $f \in L^p$ iff (a) $p_0 < p < p_1$ (b) $p_0 \leq p \leq p_1$ (c) $p = p_0$. (Consider functions of the form $f(x) = x^{-a}|\log x|^b$.)

PROOF. Let us deal with the cases.

(a) We give an f s.t. $f \in L^p$ iff $p_0 < p < p_1$. Notice that

$$\int \chi_{(2,\infty)} \frac{1}{x^{\frac{p}{p_0}}} dx \qquad \& \qquad \int \chi_{[0,2)} \frac{1}{x^{\frac{p}{p_1}}} dx \tag{280}$$

converge iff $p_0 < p < p_1$. Therefore, $f(x) := \chi_{[0,2)} \frac{1}{x^{\frac{p}{p_1}}} + \chi_{[0,2)} \frac{1}{x^{\frac{p}{p_1}}} \in L^p$ iff $p_0 < p < p_1$.

(b) Notice that

$$\int \chi_{(0,\frac{1}{e})} \frac{1}{(x^2|\log x|)^{\frac{p}{p_1}}} < \infty \tag{281}$$

iff $p \leq p_1$. The reason is that if $p \leq p_1$, the integrand is bounded above by $\chi_{(0,\frac{1}{e})} \frac{1}{x^{1-\frac{1}{p_1}}}$ which converges while when $p > p_1$, the integrand is bounded from below by $\chi_{(0,\frac{1}{e})} \frac{1}{x}$ which diverges.

Notice that

$$\int \chi_{(8,\infty)} \frac{1}{(x|\log x|^2)^{\frac{p}{p_0}}} dx \tag{282}$$

converges iff $p_0 \leq p$. When $p = p_0$, then the integral is equal to $\frac{1}{\ln(8)}$ by undergraduate analysis. When $p_0 < p$, we can bound the integrand from above by $\chi_{(8,\infty)} \frac{1}{(x|\log x|^2)}$ which converges which is only possible when x is s.t. $x|\log x|^2 \geq 1$ and we chose $(8, \infty)$ specifically for this purpose. On the other hand, when $p < p_0$, we can bound the integral from below by $\chi_{(8,\infty)} \frac{1}{x^{\frac{p}{p_0}}}$. Therefore, our desired function is

$$f := \chi_{(0,\frac{1}{e})} \frac{1}{(x^2|\log x|)^{\frac{1}{p_1}}} + \chi_{(8,\infty)} \frac{1}{(x|\log x|^2)^{\frac{1}{p_0}}} dx \tag{283}$$

and $f \in L^p$ iff $p_0 \leq p \leq p_1$.

(c) Given p_0. Now let $f := \frac{1}{x^{\frac{1}{p_0}}(|\ln x|^2+1)}$. We claim $f \in L^p$ and only when $p = p_0$.

Suppose $p > p_0$. Then $\frac{p}{p_0} > 1$ and we can bound $f^p \geq \frac{1}{x^{\frac{p}{p_0}}} \chi_{(0,1)}$ and since the integral of the latter term diverges, $f \notin L^p$ if $p > p_0$. Similarly, suppose $p < p_0$. Then we can bound $f^p > \frac{1}{x^{\frac{p}{p_0}}} \chi_{(3,\infty)}$ and since $\frac{p}{p_0} < 1$, the integral of the latter function diverges.

$\square$

Folland Exercise 6.7 If $f \in L^p \cap L^\infty$ for some $p < \infty$, so that $f \in L^q$ for all $q > p$, then $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$.

PROOF. This is Homework 8 on Homework 1 for Math 240B. We showed on the Homework that such a limit occurs iff $f \notin L^\infty \setminus \bigcup_{p \in [1,\infty)} L^p$. $\square$

Folland Exercise 6.8 Suppose $\mu(X) = 1$ and $f \in L^p$ for some $p > 0$, so that $f \in L^q$ for $0 < q < p$.

a. $\log \|f\|_q \geq \int \log |f|$. (Use Exercise 42 d in §3.5, with $F(t) = e^t$.).

b. $\left(\int |f|^q - 1\right)/q \geq \log \|f\|_q$, and $\left(\int |f|^q - 1\right)/q \to \int \log |f|$ as $q \to 0$.
c. $\lim_{q\to 0} \|f\|_q = \exp\left(\int \log |f|\right)$.

PROOF. a. By Jensen's inequality,

$$\exp\left(\int \log |f|\right)^q \leq \int \exp((\log |f|)^q) = \int |f|^q. \tag{284}$$

Taking qth roots, and taking the logarithm, $\int \log |f| \leq \log \|f\|_q$ as desired.

b. Note that $x - 1 \geq \log x$ for all $x \geq 0$. Set $x := \int |f|^q$. Then,

$$\int |f|^q - 1 \geq \log\left(\int |f|^q\right) \qquad \Longrightarrow \qquad \frac{\int |f|^q - 1}{q} \geq \log \|f\|_q \tag{285}$$

where the latter followed by properties of logarithms.

The last part, which is only sketched here, follows from noting how $x - 1 \to q \log x$ for a.e. x.

c. By part (b) and (a), we deduce that

$$\exp\left(\frac{\int |f|^q - 1}{q}\right) \geq \|f\|_q \geq \exp\left(\int \log |f|\right) \tag{286}$$

and part (b) shows that the LHS converges to $\exp\left(\int \log |f|\right)$ as $q \to 0$, hence, $\|f\|_q \to \exp\left(\int \log |f|\right)$ as $q \to 0$.

$\square$

Folland Exercise 6.9 Suppose $1 \leq p < \infty$. If $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n, then $\|f_n - f\|_p \to 0$

PROOF. Suppose $\|f_n - f\|_p \to 0$. Then $\int |f_n - f|^p \to 0$ as $n \to \infty$. Set $E_{n,\epsilon} := \{x \in X : |f_n - f|^p \geq \epsilon\}$. Then,

$$\int |f_n - f|^p \geq \int_{E_{n,\epsilon}} |f_n - f| \geq \int_{E_{n,\epsilon}} \epsilon = \epsilon\mu(E_{n,\epsilon}). \tag{287}$$

As $n \to \infty$, the LHS goes to zero. Hence, $\mu(E_{n,\epsilon}) \to 0$. Since $\epsilon > 0$ was arbitrary and $|f_n - f|^p \geq \epsilon$ is equivalent to $|f_n - f| \geq \epsilon^{1/p}$, we deduce $f_n \to f$ in measure. The statement about a subsequence converging a.e., use Theorem 2.30.

Suppose $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n. Then, $|f_n - f| \to 0$ in measure and $|f_n - f|^p \to 0$. So, we choose a subsequence s.t. $\int |f_{n_j} - f|^p \to \lim_{n\to\infty} \int |f_{n_j} - f|^p$. Taking another subsequence by Theorem 2.30, $|f_{n_{j_i}} - f|^p \to 0$ a.e.. Next, notice that $|f_{n_{j_i}} - f|^p \leq 2^{p+1}|g|^p \in L^1$ since $g \in L^p$.. We are justified in applying the DCT; $\lim_{i\to\infty} \int |f_{n_j} - f|^p = \int \lim_{i\to\infty} |f_{n_{j_i}} - f|^p = 0$. Therefore,

$$\lim_{n\to\infty} \int |f_n - f|^p = \lim_{j\to\infty} \int \left|f_{n_j} - f\right|^p = \lim_{i\to\infty} \int \left|f_{n_{j_i}} - f\right|^p = 0 \qquad \Longrightarrow \qquad \|f_n - f\|_p \to 0.$$

$\square$

Folland Exercise 6.10 (without using Exercise 2.20) Suppose $1 \leq p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$. (Use Exercise 20 in §2.3.)

PROOF. ($\implies$): Since the triangle inequality holds for $p \geq 1$, the reverse triangle inequality also holds by a standard undergraduate analysis proof; $|\|f_n\|_p - \|f\|_p| \leq \|f_n - f\|_p$. Since the RHS goes to zero as $p \to \infty$, $\|f_n\|_p \to \|f\|_p$.

($\impliedby$): It suffices to show $\int |f_n - f|^p \to 0$ as $n \to \infty$. Because $0 \leq |f_n - f|^p \leq 2^p(|f_n|^p + |f|^p)$,

$$0 \leq 2^p(|f_n|^p + |f|^p) \pm |f_n - f|^p.$$

Set $g_n := 2^p(|f_n|^p + |f|^p)$ and then, $g := \lim_{n\to\infty} g_n = 2^{p+1}|f|^p$ a.e. since $f_n \to f$ a.e.. Set $h_n := |f_n - f|^p$ and then, $h := \lim_{n\to\infty} h_n = 0$ a.e. since $f_n - f \to 0$ a.e.. Notice that $h_n \in L^1$ since $g_n \in L^1$. Then,

$$\int g + \int h = \int (g+h) \leq \int g + \liminf \int h_n$$
$$\int g - \int h = \int (g-h) \leq \liminf \left(\int g - h_n \right) \leq \int g - \limsup \int h_n.$$

Subtracting $\int g$ from both sides of the two inequalities above, and rewriting the second inequality, we have

$$\limsup \int h_n \leq \int h \leq \liminf \int h_n,$$

whereby, $\int h = \lim_{n\to\infty} \int h$. But $\int h = 0$ and from the definition of h_n, $\lim_{n\to\infty} \int |f_n - f|^p = 0$ as desired. □

Folland Exercise 6.11 If f is a measurable function on X, define the essential range R_f of f to be the set of all $z \in \mathbb{C}$ such that $\{x : |f(x) - z| < \epsilon\}$ has positive measure for all $\epsilon > 0$

a. R_f is closed.

b. If $f \in L^\infty$, then R_f is compact and $\|f\|_\infty = \max\{|z| : z \in R_f\}$.

PROOF. □

Folland Exercise 6.12 If $p \neq 2$, the L^p norm does not arise from an inner product on L^p, except in trivial cases when $\dim(L^p) \leq 1$. (Show that the parallelogram law fails.)

PROOF. Assume $p \neq 2$ and $\dim(L^p) > 1$. Let E be a set of finite measure and assume it is not of measure zero. Then set $f := \chi_E$ and $g = f$. Then the Parallelogram Law gives, after some algebraic manipulations, $\mu(E)^{2/p} = \mu(E)$ which is true iff $\mu(E) = 0, 1$ when $p \neq 2$. Suppose $\mu(E) = 1$ were the case. Then take $f := 2\chi_E$ and $g := \chi_E$. Then the Parallelogram Law would given, after some algebraic manipulations and simplifications, $9 = 10$. Contradiction.

Remark: The professor discussed this in class and noted that L^2 is special since it is the only L^p space that is a Hilbert space. Obviously, we ignore the trivial case where $\dim(L^p) \leq 1$. □

Folland Exercise 6.13 $L^p(\mathbb{R}^n, m)$ is separable for $1 \leq p < \infty$. However, $L^\infty(\mathbb{R}^n, m)$ is not separable. (There is an uncountable set $\mathcal{F} \subset L^\infty$ such that $\|f - g\|_\infty \geq 1$ for all $f, g \in \mathcal{F}$ with $f \neq g$.)

PROOF. We exhibit a countable dense subset of $L^p(\mathbb{R}^n, m)$. We know that the simple functions $\mathcal{S}$ with characteristic functions on finite measure sets are dense in $L^p(\mathbb{R}^n, m)$ by Theorem 6.7. Since being dense is transitive, we exhibit a countable dense subset of the collection of such simple functions. By Theorem 2.40c, if $m(E) < \infty$, there is a finite sequence of disjoint rectangles whose sides are intervals $\{R_j\}_1^N$ s.t. $m(E\Delta \cup_1^N R_j) < \epsilon$ and WLOG, we require the intervals to be open intervals. Define,

$$\mathcal{D} := \{ \text{ rectangles } S \mid \text{ centered at } c \in \mathbb{Q}^n \text{ with rational vertices and open interval sides}\}$$

The collection $\mathcal{D}$ approximates the collection of rectangles i.e. if R is an arbitrary rectangle, for any $\epsilon > 0$, we can choose an $S \in \mathcal{D}$ s.t. $m(R \setminus S) < \epsilon$. Also, $\mathcal{D}$ is a countable collection since there are countably many vertices rectangles in $\mathcal{D}$ can have. We show the first statement is possible.

Suppose $\epsilon > 0$. For each side $R_i := (a_i, b_i)$ of R, we can choose $S_i := (r_i, q_i)$ where $r_i, q_i \in \mathbb{Q}$, $S_i \subseteq R_i$ and $m(R_i \setminus S_i) = (b_i - q_i) + (a_i - r_i) < \epsilon^{\frac{1}{n}}$ by choosing q_i arbitrarily close to b_i and r_i arbitrarily close to a_i. The center of S_i is obviously a rational number. Set $S := \prod_{i=1}^n S_i$ which will also have a rational center. Then, $m(R \setminus S) = \prod_{i=1}^n m(R_i \setminus S_i) = \epsilon$.

Since arbitrary rectangles approximate Lebesgue measurable sets of finite measure, the elements of $\mathcal{D}$ approximate Lebesgue measurable sets of finite measure too as the following argument shows. If E is Lebesgue measurable, choose rectangles $\{R_j\}_1^N$ s.t. $m(E\Delta \cup_1^N R_j) < \epsilon$. Then, choose rectangles $S_j \in \mathcal{D}$ s.t. $m(R_j \setminus S_j) < \epsilon$. By Folland Exercise 1.12 and exploiting the fact that $S_j \subseteq R_j$,

$$m(E\Delta \cup_1^N S_j) \leq m(E\Delta \cup_1^N R_j) + m(\cup_1^N S_j \Delta \cup_1^N R_j) \leq \epsilon + m((\cup_1^N S_j) \setminus (\cup_1^N R_j)) + m((\cup_1^N R_j) \setminus (\cup_1^N S_j))$$

$$= \epsilon + ((\cup_1^N R_j) \setminus \cup_1^N S_j)) \leq \epsilon + \sum_1^N m(R_j \setminus S_j)) < (N+1)\epsilon.$$

Next, define

$$\mathcal{G} := \left\{ \sum_{i=1}^k a_i \chi_{R_i} \mid a_i \in \mathbb{Q}, R_i \in \mathcal{D} \right\}.$$

Define $\mathcal{G}_k$ as the subcollection of $\mathcal{G}$ consisting of finite sums where the k is fixed. Then $\mathcal{G}_k$ is countable since $\mathbb{Q}$ and $\mathcal{D}$ are countable, there are countably many pairs $a_i \chi_{R_i}$, and it is a finite sum. Then, $\mathcal{G}$ is countable as $\mathcal{G} = \bigcup_{k=1}^\infty \mathcal{G}_k$ is a countable union of countable sets. To show $\mathcal{G}$ is dense in $\mathcal{S}$, let $\phi \in \mathcal{S}$. WLOG it suffices to approximate simple functions of the form $\phi = b\chi_E$ for $m(E) < \infty$ as elements of $\mathcal{S}$ are sums of simple functions of that form. Choose finitely many disjoint $R_i \in \mathcal{D}$ s.t. $m(E\Delta \cup_{i=1}^N R_i) < \epsilon$. Choose $q \in \mathbb{Q}$ s.t. $|b - q| < \sqrt[p]{\epsilon}$. Then,

$$\left\| b\chi_E - \sum_{i=1}^N q\chi_{R_i} \right\|_p^p = \int \left| b\chi_E - \sum_{i=1}^N q\chi_{R_i} \right|^p = \int |b-q|^p \chi_{\cap i=1^N E \setminus R_i} = |b-q|^p m(\cap_{i=1}^N E \setminus R_i) < \epsilon^2$$

and taking pth roots shows that the p-norm of ϕ is arbitrarily approximated by elements of $\mathcal{G}$. At last, $\mathcal{G} \subseteq \mathcal{S} \subseteq L^p(\mathbb{R}^n, m)$, taking the closure $\overline{\mathcal{G}} = \mathcal{S}$, and since $\overline{\mathcal{G}}$ is closed, $\overline{\mathcal{G}} = \overline{\mathcal{S}} = L^p(\mathbb{R}^n, m)$ as desired.

We show that $L^\infty(\mathbb{R}^n, m)$ is not separable. Define $\mathcal{F}$ to be the collection of functions of the form $f_r(x) := \chi_{(B_r(0))^c}$ for $r > 0$. Then for $r \neq s$, we have

$$|f_r(x) - f_s(x)| = |\chi_{B_r(x)^c \setminus B_s(x)^c}| \tag{288}$$

and $\|f_r - f_s\|_\infty \geq 1$ is immediate since $m(\{x : |f_r(x) - f_s(x)| > 1\}) = 0$ while $m(\{x : |f_r(x) - f_s(x)| > 1\})$ is equal to the volume of the annular region remaining. This shows that $L^\infty(\mathbb{R}^n, m)$ is not separable since a countable dense subset of $L^\infty(\mathbb{R}^n, \infty)$ would need to approximate all of the functions in $\mathcal{F}$. Let us explain in more details.

Suppose S were a countable subset. For each $f \in \mathcal{F}$, consider a ball $B_{\frac{1}{3}}(f)$ about f w.r.t to the metric induced by the norm. Then $B_{\frac{1}{3}}(f)$ does not contain any other $g \in \mathcal{F}$ since $\|f - g\|_\infty \geq 1$. Since S is countable, the elements of S only lie in countably many balls. But $\mathcal{F}$ is uncountable so there is an $h \in \mathcal{F}$ which cannot be approximated by elements of S. Hence, S cannot be dense. □

Folland Exercise 6.14 If $g \in L^\infty$, the operator T defined by $Tf = fg$ is bounded on L^p for $1 \leq p \leq \infty$. Its operator norm is at most $\|g\|_\infty$, with equality if μ is semifinite.

Proof. The hypotheses give us

$$\int |fg|^p \leq \|g\|_\infty^p \int |f|^p \tag{289}$$

by Theorem 6.8a. Taking pth root gives $\|Tf\|_p = \|fg\|_p \leq \|g\|_\infty \|f\|_p$. This shows that $T : L^p \to L^p$ is a bounded linear operator. This also shows that the operator norm is at most $\|g\|_\infty$ since the definition of the operator norm has an infimum.

We claim equality occurs when μ is semifinite. Define $A := \{x : |g(x)| > \|g\|_\infty - \epsilon\}$. By semifiniteness, choose $B \subsetneq A$ s.t. $0 < \mu(B) < \infty$ (since A has nonzero measure by definition). Define $f := \chi_B \frac{1}{\sqrt[p]{\mu(B)}}$. Then,

$$\int |fg|^p = \int_B |g|^p \mu(B) \geq \mu(B)^{-1} \int_B \|g\|_\infty - \epsilon = (\|g\|_\infty^p - \epsilon) \tag{290}$$

and as $\epsilon \to 0$, this means $\int |fg|^p \geq \|g\|_\infty^p$. Taking pth roots, shows $\|Tf\|_p \geq \|g\|_\infty$. Therefore, $\|Tf\|_p \geq \|g\|_\infty \|f\|_p$ for all $\|f\|_p = 1$ which means $\|T\| \geq \|g\|_\infty$. Since the other direction was already proven, this completes the proof. □

Folland Exercise 6.15 (The Vitali Convergence Theorem) Suppose $1 \leq p < \infty$ and $\{f_n\}_1^\infty \subset L^p$. In order for $\{f_n\}$ to be Cauchy in the L^p norm it is necessary and sufficient for the following three conditions to hold: (i) $\{f_n\}$ is Cauchy in measure; (ii) the sequence $\{|f_n|^p\}$ is uniformly integrable (see Exercise 11 in §3.2); and (iii) for every $\epsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \infty$ and $\int_{E^c} |f_n|^p < \epsilon$ for all n. (To prove the sufficiency: Given $\epsilon > 0$, let E be as in (iii), and let $A_{mn} = \{x \in E : |f_m(x) - f_n(x)| \geq \epsilon\}$. Then the integrals of $|f_n - f_m|^p$ over $E \backslash A_{mn}, A_{mn}$, and E^c are small when m and n are large − for three different reasons.)

Folland Exercise 6.16 If $0 < p < 1$, the formula $\rho(f, g) = \int |f - g|^p$ defines a metric on L^p that makes L^p into a complete topological vector space. (The proof of Theorem 6.6 still works for $p < 1$ if $\|f\|_p$ is replaced by $\int |f|^p$, as it uses only the triangle inequality and not the homogeneity of the norm.)

Proof. Checking the axioms for a metric space is easy. Use the fact that $|f - h|^p \leq |f - g|^p + |g - h|^p$ when $0 < p < 1$. This metric then defines a topology on the vector space. We must check that it is a complete topological space. Replace the instances of $\|f\|_p$ with $\int |f|^p$ in the proof of Theorem 6.6. This alleviates the concerns that arose on p. 182. Of course, this just shows that L^p is a complete metric space since $\int |f|^p$ is not necessarily a norm (one cannot factor our scalars: $\int |\alpha f|^p \neq \alpha \int |f|^p$ in general). □

Folland Exercise 6.17 With notation as in Theorem 6.14, if μ is semifinite, $q < \infty$, and $M_q(g) < \infty$, then $\{x : |g(x)| > \epsilon\}$ has finite measure for all $\epsilon > 0$ and hence S_g is σ -finite.

Proof. Recall in Folland Exercise 1.14 we showed that if μ is semifinite and E is a set with infinite measure, then for all $C > 0$, there exists $F \subset E$ s.t. $C < \mu(F) < \infty$.

So suppose not and that $\{x : |g(x)| > \epsilon\}$ has infinite measure for some $\epsilon > 0$. Given $C > 0$, we can find a set F and $C < \mu(F) < \infty$ as before and then set $f := \chi_F$. Therefore,

$$\left|\int_X fg\right| > \left|\int_F g\right| > \mu(F)\epsilon > C\epsilon.$$

Since this holds for all $C > 0$, we deduce that $M_q(g) = \infty$ which is a contradiction. $\square$

Folland Exercise 6.18 The self-duality of L^2 follows from Hilbert space theory (Theorem 5.25) , and this fact can be used to prove the Lebesgue-Radon-Nikodym theorem by the following argument due to von Neumann. Suppose that μ, ν are positive finite measures on $(X, \mathcal{M})$ (the σ -finite case follows easily as in §3.2), and let $\lambda = \mu + \nu$

a. The map $f \mapsto \int f d\nu$ is a bounded linear functional on $L^2(\lambda)$, so $\int f d\nu = \int fg d\lambda$ for some $g \in L^2(\lambda)$. Equivalently, $\int f(1-g)d\nu = \int fg d\mu$ for $f \in L^2(\lambda)$.

b. $0 \leq g \leq 1\lambda$ -a.e., so we may assume $0 \leq g \leq 1$ everywhere.

c. Let $A = \{x : g(x) < 1\}, B = \{x : g(x) = 1\}$, and set $\nu_a(E) = \nu(A \cap E)$ $\nu_s(E) = \nu(B \cap E)$. Then $\nu_s \perp \mu$ and $\nu_a \ll \mu$; in fact, $d\nu_a = g(1-g)^{-1}\chi_A d\mu$.

Proof. If we have the result for finite positive measures then the σ-finite case is a consequence of the same argument as in Section 3.2 as stated.
a. Clearly, the map $f \mapsto \int f d\nu$ is a linear functional on $L^2(\lambda)$ so we just show that it is bounded. Suppose $\|f\|_{L^2(\lambda)} = 1$. Then,

$$\left|\int f d\nu\right| \leq \|f\|_{L^2(\nu)}\|1\|_{L^2(\nu)} = \|1\|_{L^2(\nu)}\left(\int |f|^2 d\nu\right)^{1/2} \leq \|1\|_{L^2(\nu)} \leq \|1\|_{L^2(\nu)}\left(\int |f|^2 d\lambda\right)^{1/2} = \nu(X)\|f\|_{L^2(\lambda)}$$

By the Riesz-Representation Theorem and recalling the definition of the inner product on $L^2(\lambda)$, we deduce that there exists a $g \in L^2(\lambda)$ in which $\int f d\nu = \langle f, g\rangle = \int fg d\lambda$. Equivalently, this means there exists $g \in L^2(\lambda)$ s..t $\int f(1-g)d\nu = \int fg d\mu$ for $f \in L^2(\lambda)$.
b. Let $E := \{x : g(x) < 0\}$. Our goal is to show that $\lambda(E) = 0$. Notice that setting $f = \chi_E$, we obtain

$$\nu(E) \leq \int f(1-g)d\nu = \int_E (1-g)d\nu = \int_E g d\mu \leq 0.$$

This implies $\nu(E) = 0$ by positivity of ν. Furthermore,

$$0 = \int_E (1-g)d\nu = \int_E g d\mu \qquad \Longrightarrow \qquad \mu(E) = 0.$$

Let $F := \{x : g(x) > 1\}$. Setting $f := \chi_F$,

$$\mu(E) \leq \int fg d\mu = \int_E g d\mu = \int_E (1-g)d\nu \leq 0.$$

This implies $\mu(E) = 0$. Also,

$$0 = \int_F g d\mu = \int_F (1-g)d\nu \qquad \Longrightarrow \qquad \nu(F) = 0.$$

Therefore, $\lambda(F) = 0$.

c. Assume $0 \le g \le 1$ everywhere. Clearly, $A \cap B = \emptyset$ and $A \cup B = X$.

To show $\nu_s \perp \mu$, notice that $\nu_a(B) = \nu(A \cap B) = \nu(\emptyset) = 0$. So we show $\mu(A) = 0$. Indeed,

$$\mu(A) = \int \chi_A d\mu = \int \chi_A (1-1) d\nu = 0.$$

To show $\nu_a \ll \mu$, suppose $\mu(E) = 0$. We want to show $\nu_a(E) = \nu(A \cap E) = 0$. Indeed,

$$\nu(A \cap E) = \int_{A \cap E} d\nu = \int \chi_{A\cap E}(1-0) d\nu = \int \chi_{A \cap E} d\mu = \mu(A \cap E) = 0.$$

For the last statement, set $f = \frac{1}{1-g}$ to get

$$\nu_a(E) = \nu(A \cap E) = \int_{A\cap E} \left(\frac{1}{1-g}\right)(1-g) d\nu = \int_{A \cap E} \frac{g}{1-g} d\mu \qquad \Longrightarrow \qquad d\nu_a = g(1-g)^{-1}\chi_A d\mu.$$

Because $\nu_a + \nu_s = \nu$, we have our desired decomposition into mutually singular and absolutely continuous parts. Furthermore, the last statement shows that the f in Theorem 3.8 is our $\frac{g}{1-g}\chi_A$ here.

□

Folland Exercise 6.19 Define $\phi_n \in (l^\infty)^*$ by $\phi_n(f) = n^{-1}\sum_1^n f(j)$. Then the sequence $\{\phi_n\}$ has a weak* cluster point ϕ, and ϕ is an element of $(l^\infty)^*$ that does not arise from an element of l^1

Proof. □

Folland Exercise 6.20 Suppose $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ a.e.

a. If $1 < p < \infty$, then $f_n \to f$ weakly in L^p. (Given $g \in L^q$, where q is conjugate to p, and $\epsilon > 0$, there exist (i) $\delta > 0$ such that $\int_E |g|^q < \epsilon$ whenever $\mu(E) < \delta$, (ii) $A \subset X$ such that $\mu(A) < \infty$ and $\int_{X\setminus A} |g|^q < \epsilon$, and (iii) $B \subset A$ such that $\mu(A \setminus B) < \delta$ and $f_n \to f$ uniformly on B.)

b. The result of (a) is false in general for $p = 1$. (Find counterexamples in $L^1(\mathbb{R}, m)$ and l^1.) It is, however, true for $p = \infty$ if μ is σ -finite and weak convergence is replaced by weak* convergence.

Proof. **a.** We justify the three claims (i), (ii), and (iii). Given $\epsilon > 0$.

Proving (i): Because $g \in L^q$, we know $|g|^q \in L^1$. By HW 4 Problem 7 from Math 240A, there is a $\delta > 0$ s.t. for any $\mu(E) < \delta$, we get $\int_E |g|^q < \epsilon$.

Proving (ii): Again, $|g|^q \in L^1$ and we can apply Exercise 2.16 to deduce that there exists a measurable set A s.t. $\mu(A) < \infty$ and $\int_A f > \int_X f - \epsilon$ which is equivalent to $\int_{X\setminus A} f < \epsilon$.

Proving (iii): Note that $\mu(A) < \infty$ since it is the A in the statement of (ii). Since $f_1, f_2, \ldots$ and f are measurable and $f_n \to f$ a.e., use Egoroff's Theorem (Theorem 2.33) to find $(A \setminus B) \subseteq A$ s.t. $\mu(A \setminus B) < \delta$ and $f_n \to f$ uniformly on B.

Since $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ a.e., we know $\|f_n - f\|_p \le C$ for some constant $C > 0$. By Theorem 6.15, any $T \in (L^p)^*$ is of the form $T := \phi_g$ for some $g \in L^q$. For A, B, and

$\delta > 0$ as above and an application of Hölder's inequality,

$$\left|\int gf_n - \int gf\right| \leq \int_{X\setminus A} |g||f_n - f| + \int_{A\setminus B} |g||f_n - f| + \int_B |g||f_n - f| \tag{291}$$

$$\leq \left[\|g\|_{L^q(X\setminus A)} + \|g\|_{L^q(A\setminus B)}\right] \|f_n - f\|_{L^p(X)} + \int_B |g||f_n - f| \tag{292}$$

$$\leq 2\sqrt[q]{\epsilon}\|f_n - f\|_{L^p(X)} + \int_B |g||f_n - f| \leq 2C\sqrt[q]{\epsilon} + \int_B |g||f_n - f|. \tag{293}$$

Choose N sufficiently large s.t. $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and $x \in B$. Then,

$$\int_B |g||f_n - f| \leq \|g\|_{L^q(X)} \left(\int_B |f_n - f|^p\right)^{1/p} \leq \|g\|_q \mu(B)^{1/p}\epsilon.$$

Therefore, $\left|\int gf_n - gf\right| < 2C\sqrt[q]{\epsilon} + \|g\|_q\mu(B)^{1/p}\epsilon$ so $\phi_g f_n \to \phi_g f$ which means $f_n \to f$ weakly in L^p.

b. *Counterexample in* $L^1(\mathbb{R}, m)$*:* Let $f_n := n\chi_{[0,1/n]}$. Clearly, $\|f_n\|_1 = 1$ for all n, $\sup_n \|f_n\| = 1$, and $f_n \to 0 =: f$ a.e.. Let $g := \chi_{[0,1]}$, and define $\phi : L^1(\mathbb{R}, m) \to \mathbb{C}$ by $h \mapsto \int hg$. Then

$$\int f_n g = 1,\ \forall n \in \mathbb{N} \qquad \& \qquad \int fg = 0 \qquad \Longrightarrow \qquad \int f_n g \not\to \int fg \qquad \Longrightarrow \qquad \phi f_n \not\to \phi f.$$

Counterexample in $\ell^1(\mathbb{N})$*:* Define f_n by $f_n(m) = 1$ for $m = n$ and $f_n(m) = 0$ for all $m \neq n$. Then, $\|f_n\|_{\ell^1} = 1$ for all n, $\sup_n \|f_n\|_1 = 1$, and $f_n \to 0 =: f$ a.e.. Define $g \in \ell^\infty$ by $g(m) = 1$ for all $m \in \mathbb{N}$ and so g defines $\phi_g \in (\ell^1)^*$ since the counting measure is σ-finite on $\mathbb{N}$ so Theorem 6.15 applies. However, $f_n \not\to f$ weakly in ℓ^1 because

$$\sum_{m\in\mathbb{N}} f_n(m)g(m) = 1,\ \forall n \in \mathbb{N} \qquad \& \qquad \sum_{m\in\mathbb{N}} f(m)g(m) = 0 \qquad \Longrightarrow \qquad \phi_g(f_n) \not\to \phi_g(f)$$

$f_n \to f$ weakly for $p = \infty$ and if μ is σ-finite:* In this situation, we are viewing $L^\infty = (L^1)^*$ as the dual space and elements $f \in L^\infty$ as defining bounded linear functionals $\phi_f(g) = \int fg d\mu$ for $g \in L^1$ which is justified by Theorem 6.15 with μ σ-finite. To show $f_n \to f$ weakly*, we show $\phi_{f_n}(g) \to \phi_f(g)$. For any $g \in L^1$,

$$|\phi_{f_n}(g) - \phi_f(g)| = \left|\int f_n g - \int fg d\mu\right| \leq \int |f_n - f||g| d\mu \leq \|f_n - f\|_\infty \|g\|_1 \to 0$$

where the convergence to zero is justified by Theorem 6.8c. The hypothesis of the Theorem 6.8c requires $f_n \to f$ uniformly on a set of full measure. In this case, we take the set of full measure E where $f_n \to f$ because $f_n \to f$ a.e. by hypothesis. To see that $f_n \to f$ uniformly on E, we note that $\sup_n \|f_n\|_\infty < \infty$ and therefore, $\sup_n \|f_n - f\|_\infty < \infty$ which ensures that $f_n \to f$ uniformly on E if we replace E as needed. $\square$

Folland Exercise 6.21 If $1 < p < \infty, f_n \to f$ weakly in $l^p(A)$ iff $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ pointwise.

Proof. ($\Longrightarrow$) We may view each f_n as an operator on ℓ^q defined by mapping $g \in \ell^q$ to $\int gf_n \in \mathbb{C}$ and this is an immediate consequence of Theorem 6.15 and $1 < p < \infty$. Next, $\sup_n \|\phi_{f_n}(g)\| < \infty$ for all $g \in \ell^q(A)$ since $f_n \to f$ weakly in $\ell^p(A)$. By the Uniform Boundedness Principle, as ℓ^q is Banach, $\sup_n \|\phi_{f_n}\| = \sup_n \|f_n\| < \infty$.

For convergence pointwise, we show $|f_n(a) - f(a)| \to 0$ for all $a \in A$ as $n \to \infty$. Define $g \in \ell^q(A)$ by $g(b) = 0$ for all $b \neq a$ and $g(b) = 1$ when $b = a$ and clearly $\|g\|_q < \infty$. Since $f_n \to f$ weakly, this means

$$|\phi_g(f_n) - \phi_g(f)| \to 0 \qquad \Longrightarrow \qquad |f_n(a) - f(a)| \to 0$$

which is precisely what we wanted.

($\Longleftarrow$): **First proof:** Given $T \in (\ell^p(A))^*$ and assume $T := \phi_g$ for some $g \in \ell^p(A)$ by Theorem 6.15 once more. We need to show $\phi_g(f_n) \to \phi_g(f)$ which is equivalent to showing $\sum_{a\in A} f_n(a)g(a) \to \sum_{a\in A} f(a)g(a)$. However, notice that this is then equivalent to showing $\phi_{f_n}(g) \to \phi_f(g)$. Recast in this way, $\phi_{f_n} : \ell^q(A) \to \mathbb{C}$ defines a sequence of linear operators and we are trying to show that $\phi_{f_n} \to \phi_f$ strongly. Since $f_n \to f$ pointwise, we deduce that $\phi_f : \ell^q(A) \to \mathbb{C}$ is actually a linear operator.

By Proposition 6.13, we know $\|\phi_{f_n}\|_p = \|f_n\|_p$ so the hypotheses implies $\sup_n \|\phi_{f_n}\|_p \leq \infty$.

We show that $\phi_f \in L(\ell^q(A), \mathbb{C})$ and by the above, we only need to show it is a bounded operator. It suffices to show $f \in \ell^p(A)$ because we would then get $|\phi_f(g)| \leq \|g\|_q \|f\|_p$ by Hölder's inequality. Indeed, by Fatou's Lemma, noting that summands are positive,

$$\sum_{a\in A} |f(a)|^p = \sum_{a\in A} \liminf_{n\to\infty} |f(a)|^p \leq \liminf_{n\to\infty} \sum_{a\in A} |f_n(a)|^p < \sup_n \|f_n\|_p^p < \infty.$$

At last, we have $\{\phi_{f_n}\}_{n=1}^\infty \subset L(\ell^q(A), \mathbb{C})$, $\sup_n \|\phi_{f_n}\| = \sup_n \|f\|_p < \infty$, and $\phi_f \in L(\ell^q(A), \mathbb{C})$ which fulfills the necessary hypothesis of Proposition 5.17. Let D denote the subset of $\ell^p(A)$ consisting of simple functions with finite support and D is dense in ℓ^p by Proposition 6.7. Let $h \in D$ and assume h is supported on the finite set $B \subseteq A$. Then,

$$\|\phi_{f_n}(h) - \phi_f(h)\| \leq \sum_{a\in A} |f_n(a)h(a) - f(a)h(a)| = \sum_{a\in B} |f_n(a)h(a) - f(a)h(a)|$$

and we have $\lim_{n\to\infty} \sum_{a\in B} |f_n(a)h(a) - f(a)h(a)| = \sum_{a\in B} h(a) \lim_{n\to\infty} |f_n(a) - f(a)| = 0$ because we have a finite sum so interchange of limit with the sum is permitted. Invoking Proposition 5.17, we know $\phi_{f_n} \to \phi_f$ strongly and recalling what we said above, this is equivalent to $f_n \to f$ weakly in $\ell^p(A)$.

Second proof:[16] Because $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ pointwise, which certainly implies $f_n \to f$ a.e., we use Exercise 6.20a., and deduce that $f_n \to f$ weakly in $\ell^p(A)$. $\square$

Folland Exercise 6.31 *(A Generalized Hölder Inequality)* Suppose that $1 \leq p_j \leq \infty$ and $\sum_1^n p_j^{-1} = r^{-1} \leq 1$. If $f_j \in L^{p_j}$ for $j = 1, \ldots, n$, then $\prod_1^n f_j \in L^r$ and $\|\prod_1^n f_j\|_r \leq \prod_1^n \|f_j\|_{p_j}$. (First do the case $n = 2$.)

Proof. Consider the base case of $n = 2$. Assume $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} \leq 1$ and $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$. Then, apply Hölder's inequality with $\frac{1}{\frac{p_1}{r}} + \frac{1}{\frac{p_2}{r}} = 1$ (as this equation shows they are conjugate exponents) to get

$$\|f_1 f_2\| = \left(\int |f_1|^r |f_2|^r\right)^{1/r} \leq \left(\||f_1|^r\|_{p_1/r} \||f_2|^r\|_{p_2/r}\right)^{1/r} = \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

[16] Credits go to Jeb Runnoe for pointing that this should have been much more straightforward.

We must take note that the hypotheses of Hölder's inequality are fulfilled in this situation. Because $1 \leq p_1, p_2 \leq \infty$ and $\frac{1}{r} \leq 1$, for $i \in \{1,2\}$ we obtain

$$\frac{1}{p_i} \leq \frac{1}{r} \leq 1 \qquad \Longrightarrow \qquad 1 \leq \frac{p_i}{r} \leq p_i \qquad \Longrightarrow \qquad \frac{p_i}{r} \in [1,\infty].$$

Now we induct on n. First, assume the result holds for $n \in \mathbb{N}$ and we prove that it holds for $n+1$ as well. Let $\beta := \frac{1}{\sum_{i=1}^{n} p_i^{-1}}$ and then $\sum_{i=1}^{n+1} \frac{1}{p_i} = \frac{1}{r}$ is equivalent to $\frac{1}{\beta} + \frac{1}{p_{n+1}} = \frac{1}{r}$ and clearly $\frac{1}{r} \leq 1$. By the same reasons as above, we know $\beta, p_{n+1} \in [1,\infty]$. Then applying the above result and then the inductive hypothesis,

$$\left\| \prod_1^{n+1} f_j \right\|_r \leq \|f_{n+1}\|_{p_{n+1}} \left\| \prod_1^{n} f_j \right\|_\beta \leq \|f_{n+1}\|_{p_{n+1}} \prod_1^{n} \|f_j\|_{p_j} = \prod_1^{n+1} \|f_j\|_{p_j}.$$

$\square$

Folland Exercise 6.32 Suppose that $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ are σ -finite measure spaces and $K \in L^2(\mu \times \nu)$. If $f \in L^2(\nu)$, the integral $Tf(x) = \int K(x,y)f(y)d\nu(y)$ converges absolutely for a.e. $x \in X$; moreover, $Tf \in L^2(\mu)$ and $\|Tf\|_2 \leq \|K\|_2 \|f\|_2$.

Proof. **Converges absolutely:** By an application of Hölder's inequality,

$$\int |K(x,y)f(y)|dy \leq \|K(x,\cdot)\|_2 \|f(\cdot)\|_2 < \infty$$

and this shows absolute convergence a.e. x since $\int |K(x,y)|^2 dy$ converges for a.e. x by the Fubini-Tonelli Theorem and $K \in L^2(\mu \times \nu)$ and we had assumed $f \in L^2(\nu)$.

$Tf \in L^2(\mu)$ and $\|Tf\|_2 \leq \|K\|_2\|f\|_2$: We first justify the hypotheses of Minkowski's inequality for integrals. We need to show that $y \mapsto \|K(\cdot,y)f(y)\|_2$ is in $L^1(\nu)$ and we note that it is clearly measurable since $K(\cdot,y)f(y)$ is measurable, raising to the second power doesn't affect measurability, neither does integrating. For the hypothesis,

$$\begin{aligned}
\int |\|K(\cdot,y)f(y)\|_2| dy &= \int \left| \int |K(x,y)f(y)|^2 dx \right|^{1/2} dy = \int \|K(\cdot,y)\|_2 |f(y)| dy \\
&\leq \left(\int |\|K(\cdot,y)\|_2|^2 dy \right)^{1/2} \left(\int |f(y)|^2 dy \right)^{1/2} && \text{(Hölder's Inequality)} \\
&= \left(\int \left| \int |K(x,y)|^2 dx \right| dy \right)^{1/2} \|f(\cdot)\|_{L^2(\nu)} \\
&= \left(\int \int |K(x,y)|^2 dx dy \right)^{1/2} \|f(\cdot)\|_{L^2(\nu)} \\
&= \|K(\cdot,\cdot)\|_{L^2(\mu\times\nu)} \|f(\cdot)\|_{L^2(\nu)} < \infty && \text{(Fubini-Tonelli).}
\end{aligned}$$

The finiteness follows from the hypotheses of the problem. This shows that the hypothesis of Minkowski's Inequality is fulfilled. Then, if we use the above inequality for the last inequality below,

$$\begin{aligned}
\|Tf(\cdot)\|_{L^2(\mu)} &= \left\| \int K(\cdot,y)f(y)dy \right\|_2 \leq \int \|K(\cdot,y)f(y)\|_2 dy \qquad \text{(Minkowski's Inequality)} \\
&\leq \|f(\cdot)\|_{L^2(\nu)} \|K(\cdot,\cdot)\|_{L^2(\mu\times\nu)}.
\end{aligned}$$

This establishes $\|Tf\|_2 \le \|K\|_2\|f\|_2$ and by the hypotheses, we know $\|K\|_2 < \infty$ and $\|f\|_2 < \infty$ which implies $\|Tf\|_2 < \infty$ so $Tf \in L^2(\mu)$. Furthermore, this establishes $\|Tf\|_2 \le \|K\|_2\|f\|_2$.

$\square$

Folland Exercise 6.38 $f \in L^p$ iff $\sum_{-\infty}^{\infty} 2^{kp}\lambda_f\left(2^k\right) < \infty$.

Professor Zlatoš's comments: Assume $p \in (0,\infty)$.

Proof. ($\Longrightarrow$): Assume $f \in L^p$ so that $\|f\|_p^p < \infty$. Then,

$$\begin{aligned} p\sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f(2^k) &= p\sum_{k=-\infty}^{\infty} 2^k 2^{k(p-1)}\lambda_f(2^k) = p\sum_{k=-\infty}^{\infty} (2^{k+1}-2^k)2^{k(p-1)}\lambda_f(2^k) \\ &\le p\sum_{k=-\infty}^{\infty}\int_{\alpha\in[2^k,2^{k+1}]} \alpha^{p-1}\lambda_f(\alpha)d\alpha = p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha \\ &= \|f\|_p^p < \infty \end{aligned}$$

where the last equality follows by Proposition 6.24. In case it is not obvious, the second inequality follows by $2^{k+1} = 2^k + 2^k$ which means $2^{k+1} - 2^k = 2^k$. The inequality follows by definition of $\lambda_f(2^k)$, the preceding fact, and just observing what the upper bound for the summand is.

($\Longleftarrow$): Let $F_k := \{x : 2^k < |f(x)| \le 2^{k+1}\}$. Then, we deduce $\sum_{-\infty}^{\infty} 2^{kp}\mu(F_k) < \infty$ since $\mu(F_k) \le \lambda_f(2^k)$. Therefore, noting that the $+1$ shift in index does not affect convergence of the sum,

$$\|f\|_p^p = \int_X |f|^p = \sum_{k=-\infty}^{\infty}\int_{F_n}|f|^p \le \sum_{k=-\infty}^{\infty} 2^{(k+1)p}\mu(F_n) < \infty.$$

$\square$

Folland Exercise 6.40 (a,b,c,d) If f is a measurable function on X, its decreasing rearrangement is the function $f^* : (0,\infty) \to [0,\infty]$ defined by

$$f^*(t) = \inf\{\alpha : \lambda_f(\alpha) \le t\} \qquad (\text{where } \inf\ \varnothing = \infty)$$

a. f^* is decreasing. If $f^*(t) < \infty$ then $\lambda_f\left(f^*(t)\right) \le t$, and if $\lambda_f(\alpha) < \infty$ then $f^*\left(\lambda_f(\alpha)\right) \le \alpha$

b. $\lambda_f = \lambda_{f^*}$, where λ_{f^*} is defined with respect to Lebesgue measure on $(0,\infty)$.

c. If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and $\lim_{\alpha\to\infty}\lambda_f(\alpha) = 0$ (so that $f^*(t) < \infty$ for all $t > 0$), and ϕ is a nonnegative measurable function on $(0,\infty)$, then $\int_X \phi\circ|f|d\mu = \int_0^\infty \phi\circ f^*(t)dt$. In particular, $\|f\|_p = \|f^*\|_p$ for $0 < p < \infty$

d. If $0 < p < \infty, [f]_p = \sup_{t>0} t^{1/p}f^*(t)$.

Proof. **a.** *f^* is decreasing:* Assume $s < t$. Then we have

$$f^*(s) = \inf\{\alpha : \lambda_f(\alpha) \le s\} \ge \inf\{\alpha : \lambda_f(\alpha) \le t\} = f^*(t)$$

because

$$\{\alpha : \lambda_f(\alpha) \le s\} \subseteq \{\alpha : \lambda_f(\alpha) \le t\}.$$

If $f^*(t) < \infty$ then $\lambda_f(f^*(t)) \le t$: Because $f^*(t) < \infty$ we know $\{\alpha : \lambda_f(\alpha) \le t\} \ne \emptyset$. Then,

$$\lambda_f(f^*(t)) = \mu(\{x : |f(x)| \ge f^*(t)\}) = \mu(\{x : |f(x)| \ge \inf\{\beta : \lambda_f(\beta) \le t\}) \le t$$

if we just observe the definition of $f^*(t)$ and that the infimum is at most t.

If $\lambda_f(\alpha) < \infty$, then $f^*(\lambda_f(\alpha)) \leq \alpha$: Because $\lambda_f(\alpha) < \infty$, $f^*(\lambda_f(\alpha))$ is well-defined and the set $\{\beta : \lambda_f(\beta) \leq \lambda_f(\alpha)\}$ is nonempty and the infimum is well-defined. Then,

$$f^*(\lambda_f(\alpha)) = \inf\{\beta : \lambda_f(\beta) \leq \lambda_f(\alpha)\} \leq \alpha$$

because clearly $\alpha \in \{\beta : \lambda_f(\beta) \leq \lambda_f(\alpha)\}$.

b. First, we prove the result for characteristic functions, then for simple functions, and then generalize to arbitrary measurable functions.

Case f is characteristic: Suppose $f := \chi_E$ for some measurable $E \subseteq (0, \infty)$. Then,

$$\lambda_f(\alpha) = \mu(\{x : \chi_E(x) > \alpha\}) = \begin{cases} \mu(E) & \text{if } \alpha < 1 \\ 0 & \text{if } \alpha \geq 1 \end{cases}$$

On the other hand,

$$\lambda_{f^*}(\alpha) = m(\{x : |f^*(x)| > \alpha\}) = \begin{cases} m((0, \mu(E)) & \text{if } \alpha < 1 \\ 0 & \text{if } \alpha \geq 1 \end{cases} = \begin{cases} \mu(E) & \text{if } \alpha < 1 \\ 0 & \text{if } \alpha \geq 1 \end{cases}$$

This might be tricky to see so we provide some exposition.

Suppose $\alpha < 1$. We claim that $x \in (0, \mu(E))$ satisfies $|f^*(x)| > \alpha$. By definition, $f^*(x) = \inf\{\beta : \lambda_f(\beta) \leq x\}$. The only way for $|f^*(x)| > \alpha$ when $\alpha < 1$ is for the infimum β to be bigger than α. We can use the above computation for $\lambda_f(\alpha)$. In this case, $\lambda_f(\beta) \leq x \in (0, \mu(E))$ iff $\beta \geq 1$ and so the infimum is 1. This means $|f^*(x)| = 1 > \alpha$ which is what we wanted.

Suppose $\alpha \geq 1$. We claim there is no value of x that fulfills $|f^*(x)| > \alpha$. We want the infimum β such that $\lambda_f(\beta) \leq x$ to be larger than α. If $x \in (0, 1)$, then $\lambda_f(\beta) \leq x$ is fulfilled with $\beta = 1$. But then $|f^*(x)| \not> \alpha$. Now if $x \in [1, \infty)$, then $\lambda_f(\beta) \leq x$ is fulfilled with $\beta = 1$ and possibly $\beta = 0$ if x large enough. But again, that means $|f^*(x)| \not> \alpha$.

Case f is simple: The case of f simple is complicated to write down, but it essentially follows from the analysis we did above. We demonstrate determining the values for λ_f when $f := \sum_{n=1}^{N} a_n \chi_{E_n}$ and $|a_1| \leq |a_2| \leq \cdots \leq |a_N|$. We see $\lambda_f(\alpha) = \mu(X \setminus (E_1 \cup \cdots \cup E_m))$ when $\alpha < |a_m|$. It then readily follows that for the same α, we get $\lambda_{f^*}(\alpha) = m((0, \mu(X \setminus E_1 \cup \cdots \cup E_m)) = \lambda_f(\alpha)$.

General case of f measurable: Choose a sequence f_n of simple functions such that $f_n \to f$ pointwise and $|f_n|$ increases to $|f|$. Proposition 6.22 shows that $\lambda_{f_n} \to \lambda_f$ increasingly. But then we know $\lambda_{f_n^*} \to \lambda_f$ increasingly. All that remains is to show that $\lambda_{f_n^*} \to \lambda_f$ pointwise and increasingly.

First off, if $|g| \leq |h|$, then Proposition 6.22c shows $\lambda_g \leq \lambda_h$. But that means

$$\{x : |g^*(x)| > \alpha\} \subseteq \{x : |h^*(x)| > \alpha\}$$

because we can observe the definition of h^* and g^* and compare them using λ_g and λ_h. But by the same argument as in Proposition 6.22, we deduce that $\lambda_{f_n^*} \to \lambda_{f^*}$ increasingly because the above set containment shows we just have an increasing union of sets.

c. *First statement:* Notice that the hypotheses for the first statement fulfill the hypotheses we need to apply Proposition 6.24 twice to get

$$\int_0^\infty \phi \circ f^*(t) dt = -\int_0^\infty \phi(\alpha) d\lambda_{f^*}(\alpha) = -\int_0^\infty \phi(\alpha) d\lambda_{f^*}(\alpha) = -\int_0^\infty \phi(\alpha) d\lambda_f(\alpha) = \int_X \phi \circ |f| d\mu.$$

Second statement on inequality: We first must note that the p-norm that $\|f\|_p$ refers to is with regards to μ while the p-norm $\|f^*\|_p$ refers to is with the Lebesgue measure.

Set $\phi(x) = x^p$ and applying the preceding result we just proved. We obtain

$$\int_X |f|^p d\mu = \int_0^\infty (f^*(t))^p dt = \int_0^\infty |f^*(t)|^p dt$$

because $f^*(t)$ is nonnegative. Taking the pth root, we get $\|f\|_p \le \|f^*\|_p$ as desired.

d. First, we show for all $t_0 > 0$, there exists $\alpha > 0$ s.t. $t_0^{1/p} f^*(t_0) \le \alpha \lambda_f(\alpha)^{1/p}$. Given a fixed $t_0 > 0$.

Since $f^*(t_0) = \sup\{\beta : \lambda_f(\beta) > t_0\}$, for any $\epsilon > 0$, there exists $\gamma > 0$ such that

$$f^*(t_0) - \epsilon \le \gamma \qquad \& \qquad \lambda_f(\gamma) > t_0.$$

Hence,

$$t_0^{1/p} f^*(t_0) \le \lambda_f(\gamma)^{1/p}(\gamma + \epsilon)$$

and letting $\epsilon \to 0$, we get the result. This establishes that $\sup_{t>0} t^{1/p} f^*(t) \le [f]_p$. So, we proceed to establish the reverse inequality.

For the reverse direction, we show that for every $\alpha_0 > 0$, there is an $\epsilon > 0$ sufficiently small s.t. $[f]_p \le \sup_{t>0} t^{1/p} f^*(t) + \epsilon$. First, assume $\lambda_f(\alpha_0) \ne 0$ because if it were zero, there is nothing nontrivial to show. Choose the infimum possible $\delta > 0$ in which we get

$$f^*(\lambda_f(\alpha_0) - \delta) \ge \alpha_0.$$

Then, let $\epsilon > 0$ be sufficiently small so that $\alpha_0 \lambda_f(\alpha_0)^{1/p} - \epsilon \le \alpha_0(\lambda_f(\alpha_0) - \delta)^{1/p}$. We have

$$\alpha_0 \lambda_f(\alpha_0)^{1/p} - \epsilon \le \alpha_0(\lambda_f(\alpha_0) - \delta)^{1/p} \le f^*(\lambda_f(\alpha_0) - \delta)(\lambda_f(\alpha_0) - \delta)^{1/p} \le \sup_{t>0} t^{1/p} f^*(t).$$

Taking the supremum over all α_0 and as $\epsilon \to 0$, we get $[f]_p \le \sup_{t>0} t^{1/p} f^*(t)$. □

Folland Exercise 6.41 Suppose $1 < p \le \infty$ and $p^{-1} + q^{-1} = 1$. If T is a bounded operator on L^p such that $\int (Tf)g = \int f(Tg)$ for all $f, g \in L^p \cap L^q$, then T extends uniquely to a bounded operator on L^r for all r in $[p, q]$ (if $p < q$) or $[q, p]$ (if $q < p$).

Professor Zlatoš's comments: "bounded operator on L^p" means "bounded linear operator from $L^p(\mu)$ to $L^p(\mu)$" Also assume μ is semifinite if $p = \infty$.

Proof. Assume WLOG that $p < q$ and $r \in [p, q]$. The argument would be done with p and q interchanged if $p > q$. Because the function $\frac{1}{x}$ is strictly decreasing on $(0, \infty]$, we know $\frac{1}{q} < \frac{1}{r} < \frac{1}{p}$ so there exists $t \in (0, 1)$ such that $\frac{1}{r} = \frac{1-t}{p} + \frac{t}{q}$ by convexity of $\left[\frac{1}{q}, \frac{1}{p}\right]$. We also note that $T : L^p \to L^p$ since it is stated to be a bounded operator on L^p.

Case where $p \in (1, \infty)$: First, we extend the domain of T to L^q. From the hypotheses, we have

$$\left| \int (Tf)g \right| \le \int |(Tf)||g| \le \|Tf\|_p \|g\|_q \le \|T\| \|f\|_p \|g\|_q$$

where we applied Hölder's inequality for the second inequality. What the above shows is that the linear functional $f \mapsto \int (Tf)g$ is a bounded linear functional with operator norm bounded above by $\|T\|\|g\|_q$. By Theorem 6.15, we know there is an $g^* \in L^q$ that corresponds to the bounded linear functional as ϕ_{g^*}. So, for all $f \in L^p$, we get $\int fg^* = \int (Tf)g = \int f(Tg)$ and the second equality follows by the hypothesis if we use the fact that $L^p \cap L^q$ is dense in L^q and L^p respectively so that we can pass the T through for any $f \in L^p$ and $g \in L^q$. So, we take $Tg = g^*$.

Next, we make some trivial verifications. Namely, we check that $T : L^q \to L^q$ is linear and well-defined. We check linearity because well-definedess follows by the fact that $(L^p)^* \cong L^q$ is an isometric isomorphism so the choice of g^* is uniquely determined. For linearity, let $\alpha, \beta \in \mathbb{C}$ and $g, h \in L^q$. Then linearity follows immediately from the fact that $\int f(\alpha g + \beta h) = \int \alpha \int fg + \beta \int fh$.

Then, we define $T : L^p + L^q \to L^p + L^q$ by $T(f+g) = Tf + Tg$ for $f \in L^p$ and $g \in L^q$. Notice that $L^r \subseteq L^p + L^q$ by Proposition 6.9. Next, if $f + g = f' + g'$, we know $f - f' + g - g' = 0$ and

$$0 = T(f-f'+g-g') = T(f)-T(f')+(g-g')^* = T(f)-T(f')+g^*-(g')^* \implies T(f)+g^* = T(f')+(g')^*$$

which implies $T(f+g) = T(f'+g')$. This shows $T : L^p + L^q \to L^p + L^q$ is well-defined.

Moving on, we employ Theorem 6.27 to show that $T : L^r \to L^r$ is actually bounded linear. Checking the hypotheses, we know $T : L^p + L^q \to L^p + L^q$ is a linear map satisfying $\|Tf\|_p \le M_p\|f\|_p$ and $\|Tf\|_q \le M_q\|f\|_q$ and clearly $p, q \in [1, \infty]$. For $t \in (0,1)$ and $\frac{1}{r} = \frac{1-t}{p} + \frac{t}{q}$, we have $\|Tf\|_r \le M_p^{1-t} M_q^t \|f\|_r$ which shows T is a bounded linear operator on L^r.

Now, we proceed to show that the extension T is unique. For clarity, suppose S were another extension of T. Therefore, $S : L^r \to L^r$ and $S|_{L^p} = T$. Let $h \in L^r$ be arbitrary and since $L^r \subset L^p + L^q$, write $h := f + g$ for $f \in L^p$ and $g \in L^q$ and we show $S(f+g) = T(f+g)$. By linearity, we only need to show $S(g) = T(g)$. Choose a sequence of simple functions g_n with finite support such that $g_n \to g$ w.r.t the r-norm. Because of how simple functions are defined, we also have $g_n \to g$ w.r.t the p-norm and q-norm. Because S is a continuous extension of T, we deduce that $\lim_{n\to\infty} S(g_n) = S(g)$. But for simple functions, we know $g_n \in L^q$ and therefore, $S(g_n) = T(g_n)$ and by continuity of T, we deduce $\lim_{n\to\infty} T(g_n) = T(g)$. Hence, $S(g) = T(g)$.

Case where $p = \infty$: In this case, we assume the measure is semifinite. If we scan through our proof, there are two instances that we need the hypothesis. One was to employ Theorem 6.15 as we needed the measure to be at least σ-finite because if $p = \infty$, then $q = 1$. Two, we needed semifiniteness for Proposition 6.13 to ensure $\|g^*\|_q = \|g\|_q = \|\phi_g\|$. Otherwise, we could not apply Theorem 6.27 with our extension.

A few other things to note. To apply Theorem 6.27, notice that we do not need semifiniteness unless we have *both* $p = \infty$ and $q = \infty$. So our proof is still valid in this situation. Furthermore, the remark on convergence of simple functions is still valid because our simple functions were required to have finite support. Additionally, the initial application of Hölder's inequality is valid for $p = \infty$ because of Theorem 6.8.

□

Folland Exercise 6.42 Prove the Marcinkiewicz theorem in the case $p_0 = p_1$. (Setting $p = p_0 = p_1$, we have $\lambda_{Tf}(\alpha) \le (C_0\|f\|_p/\alpha)^{q_0}$ and $\lambda_{Tf}(\alpha) \le (C_1\|f\|_p/\alpha)^{q_1}$. Use whichever estimate is better, depending on α, to majorize $q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha$.)

Proof. Assume $p_0 = p_1$. Then clearly $p = p_0 = p_1$ by the hypotheses of the Marcinkiewicz theorem. First step for us is to justify the hint and it folllows immediately from the weak type estimates since T is weak type (p, q_0) and (p, q_1). With the estimates justified, we proceed to bound $\|Tf\|_q^q$. We have

$$\|Tf\|_q^q = \int |Tf|^q = q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha \le qC_0^{q_0}\|f\|_p^{q_0}\int_0^A \alpha^{q-q_0-1}d\alpha + qC_1^{q_1}\|f\|_p^{q_1}\int_A^\infty \alpha^{q-q_1-1}d\alpha$$

for $A > 0$. The inequality follows from splitting the integral in two, and then bounding above by the estimates in the hint. WLOG assume $q_0 < q_1$. The reason we may assume WLOG is because if $q_0 > q_1$, then we would just switch our choice of estimates above. Then $q_0 \le q \le q_1$ which means $0 < q - q_0 < q_1 - q_0$ and $q_0 - q_1 < q - q_1 < 0$. Note $0 < q_1 - q$ as well. Then, we evaluate the integrals (using the inequalities just stated to ensure they are well-defined),

$$\int_0^A \alpha^{q-q_0-1} d\alpha = \left[\frac{\alpha^{q-q_0}}{q-q_0}\right]_0^A = \frac{A^{q-q_0}}{q-q_0} \qquad \& \qquad \int_A^\infty \alpha^{q-q_1-1} d\alpha = \left[\frac{\alpha^{q-q_1}}{q-q_1}\right]_A^\infty = \frac{A^{q-q_1}}{q_1-q}.$$

Then from the bound on $\|Tf\|_q^q$,

$$\|Tf\|_q^q \le qC_0^{q_0}\|f\|_p^{q_0}\frac{A^{q-q_0}}{q-q_0} + qC_1^{q_1}\|f\|_p^{q_1}\frac{A^{q-q_1}}{q_1-q}.$$

If we take $A := \|f\|_p$, then the above reduces to

$$\|Tf\|_q^q \le \left(qC_0^{q_0}\frac{1}{q-q_0} + qC_1^{q_1}\frac{1}{q_1-q}\right)\|f\|_p^q.$$

Taking the qth root, we see that the constant we wanted is $B_p := \left(qC_0^{q_0}\frac{1}{q-q_0} + qC_1^{q_1}\frac{1}{q_1-q}\right)^{1/q}$. Notice that this constant depends only on p, q_j (since q is written in terms of q_j and we allowed to depend on t), C_0, and C_1. $\square$

7. Radon Measures

Folland Exercise 7.1 Let X be an LCH space, Y a closed subset of X (which is an LCH space in the relative topology), and μ a Radon measure on Y. Then $I(f) = \int (f \mid Y) d\mu$ is a positive linear functional on $C_c(X)$, and the induced Radon measure ν on X is given by $\nu(E) = \mu(E \cap Y)$.

PROOF. [17] First, we check linearity. Let $f, g \in C_c(X)$ and $\alpha, \beta \in \mathbb{C}$ and so by linearity of the integral,

$$I(\alpha f + \beta g) = \int (\alpha f + \beta g)|_Y d\mu = \int \alpha f|_Y + \beta g|_Y d\mu = \alpha \int f|_Y d\mu + \beta \int g|_Y d\mu = \alpha I(f) + \beta I(g).$$

Next, if $f \in C_c(X)$, then $f|_Y \in C_c(Y)$ so that $I(f) = \int f|_Y d\mu$ is well-defined. This is because $f|_Y$ is definitely continuous and if $\operatorname{supp}(f)$ is compact, then $\operatorname{supp}(f|_Y) = \operatorname{supp}(f) \cap Y$ is compact since Y is closed. Then, the comment proceeding Proposition 7.1 imply that I mapping $f' \in C_c(Y)$ to $\int f' d\mu$ is a positive linear functional. In particular, if $f \ge 0$ and $f \in C_c(X)$, then $f|_Y \in C_c(Y)$ and

$$I(f) = \int (f|Y) d\mu = \int_{\operatorname{supp}(f)} (f|Y) d\mu = \int_{\operatorname{supp}(f)\cap Y} f d\mu \ge 0.$$

[17]NTS: Think about this some more. If we show $I(f) = \int f d\mu_Y = \int f d\nu$, then uniqueness of the Riesz Representation Theorem implies $\mu_Y = \nu$. What goes wrong is that μ_Y may not be a Radon measure on X itself even though it is a Radon measure on Y (inner regularity may fail).

Define $\mu_Y(E) := \mu(E \cap Y)$ and it is a Borel measure by Exercise 1.10 as it is a restriction of a Borel measure and Y is closed. To show $\nu = \mu_Y$, we need to show $\nu(E) = \mu_Y(E)$ for all $E \in \mathcal{B}_X$. Let us show μ_Y is outer regular on $\mathcal{B}_X$. Let $E \in \mathcal{B}_X$. The inequality

$$\mu_Y(E) = \mu(E \cap Y) \leq \inf\{\mu(U \cap Y) : E \cap Y \subseteq U \text{ open in } X\} =: \eta$$

is immediate from monotonicity of measures. Now we show the other direction. By μ being Radon on Y and definition of the subspace topology,

$$\mu(E \cap Y) = \inf\{\mu(W) : E \cap Y \subseteq W,\ W \text{ open in } Y\}$$

If W is open in Y, there is an open subset $V \subseteq X$ s.t. $W = V \cap Y$. In particular, we can choose $V = W \cup Y^c$ which is an open subset of X because

$$W \cup Y^c = (V \cap Y) \cup Y^c = V \cap Y^c = (Y \cap V^c)^c$$

is the complement of $Y \cap V^c$ which is open. Therefore,

$$\mu(E \cap Y) = \inf\{\mu((W \cup Y^c) \cap Y) : E \cap Y \subseteq (W \cup Y^c) \cap Y,\ W \text{ open in } Y\}.$$

But since $W \cup Y^c$ is open in X, we deduce from bounding the RHS above from below,

$$\mu(E \cap Y) \geq \inf\{\mu(U \cap Y) : E \cap Y \subseteq U,\ U \text{ open in } X\}$$

Since μ_Y and ν are outer regular $\mathcal{B}_X$, to show equality on $\mathcal{B}_X$, it suffices to show equality on open sets.

Let $U \subseteq X$ be open. Let $\epsilon > 0$ and by Theorem 7.2, there exists an $f \in C_c(X)$ s.t. $f \prec U$ and $\nu(U) - \epsilon \leq I(f)$ and therefore,

$$\nu(U) - \epsilon \leq I(f) = \int f|_Y d\mu \leq \int_U 1|_Y d\mu = \mu(U \cap Y).$$

Since $\epsilon > 0$ was arbitrary, $\nu(U) \leq \mu(U \cap Y)$. For the other direction, suppose $K \subseteq U \cap Y$ were a compact set and by Lemma 4.32, there exists a $g \in C_c(X)$ s.t. $g|_K = 1$ and $g \prec U$. Then by Theorem 7.2, we have

$$\begin{aligned}
\nu(U) &= \sup\{I(f) : f \in C_c(X), f \prec U\} \geq \sup\left\{\int \chi_K|_Y d\mu : K \subseteq U \cap Y,\ K \text{ compact}\right\} \\
&= \sup\{\mu(K \cap Y) : K \subseteq U \cap Y,\ K \text{ compact}\} = \sup\{\mu_Y(K) : K \subseteq U \cap Y,\ K \text{ compact}\} \\
&= \mu(U \cap Y) = \mu_Y(U).
\end{aligned}$$

where the existence of $g \in C_c(X)$ satisfying the stated conditions gives the inequality, the equality in the second line follows from $K \cap Y$ being compact as Y is closed[18] and the equality from the second to third line follows from μ being Radon on Y and $U \cap Y \subseteq Y$ being an open subset of Y. $\square$

Folland Exercise 7.2 Let μ be a Radon measure on X.

a. Let N be the union of all open $U \subset X$ such that $\mu(U) = 0$. Then N is open and $\mu(N) = 0$. The complement of N is called the support of μ.

b. $x \in \text{supp}(\mu)$ iff $\int f d\mu > 0$ for every $f \in C_c(X, [0,1])$ such that $f(x) > 0$.

[18] While discussing this with James Wong, Y being closed is not necessarily needed here.

Proof. **a.** Because N is a union of open sets, N is necessarily open as well.

If $K \subseteq N$ is a compact subset of N, there is a finite subcover $K \subseteq \bigcup_{k=1}^n U_k$ where $\mu(U_k) = 0$. Then,

$$\mu(K) \leq \mu\left(\bigcup_{k=1}^n U_k\right) \leq \sum_{k=1}^n \mu(U_k) = \sum_{k=1}^n 0 = 0.$$

That is, any compact subset $K \subseteq N$ has measure zero. Because N is open, by definition, μ is inner regular on N. Therefore,

$$\mu(N) = \sup\{\mu(K) : K \subseteq N,\ K \text{ compact}\} = 0.$$

We note that $\operatorname{supp}(\mu) = N^c$.

b. ($\Longrightarrow$): Assume $x \in \operatorname{supp}(\mu)$. Let $f \in C_c(X, [0,1])$ and assume $f(x) > 0$. Because $x \in \operatorname{supp}(\mu)$, there is an open $V \subseteq X$ s.t. $x \in V$ and $\mu(V) > 0$. Because f is continuous, there is a $W \subseteq V$ s.t. $\mu(W) > 0$ and $f|_W > 0$. Then,

$$\int f d\mu > \int_W f d\mu > \mu(W)\delta > 0$$

where $\delta > 0$ is sufficiently small so that $f|_W > \delta > 0$.

($\Longleftarrow$)[19]: Suppose not and $x \notin \operatorname{supp}(\mu)$. Then there exists an open set $U \ni x$ s.t. $\mu(U) = 0$. Because $\{x\}$ is a compact subset and X Hausdorff, we know $\{x\}$ is in particular, closed. Let U be some open set containing x and by Urysohn's Lemma 4.32, there exists a continuous f s.t. $f|_{\{x\}} = 1$, $0 \leq f \leq 1$, and $\operatorname{supp}(f) \subseteq X$. Then $f(x) = 1 > 0$ and since $x \notin \operatorname{supp}(\mu)$, we know $\mu(\{x\}) = 0$. By the Riesz-Representation Theorem (7.3), if I is given by $g \mapsto \int g d\mu$ for $g \in C_c(X)$, we know

$$0 = \mu(U) = \sup\{I(f) : f \in C_c(X), f \prec U\} \geq \int f d\mu > 0$$

where the last inequality follows by hypothesis. So $0 > 0$ which is a contradiction.

□

Folland Exercise 7.3 Let X be the one-point compactification of a set with the discrete topology. If μ is a Radon measure on X, then $\operatorname{supp}(\mu)$ (see Exercise 2) is countable.

Proof. [20] Use Exercise 7.2b as needed.

For all $x \in X \setminus \{\infty\}$, we have $x \in \operatorname{supp}(\mu)$ iff $\mu(\{x\}) > 0$. If $\mu(\{x\}) > 0$, then for all $f \in C_c(X, [0,1])$, we know $\int f \geq f(x)\mu(x) > 0$ which implies $x \in \operatorname{supp}(\mu)$. On the other hand, if $x \in \operatorname{supp}(\mu)$, consider $h(y)$ defined by $h(x) = 1$ and $h = 0$ elsewhere. For any

[19]NTS: Another possible proof, but there might be an error somewhere.
($\Longleftarrow$): Suppose not and $x \notin \operatorname{supp}(\mu)$. So there is an open U s.t. $x \in U$ where $\mu(U) = 0$. By Proposition 4.30, there is a compact set $x \in M \subseteq U$. By Urysohn's Lemma 4.32, there is an $f \in C(X, [0,1])$ s.t. $f = 1$ on M and $\operatorname{supp}(f) \subseteq U$. Then,

$$\int f d\mu = \int_{\operatorname{supp}(f)} f d\mu \leq (\sup_{y \in U} f(u))\mu(\operatorname{supp}(f)) = 0$$

since $\mu(\operatorname{supp}(f)) \leq \mu(U) = 0$ (note that $\operatorname{supp}(f)$ is compact and since X is Hausdorff, $\operatorname{supp}(f)$ is closed). Therefore, from the hypothesis and $f(x) = 1 > 0$, we obtain

$$0 < \int f d\mu = 0$$

which means $0 < 0$. Contradiction.

[20]This is the correct proof provided by Yunyi Zhang for Math 200C.

open set $U \subseteq \mathbb{C}$, if $0 \in U$ and $1 \notin U$, $h^{-1}(U) = \{x\}^c$ is open. If $1 \in U$ and $0 \notin U$, then $h^{-1}(U) = \{x\}$ is open. If $0, 1 \in U$, then $h^{-1}(U) = X$ is also open. Finally, if $0, 1 \notin U$, then $h^{-1}(U) = \emptyset$ is open. This means h is continuous. Also, it is clearly compactly supported. But then $0 < \int h d\mu = h(x)\mu(\{x\}) = \mu(\{x\})$.

For all $m \in \mathbb{N}$, $\infty > \mu(X) \geq \sum_{\mu(\{x\}) \geq \frac{1}{m}} \mu(\{x\})$ implies that only finitely many x have $\mu(\{x\}) \geq \frac{1}{m}$. Also, $\operatorname{supp}(\mu) \subseteq \{\infty\} \cup \bigcup_{\mu(\{x\}) > \frac{1}{m}} \{x\}$ is countable. □

Folland Exercise 7.4 Let X be an LCH space.

a. If $f \in C_c(X, [0, \infty))$, then $f^{-1}([a, \infty))$ is a compact G_δ set for all $a > 0$.

b. If $K \subset X$ is a compact G_δ set, there exists $f \in C_c(X, [0, 1])$ such that $K = f^{-1}(\{1\})$.

c. The σ -algebra $\mathcal{B}_X^0$ of Baire sets is the σ -algebra generated by the compact G_δ sets.

Proof. **a.** Set $K := f^{-1}([a, \infty)$ which is a closed set because it is the preimage of a closed set under a continuous function. First off, $f^{-1}([a, \infty)) = \bigcap_{n=1}^{\infty} f^{-1}\left(\left(a - \frac{1}{n}, \infty\right)\right) \in G_\delta$ because $f^{-1}\left(\left(a - \frac{1}{n}, \infty\right)\right)$ is the preimage of an open set and f is continuous. Also, K is compact since $a > 0$ implies $K \subseteq \operatorname{supp}(f)$, and because K is closed and $\operatorname{supp}(f)$ is compact, Proposition 4.22 shows K is compact..

b. Let $K := \bigcap_{n=1}^{\infty} U_n$ be a countable intersection of open subsets $U_n \subseteq X$. By replacing U_k with $\bigcap_{i=1}^{k} U_i$, we may assume $U_i \supseteq U_{i+1}$ for all i. By Proposition 4.31, there is a precompact $V \subseteq X$ s.t. $K \subseteq V \subseteq \overline{V} \subseteq U_1$. Set $V_i := U_i \cap V$ and then, $\bigcap_{i=1}^{\infty} V_i = \bigcap_{i=1}^{\infty} U_i = K$. By Urysohn's Lemma 4.32, we can find continuous f_i s.t. $f_i|_K = 1$, $0 \leq f_i \leq 1$, and $\operatorname{supp}(f_i) \subseteq V_i$. Define $f(x) := \sum_{i=1}^{\infty} \frac{f_i(x)}{2^i}$ and this is is a well-defined sum since

$$|f(x)| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

for all $x \in X$. Next, we claim $f(x)$ is continuous. Indeed, the summands $\frac{f_i}{2^i}$ are bounded and continuous and because $BC(X, [0, 1])$ is a Banach space w.r.t to the uniform norm $\|f\|_u \leq \sum_{i=1}^{\infty} \frac{\|f_i\|_u}{2^i} \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$ shows $\sum_{i=1}^{\infty} \frac{\|f_i\|_u}{2^i}$ is an absolutely convergent series and therefore, the series that defines f converges in $BC(X, [0, 1])$. Next, $\operatorname{supp}(f) \subseteq \overline{\bigcup_{i=1}^{\infty} \operatorname{supp}(f_i)} \subseteq \overline{\bigcup_{i=1}^{\infty} V_i} \subseteq \overline{V}$ which means $\operatorname{supp}(f)$ is a closed subset of a compact set and therefore, it is compact.[21] Therefore, $f \in C_c(X)$.

Finally, we show $K = f^{-1}(\{1\})$. If $x \in K$, then $f(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$ by construction of the f_i. Conversely, if $x \in f^{-1}(\{1\})$, then we must have $f_i(x) = 1$ for all i because the summands $\frac{f_i(x)}{2^i}$ are nonnegative. But that means $x \in \operatorname{supp}(f_i) \subseteq V_i$ for all i and hence, $x \in \bigcap_{i=1}^{\infty} V_i = K$. We conclude that $K = f^{-1}(\{1\})$ and that f is the desired function.

c. Note $\mathcal{B}_X^0 := \mathcal{M}(\mathcal{E})$ is the σ-algebra generated by $\mathcal{E} := \{f^{-1}(E) : f \in C_c(X),\ E \subseteq \mathbb{C} \text{ measurable}\}$. The goal is to show $\mathcal{B}_X^0 = \mathcal{M}(\mathcal{F})$ where $\mathcal{F}$ is the collection of compact G_δ sets. We note that is suffices to restrict attention to real-valued functions $f \in C_c(X, \mathbb{R})$ by taking real and imaginary parts. Furthermore, we can restrict attention to the nonnegative part by decomposing $f = f^+ - f^-$ and noticing that if $f \in C_c(X)$, then $f^+, f^- \in C_c(X)$ as well. So we reduce to assuming

$$\mathcal{E} := \{f^{-1}(E) : f \in C_c(X, [0, \infty),\ E \subseteq [0, \infty) \text{ measurable}\}.$$

[21] This is the where the choice of V is important. Had we chosen f_i s.t. $f_i \subseteq U_i$ instead, we would not be able to deduce that $\operatorname{supp}(f)$ was indeed compact.

Showing $\mathcal{B}_X^0 \subseteq \mathcal{M}(\mathcal{F})$: Since we want to show $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$, we can use Lemma 1.1 to reduce to showing $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$.

Let $f^{-1}(E) \in \mathcal{E}$. Because $\mathcal{B}_{[0,\infty)}$ is generated[22] by the half-open intervals of form $[a, \infty)$ for $a > 0$, we reduce to showing $f^{-1}([a,\infty)) \in \mathcal{M}(\mathcal{F})$. By part a., shows that $f^{-1}([a,\infty) \in \mathcal{F}$ so $f^{-1}([a,\infty)) \in \mathcal{M}(\mathcal{F})$.

Showing $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{B}_X^0$: By Lemma 1.1, we show $\mathcal{F} \subseteq \mathcal{M}(\mathcal{E})$. Let $K \in \mathcal{F}$ and by part b., there exists $f \in C_c(X, [0,1])$ s.t. $K = f^{-1}(\{1\})$. But $\{1\} \subseteq [0,\infty]$ is measurable so this means $K \in \mathcal{E}$. Hence, $K \in \mathcal{M}(\mathcal{E})$. □

Folland Exercise 7.5 Let X be a second countable LCH space.

a. Every compact subset of X is a G_δ set.

b. $\mathcal{B}_X = \mathcal{B}_X^0$.

Proof. **a.** Let K be compact and $\{B_i\}_{i=0}^\infty$ be a countable basis of the topology. For every $\overline{B_i} \cap K = \emptyset$, there exist disjoint open sets $U_i \supseteq \overline{B_i}$ and $V_i \supseteq K$.

We claim $K = \bigcap_{i=0}^\infty V_i$. Clearly, $K \subseteq \bigcap_{i=0}^\infty V_i$. For the other direction, suppose not and there was a $v \in \bigcap_{i=0}^\infty V_i$ s.t. $v \notin K$. Since $\{v\}$ is a closed set, there are disjoint open sets $U \supseteq \{v\}$ and $V \supseteq K$. Then, there is a basis open set B_i s.t. $\{v\} \subseteq B_i \subseteq V$ and $\overline{B_i} \cap K = \emptyset$. This means, there is an i s.t. we have disjoint open sets $U_i \supseteq \overline{B_i}$ and $V_i \supseteq K$. However, we would have $v \in V_i$ and $v \in U_i$ which means they cannot be disjoint. Contradiction.

b. Clearly, $\mathcal{B}_X \supseteq \mathcal{B}_X^0$. For the other direction, we recall Folland Exercise 4.55 which says every open set in a second countable LCH space is σ-compact. Therefore, if U is open in X, we can write $U = \bigcup_{n=1}^\infty K_n$ as a union of compact sets. By a., we know the K_n are all G_δ sets and by Folland Exercise 7.4c. we know the K_n are in the generating set of $\mathcal{B}_X^0$. Therefore, $U \in \mathcal{B}_X^0$. Since the open sets generate $\mathcal{B}_X$, we deduce that $\mathcal{B}_X \subseteq \mathcal{B}_X^0$. □

Folland Exercise 7.6 Let X be an uncountable set with the discrete topology, or the one-point compactification of such a set. Then $\mathcal{B}_X \neq \mathcal{B}_X^0$

Proof. **[Incomplete]**

Suppose X is an uncountable set with the discrete topology. We must show $\mathcal{B}_X \not\subseteq \mathcal{B}_X^0$. A $C_c(X)$ function $f(x)$ must be nonzero on a closed finite set. However, the set where $f(x)$ is zero is not a compact G_δ set and therefore, $f(x)$ is not measurable. Take for example, $f := \chi_{X \setminus \{x\}}$ which is $(\mathcal{B}_X, [0,1])$ measurable, but it is not $(\mathcal{B}_X^0, [0,1])$ measurable since its support is not compact.

The one-point compactification of X does not affect this fact. □

Folland Exercise 7.7 If μ is a σ -finite Radon measure on X and $A \in \mathcal{B}_X$, the Borel measure μ_A defined by $\mu_A(E) = \mu(E \cap A)$ is a Radon measure. (See also Exercise 13.)

Proof.
(1) First off, μ_A is a Borel measure on $\mathcal{B}_X$ because $A \in \mathcal{B}_X$ and by Exercise 1.10.
(2) Let $K \subseteq A$ be a compact set. Since μ is a Radon measure, $\mu_A(K) = \mu(K \cap A) \leq \mu(K) < \infty$.

[22]First off, $\mathcal{B}_\mathbb{R}$ is generated by half-open intervals of form $[a,\infty)$ for $a \in \mathbb{R}$. Restricting to $\mathcal{B}_{[0,\infty)} := \{E \cap [0,\infty) : E \in \mathcal{B}_\mathbb{R}\}$ means that $\mathcal{B}_{[0,\infty)}$ is now generated by the half-open intervals $[a,\infty) \cap [0,\infty)$. So, $\mathcal{B}_{[0,\infty)}$ is generated by the half-open intervals $[a,\infty)$ for $a \geq 0$. We can require $a > 0$ because if $a = 0$, then we just write $[0,\infty) = [a,\infty)^c \cup [a,\infty)$ so we do not need it in the generating set.

(3) [23] [24]

Let $E \in \mathcal{B}_X$ and then $A \cap E$ is Borel. Let $\epsilon > 0$. By outer regularity of μ, there exists an open $U \supseteq A \cap E$ s.t. $\mu(U) < \mu(A \cap E) + \epsilon$. By Proposition 7.7, since μ is σ-finite and A^c is Borel, there exists a closed F and open V s.t. $F \subseteq A^c \subseteq V$ and $\mu(V \setminus F) < \epsilon$. Note that $F \subseteq A^c$ implies $A \subseteq F^c$. Then,

$$\begin{aligned}\mu_A(U \cup V) &= \mu((A \cap U) \cup (A \cap V)) \leq \mu(A \cap U) + \mu(A \cap V) \leq \mu(U) + \mu(A \cap V) \\ &< \mu(A \cap E) + \epsilon + \mu(V \cap F^c) = \mu(A \cap E) + \epsilon + \mu(V \setminus F) < \mu(A \cap E) + 2\epsilon = \mu_A(E) + 2\epsilon.\end{aligned}$$

Since $U \cup V$ is open, $E \subseteq (E \cap A) \cup A^c \subseteq U \cup V$ and $\epsilon > 0$ was arbitrary, we are done by definition of outer regularity.

(4) Let $U \subseteq X$ be an open set. To prove inner regularity on U, we show two inequalities. First,

$$\mu_A(U) = \mu(U \cap A) = \sup\{\mu(K) : K \subseteq U \cap A,\ K \text{ compact}\} \geq \sup\{\mu_A(K) : K \subseteq U,\ K \text{ compact}\}$$

and the second equality is by Proposition 7.5 and the fact that μ is σ-finite while the inequality is due to the fact that if $K \subseteq U \cap A$, then $K \subseteq U$ and $\mu(K) \geq \mu(K \cap A)$ by monotonicity.

Now, we show the other inequality. We split into two cases.

Case $\mu_A(U) < \infty$: In this case, $\mu(U \cap A) < \infty$ by Proposition 7.5, there is a compact set $K \subseteq U \cap A$ s.t. $\mu(U \cap A) < \mu(K) + \epsilon$. But then $K \subseteq U \cap A \subseteq U$ so we have a compact set $K \subseteq U$ which has $\mu_A(U) < \mu_A(K) + \epsilon$. So this establishes the other inequality[25].

Case $\mu_A(U) = \infty$: We show that there are compact subsets of U with arbitrarily large μ_A measure.

Let $U := \bigcup_{n=1}^\infty F_n$ be a countable union of finite measure sets and assume the F_n are disjoint. Let $M \in \mathbb{N}$. We have $\infty = \mu(U \cap A) = \sum_{n=1}^\infty \mu(A \cap F_n)$. So there is an $N \in \mathbb{N}$ s.t. $\sum_{n=1}^N \mu(A \cap F_n) > M + 1$. Since $\mu(A \cap F_n)$ is of finite measure, from Proposition 7.5., there is a compact set $K_n \subseteq A \cap F_n$ s.t. $\mu(A \cap F_n) < \mu(K_n) + \frac{1}{N} = \mu_A(K_n) + \frac{1}{N}$. Then $\bigcup_{n=1}^N K_n \subseteq \bigcup_{n=1}^N A \cap F_n \subseteq U \cap A \subseteq U$ and the finite union of compact sets is still compact. But we find that

$$\mu_A\left(\bigcup_{n=1}^N K_n\right) = \sum_{n=1}^N \mu_A(K_n) > \sum_{n=1}^N \left(\mu(A \cap F_n) - \frac{1}{N}\right) > M + 1 - 1 = M.$$

This shows that there are compact subsets of U of arbitrarily large measure and therefore, $\sup\{\mu_A(K) :\ U \supseteq K \text{ compact}\} = \infty = \mu_A(U)$ as desired.

□

Folland Exercise 7.8 Suppose that μ is a Radon measure on X. If $\phi \in L^1(\mu)$ and $\phi \geq 0$, then $\nu(E) = \int_E \phi d\mu$ is a Radon measure. (Use Corollary 3.6.)

[23] NTS: I initially gave an incorrect argument using inequalities of infimums of sets. Try to be careful since inequalities in those situations can easily go awry.

[24] James Wong pointed out a different approach using a $\frac{\epsilon}{2^n}$ argument. It is essentially just a repeat of Proposition 7.7: If $E \in \mathcal{B}_X$, let $E := \bigcup_{k=1}^\infty E_k$ be a countable union of disjoint finite measure sets. For each E_k, choose open $U_k \supseteq E_k$ s.t. $\mu(U_k \setminus E_k) < \frac{\epsilon}{2^k}$ by Proposition 7.5. Set $U := \bigcup_{k=1}^\infty U_k$. Then, $\mu(U) = \sum_{k=1}^\infty \mu(U_k) < \sum_{k=1}^\infty \mu(E_k) + \frac{\epsilon}{2^k} = \mu(E) + \epsilon$.

[25] This actually establishes equality as well. We could have gone without showing two inequalities.

PROOF. (1) First off, $\nu : \mathcal{B}_X \to [0,\infty]$ defines a Borel measure by Exercise 2.13.

(2) It suffices to show that ν is a finite measure since it implies finiteness on all compact sets and indeed,
$$\nu(X) = \int_X \phi d\mu = \int_X |\phi| d\mu = \|\phi\|_{L^1(\mu)} < \infty.$$

(3) [26] To show inner regularity on open sets, it suffices to show inner regularity on all Borel sets. Let $E \in \mathcal{B}_X$ and $\epsilon > 0$. Let $\delta > 0$ be as in Corollary 3.6 with $\epsilon > 0$. The set $\phi^{-1}((0,\infty))$ is σ-finite by Proposition 2.23 and nonnegativity. So, let $\phi^{-1}((0,\infty)) = \bigcup_{k=1}^\infty E_k$ a countable union of finite measure sets and WLOG[27], assume that $E_k \subseteq E_{k+1}$ for all k. Because $\mu(E_k) < \infty$, Proposition 7.5 tells us that μ is inner regular on E_k. So, there exists a compact set K_k s.t. $K_k \subseteq E_k$ and $\mu(E_k \setminus K_k) < \delta$. Therefore, $\nu(E_k \setminus K_k) < \epsilon$. Since ν is a finite measure by part 1., we know $\nu(\bigcup_{k=1}^\infty K_k) < \infty$. So there is an $N \in \mathbb{N}$ sufficiently large s.t. $\nu(\bigcup_{k=N+1}^\infty K_k) < \epsilon$. Now,
$$\nu\left(E_n \setminus \bigcup_{k=1}^N K_k\right) \le \nu\left(E_n \setminus \left(\bigcup_{k=1}^\infty K_k\right)\right) + \nu\left(\bigcup_{k=N+1}^\infty K_k\right) < \nu(E_n \setminus K_n) + \epsilon < 2\epsilon.$$

We want to compute that measure of E. We have
$$\nu(E) = \int_E \phi d\mu = \int_{E\cap\{\phi>0\}} \phi d\mu = \int_{\bigcup_{k=1}^\infty E_k} \phi d\mu = \nu\left(\bigcup_{k=1}^\infty E_k\right).$$

Since $\{E_k\}_{k=1}^\infty$ is an ascending sequence of measurable sets, we know that $\nu(E) = \lim_{k\to\infty} \nu(E_k)$. From this fact, $\nu\left(E \setminus \bigcup_{k=1}^N K_k\right) = \lim_{n\to\infty} \nu\left(E_n \setminus \bigcup_{k=1}^N K_k\right) \le 2\epsilon$. But $\bigcup_{k=1}^N K_k$ is a finite union of compact sets and is therefore compact. This proves that ν is inner regular on *all* Borel sets.

(4) Now for outer regularity on all Borel sets. Let $E \in \mathcal{B}_X$ and let $\epsilon > 0$. Since E Borel, E^c is Borel and then there exists a compact set K s.t. $\nu(E^c \setminus K) < \epsilon$ or equivalently, $\nu(E^c) - \epsilon < \nu(K)$ since ν finite. Since K is compact, it is closed and K^c is open. Since $K \subseteq E^c$, we get $E \subseteq K^c$. All that is left is the inequality, and we use finiteness of ν,
$$\nu(K^c) = \nu(X) - \nu(K) \le \nu(X) - (\nu(E^c) - \epsilon) = \nu(X) - \nu(E^c) + \epsilon = \nu(X \setminus E^c) + \epsilon = \nu(E) + \epsilon.$$

This last inequality implies that $\nu(E) := \inf\{\nu(U) : E \subseteq U, \text{ open}\}$.

□

Folland Exercise 7.9 Suppose that μ is a Radon measure on X and $\phi \in C(X,(0,\infty))$. Let $\nu(E) = \int_E \phi d\mu$, and let ν' be the Radon measure associated to the functional $f \mapsto \int f\phi d\mu$ on $C_c(X)$.

a. If U is open, $\nu(U) = \nu'(U)$. (Apply Corollary 7.13 to $\phi\chi_U$).

b. ν is outer regular on all Borel sets. (Hint: The open sets $V_k = \left\{x : 2^k < \phi(x) < 2^{k+2}\right\}, k \in \mathbb{Z}$, cover X.).

c. $\nu = \nu'$, and hence ν is a Radon measure. (See also Exercise 13.)

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[26] A hint was provided by James Wong. In particular, find a way to exploit σ-finiteness since $\phi \in L^1(\mu)$.

[27] In particular, one can take $E_k := \phi^{-1}(\frac{1}{k},\infty))$.

Proof. **a.**[28] Since ϕ is continuous and χ_U is LSC by Proposition 7.11, $\phi\chi_U$ is LSC. So by Corollary 7.13, since $\phi\chi_U$ is nonnegative and LSC and μ is Radon,

$$\nu(U) = \int \phi\chi_U = \sup\left\{\int g d\mu : g \in C_c(X), 0 \le g \le \phi\chi_U\right\}.$$

We also have $\sup\left\{\int g d\mu : g \in C_c(X), 0 \le g \le \phi\chi_U\right\} = \sup\left\{\int h\phi d\mu : h \in C_c(X), 0 \le h \le \chi_U\right\}$: $h \in C_c(X)$ iff $h\phi \in C_c(X)$ and this fact is just a consequence of the fact that $\operatorname{supp}(h\phi) = \operatorname{supp}(h)$ by ϕ nonzero and $0 \le h \le \chi_U$ iff $0 \le h\phi \le \phi\chi_U$. Essentially, if g is in the LHS set, then $g\phi$ is in the RHS set and conversely, if h is in the RHS set, then $h\phi$ is in the LHS set.

Since $h\phi \in C_c(X)$, we can use the definition ν' to deduce that $\int h\phi d\mu = \int h d\nu'$. So we get

$$\nu(U) = \sup\left\{\int h\phi d\mu : h \in C_c(X), 0 \le h \le \chi_U\right\} = \sup\left\{\int h d\nu' : h \in C_c(X), 0 \le h \le \chi_U\right\} = \int \chi_U d\nu' =$$

and the second to last equality follows from Corollary 7.13 (ν' is Radon by hypothesis) and the last equality is from definition.

b. Let E be a Borel set. First off, $\nu(E) \le \inf\{\nu(U) : E \subseteq U,\ U \subseteq X \text{ open}\}$ is clear by monotonicity of measures. We need to show the converse. Let $V_k := \phi^{-1}((2^k, 2^{k+2}))$ and these sets cover X because $\bigcup_{k\in\mathbb{Z}} V_k = \phi^{-1}((0,\infty))$ (note that we have $k+2$ which is crucial).

Suppose we showed that for all $k \in \mathbb{Z}$, we could find a $U_k \supseteq V_k \cap E$ s.t. $\nu(U_k) < \nu(V_k \cap E) + \frac{\epsilon}{2^{|k|}}$. Then we can take $U := \bigcup_{k\in\mathbb{Z}} U_k$ and then, $U \supseteq \bigcup_{z\in\mathbb{Z}} V_k \cap E = E$ while $\nu(U \setminus E) < 3\epsilon$ because $\nu(\bigcup_{k\in\mathbb{Z}} U_k \setminus V_k \cap E) \le \sum_{k\in\mathbb{Z}} \frac{\epsilon}{2^{|k|}} = 3\epsilon$.

By replacing U_k with the open set $U_k \cap V_k$, we may assume $U_k \subseteq V_k$. Then that means we want to find a $U_k \subseteq V_k$ s.t. $U_k \supseteq V_k \cap E$ and U_k satisfies the condition above.

From the hint,

$$2^k \mu(U_k \setminus E \cap V_k) \le \int \phi\chi_{U_k\setminus E\cap V_k} d\mu \le 2^{k+2}\mu(U_k \setminus E \cap V_k)$$

and since $\nu(U_k \setminus E \cap V_k) = \int \phi\chi_{U_k\setminus E\cap V_k} d\mu$, we need to choose $\mu(U_k \setminus E \cap V_k)$ sufficiently small. From μ Radon, we can find $U_k \supseteq E \cap V_k$ s.t. $\mu(U_k) < \mu(E \cap V_k) + \frac{\epsilon}{2^{k+2}2^{|k|}}$ and this means $\mu(U_k \setminus E \cap V_k) < \frac{\epsilon}{2^{k+2}2^{|k|}}$. From the above inequality, this means

$$\nu(U_k \setminus E \cap V_k) \le 2^{k+2}\mu(U_k \setminus E \cap V_k) < \frac{\epsilon}{2^{|k|}}$$

as desired.

c. Since $\nu(U) = \nu'(U)$ for all open sets, ν' is outer regular on all Borel sets by definition, and ν is outer regular on Borel sets by part b., we deduce that $\nu(E) = \nu'(E)$ for all Borel sets and so $\nu = \nu'$. Since ν' is a Radon measure, ν is Radon as well. □

Folland Exercise 7.10 If μ is a Radon measure and $f \in L^1(\mu)$ is real-valued, for every $\epsilon > 0$ there exist an LSC function g and a USC function h such that $h \le f \le g$ and $\int (g-h)d\mu < \epsilon$.

Proof. Let us assume[29] that $f \ge 0$ first. Then by Proposition 2.23, $\{x : f(x) > 0\}$ is σ-finite and by hypothesis, f is Borel measurable. Then we can characterize $\int f d\mu$ by the infimum and supremum in Proposition 7.14. Let $\epsilon > 0$. Then there exists a USC function h

[28]NTS: A difficult and impossible to savlage proof I initially went with entailed using Lusin's Theorem on the cases where U was finite or nonfinite μ-measure. It worked for finite measured U, but is impossible to work with when infinite measured because there is no "good" upper bound.

[29]This is not necessary, actually, but was a good starting point when working on the problem.

s.t. $0 \le h \le f$ and $\int hd\mu + \frac{\epsilon}{2} > f$ and a LSC function g s.t. $\int gd\mu - \frac{\epsilon}{2} > f$. Subtracting from one another,

$$\int h - gd\mu + \epsilon = \int hd\mu + \frac{\epsilon}{2} - \left(\int gd\mu - \frac{\epsilon}{2}\right) > f - f = 0 \qquad \Longrightarrow \qquad \int (g-h)d\mu < \epsilon.$$

Let $\epsilon > 0$. Assume f is real-valued and by Proposition 2.23 again, $\{x : f^+(x) > 0\}$ and $\{x : f^-(x) < 0\}$ are σ-finite where $f = f^+ - f^-$ is the positive and negative parts decomposition from Chapter 2. This enables us to apply Proposition 7.14 four times[30]: there exists LSC functions g_1, g_2 s.t. $g_1 \ge f^+$, $g_2 \ge f^-$, $\int g_1 < \int f^+ + \epsilon$, and $\int g_2 < \int f^- + \epsilon$ and there exists USC functions h_1, h_2 s.t. $h_1 \le f^+$, $h_2 \le f^-$, $\int h_1 > \int f^+ - \epsilon$, and $\int h_2 > \int f^- - \epsilon$. Putting this together, we know

$$h_1 - g_2 \le f^+ - f^- \le g_1 - h_2$$

and in case it is not clear, the fact that $-g_2 \le -f^-$ and $-f^- \le -h_2$ is used. Because $f = f^+ - f^-$, we obtain $h_1 - g_2 \le f \le g_1 - h_2$. Since g_2 is LSC, $-g_2$ is USC and $h_1 - g_2 = h_1 + (-g_2)$ is the sum of USC continuous which is then USC. On the other hand, h_2 is USC implies $-h_2$ is LSC and therefore, $g_1 - h_2 = g_1 + (-h_2)$ is a sum of LSC functions meaning it is LSC. This gives the desired lower bound by a USC function and upper bound by an LSC function. For the integral bound,

$$\begin{aligned} \int (g_1 - h_2) - (h_1 - g_2)d\mu &= \int g_1 d\mu - \int h_2 d\mu - \int h_1 d\mu + \int g_2 d\mu \\ &< \int f^+ d\mu + \epsilon - \int f^- d\mu + \epsilon + - \int f^+ d\mu + \epsilon + \int f^- d\mu + \epsilon \\ &= 4\epsilon. \end{aligned}$$

Since we can scale the our choice of ϵ to $\frac{\epsilon}{4}$, we are done. ☐

Folland Exercise 7.11 Suppose that μ is a Radon measure on X such that $\mu(\{x\}) = 0$ for all $x \in X$, and $A \in \mathcal{B}_X$ satisfies $0 < \mu(A) < \infty$. Then for any α such that $0 < \alpha < \mu(A)$ there is a Borel set $B \subset A$ such that $\mu(B) = \alpha$.

Proof. [31] Define a set $\mathfrak{S} := \{B \subseteq A : \mu(B) \le \alpha, U \cap A = B \text{ with } U \subseteq X \text{ open}\}$ which partially order by inclusion i.e. $B \le B'$ whenever $B \subseteq B'$.

First, establish that $\mathfrak{S}$ is nonempty. Because $0 < \mu(A) < \infty$ is nonempty and μ is zero for all one point sets, we can choose $a \in A$, use outer regularity to find an open set $U \supseteq \{a\}$ s.t. $\mu(U) < \alpha$ and take $B := U \cap A$ which then satisfies the conditions to be in $\mathfrak{S}$.

Now for the chain condition of Zorn's Lemma. Let $\{B_i\}_{i \in I}$ be a chain inside $\mathfrak{S}$. We claim the upper bound is $\bigcup_{i \in I} B_i$. Certainly, if $B_i = U_i \cap A$, then $\bigcup_{i \in I} B_i = \left(\bigcup_{i \in I} U_i\right) \cap A$ which is an intersection of an open set with A. Next, to compute $\mu(\bigcup_{i \in I} B_i)$, we note that $\bigcup_{i \in I} B_i \subseteq A$ which allows us to use Exercise 1.15 (the measure μ is semifinite on A since A has finite measure). This means that $\mu(\bigcup_{i \in I} B_i) = \sup\{\mu(F) : F \subseteq \bigcup_{i \in I} B_i,\ \mu(F) < \infty\}$ and since every $F \subseteq \bigcup_{i \in I} B_i$ is in some B_j, we deduce that $\mu(\bigcup_{i \in I} B_i) \le \alpha$. Invoking Zorn's Lemma, we know there exists such a set $B \in \mathfrak{S}$ that is maximal w.r.t partial ordering.

[30] The hypotheses of Proposition 7.14 are fulfilled since $f^{\pm}$ are nonnegative and Borel measurable.

[31] NTS: Is it possible to do this construction without Zorn's Lemma? It looks impossible to do without it because there are possibly uncountably many measurable sets and we would need to choose a candidate. However, it is likely possible if we had semifiniteness of measures.

To establish that B is the set we want, show $\mu(B) = \alpha$. Suppose not and that we had strict inequality $\mu(B) < \alpha$. Then $0 < \alpha - \mu(B)$ and then $\mu(A \setminus B) > 0$. So, $A \setminus B \neq \emptyset$. Choose an $a \in A \setminus B$ which makes $\{a\}$ a closed set. Since $\infty > \mu(A \setminus B) > 0$, the measure $\mu_{A\setminus B}$ is σ-finite and thereby a Radon measure. By outer regularity, there exists an open set V s.t. $V \supseteq \{x\}$ and $\mu_{A\setminus B}(V) < \mu_{A\setminus B}(\{x\}) + \alpha - \mu(B) = \alpha - \mu(B)$ by hypothesis. Then, we get

$$\mu((V \cap A) \cup B) \leq \mu((V \cap A) \cup B) \leq \mu(B) + \mu(V \cap A) \leq \mu(B) + \alpha - \mu(B) = \alpha.$$

Notice that $B \subsetneq (V \cap A) \cup B$ because V contains the point $x \in A \setminus B$. To contradict maximality of B, we must establish that $(V \cap A) \cup B \in \mathfrak{S}$. Indeed, if $B = U \cap A$ for $U \subseteq X$ open, then $(V \cap A) \cup B = (V \cup U) \cap A$ and is therefore in $\mathfrak{S}$. Contradiction. $\square$

Folland Exercise 7.12 Let $X = \mathbb{R} \times \mathbb{R}_d$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ with the discrete topology. If f is a function on X, let $f^y(x) = f(x, y)$; and if $E \subset X$, let $E^y = \{x : (x, y) \in E\}$

a. $f \in C_c(X)$ iff $f^y \in C_c(\mathbb{R})$ for all y and $f^y = 0$ for all but finitely many y.

b. Define a positive linear functional on $C_c(X)$ by $I(f) = \sum_{y\in\mathbb{R}} \int f(x, y)dx$ and let μ be the associated Radon measure on X. Then $\mu(E) = \infty$ for any E such that $E^y \neq \varnothing$ for uncountably many y.

c. Let $E = \{0\} \times \mathbb{R}_d$. Then $\mu(E) = \infty$ but $\mu(K) = 0$ for all compact $K \subset E$.

Proof. **a.** ($\Longrightarrow$): Suppose $f \in C_c(X)$. If we fix $y \in \mathbb{R}$ and consider f^y, then f^y is certainly continuous by definition of the product topology. Also, $\text{supp}(f^y) \subseteq \text{supp}(f)$ which means it is a closed subset of a compact set and therefore, compact. For the last statement, suppose not and that $f^y \neq 0$ for infinitely many y. Let Y be the set of those y s.t. $f^y \neq 0$. Then $(f^y)^{-1}(\mathbb{R}) \times \{y\}$ is open in X since $(f^y)^{-1}(\mathbb{R}) \subseteq \mathbb{R}$ is open and by definition of the product topology. But then $\text{supp}(f) \subseteq \bigcup_{y\in Y} (f^y)^{-1}(\mathbb{R}) \times \{y\}$ and there is no finite subcover because of our assumption. Contradiction. Therefore, $|Y| < \infty$.

($\Longleftarrow$): This direction is trickier[32] so we are more detailed here. Suppose $U \subseteq \mathbb{C}$ were an open set. If $0 \in U$, then $f^{-1}(U) = \bigcup_{y\in\mathbb{R}_d} (f^y)^{-1}(U) \times \{y\}$ is a union of open sets with Y being the set of y s.t. $f^y \not\equiv 0$. If $0 \notin Y$, then $f^{-1}(U) = \bigcup_{y\in Y} (f^y)^{-1}(U) \times \{y\}$ where Y is the set of y s.t. $f^y \neq 0$. Therefore, $f^{-1}(U)$ is open. To show that f has compact support, we have

$$\text{supp}(f) = \overline{\bigcup_{y\in Y} (f^y)^{-1}(\mathbb{C} \setminus \{0\})) \times \{y\}} = \bigcup_{y\in Y} \overline{(f^y)^{-1}(\mathbb{C} \setminus \{0\}))} \times \{y\} = \bigcup_{y\in Y} \text{supp}(f^y) \times \{y\}$$

and the first equality is by definition of the support, the second equality is because the union is finite and the closure of a product is the closure of the individual parts, and the last is by definition of the support.

Because $\{y\}$ is a compact set, Tychonoff's Theorem shows that $\text{supp}(f^y) \times \{y\}$ is compact. The finite union of compact spaces is necessarily compact. Hence, $\text{supp}(f)$ is compact.

b. First, we verify that $I(f)$ is positive and a linear functional $I : C_c(X) \to \mathbb{C}$ as claimed. Certainly, I is linear because the integral and sum is linear and the sum is a finite sum when $f \in C_c(X)$ by part a.. For positivity, suppose $f \geq 0$ and then, $I(f)$ is a finite sum of

[32]Continuity of functions from product spaces can be tricky and we recall that it is not true that $f(x, y) : X \times Y \to Z$ is continuous when $f^y(x)$ is continuous for all y and $f^x(y)$ is continuous for all x. The standard counterexample is $f(x, y) = \frac{xy}{x^2+y^2}$ with $f(0, 0) = 0$.

positive summands and is thereby positive. From the Riesz-Representation Theorem, there is an associated Radon measure μ i.e. $I(f) = \int f d\mu$ for $f \in C_c(X)$.

Let E be s.t. $E^y \neq \emptyset$ for uncountably many y. By outer regularity, $\mu(E) = \inf\{\mu(U) : E \subseteq U \text{ open}\}$. It suffices to show that $\mu(U) = \infty$ for all $E \subseteq U$.

To compute $\mu(U)$, the Riesz Representation Theorem tells us that $\mu(U) = \sup\{I(f) : f \in C_c(X),\ f \prec U\}$. Since $U \supseteq E$, we know that $U^y \neq \emptyset$ for uncountably many y.

Define an $f_m \in C_c(X)$ as follows. First, choose a finite sequence $y_1, \ldots, y_m \in Y$ s.t. $E^{y_i} \neq \emptyset$. For each i, choose a point $x_i \in E^{y_i}$ and apply Urysohn's Lemma to find a function $g_i \in C_c(\mathbb{R})$ s.t. $g_i(x_i) = 1$, $\text{supp}(g_i) \subseteq U^i$. Then, set $f_m := g_i$ on $\mathbb{R} \times \{y_i\}$ and for all $y \notin \{y_1, \ldots, y_m\}$, set $f_m = 0$ for $\mathbb{R} \times \{y\}$. From part a., we know $f_m \in C_c(X)$ and by construction, $f_m \prec U$.

Now we integrate and note that by construction [33], $\int f_m(x, y_i)dx > 0$ for all i,

$$I(f_m) = \sum_{y \in \mathbb{R}} \int f_m(x, y)dx = \int f_m(x, y_1)dx + \cdots \int f_m(x, y_m)dx.$$

Let F be defined by setting $F^y = g$ for y s.t. $E^y \neq \emptyset$ and g is obtained from Urysohn's Lemma by $\{x_y\} \subseteq E^y$, $g(x_y) = 1$, $0 \leq g \leq 1$, and $\text{supp}(g) \subseteq U^y$. Now we take the supremum when computing $\mu(U)$ and therefore, our $\mu(U)$ is greater than or equal to the supremum over all $I(f_m)$. Therefore,

$$\mu(U) \geq \sup\{I(f_m) : f_m \in C_c(X) \text{ as above}\} = \sum_{y \in \mathbb{R}} \int F(x, y)dx = \infty.$$

The equality of the supremum with the uncountable sum follows from the fact that an uncountable sum is defined[34] as the supremum over the finite sums[35]. The last equality follows since the RHS sum is an uncountable sum of positive values.

c. Let $E := \{0\} \times \mathbb{R}_d$. Then $E^y \neq \emptyset$ for all $y \in \mathbb{R}_d$ because $E^y = \{0\} \times \{y\}$. Therefore, by part b., $\mu(E) = \infty$. Now suppose $K \subseteq E$ is compact and so $K := \{0\} \times \{y_1, \ldots, y_n\}$ for $n \in \mathbb{N}_0$ with $n = 0$ meaning $K = \{0\} \times \emptyset$. Notice that if $K := \{0\} \times \emptyset$, then the zero function is in the set on the RHS so $\mu(K) = 0$ is obvious. So assume $n \in \mathbb{N}$. To show $\mu(K) = 0$, apply the Riesz-Representation Theorem to get $\mu(K) = \inf\{I(f) : f \in C_c(X), f \geq \chi_K\}$.

Let $m \in \mathbb{N} \setminus \{1\}$. Define $f_m(x, y)$ by setting $f_m^{y_k}(x)$ to be the function where $f^{y_k}|_{[0,1/m^2]} = 1$, $\text{supp}(f^{y_k}) \subseteq (-1/m, 1/m)$, and $f^{y_k} \in C_c(\mathbb{R})$ which is obtained from Urysohn's Lemma and then

$$f_m(x, y) := \begin{cases} f^{y_k}(x) & y_k,\ 1 \leq k \leq n, \\ 0 & y \notin \{y_1, \ldots, y_n\}. \end{cases}$$

Then $f_m \in C_c(X)$, because of part a., $f_m \geq \chi_K$, and $f_m \geq 0$. Moreover,

$$0 \leq I(f_m) = \sum_{y \in \mathbb{R}} \int f_m(x, y)dx = \sum_{k=1}^{n} \int f_m^{y_k}(x)dx \leq \sum_{k=1}^{n} \frac{2}{m} = \frac{2n}{m}.$$

[33] We know $f_m(x, y_i) = g(x)$. But we also know $0 \leq g \leq 1$, $g(x)$ is continuous, and $g(x_i) = 1$. So there is an open set $W \ni x_i$ on which $g|_W > 0$. But every nonempty open set of $\mathbb{R}$ has positive Lebesgue measure so $\int f_m(x, y_i)dx > 0$.

[34] See the remarks on p. 11 of Folland.

[35] In this case, finite subsets $\{y_1, \ldots, y_m\}$ in which $E^{y_i} \neq \emptyset$.

Since $m \in \mathbb{N}$ was arbitrary, we have a sequence $\{f_m\}_{m=1}^{\infty}$ s.t. $f_m \in C_c(X)$, $f_m \geq \chi_K$, and $I(f_m) \to 0$. These are candidates for the infimum and so, $\mu(K) = 0$. □

Folland Exercise 7.13 In the setting of Exercise 12, let $A = (\mathbb{R}\backslash\{0\}) \times \mathbb{R}_d$ and $\phi(x, y) = |x|$. Then the measures $\mu_A(E) = \mu(A \cap E)$ and $\nu(E) = \int_E \phi d\mu$ are not Radon. (Thus, the hypotheses that μ be σ -finite in Exercise 7 , that $\phi \in L^1(\mu)$ in Exercise 8 , and that $\phi > 0$ in Exercise 9, cannot be dropped.

Proof. **Failure of μ_A:** [36] Let $E := \{0\} \times \mathbb{R}_d$. Then $E \cap A = \emptyset$ and therefore, $\mu_A(E) = 0$. We claim that outer regularity fails. Let $U \supseteq E$ be an open set containing E. By definition of the product topology and by the basis of the topology on $\mathbb{R}$, there exists an interval (a, b) that contains 0 s.t. $E \subseteq (a, b) \times \mathbb{R}_d \subseteq U$. Now choose an element $x \in (a, b) \times \mathbb{R}_d$ which is not equal to 0. Then, $\{x\} \times \mathbb{R}_d \subseteq (a, b) \times \mathbb{R}_d$. From Exercise 7.12b, we know that $\mu(\{x\} \times \mathbb{R}_d) = \infty$. But $\{x\} \times \mathbb{R}_d \subseteq A$ implies that $\mu_A(\{x\} \times \mathbb{R}_d) = \infty$. But this implies that $\mu_A(U) = \infty$ by monotonicity of measures.

Since U was an arbitrary open set, we deduce that every open set containing E has $\mu_A(U) = \infty$. But this means $\inf\{\mu_A(U) : E \subseteq U, \text{ open}\} = \infty$ and hence, $\inf\{\mu_A(U) : E \subseteq U, \text{ open}\} \neq \mu_A(E)$. So outer regularity on Borel sets fails.

Failure of ν: Let $E := \{0\} \times \mathbb{R}_d$ and E is Borel. We know $\mu(E) = \infty$ from Exercise 7.12c. We also know that $\phi = 0$ on E by definition. Therefore, $\int_E \phi d\mu = 0$. Suppose we had an open $U \supseteq E$ and by taking $n \in \mathbb{N}$ sufficiently large, we have $U \supseteq (-1/n, 1/n) \times \mathbb{R}_d \supseteq E$.

Let $Y := \{y_1, \ldots, y_N\} \subseteq \mathbb{R}_d$ be some arbitrary finite subset of $\mathbb{R}_d$. From Corollary 7.13, since ϕ is lower semicontinuous, nonnegative, and μ is Radon, we can approximate $\phi|_{(-1/n,1/n)\times\mathbb{R}_d} \geq h(x, y)$ where

$$h(x, y) := \begin{cases} -x + \frac{1}{(n+1)} & \frac{1}{2(n+1)} \leq x \leq \frac{1}{n+1} \\ x & 0 \leq x \leq \frac{1}{2(n+1)} \\ 0 & \text{elsewhere} \end{cases} \quad \text{for } y \in Y \qquad \& \qquad h(x, y) = 0 \text{ for } y \notin Y.$$

and note that $h(x, y) \in C_c(X)$ because $h(x, y) \neq 0$ for only finitely many y and $h(x, y) \in C_c(\mathbb{R})$ for those y in which it is nonzero. The support of $h(x, y)$ in $\mathbb{R}$ when it is nonzero is $[0, \frac{1}{n+1}]$ and it is continuous because of how we defined it. Then,

$$\begin{aligned} \int_U \phi d\mu &\geq \int_{(\frac{-1}{n},\frac{1}{n})\times\mathbb{R}_d} \phi d\mu \geq \int_{(\frac{-1}{n},\frac{1}{n})\times\mathbb{R}_d} h(x, y) d\mu \geq \int_{[\frac{-1}{n+1},\frac{1}{n+1}]\times Y} h(x, y) d\mu \\ &= \sum_{i=1}^{N} \int_{-1/(n+1)}^{1/(n+1)} h(x, y) dx = N \int_{-1/(n+1)}^{1/(n+1)} h(x, y) dx. \end{aligned}$$

The entirety of the first line is clear because of our approximation and the second inequality is because we restricted to a smaller set. The first equality on the second line follows from the definition in Exercise 7.12b and the fact that $h(x, y) \in C_c(X)$. Note further that

[36] For this part of the proof, I discussed the material with Professor Ebenfelt, Frederick Rajasekaran, and with Scotty Tilton and Jeb Runnoe.
NTS: I was under the impression that the problem statement was wrong, in part of reading a discussion about why restrictions don't work well with Radon measures. I wrote my own proof "falsifying" Folland's problem. However, I was not careful in writing the proof that "falsified" Folland's statement. A crucial error I made was in the my proof that μ_A was of outer regular on Borel sets. I used an $\epsilon > 0$ argument which I purported gave me the nontrivial inequality. However, that does not given the way I used it.

$\int_{-1/(n+1)}^{1/(n+1)} h(x,y)dx > 0$. But our choice of N was arbitrary so if we let $N \to \infty$, we deduce that $\int_U \phi d\mu = \infty$. This establishes $\mu(U) = \infty$ whenever $U \supseteq E$. Since $\mu(E) = 0$ and the infimum of only infinite values is just infinity $\mu(E)$ cannot possibly be outer regular. $\square$

Folland Exercise 7.14 Let μ be a Radon measure on X, and let μ_0 be the semifinite part of μ (see Exercise 15 in §1.3).

a. μ_0 is inner regular on all Borel sets.

b. μ_0 is outer regular on all Borel sets E such that $\mu(E) < \infty$.

c. $\int f d\mu = \int f d\mu_0$ for all $f \in C_c(X)$

d. If μ is the measure of Exercise 12 and m is Lebesgue measure on $\mathbb{R}$, then $\mu_0(E) = \sum_{y\in\mathbb{R}} m(E^y)$ for any Borel set E.

Proof. Let $\mu = \mu_0 + \nu$ be the decomposition from Exercise 1.15.

a. Let $E \in \mathcal{B}_X$ and $\epsilon > 0$. The there exists a finite measure $F \subseteq E$ s.t.

$$\mu_0(E) - \epsilon < \mu(F).$$

Then, by inner regularity of μ on σ-finite sets, there is a $K \subseteq F$ compact s.t. $\mu(F) - \epsilon < \mu(K)$. Since $\mu(K) < \infty$, we know $\mu(K) = \mu_0(K)$. Then,

$$\mu_0(E) - 2\epsilon < \mu(F) - \epsilon < \mu(K) = \mu_0(K)$$

as desired.

b. Assume $\mu(E) < \infty$. Let $\epsilon > 0$ and choose $U \supseteq E$ s.t. U is open and $\mu(U) < \mu(E) + \epsilon$. Then,

$$\mu_0(U) + \nu(U) < \mu_0(E) + \nu(E) + \epsilon \qquad \Longrightarrow \mu_0(U) < \mu_0(E) + \nu(E) - \nu(U) + \epsilon \leq \mu_0(E) + \epsilon$$

which gives us what we wanted. Finiteness of $\mu(E)$ is needed since otherwise, $\nu(E) = \infty$.

c. [Incomplete]

d. Approximate χ_E by a $C_c(X)$ function f. Then $\int f d\mu_0 = \int f d\mu = \sum_{y\in\mathbb{R}} \int f(x,y) dy$ implies that $\mu_0(E) = \sum_{y\in\mathbb{R}} \int \chi_E^y dx = \sum_{y\in\mathbb{R}} m(E^y)$. $\square$

Folland Exercise 7.16 Suppose that $I \in C_0(X,\mathbb{R})^*$ and I^+, I^- are the functionals constructed in the proof of Lemma 7.15. If μ is the signed Radon measure associated to I, then the positive and negative variations of μ are the Radon measures associated to I^+ and I^-

Proof. By Theorem 3.4, choose P and N s.t. $P \cap N = \emptyset$, $P \cup N = X$, $\nu^-(P) = 0$, and $\nu^+(N) = 0$. Let $f \in C_0(X,\mathbb{R})$ and $I^+(f) = \sup\{I(g) : g \in C_0(X,\mathbb{R}), 0 \leq g \leq f\}$ by construction[37]. So, if we expand $I(g) = \int g d\mu = \int g d\mu^+ - \int g d\mu^-$, and note $\int g d\mu^- \geq 0$,

$$I^+(f) = \sup\left\{\int g d\mu^+ - \int g d\mu^- : g \in C_0(X,\mathbb{R}), 0 \leq g \leq f\right\} \leq \sup\left\{\int g \mu^+ : g \in C_0(X,\mathbb{R}), 0 \leq g \leq f\right\} \leq$$

For the other inequality, let $\epsilon > 0$. Also, $f\chi_P$ is[38] a measurable function vanishing outside a set of finite $|\mu|$-measure. By Theorem 7.10, choose a $\phi \in C_c(X)$ s.t. $\phi = f\chi_P$ except on a set E_ϵ s.t. $|\mu|(E_\epsilon) < \epsilon$ and since f is bounded, assume $\|\phi\|_u \leq \|f\|_u$. WLOG, we may take the positive part of the Re (ϕ) to get $\|\phi\|_u \leq \|f\|_u$ and ϕ still differs from f on a set of $< \epsilon$ measure. Since $\phi \in C_0(X)$,

[37] See p. 221 of Folland

[38] Since $f \in C_0(X,\mathbb{R})$ it is a measurable function and the product of measurable functions is still measurable.

$$
\begin{aligned}
I^+(f) &= \sup\{I(g) : g \in C_0(X, \mathbb{R}), 0 \le g \le f\} \ge I(\phi) = \int \phi d\mu = \int_{E_\epsilon^c} \phi d\mu + \int_{E_\epsilon} \phi d\mu \\
&= \int_{E_\epsilon^c} \phi d\mu^+ + \int_{E_\epsilon} \phi d\mu^+ - \int_{E_\epsilon^c} \phi d\mu^- - \int_{E_\epsilon} \phi d\mu^- \\
&\ge \int_{E_\epsilon^c} \phi d\mu^+ - \int_{E_\epsilon^c} \phi d\mu^- - \int_{E_\epsilon} \phi d\mu^- \\
&\ge \int_{E_\epsilon^c} f\chi_P d\mu^+ - \int_{E_\epsilon^c} f\chi_P d\mu^- - \int_{E_\epsilon} \phi d\mu^- && (\phi = f\chi_E \\
&\ge \int_{E_\epsilon^c} f\chi_P d\mu^+ - \int_{E_\epsilon^c} f\chi_P d\mu^- - \int_{E_\epsilon} \|f\|_u d\mu^- && (\|\phi\|_u \le \\
&\ge \int_{E_\epsilon^c} f\chi_P d\mu^+ - \int_{E_\epsilon^c} f\chi_P \chi_N d\mu - \int_{E_\epsilon} \|f\|_u d\mu^- && (\text{def.} \\
&= \int_{E_\epsilon^c} f\chi_P d\mu^+ - \int_{E_\epsilon} \|f\|_u d\mu^- && (P \cap \\
&= \int_{E_\epsilon^c} f\chi_P d\mu^+ + \int_{E_\epsilon} f\chi_P d\mu^+ - \int_{E_\epsilon} f\chi_P d\mu^+ - \int_{E_\epsilon} \|f\|_u d\mu^- \\
&= \int_X f d\mu^+ - \int_{E_\epsilon} \|f\|_u d\mu^+ - \int_{E_\epsilon} \|f\|_u d\mu^- && (\text{def. of } \mu^+ \ \& \ \|\phi\|_u \le \\
&\ge \int_X f d\mu^+ - \|f\|_u |\mu|(E_\epsilon) - \|f\|_u |\mu|(E_\epsilon) && (\mu^- \\
&\ge \int_X f d\mu^+ - 2\|f\|_u \epsilon && (|\mu|(E
\end{aligned}
$$

Because $f \in C_0(X, \mathbb{R})$ was arbitrary and $\epsilon > 0$ was arbitrary, we deduce that $I^+(f) \ge \int f d\mu^+$ for all $f \in C_0(X, \mathbb{R})$. So by the uniqueness part of Theorem 7.17, I^+ has associated Radon measure μ^+.

On the other hand, for all $f \in C_0(X, \mathbb{R})$, we have

$$I^-(f) = I(f) - I^+(f) = \int f d\mu - \int f d\mu^+ = \int f d\mu^-$$

and therefore, I^- has associated Radon measure μ^- the uniqueness part of by Theorem 7.17. □

Folland Exercise 7.17 If μ is a positive Radon measure on X with $\mu(X) = \infty$, there exists $f \in C_0(X)$ such that $\int f d\mu = \infty$. Consequently, every positive linear functional on $C_0(X)$ is bounded.

PROOF.[39] By inner regularity, choose a compact subset $K_1 \subseteq X$ s.t. $\mu(K_1) > 2^1 + 1$. Then $X \setminus K_1$ is open and $\mu(X \setminus K_1) = \infty$ so we can apply inner regularity to find a K_2 s.t.

[39]There is an error in the construction which has not been fixed. I was deducted one point since my construction cannot ensure the f_n do not interfere outside the set K_n. Not too sure what the issue, maybe th uniform convergence?

$\mu(K_2) > 2^2+1$ and $K_2 \subseteq X \setminus K_1$. Inductively, we can find a compact K_n s.t. $\mu(K_n) > 2^n+1$ and K_n is disjoint from $K_1, \ldots, K_{n-1}$.

Define $F := \sum_{n=1}^{\infty} \frac{f_n}{2^n}$. The sequence of partial sums $F_N := \sum_{n=1}^{N} f_n$ converge to F because $\|F\|_u \leq \sum_{n=1}^{\infty} \frac{\|f_n\|_u}{2^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. Furthermore, the partial sums are a finite sum of continuous functions and hence, continuous while the support $\operatorname{supp}(F_n) = \overline{\bigcup_{n=1}^{N}\{x \in X : f_n(x) \neq 0\}} = \bigcup_{n=1}^{N} \operatorname{supp}(f_n)$ is compact[40] and therefore $F_N \in C_c(X)$. Since $F_N \to F$ w.r.t. the uniform norm and $C_0(X)$ is the closure of $C_c(X)$ under $\|\cdot\|_u$ by Proposition 4.35, we get $F \in C_0(X)$. Now, we show $\int f d\mu = \infty$ by estimating from below. By Theorem 2.15,

$$\int f d\mu = \sum_{n=1}^{\infty} \int_X \frac{f_n}{2^n} d\mu \geq \sum_{n=1}^{\infty} \int_K \frac{f_n}{2^n} d\mu \geq \sum_{n=1}^{\infty} \frac{\mu(K)}{2^n} > \sum_{n=1}^{\infty} \frac{2^n+1}{2^n} = \infty$$

since the RHS sum has nonnegative terms that converge to 1 as $n \to \infty$.

Suppose I were a positive linear functional on $C_0(X)$ that was unbounded. Restricting I to $I|_{C_c(X)}$ gives a positive linear functional on $C_c(X)$ and then $I_{C_c(X)}$ is given by integration against a positive Radon measure μ on X by Theorem 7.2. Furthermore, $\mu(X) < \infty$. Suppose not and $\mu(X) = \infty$ so there exists a sequence of F_N as the one above. Let $G_N := \frac{F_N}{\|F_N\|_u}$ and note that $\|F_N\|_u \leq \|F\|_u = 1$. Then,

$$I_{C_c(X)}(G_N) = \frac{I(F_N)}{\|F_N\|_u} \geq \frac{I(F_N)}{1} = I(F_N)$$

and as $N \to \infty$, we get $I_{C_c(X)}(G_N) \to \infty$. This implies that $\|I_{C_c(X)}\| = \sup\{I(f) : f \in C_c(X), \|f\|_u = 1\} = \infty$ and so, $I_{C_c(X)}$ is unbounded. From the remarks on p. 221, this means $I_{C_c(X)}$ cannot be extended continuously to a functional on $C_0(X)$. However, I itself is an extension of $I_{C_c(X)}$ that is continuous which is a contradiction. We conclude that $\mu(X) < \infty$. By requiring $\mu(X) < \infty$, we deduce that I itself is bounded because $|\int f d\mu| \leq \|f\|_u \mu(X) = \|f\|_u \|\mu\|$ for all $f \in C_0(X)$. $\square$

Folland Exercise 7.18 If μ is a σ-finite Radon measure on X and $\nu \in \mathcal{M}(X)$, let $\nu = \nu_1 + \nu_2$ be the Lebesgue decomposition of ν with respect to μ. Then ν_1 and ν_2 are Radon. (Use Exercise 8.)

Proof. The Lebesgue Decomposition of $\nu = \nu_1 + \nu_2$ is obtained from the Lebesgue-Radon-Nikodym Theorem and WLOG, there is an $f \in L^1(\mu)$ s.t. $d\nu = d\nu_1 + f d\mu$ and $d\nu_2 = f d\mu$ and $\nu_1 \perp \lambda$.

First Approach:[41] By the Lebesgue-Radon-Nikodym Theorem, there is an $f \in L^1(\mu)$ s.t. $d\nu = d\nu_1 + f d\mu$, $d\nu_2 = f d\mu$, and $\nu_1 \perp \mu$.

We know ν_2 is Radon iff $|\nu_2|$ is Radon and since $d\nu_2 = f d\mu$, we know $d|\nu_2| = |f| d\mu$. From Exercise 7.8, we know $|\nu_2|$ is Radon since $|\nu_2|(E) = \int_E |f| d\mu$, μ is Radon, $|f| \in L^1(\mu)$, and $|f| \geq 0$.

Since $M(X)$ is a vector space, the difference of Radon measures is also a Radon measure. Since ν is Radon, we see that $\nu_2 = \nu - \nu_1$ must also be Radon.

Second Approach: Since ν_1 is a complex measure, $\nu_1 = \nu_{1,r} + i\nu_{1,i}$ can be split into its real and imaginary parts. Decompose $\nu_{1,r} = \nu_{1,r}^{+} - \nu_{1,r}^{-}$ into its positive and negative variations

[40] The finite union of compact sets is compact and the closure distributes over finite unions.

[41] This first approach was suggested by Jeb which is fast while the second approach was my original (longer) approach.

which are just restrictions $\nu_{1,r,P}$, $\nu_{1,r,N}$ of $\nu_{1,r}$ to a positive and negative set P, N respectively s.t. $P \cap N = \emptyset$ and $P \cup N = X$ from the Hahn-Jordan Decomposition. Since ν was σ-finite, so are ν_r and ν_i as well as their positive and negative variations. Since $\nu_{1,r}^+ = \nu_{1,r,P}$, Exercise 7.7 tells us that $\nu_{1,r}^+$ is Radon as well. Repeating the same argument for $\nu_{1,r}^-$, $\nu_{1,i}^+$, and $\nu_{1,i}^-$, we know ν_1 is a Radon measure.

Deal with the case $f \geq 0$ and $f \in L^1(\mu)$ first. In this situation, $fd\mu$ is a Radon measure by Exercise 7.8. Because $d\nu_2 = fd\mu = f_r d\mu + f_i d\mu = f_r^+ d\mu - f_r^- d\mu + (f_i^+ d\mu - f_i^- d\mu)i$, we can apply this to each part to deduce that $f_r^+ d\mu$, $f_r^- d\mu$, f_i^+, and f_i^- are Radon measures. But these are the positive and negative variations of the real and imaginary parts of ν_2 respectively. Hence, $f_r d\mu$ and $f_i d\mu$ are Radon measures which implies ν_2 is a Radon measure as well.

□

Folland Exercise 7.22 A sequence $\{f_n\}$ in $C_0(X)$ converges weakly to $f \in C_0(X)$ iff sup $\|f_n\|_u < \infty$ and $f_n \to f$ pointwise.

PROOF. ($\Longrightarrow$): For $f_n \to f$ pointwise, let $x \in X$ and let μ_x be the point-mass measure at $x \in X$ and this is a Radon measure [42]. Also, $\int g d\mu_x = g(x)$ for any $g \in C_0(X)$. Then, $f_n \to f$ weakly in $C_0(X)$ implies $f_n(x) = \int f_n d\mu_x \to \int f d\mu_x = f(x)$ and since x was arbitrary, $f_n \to f$ pointwise on X.

To show $\sup \|f_n\|_u < \infty$, we use the Uniform Boundedness Principle. Since $f_n \to f$ weakly, we know $\widehat{f_n} \circ I \to \widehat{f} \circ I$ for every $I \in C_0(X)^*$. Therefore, $\sup_n \|f_n \circ I\| < \infty$ for all $I \in C_0(X)^*$ and the Uniform Boundedness Principle implies $\sup_n \|f_n\|_u = \sup_n \|\widehat{f_n}\| < \infty$.

Alternatively, to show $\sup \|f_n\|_u < \infty$, we directly use Exercise 5.47b. We know $C_0(X)$ is a Banach space w.r.t the uniform norm and therefore, every weakly convergent sequence is norm bounded. That means $\|f_n\|_u < \infty$ for all $n \in \mathbb{N}$ and hence, $\sup \|f_n\|_u < \infty$.

($\Longleftarrow$): Let $I \in C_0(X)^*$ and by the Riesz-Representation Theorem, there is a $\mu \in M(X)$ s.t. $I = I_\mu$. We need to show $I_\mu(f_n) \to I_\mu(f)$. Let $g := \sup_n \|f_n\|_u$ and notice that $g \in L^1(\mu) = L^1(|\mu|)$ because $|\mu|$ is a finite measure and g is bounded and by applying Proposition 3.13c. Since $f_n \to f$ pointwise, we certainly have $f_n \to f$ a.e.. Also, $|f_n| \leq g$ a.e. and for all n from the definition of g. By the Dominated Convergence Theorem,

$$\lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n(x) d\mu(x) = \int f(x) d\mu(x) = I(f)$$

as desired. Hence, $f_n \to f$ weakly. □

Folland Exercise 7.26 If $\{\mu_n\} \subset M(X)$, $\mu_n \to \mu$ vaguely, and $\|\mu_n\| \to \|\mu\|$, then $\int f d\mu_n \to \int f d\mu$ for every $f \in BC(X)$. (If $\mu = 0$ the result is trivial. Otherwise, there exists $g \in C_c(X)$ with $\|g\|_u \leq 1$ such that $\int g d\mu > \|\mu\| - \epsilon$, and $\int g f d\mu_n \to \int g f d\mu$ for $f \in BC(X)$.) Moreover, the hypothesis $\|\mu_n\| \to \|\mu\|$ cannot be omitted.

[42] Clearly, it is finite on compact sets because it is a finite measure. It is also outer regular because if $x \notin E$, then $\mu_x(E) = 0$ and the infimum measure of open sets containing E is zero. If $x \in E$, then μ_x is outer regular on E. If U is an open set and it contains x, there is a compact set $\{x\} \subseteq \overline{V} \subseteq U$ by Proposition 4.31 and $\mu_x(\overline{V}) = 1$ which shows inner regularity in this case. If $x \notin U$, then compact sets in U all have μ_x measure zero and $\mu_x(U) = 0$ and inner regularity still holds.

PROOF. [43] Suppose $\mu = 0$. Let $f \in BC(X)$ and so $\int f d\mu = 0$. Then $\left|\int f d\mu_n\right| \leq \int |f| d|\mu_n| \leq \|f\|_u \|\mu_n\| \to \|f\|_u \cdot 0 = 0$ and we are done in this case.

Suppose $\mu \neq 0$. We show[44] there exists a compact set $K \subseteq X$ and an $n \in \mathbb{N}$ s.t. for all $n \geq N$, one has $\mu_n(X \setminus K) < \epsilon$. Assuming $\|\mu\| > 0$, we can find a $g \in C_c(X)$ s.t. $\|g\|_u \leq 1$ and $|\int g d\mu| > \|\mu\| - \epsilon$ which is equivalent to $\epsilon > \|\mu\| - |\int g d\mu|$. Since $\|\mu_n\| \to \|\mu\|$ and $\mu_n \to \mu$ vaguely, for large enough $n \in \mathbb{N}$, we have $|\|\mu\| - \|\mu_n\|| < \epsilon$ and so $\|\mu\| < \epsilon + \|\mu_n\|$ as well as $||\int g d\mu| - |\int g d\mu_n|| < \epsilon$ and so $|\int g d\mu| - \epsilon < |\int g d\mu_n|$. These estimates give,

$$\|\mu_n\| - \left|\int g d\mu_n\right| = \|\mu_n\| - \|\mu\| + \|\mu\| - \left|\int g d\mu_n\right| < \epsilon + \|\mu\| - \left|\int g d\mu_n\right| < \epsilon + \|\mu\| - \left|\int g d\mu\right| + \epsilon < 3\epsilon.$$

Then, writing $\|\mu_n\| = \int 1 d|\mu_n|$ and using the fact that $|\int g d\mu_n| \leq \int |g| d|\mu_n|$ since $g \in L^1(\mu_n)$,

$$\int 1 - |g| d|\mu_n| = \|\mu_n\| - \int |g| d|\mu_n| \leq \|\mu_n\| - \left|\int g d\mu_n\right| < 3\epsilon.$$

Since $g \in C_c(X)$, we know $g \in C_0(X)$ and the set $K := \{x \in X : |g(x)| \geq 1/3\}$ is compact. Next, $K^c = \{x \in X : |g(x)| < 1/3\} = \{x \in X : 2/3 < 1 - |g(x)|\}$. Then, for n large enough,

$$\begin{aligned} ||\mu_n|(X \setminus K)| &= \left|\frac{3}{2}\int_{K^c} \frac{2}{3} d\mu_n\right| < \left|\frac{3}{2}\int_{K^c} (1 - |g|) d\mu_n\right| \leq \frac{3}{2}\int_{K^c} |1 - |g|| d|\mu_n| \\ &= \frac{3}{2}\int_{X \setminus K} (1 - |g|) d|\mu_n| \\ &\leq \frac{3}{2}\int_X (1 - |g|) d|\mu_n| < \frac{9}{2}\epsilon. \end{aligned}$$

So the set K is our desired set to get $\mu_n(X \setminus K) < \epsilon$ (after scaling our ϵ)) provided n is sufficiently large, say $n \geq N$. Since there are only finite many n s.t. $1 \leq n \leq N$, and inner regularity allows us to always find a choice of K_n for each of those values, $\mu_n(X \setminus K_n) < \epsilon$ for $1 \leq n \leq N$, and so we can increase the size of the set K so that $\mu_n(X \setminus K) < \epsilon$ holds for all $n \in \mathbb{N}$ by taking the union of our choices of compact sets and using the fact that a finite union of compact sets is still finite.

Let $f \in BC(X)$. Now, choose a K as above, since it exists, s.t. $\mu_n(X \setminus K) < \epsilon$ for all $n \in \mathbb{N}$. Then, by inner regularity on open sets, choose a compact set J s.t. $\mu(X \setminus J) < \epsilon$. Set $L := J \cup K$ so that the estimates $|\mu_n|(X \setminus L) < \epsilon$ holds for all n and for μ in place of μ_n. By Urysohn's Lemma, choose a $g \in C_c(X)$ s.t. $g = 1$ on L. Then by vague convergence, since $\text{supp}(fg) \subseteq \text{supp}(g)$ and g continuous means $fg \in C_c(X)$, choose an N s.t. for all $n \geq N$,

[43]NTS: You had an argument where you split $f = (1 - g)f + gf$. The argument you wrote failed because your choice of a $C_c(X)$ function for the approximation is not actually $C_c(X)$.

[44]A hint was provided by James Wong.

$|\int gfd\mu_n - \int gfd\mu| < \epsilon$. Then, we have estimates for n large,

$$\begin{aligned}\left|\int fd\mu_n - \int fd\mu\right| &\leq \left|\int_L fd\mu_n - \int_L fd\mu\right| + \left|\int_{L^c} fd\mu_n - \int_{L^c} fd\mu\right| \\ &= \left|\int_L gfd\mu_n - \int_L gfd\mu\right| + \left|\int_{L^c} fd\mu_n - \int_{L^c} fd\mu\right| \\ &< \epsilon + \left|\int_{L^c} fd\mu_n - \int_{L^c} fd\mu\right| \leq \epsilon + \|f\|_u \left|\int_{L^c} d\mu_n - \int_{L^c} d\mu\right| \leq \epsilon + \|f\|_u(|\mu_n|(L^c)| + |\mu|(L \\ &\leq \epsilon + \|f\|_u(|\mu_n|(L^c) + |\mu|(L^c)) \leq \epsilon + \|f\|_u(2\epsilon) < \epsilon(1 + 2\|f\|_u).\end{aligned}$$

Necessity of hypothesis: We show that hypothesis $\|\mu_n\| \to \|\mu\|$ cannot be dropped. Let $X := \mathbb{R}$ and $\mu_n = \delta_{-n}$ be the point mass at $-n \in \mathbb{R}$. Then $\|\mu_n\| = 1$ for all n and $\mu_n \to 0 =: \mu$ vaguely because $\int f(x)d\mu_n = f(-n) \to 0$ for all $f \in C_0(X)$ and $\int f(x)d\mu = 0$. Now, let $f(x) := 1$ the constant function which is bounded and continuous. However, $\int fd\mu_n = f(-n) = 1$ for all n which means $\int fd\mu_n \not\to \int fd\mu$ since $\int fd\mu = 0$.

□

Folland Exercise 7.30 Let μ and ν be Radon measures on X and Y, not necessarily σ -finite. If f is a nonnegative LSC function on $X \times Y$, then $x \mapsto \int f_x d\nu$ and $y \mapsto \int f^y d\mu$ are Borel measurable and $\int fd(\mu \widehat{\times} \nu) = \iint fd\mu d\nu = \iint fd\nu d\mu$

Proof. **Borel measurability:** We establish Borel measurability of $\Phi(x) = \int f_x d\nu$ and $\Psi(y) = \int f^y d\mu$. Since f is nonnegative and LSC on $X \times Y$, we know f_x and f^y are nonnegative and LSC functions on Y and X respectively.

By Corollary 7.13,

$$\Phi(x) = \int f_x d\nu = \sup\left\{\int gd\nu : g \in C_c(Y), 0 \leq g \leq f_x\right\} = \sup\left\{\int g_x d\nu : g \in C_c(X \times Y), 0 \leq g_x \leq f_x\right\},$$

and it is clear that the set on the RHS is contained in the set on the LHS, so we at least have $\geq$, but it is necessary to show the other inequality to establish equality above. Let C_1 be the supremum on the LHS while C_2 is the supremum on the RHS and we have $C_1 \geq C_2$.

Let $\epsilon > 0$. There exists a $g \in C_c(Y)$ s.t. $0 \leq g \leq f_x$ and $C_1 - \epsilon \leq \int gd\nu$. Since $g \in C_c(Y)$, we know supp(g) is compact so $g \in C(\{x\} \times \text{supp}(g))$ if we define $g(x, y) = g(y)$ as is. By the Tietze Extension Theorem 4.34, we may extend g to a $G \in C_C(X \times Y)$ s.t. $G|_{\{x\}\times Y} = g$. Then, G is in the set on the RHS since $G_x = g$ means $0 \leq G_x \leq f_x$ and $G \in C_c(X \times Y)$. But $\int gd\nu = \int G_x d\nu$ so that means $C_1 - \epsilon \leq \int gd\nu \leq C_2$. Since $\epsilon > 0$ was arbitrary, we get $C_1 \leq C_2$ and we already know $C_1 \geq C_2$ which means $C_1 = C_2$.

Choose a sequence $g_n \in C_c(X \times Y)$ s.t. $\int g_{n,x} d\nu \to C_2$ increasingly. Define $\Phi_n(x) = \int g_{n,x} d\nu$ and $\Phi_n(x)$ is Borel measurable for each n because Lemma 7.24 establishes that they are continuous and continuous functions are necessarily Borel measurable. But then $\Phi(x) = \sup_n \Phi_n(x) = \lim_{n\to\infty} \Phi_n(x)$ which implies that $\Phi(x)$ is also Borel measurable by Corollary 2.9.

The result for $\Psi(y)$ follows by switching x with y, X with Y, f_x with f^y and modifying the above proof in the obvious way. Thus, $\Phi(x)$ and $\Psi(y)$ are Borel measurable functions on X and Y respectively.

Iterated Integrals: Since f is a nonnegative LSC function, Corollary 7.13 can be applied. If g were a $C_c(X \times Y)$ function, we know $f \in L^1(\mu \times \nu)$ by Proposition 7.22 since

μ and ν are Radon measures. So, $g \in L^1(\mu \times \nu)$. But by definition of the Radon product, $\int g d(\mu \widehat{\times} \nu) = \int g d(\mu \times \nu)$. Putting this all together

$$\begin{aligned}
\int f d(\mu \widehat{\times} \nu) &= \sup \left\{ \int g d(\mu \widehat{\times} \nu) : g \in C_c(X \times Y), 0 \leq g \leq f \right\} \\
&= \sup \left\{ \int g d(\mu \times \nu) : g \in C_c(X \times Y), 0 \leq g \leq f \right\} \\
&= \sup \left\{ \iint g d\mu d\nu : g \in C_c(X \times Y), 0 \leq g \leq f \right\} \\
&= \int \sup \left\{ \int g_y d\mu : g \in C_c(X \times Y), 0 \leq g \leq f \right\} d\nu \\
&= \iint f d\mu d\nu
\end{aligned}$$

and to reiterate, the first equality is by Corollary 7.13, the second is by definition of $\mu \widehat{\times} \nu$, the third is by Proposition 7.22, and the fourth is by an application of Proposition 7.12. The last equality then follows since the supremum is just equal to the integral $\int f d\mu$. Changing which iteration we took in the third equality, $\int f d(\mu \widehat{\times} \nu) = \iint f d\mu d\nu = \iint f d\nu d\mu$ as desired. □

8. Elements of Fourier Analysis

Folland Exercise 8.1 Prove the product rule for partial derivatives as stated in the text. Deduce that

$$\partial^\alpha \left(x^\beta f \right) = x^\beta \partial^\alpha f + \sum c_{\gamma\delta} x^\delta \partial^\gamma f, \quad x^\beta \partial^\alpha f = \partial^\alpha \left(x^\beta f \right) + \sum c'_{\gamma\delta} \partial^\gamma \left(x^\delta f \right)$$

for some constants $c_{\gamma\delta}$ and $c'_{\gamma\delta}$ with $c_{\gamma\delta} = c'_{\gamma\delta} = 0$ unless $|\gamma| < |\alpha|$ and $|\delta| < |\beta|$.

Proof. Recall the product rule is $\partial^\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (\partial^\beta f)(\partial^\gamma g)$. It is easier to observe the rule inductively on n $\alpha = (\alpha_1, \ldots, \alpha_n)$ i.e. show the result WLOG for one coordinate being nonzero. Certainly when $\alpha_1 \neq 0$ and $\alpha_i = 0$ for $i \neq 1$ one has

$$\partial_1(fg) = \sum_{\beta_1+\gamma_1=\alpha} \frac{\alpha_1!}{\beta_1!\gamma_1!} \partial^{\beta_1}(f) \partial^{\gamma_1}(g)$$

from the usual Leibniz Rule[45] The result just carries over by noting that we need to differentiate in each coordinate.

The first equation follow from applying this rule directly and considering the term where $\beta = 0$ and $\gamma = \alpha$. **[Incomplete]** □

Folland Exercise 8.2 Observe that the binomial theorem can be written as follows:

$$(x_1 + x_2)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha \quad (x = (x_1, x_2), \alpha = (\alpha_1, \alpha_2))$$

[45]See this page for instance.

Prove the following generalizations: a. The multinomial theorem: If $x \in \mathbb{R}^n$,

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha.$$

b. The n -dimensional binomial theorem: If $x, y \in \mathbb{R}^n$,

$$(x + y)^\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} x^\beta y^\gamma.$$

PROOF. **a.** The multinomial theorem is an easy consequence of induction and in fact, one can deduce the result quite efficiently by use of combinatorial techniques.

b. This generalization follows by consider the binomial theorem in each coordinate. $\square$

Folland Exercise 8.4 If $f \in L^\infty$ and $\|\tau_y f - f\|_\infty \to 0$ as $y \to 0$, then f agrees a.e. with a uniformly continuous function. (Let $A_r f$ be as in Theorem 3.18. Then $A_r f$ is uniformly continuous for $r > 0$ and uniformly Cauchy as $r \to 0$.)

9. Elements of Distribution Theory

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