Notes on Commutative Algebra

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Contents

CHAPTER 1

An Introduction to Commutative Algebra by Atiayh and Macdonald

The text here is [1]. I have not bothered to write out the details of every exercise because they may either be straightforward or far too tedious to do so. A significant portion of the exercise were done for homework when taking Math 200C at UCSD, during my directed reading course with Michael Loper, and in my spare time as I worked through [3]. Throughout, we shall often refer to Atiyah-Macdonald as AM.

Review. AM is truly a marvelous book. It is only 128 pages long (137 pages if one demands the count of the nine pages of introductions and the title page) and provides a very rapid introduction to commutative algebra. Though it largely ignores homological techniques in the main exposition, the text is wonderful for those students with broad interests and are taking an algebraic topology, differential geometry, or homological algebra course alongside the text.

It is clear that the crowning jewel of the text is in its exercises. Any student working through the exercises would be trained deeply in the basic ideas of commutative algebra, homological algebra, algebraic geometry, and algebraic number theory. However, no student should expect to work through all of the exercises in one attempt. Indeed, I have personally gained much from many other subjects which force me to return to this little book of Atiyah and Macdonald. The book is small enough to fit in one's bags and carry around without much weight. Indeed, it is one of the few books I always have on my person. There have been moments where I recalled a result in Atiyah-Macdonald and having the convenience of it in my bag has swayed the another person to my side about a step in a proof.

The book is also a wonderful reference for *excellent mathematical writing*. I hope that one day, I shall be capable of writing mathematics as clear, concise, and pedagogically sound as AM.

1. Rings and Ideals

Exercise 1.1 Let x be a nipotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Proof. Let x be nilpotent element of A. Let n be s.t. $x^n = 0$. I claim that $1 + x$ is a unit. Indeed, x lies in the Jacobson radical and Proposition 1.9 says that $1 - xy$ is a unit for all $y \in A$. Let $y = -1$.

Alternatively, consider $\frac{1}{1+x}$. This should be the inverse to $1 + x$. We have that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots \pm x^{n-1}.$$

This is an element of A.

Let x be nilpotent and y be a unit. Then, we can write $x+y$ as $y^{-1}(xy^{-1}+1)$. By the above, $xy^{-1}+1$ is a unit since xy^{-1} is nilpotent. Thus, $y^{-1}(xy^{-1}+1)$ is a product of units. Hence, $x+y$ is a unit. $\square$

Exercise 1.2

PROOF. Omitted; you did this exercise for Math 200A. $\square$

Exercise 1.3 Generalize the results of Exercise 2 to a polynomial ring $A[x_1,\dots,x_r]$ in several indeterminates.

PROOF. Omitted; the generalization is relatively straightforward. $\square$

Exercise 1.4 In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

PROOF. In general, the nilradical is contained in the Jacobson radical. So let $f \in \mathfrak{J}$ be in the Jacobdon radical. Set $f = \sum_{i=0}^n a_i x^i$ and AM Proposition 1.9i) says that $1-xf$ is a unit in $A[x]$. In particular, this means $1-$ is a unit and each of $a_0,\dots,a_n$ are nilpotent. But this means f had all nilpotent coefficients. So, AM Proposition 1.9ii) says that $f \in \mathfrak{N}$. So, $\mathfrak{J} \subseteq \mathfrak{N}$ as desired. $\square$

Exercise 1.5

PROOF. Omitted. You essentially did this exercise for Math 200A. $\square$

Exercise 1.6

PROOF. Clearly, the nilradical is contained in the Jacobson radical. So we show the other containment.

Suppose not and that Jacobson radical is not contained in the nilradical. Then the Jacobson radical has a nonzero idempotent by hypothesis say e. Since e is in the Jacobson radical, Proposition 1.9 says that $1-ey$ is a unit for all $y \in A$. In particular, $1-e$ is a unit in A. Therefore, there is an element $f \in A$ for which $(1-e)f = 1$. But that means $f - ef = 1$ and $-ef = 1-f$. Multiply through by e to get $-ef = e - ef$. Therefore, $e = 0$. Contradiction. $\square$

Exercise 1.7

PROOF. Let A be a ring in which every x is satisfies: $x^n = x$ for some $n > 1$. We claim every prime ideal is maximal.

It suffices to show that $A/\mathfrak{p}$ is a field for any prime ideal $\mathfrak{p}$. This comes down to showing that nonzero elements of $A/\mathfrak{p}$ have inverses. Suppose $f+\mathfrak{p} \in A/\mathfrak{p}$ is a nonzero element. By hypothesis, there is an n for which $f^n = f$. To find an inverse, we want to get a $g+\mathfrak{p}$ s.t.

$$(f+\mathfrak{p})(g+\mathfrak{p}) = fg+\mathfrak{p} = 1+\mathfrak{p}.$$

Note that to write $f^n = f$ is the same as writing $f(f^{n-1}-1) = 0$. Because we are at least in an integral domain, this means $f = 0$ or $f^{n-1} = 1$. Because f was assumed to be nonzero, this means $f^{n-1} = 1$. But then we take $g = f^{n-1}$ and we are done. $\square$

Exercise 1.8

PROOF. This is a Zorn's Lemma proof.

We can order the set $\mathfrak{S}$ of prime ideals by inclusion $\supseteq$. The set of prime ideals is clearly nonempty since it has the zero element. Now, every chain $\{\mathfrak{p}_\alpha : \alpha \in A\}$ has a maximal element which is $\bigcap_\alpha \mathfrak{p}_\alpha$ since the intersection of prime ideals is still a prime ideal so long as one contains the other and by the fact that we have a chain, $\mathfrak{p}_\alpha \supseteq \mathfrak{p}_\beta$ for $\alpha, \beta \in A$. By Zorn's Lemma, the set of prime ideals has a maximal element w.r.t to $\supseteq$. But that is precisely a minimal element w.r.t $\subseteq$. $\square$

Exercise 1.9

PROOF. One direction is easy. If $\mathfrak{a}$ is an intersection or prime ideals, then $r(\mathfrak{a})$ is just the intersection of all prime ideals containing $\mathfrak{a}$ which would just give $\mathfrak{a}$.

Suppose $\mathfrak{a} = r(\mathfrak{a})$. This means that $\mathfrak{a}$ is equal to the intersection of all prime ideals containing $\mathfrak{a}$. But this occurs iff $\mathfrak{a}$ is already an intersection of prime ideals. $\square$

Exercise 1.10

PROOF. Prove each implication. The best way is to just go (i) $\implies$ (iii) $\implies$ (ii) $\implies$ (i).

(i) $\implies$ (ii): If A has exactly one prime ideal $\mathfrak{p}$, then that one ideal is equal to both the Jacobson radical and the nilradical. If $x \in A$, then $x \in \mathfrak{p}$ or $x \notin \mathfrak{p}$. If $x \in \mathfrak{p}$, then x is nilpotent by the definition of the nilradical.

On the other hand, suppose $x \notin \mathfrak{p}$. We will show that x is a unit. Suppose not and that x is not a unit. Proposition 1.9 says that, since $x \in \mathfrak{J}$, there is some $y \in A$ for which $1 - xy$ is not a unit. Therefore, $(1 - xy) + xy = 1$ is the sum of two nonunits which gives rise to a unit.

We show that we have a contradiction. Corollary 1.5 says that every non-unit is contained in some maximal ideal. Since there is only one prime ideal, we deduce that $(1 - xy)$ and xy are both in $\mathfrak{p}$. Therefore, $(1 - xy) + xy = 1 \in \mathfrak{p}$. But this means $\mathfrak{p} = A$ which is a contradiction.

(ii) $\implies$ (iii): Suppose every element of A is either a unit or a nilpotent. Then, $A/\mathfrak{R}$ is a field because every element in A is invertible besides zero. Indeed, if $a + \mathfrak{R} \neq \mathfrak{R}$, then we know $a \notin \mathfrak{R}$ and hence, a is invertible and has inverse a^{-1}. Therefore,

$$\begin{aligned}(a + \mathfrak{R})(a^{-1} + \mathfrak{R}) &= aa^{-1} + a\mathfrak{R} + a^{-1}\mathfrak{R} + \mathfrak{R} \\ &= aa^{-1} + \mathfrak{R} \\ &= 1 + \mathfrak{R}.\end{aligned}$$

where we use the fact that $1 + \mathfrak{R} \neq \mathfrak{R}$, and $a\mathfrak{R} = a^{-1}\mathfrak{R} = \mathfrak{R}$. The second statement follows from how multiplying a nilpotent by a unit always gives a nilpotent.

Thus, $A/\mathfrak{R}$ is a field since every nonzero element has an inverse.

(iii) $\implies$ (i): If $A/\mathfrak{R}$ is a field, we know that $\mathfrak{R}$ has to be a maximal ideal in A. By definition, $\mathfrak{R}$ is the intersection of all prime ideals in A. Therefore, $\mathfrak{R}$ has to be the intersection of only one prime ideal which is maximal in A. Therefore, A has a single prime ideal. $\square$

Exercise 1.11

PROOF. (a) We have $(x + 1)^2 = x^2 + 2x + 1 = x + 2x + 1 = 3x + 1 = x + 1$. The last equality gives $3x + 1 = x + 1$ which implies $2x = 0$ as desired.

(b) Every prime ideal is maximal because $A/\mathfrak{p}$ is always a field with two elements which is has just 1 and 0. This is because a prime ideal will contain all nonidentity elements.

(c) Let $I = (a_1, \ldots, a_n)$ be a f.g. ideal of A. Assume $a_i \neq 0$ for all i. Then, $I = (a_1^2, \ldots, a_n^2)$. Then each a_i^2 can be written as a finite R-linear combination $a_i^2 = \sum_{j=1}^n b_{ij}a_j^2$ and then

$$a_i = \sum_{j=1}^{n} b_{ij}\frac{a_j^2}{a_i}$$

is an R-linear combination in terms of $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$. This means $I = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. Repeating the argument above and since there are finitely many a_i, we deduce that $I = (a_1)$.

□

Exercise 1.12

Proof. Let A be a local ring. Then it has a unique maximal ideal.

Now if e is a nonzero and nonidentity idempotent, we know $e^2 = e$. But this implies $e(e-1) = 0$. Therefore, e and $(e-1)$ are zero-divisors. Hence, they are not units. So, e and $e-1$ lie in the maximal ideal $\mathfrak{m}$. But because they both lie in the maximal ideal, that implies $1 = e - (e-1)$ lies in the maximal ideal. Contradiction. □

Exercise 1.13

The following is the construction of the algebraic closure due to Emil Artin.

Proof. Let K be field and Σ the set of all irreducible monic polynomials $f \in K[x]$.

Let A be the polynomial ring generated by indeterminate x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for $f \in \Sigma$.

We show $\mathfrak{a} \neq (1)$. By hypothesis, all of the polynomials in Σ are irreducible and monic. Therefore, if $\mathfrak{a} = (1)$, there is a finite sum $\sum_i a_i f_i(x_{f_i}) = 1$. If we rewrite the LHS has a polynomial $g(y)$ in terms of $y := x_{f_j}$ for some fixed j, we know that g is a unit in $K[x_f][y]$ where $K[x_f]$ $K[x_f]$ is the polynomial ring over all indeterminate except $y := x_{f_j}$. Exercise 1.2.i tells us that g has a unit for its constant term and all other coefficients are nilpotent. So, the leading coefficient must be a nilpotent. However, the $f_i(x_{f_i})$ has leading coefficient equal not equal to zero.[1] By $K[x_f]$ is an integral domain and therefore, the coefficient cannot be nilpotent unless it is zero. Contradiction.

Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{a}$. Consider $K_1 := A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K. Let $\underline{K} = \bigcup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits into linear factors completely. Let $\overline{K}$ be the set of elements of L that are algebraic over K. Therefore, $\overline{K}$ is the algebraic closure of K. □

Exercise 1.14

Proof. First off, Σ is nonempty since it contains the zero ideal and we order Σ by inclusion. Now let $(\mathfrak{p}_\alpha)_{\alpha \in A}$ be a chain in Σ. The union $\bigcup_{\alpha \in A} \mathfrak{p}_\alpha$ is an upper bound for the chain since the union of an ascending chain of ideals is still an ideal and it is in Σ since all of its elements are in some $\mathfrak{p}_\alpha$ and therefore still a zero divisor. By Zorn's Lemma, Σ has a maximal element.

[1] It isn't 1 and it isn't in the field. Recall what ring we are working in.

Let $\mathfrak{m} \in \Sigma$ be a maximal element. Suppose $ab \in \mathfrak{m}$ and $a \notin \mathfrak{m}$. Then the ideal $(\mathfrak{m}, a)$ generated by $\mathfrak{m}$ and a properly contains $\mathfrak{m}$. This means $(\mathfrak{m}, a)$ contains an element that is not nilpotent, say $mn + ac$ for $m \in \mathfrak{m}$ and $n, c \in A$. But then multiplying by b gives $b(mn + ac) = bmn + abc \in \mathfrak{m}$ which means b itself must have been nilpotent. Therefore, $b \in \mathfrak{m}$.

Suppose not and $ab \in \mathfrak{m}$ but $a, b \notin \mathfrak{m}$. This means $(\mathfrak{m}, a)$ and $(\mathfrak{m}, b)$ properly contain $\mathfrak{m}$ and therefore, there exists $mn + ax$ and $m'n' + bx'$ in $(\mathfrak{m}, a)$ and $(\mathfrak{m}, b)$ respectively that are not zero divisors. But then,

$$(mn + ax)(m'n' + bx') = mnm'n' + axm'n' + bx'mn + axbx'$$

and we can observe that each of the terms on the RHS are zero divisors so chose a nonzero s s.t. multiplying the RHS by s gives zero and so,

$$s(mn + ax)(m'n' + bx') = 0.$$

But this means one of $(mn + ax)$ or $(m'n' + bx')$ is a zero-divisor. Contradiction. □

Exercise 1.15 Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

ii) $V(0) = X, V(1) = \emptyset$.

iii) if $(E_i)_{i\in I}$ is any family of subsets of A, then

$$V\left(\bigcup_{i\in t} E_i\right) = \bigcap_{i\in I} V(E_i)$$

iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{ab}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals a, b of A. These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the Zariski topology. The topological space X is called the prime spectrum of A, and is written $\operatorname{Spec}(A)$.

Proof. Let A be a ring, X the set of all prime ideals, which we denote as $\operatorname{Spec} A$, let and $E \subseteq A$. We define the Zariski topology by defining closed sets to be sets of the form $V(E)$ for some $E \subseteq A$. We verify that we have a topology.

i) Let $\mathfrak{a} = (E)$ be the ideal generated by E. The first two inclusions in

$$V(E) \subseteq V(\mathfrak{a}) \subseteq V(r(\mathfrak{a})) \subseteq V(E) \tag{1}$$

are immediate. Indeed, if a $\mathfrak{p}$ prime ideal contains E, then $\mathfrak{p}$ contains $\mathfrak{a}$ since $\mathfrak{p}$ is closed under addition and multiplication by elements of A. Furthermore, if $\mathfrak{p}$ contains $\mathfrak{a}$, then $\mathfrak{p}$ is contains $r(\mathfrak{a})$ because $r(\mathfrak{a})$ is the intersection of all prime ideals containing $\mathfrak{a}$. To finish the proof, we show the last inclusion which implies equality for all the other inclusions. Clearly, $V(r(\mathfrak{a})) \subseteq V(E)$ because $E \subseteq r(\mathfrak{a})$ so any prime ideal containing $r(\mathfrak{a})$ contains E.

ii) We know $\mathfrak{p} \in V(0)$ iff $0 \in \mathfrak{p}$ and $\mathfrak{p}$ is prime iff $\mathfrak{p} \in X$ essentially from the fact that every prime ideal contains the zero element. We know $\mathfrak{p} \in V(1)$ iff $1 \in \mathfrak{p}$ and $\mathfrak{p}$ is a prime ideal iff $\mathfrak{p}$ is not a prime ideal because prime ideals cannot be the whole ring. But this implies $\mathfrak{p} \notin V(1)$ for any $\mathfrak{p} \in \operatorname{Spec}(A)$ and so, $V(1) = \emptyset$.

iii) Let $(E_i)_{i\in I}$ be a family of subsets of A. If $\mathfrak{p} \in V(\cup_{i\in I} E_i)$, then $\mathfrak{p}$ contains each E_i. Therefore, $\mathfrak{p} \in V(E_i)$ for each E_i and so, $\mathfrak{p} \in \cap_{i\in I} V(E_i)$. For the converse, suppose

$\mathfrak{p} \in \cap_{i\in I} V(E_i)$. Then, $\mathfrak{p} \in V(E_i)$ for $i \in I$. Therefore, $\cup_{i\in I} E_i \subseteq \mathfrak{p}$ and hence, $\mathfrak{p} \in V(\cup_{i\in I} E_i)$.

iv) We have $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$ iff $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$ and $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b}$ gives $\mathfrak{p} \in V(\mathfrak{ab})$. Hence, $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{ab})$.

We have $\mathfrak{p} \in V(\mathfrak{ab})$ iff $\mathfrak{ab} \subseteq \mathfrak{p}$ which gives $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{a} \subseteq \mathfrak{p}$ since $\mathfrak{p}$ is prime and so[2], $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. Indeed, if $\mathfrak{ab} \subseteq \mathfrak{p}$ but $\mathfrak{b} \not\subseteq \mathfrak{p}$, then choose $b \in \mathfrak{b} \setminus \mathfrak{p}$ and then $ab \in \mathfrak{ab} \subseteq \mathfrak{p}$ for all $a \in \mathfrak{a}$ and this means $a \in \mathfrak{p}$ and since a was arbitrary, $\mathfrak{a} \subseteq \mathfrak{p}$. Conversely, if $\mathfrak{ab} \subseteq \mathfrak{p}$ implies $\mathfrak{a} \subseteq \mathfrak{p}$ of $\mathfrak{b} \subseteq \mathfrak{p}$, then if $ab \in \mathfrak{p}$ we can consider ideals $(a)(b) \in \mathfrak{p}$ and this means WLOG that $(a) \subseteq \mathfrak{p}$ and therefore $a \in \mathfrak{p}$ so that $\mathfrak{p}$ is prime. This work shows $V(\mathfrak{ab}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.

Suppose $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$ then WLOG assume $\mathfrak{a} \subseteq \mathfrak{p}$ and since $\mathfrak{ab} \subseteq \mathfrak{a}$, we deduce that $\mathfrak{ab} \subseteq \mathfrak{p}$ and so $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$. This shows $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$.

To conclude, we have shown that

$$V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{ab}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$$

so these containments are all equalities of sets.

□

Exercise 1.16

Proof. Have fun with this! See Mumford's treasure maps for some great pictures. □

Exercise 1.17 For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

i) $X_f \cap X_g = X_{fg}$

ii) $X_f = \varnothing \Leftrightarrow f$ is nilpotent;

iii) $X_f = X \Leftrightarrow f$ is a unit;

iv) $X_f = X_g \Leftrightarrow r((f)) = r((g))$;

v) X is quasi-compact (that is, every open covering of X has a finite subcovering).

vi) More generally, each X_f is quasi-compact.

vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.

The sets X_f are called basic open sets of $X = \operatorname{Spec}(A)$

[To prove (v), remark that it is enough to consider a covering of X by basic open sets $X_{f_i} (i \in I)$. Show that the f_1 generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i\in J} g_i f_i \quad (g_i \in A)$$

where J is some finite subset of I. Then the $X_{f_i} (i \in J)$ cover X.]

Proof. Let $\mathcal{B}$ be the collection of sets X_f for $f \in A$. Then, obviously $\cup_{f\in A} X_F$ covers $\operatorname{Spec}(A)$ because X_f denotes the set of prime ideals not containing f and we let f vary. Now suppose U is a nonempty open set in the Zariski topology. Then, $U = V(\mathfrak{a})^c$ for some ideal $\mathfrak{a}$. Consider sets X_{f_i} for $f_i \in \mathfrak{a}$. We claim $\bigcup_{f_i\in A} X_{f_i} = U$. The containment $U \subseteq \bigcup_{f_i\in A} X_{f_i}$ is clear. Now, if $\mathfrak{p} \in \bigcup_{f_i\in A} X_{f_i}$, then $\mathfrak{p} \in X_{f_i}$ for some $f_i \in A$. Therefore, $f_i \notin \mathfrak{p}$. But this means $\mathfrak{a} \not\subseteq \mathfrak{p}$ and so, $\mathfrak{p} \in U$ which establishes $\bigcup_{f_i\in A} X_{f_i} \subseteq U$. Therefore, $U = \bigcup_{f_i\in A} X_{f_i}$.

[2] This is a standard fact about prime ideals. See Lemma 26.1 of Isaac's book for instance.

Since U was arbitrary, we have shown that every open set is a union of elements in $\mathcal{B}$ so $\mathcal{B}$ is a base[3].

i) We have $\mathfrak{p} \in X_f \cap X_g$ iff $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$ iff $fg \notin \mathfrak{p}$ iff $\mathfrak{p} \in X_{fg}$.

ii) We have $X_f = \emptyset$ iff $f \in \mathfrak{p}$ for any prime ideal $\mathfrak{p} \subseteq X$ iff f is in the nilradical of A iff f is nilpotent.

iii) We have $X_f = X$ iff $V(f) = \emptyset$ iff f lies no prime ideal iff f is a unit. This last "iff" can be argued as follows. Certainly, if f is a unit, then f cannot be contained in a prime ideal while if f lies no prime ideal, in particular any maximal ideal, then Corollary 1.5 of AM tells us non-units are always in some maximal ideal so f itself must be a unit.

iv) We have $X_f = X_g$ iff prime ideals not containing f also do not contain g iff prime ideals containing f also contain g and the prime ideals containing g also contain f iff the intersection of all prime ideals containing f also contains g and the intersection of all prime ideals containing g iff $r((f)) = r((g))$.

v) Suppose we are given an open covering $\mathcal{U}$ of X. Consider instead the covering $\{X_{f_i}\}$ in by writing each U as a union of the basic open sets. If we show that there is a finite subcover $\{X_{f_{i_j}}\}$, then we can choose finitely many U_j in which $X_{f_{i_j}} \subseteq U_j$ was used to write U_j as a union and then $\{U_j\}$ forms a finite subcover.

Consider a covering by open sets X_{f_i} for $i \in I$ i.e. $\bigcup_{i \in I} X_{f_i}$. But then,

$$X = \bigcup_{i \in I} X_{f_i} = \bigcup_{i \in I} X \setminus V(f_i) = X \setminus \left(\bigcap_{i \in I} V(f_i) \right) \qquad \Longrightarrow \qquad \bigcap_{i \in I} V(f_i) = \emptyset.$$

By Exercise 1.15, we know $\emptyset = \bigcap_{i \in I} V(f_i) = V\left(\bigcup_{i \in} \{f_i\}\right) = V((f_i)_{i \in I})$ (in particular, the first equality follows from (iii) and the second from (i) where $(f_i)_{i \in I}$ denotes the ideal generated by the f_i). From Exercise 1.15 (ii), we know $V((f_i)_{i \in I}) = V(1)$ so this means $(f_i)_{i \in I} = (1)$ i.e. the f_i generate the unit ideal. So, there is a finite linear combination $\sum_{i=1}^n g_i f_i = 1$ where $g_i \in A$ and we reindexed for convenience. But then, $(f_1, \ldots, f_n) = 1$ implies $\{X_{f_i}\}_{i=1}^n$ is a finite covering of X.

vi) Consider a covering of X_f by open sets U and by the same method of reduction as in v), assume X_f is covered by open sets of the form X_{g_i} $(i \in I)$. From i), we get $X_f \cap X_{g_i} = X_{fg_i}$ and X_f is covered by these X_{fg_i}. For this to cover all of X_f, we must have $(fg_i)_{i \in I}$ generating the ideal (f). Indeed

$$\begin{aligned} X_f = \bigcup_{i \in I} X_{gf_i} = X \setminus \left(\bigcap_{i \in I} V(gf_i) \right) \\ \Longrightarrow \quad V(f) = \bigcap_{i \in I} V(gf_i) = V(\{gf_i\}_{i \in I}) \quad \Longrightarrow \quad V((f)) = V((gf_i)_{i \in I}) \end{aligned}$$

where the first implication follows from set definitions and the last follows from Exercise 1.15(i) alongside with using Exercise 1.15(iii) to get the equalities in the middle. This means $f = \sum_{i \in J} h_i g f_i$ for some finite subset $J \subseteq I$ and $h_i \in A$ and therefore, X_f is covered by the $(X_{g_i})_{i \in J}$ which is the desired finite subcovering.

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[3]This is a standard result of point-set topology. The following citation is from Folland's "Real Analysis: Modern Techniques and Applications":
"4.2 Proposition. If $\mathcal{T}$ is a topology on X and $\mathcal{E} \subset \mathcal{T}$, then $\mathcal{E}$ is a base for $\mathcal{T}$ iff every nonempty $U \in \mathcal{T}$ is a union of members of $\mathcal{E}$."

A faster way[4] to intuit that X_f, is quasi-compact is to notice that $X_f \cong \operatorname{Spec}(A_f)$ where A_f is the localization of A at f. Then, A_f is a ring and X_f is quasi-compact by the preceding part.

vii) ($\implies$): A quasi-compact open subset U of X is a finite union of X_f since such sets form a basis for open sets in the Zariski topology. Indeed, by definition of a base, we can write $U = \bigcup_{f_i \in A, i \in I} X_{f_i}$ as a union of basis elements..By quasi-compactness, there is a subcover $U = \bigcup_{f_{i_j} \in A, 1 \leq j \leq m} X_{f_{i_j}}$.

($\impliedby$): Conversely, suppose $U := \bigcup_{1 \leq i \leq n} X_{g_i}$. Suppose U has a covering $\{V_i\}$. By the same reduction from v) and vi), replace the covering by a covering $\{X_{f_j}\}_{j \in J}$. Since $X_{f_j} \cap X_{g_i} = X_{f_j g_i}$, we can consider all of the possible intersections and that forms a new covering $\{X_{f_j g_i}\}$ of X. But the collections $\{X_{f_j g_i}\}_{j \in J}$ cover each X_{g_i} which we know is quasi-compact. So choose finite subcovers $\{X_{f_{j_l} g_i}\}_{l=1}^m$ for each X_{g_i}. But there are only finitely many X_{g_i} that cover X so $\bigcup_{1 \leq i \leq n} \{X_{f_{j_l}, g_i}\}_{l=1}^m$ is a finite subcover of $\{X_{f_j}\}_{j \in J}$ of X. $\square$

Exercise 1.18 For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \operatorname{Spec}(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that

i) the set $\{x\}$ is closed (we say that x is a "closed point") in $\operatorname{Spec}(A) \iff \mathfrak{p}_x$ is maximal;

ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;

iii) $y \in \overline{\{x\}} \Leftrightarrow \mathfrak{p}_x \subseteq \mathfrak{p}_y$

iv) X is a T_0 -space (this means that if x, y are distinct points of X, then either there is a neighborhood of x which does not contain y, or else there is a neighborhood of y which does not contain x).

Proof. **i)** The set $\{x\}$ is closed iff $\overline{\{x\}} = \{x\}$ iff the only prime ideal containing $\mathfrak{p}_x$ is $\mathfrak{p}_x$ iff $\mathfrak{p}_x$ is maximal.

ii) First, $\overline{\{x\}} \subseteq V(\mathfrak{p}_x)$ because $\overline{\{x\}}$ is the smallest closed set containing x and $x \in V(\mathfrak{p}_x)$ which is a closed set. Any closed set $V(E)$ s.t. $x \in V(E)$ must contain $\mathfrak{p}_x$ and therefore, $V(\mathfrak{p}_x) \subseteq V(E)$. Since the closure $\overline{\{x\}}$ is such a closed set, we must have $V(\mathfrak{p}_x) \subseteq \overline{\{x\}}$. These two containments show $\overline{x} = V(\mathfrak{p}_x)$.

iii) We have $y \in \overline{\{x\}}$ iff $\mathfrak{p}_y \in V(\mathfrak{p}_x)$ iff $\mathfrak{p}_x \subseteq \mathfrak{p}_y$. The first "iff" is i) while the second is by definition.

iv) Since x, y are distinct, one of $\mathfrak{p}_x$ or $\mathfrak{p}_y$ is not contained in the other and so $y \notin \overline{\{x\}}$ or $x \notin \overline{\{y\}}$. So we can consider, WLOG assuming that $y \notin \overline{\{x\}}$, the open set $X \setminus \overline{\{x\}}$ which contains y but not x. $\square$

Exercise 1.19 A topological space X is said to be irreducible if $X \neq \varnothing$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that $\operatorname{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Proof. **First Statement:** Suppose X is irreducible. Let U be an arbitrary non-empty open set. Then U intersects every other non-empty open set in X. A set is dense iff it

[4]This is essentially Exercise II.2.1 of Hartshorne's text on Algebraic Geometry. However, this would be overkill since that problem deals requires the structure sheaf and we are only interested in the topological space.

intersects all nonempty open sets so U is dense. The converse is obvious from the definition of dense.

Second Statement:

($\implies$) Suppose $\mathrm{Spec}(A)$ is irreducible. Let N denote the nilradical of A. To show N is prime, suppose $fg \in N$ for $f, g \in A$ and we show this implies $f \in N$ or $g \in N$. By Exercise 1.17i), $X_f \cap X_g = X_{fg}$ and so $X_{fg} = \emptyset$ because $fg \in N$ implies $fg \in \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Spec}(A)$. Since any pair of nonempty open sets must intersect and X_f, X_g are open, WLOG, $X_f = \emptyset$. So $f \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. But then $f \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p} = N$.

($\impliedby$): Suppose $N \subseteq A$ were a prime ideal. Let U be a nonempty open set of X. Then $U = X \setminus V(E)$ for some set E and from Exercise 1.15, assume E were an ideal $\mathfrak{a} = E$. Since U is nonempty, U must contain a prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$. But a prime ideal $\mathfrak{p}$ will necessarily contain the prime ideal N since $N \subseteq \mathfrak{p}$. This means[5] $N \in U$. From Exercise 1.18ii, we know $\overline{\{N\}} = V(N) = \mathrm{Spec}\, A$ and therefore, $\overline{\{N\}} \subseteq \overline{U}$ implies $\overline{U} = \mathrm{Spec}\, A$. □

Exercise 1.20 Let X be a topological space.

i) If Y is an irreducible (Exercise 19) subspace of X, then the closure $\bar{Y}$ of Y in X is irreducible.

ii) Every irreducible subspace of X is contained in a maximal irreducible subspace.

iii) The maximal irreducible subspaces of X are closed and cover X. They are called the irreducible components of X. What are the irreducible components of a Hausdorff space?

iv) If A is a ring and $X = \mathrm{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A (Exercise 8).

Proof. **i)** Suppose not and there is a pair of open subsets $U, V \subseteq \overline{Y}$ s.t. $U \cap V = \emptyset$. Then $U \cap Y$ and $V \cap Y$ are open subsets (with the assumed topology on Y)[6] of Y and by irreducibility of Y, we get $(U \cap Y) \cap (V \cap Y) \neq \emptyset$. But $(U \cap Y) \cap (V \cap Y) \subseteq U \cap V = \emptyset$ leads to a contradiction.

ii) Let W be some irreducible subspace of X. Let $\mathfrak{S} := \{Z \subseteq X : Z \text{ irreducible subspace}, W \subseteq Z\}$ and order it by inclusion i.e. $Z \leq Z'$ iff $Z \subseteq Z'$. Clearly, $\mathfrak{S} \neq \emptyset$ since it contains W. Let $\{Z_i\}_{i \in I}$ be a chain in $\mathfrak{S}$. We show $\bigcup_{i \in I} Z_i$ is also an irreducible subspace that contains W. Certainly, $W \subseteq Z_i \subseteq \bigcup_{i \in I} Z_i \subseteq X$ so we check irreducibility. Suppose $U, V \subseteq \bigcup_{i \in I} Z_i$ were nonempty open subsets. Then $U \cap Z_i \neq \emptyset$ and $V \cap Z_j \neq \emptyset$ for some i, j and there exists a $Z_l \supseteq Z_i$ and $Z_l \supseteq Z_i$ which means $(U \cap Z_l) \cap (V \cap Z_l) \neq \emptyset$ by irreducibility of Z_l. Therefore, $U \cap V \neq \emptyset$. So all pairs of nonempty open subsets of $\bigcup_{i \in I} Z_i$ have nonempty intersection and hence, the space is irreducible. Invoking Zorn's Lemma, $\mathfrak{S}$ has a maximal element Z' and therefore, Z' is a maximal irreducible subspace of X.

iii) Closedness: Let $Z \subseteq X$ be a maximal irreducible subspace. From i), we know $\overline{Z}$ is also irreducible and since $Z \subseteq \overline{Z}$, maximality of Z implies $Z = \overline{Z}$. Hence, Z is closed.

Covering of X: It suffices to show every point $x \in X$ is contained in some maximal irreducible subspace. The one point set $\{x\}$ is certainly irreducible because its open subsets, depending on the topology of X, can only ever be $\emptyset$ or $\{x\}$. We know every irreducible subspace is contained in some maximal subspace and so, $\{x\} \subseteq M$ for M an irreducible component of X since x was arbitrary, the irreducible components cover X.

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[5]The fact that N is a prime ideal allows us to get this. In general, N may not be an element of $\mathrm{Spec}(A)$.

[6]This is where we used the fact that we are working with $\overline{Y}$. If we were working with an arbitrary set containing Y, we would have $U \cap Y$ and $V \cap Y$ open in the subspace topology which does not necessarily agree with the topology of Y.

Irreducible components of T_2-space: Let X be a Hausdorff space. We show the irreducible components are precisely the one point sets. Clearly every one point set is irreducible from the remarks in the previous part. Now we establish that every irreducible component is a one point set.

Suppose not and an irreducible component Z had more than one element (it cannot be empty by definition). Then choose distinct points $x, y \in Z$ and by X Hausdorff, choose disjoint open sets $U \ni x$ and $V \ni y$. Then $U \cap Z$ and $V \cap Z$ are open subsets of Z which are nonempty, but disjoint. Contradiction.

iv) Let $\mathfrak{p}$ be a minimal prime ideal and we show $V(\mathfrak{p})$ is an irreducible component of $\operatorname{Spec}(A)$. We establish that $V(\mathfrak{p})$ is irreducible first. We know [7] $V(\mathfrak{p}) \cong \operatorname{Spec}(A/\mathfrak{p})$ and since $\operatorname{Spec}(A/\mathfrak{p})$ corresponds to prime ideals of A that contain $\mathfrak{p}$, the nilradical is precisely $\mathfrak{p}$ or just the zero ideal. This means the nilradical of $A/\mathfrak{p}$ is prime and hence, $\operatorname{Spec}(A/\mathfrak{p})$ is irreducible by Exercise 1.19.

Now, we show $V(\mathfrak{p})$ is a maximal irreducible subspace. Suppose not and $V(\mathfrak{p}) \subsetneq V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ and $V(\mathfrak{a})$ is irreducible. This means every prime ideal that contains $\mathfrak{p}$ also contains $\mathfrak{a}$. In particular, $\mathfrak{a} \subseteq \mathfrak{p}$. To derive a contradiction, we show $\mathfrak{a} = \mathfrak{p}$. We know $V(\mathfrak{a}) \cong \operatorname{Spec}(A/\mathfrak{a})$ and since $V(\mathfrak{a})$ is irreducible, Exercise 1.19 implies the nilradical of $A/\mathfrak{a}$, which is corresponds to $\mathfrak{a}$, must be prime. Therefore, $\mathfrak{a}$ must be a prime ideal. Since $\mathfrak{p}$ is minimal, $\mathfrak{a} = \mathfrak{p}$.

Now we show that if Z is an irreducible component of $\operatorname{Spec}(A)$, then $Z = V(\mathfrak{p})$ for some minimal prime ideal $\mathfrak{p}$. Since Z is an irreducible component, Z is closed and there is an ideal $\mathfrak{p}$ s.t. $Z = V(\mathfrak{p})$. Since Z is irreducible, $V(\mathfrak{p}) \cong \operatorname{Spec}(A/\mathfrak{p})$ implies $\mathfrak{p}$ is a prime ideal. Now suppose $\mathfrak{p}$ were not a minimal prime. There exists a proper prime ideal $\mathfrak{q} \subsetneq \mathfrak{p}$ and this means $V(\mathfrak{p}) \subsetneq V(\mathfrak{q}) \cong \operatorname{Spec}(A/\mathfrak{q})$. But $V(\mathfrak{q})$ is irreducible by Exercise 1.19 and this contradicts $V(\mathfrak{p})$ being an irreducible component. So, $\mathfrak{p}$ was a minimal prime ideal to begin with. $\square$

Exercise 1.21 Let $\phi : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. If $q \in Y$, then $\phi^{-1}(q)$ is a prime ideal of A, i.e., a point of X. Hence ϕ induces a mapping $\phi^* : Y \to X$. Show that

i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$ and hence that ϕ^* is continuous.

ii) If α is an ideal of A, then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^c)$.

iii) If $\mathfrak{b}$ is an ideal of B, then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\operatorname{Ker}(\phi))$ of X. (In particular, $\operatorname{Spec}(A)$ and $\operatorname{Spec}(A/\mathfrak{N})$ (where $\mathfrak{N}$ is the nilradical of A) are naturally homeomorphic.)

v) If ϕ is injective, then $\phi^*(Y)$ is dense in X. More precisely, $\phi^*(Y)$ is dense in $X \Leftrightarrow \operatorname{Ker}(\phi) \subseteq \mathfrak{N}$.

vi) Let $\psi : B \to C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

[7] It is more efficient to do this exercise using $V(\mathfrak{p}) \cong \operatorname{Spec}(A/\mathfrak{p})$. Consider the ring homomorphism $\phi : A \to A/\mathfrak{p}$. Then $\phi^* : \operatorname{Spec}(A/\mathfrak{p}) \to \operatorname{Spec}(A)$, using notation from Exercise 1.21, is a continuous map. By the correspondence theorem for rings, ϕ^* is a bijective continuous map from $\operatorname{Spec}(A/\mathfrak{p})$ to $V(\mathfrak{p})$. On the other hand, $\phi^{*-1} : V(\mathfrak{p}) \to \operatorname{Spec}(A/\mathfrak{p})$ defines a map $\mathfrak{q} \mapsto \phi(\mathfrak{q})$ by sending $\mathfrak{q}$ to $\mathfrak{q}/\mathfrak{p}$ which is still a prime ideal by the correspondence theorem. The map ϕ^{*-1} is continuous because the preimage ϕ^* maps $(A/\mathfrak{p})_f$ to $V(\mathfrak{p}) \cap X_f$ where $X = \operatorname{Spec}(A)$. This gives the homeomorphism $V(\mathfrak{p}) \cong \operatorname{Spec}(A/\mathfrak{p})$ so any topological facts about $V(\mathfrak{p})$ are given by $\operatorname{Spec}(A/\mathfrak{p})$ and vice-versa.
I could not find or recall the reference in Hartshorne's "Algebraic Geometry". In case I want to read a more detailed proof, see here.

vii). Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \to B$ by $\phi(x) = (\bar{x}, x)$, where $\bar{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ^* is bijective but not a homeomorphism.

PROOF. [8] **i)** Let $\mathfrak{p} \in \phi^{*-1}(X_f)$. That means $\mathfrak{p} \in Y$ s.t. $\phi^*(\mathfrak{p}) \in X_f$ and therefore, $\phi^{-1}(\mathfrak{p})$ does not contain f. Then, $\phi(f) \notin \mathfrak{p}$ for if not, $\phi(f) \in \mathfrak{p}$ implies $\phi^{-1}(\mathfrak{p}) \supseteq \phi^{-1}(\phi(f)) \ni f$ which is absurd. This means $\mathfrak{p} \in Y_{\phi(f)}$.

Conversely, suppose $\mathfrak{p} \in Y_{\phi(f)}$. That means $\phi(f) \notin \mathfrak{p}$. Then $\phi^{-1}(\mathfrak{p}) \in X_f$ because if not, $\phi^{-1}(\mathfrak{p}) \supseteq \phi^{-1}(\phi(f)) \ni f$ which is absurd. But then $\phi^*(\mathfrak{p}) \in X_f$ which means $\mathfrak{p} \in \phi^{*-1}(X_f)$.

So, the preimage ϕ^{*-1} of any basis element is also a basis element and this implies the preimage of any open subset is also open. Hence, ϕ^* is continuous.

ii) One has $\mathfrak{p} \in \phi^{*-1}(V(\mathfrak{a}))$ iff $\phi^*(\mathfrak{p}) \in V(\mathfrak{a})$ iff $\phi^{-1}(\mathfrak{p}) \in V(\mathfrak{a})$ iff $\mathfrak{a} \subseteq \phi^{-1}(\mathfrak{p})$ iff $\phi(\mathfrak{a}) \subseteq \phi(\phi^{-1}(\mathfrak{p})) \subseteq \mathfrak{p}$ iff $\mathfrak{a}^e \subseteq \mathfrak{p}$ iff [9] $\mathfrak{p} \in V(\mathfrak{a}^e)$.

iii) One also has $\mathfrak{p} \in \overline{\phi^*(V(\mathfrak{b}))}$ iff $\mathfrak{p} \in V(\phi^*(V(\mathfrak{b}))$ iff $\phi^*(V(\mathfrak{b})) \subseteq \mathfrak{p}$ iff $V(\mathfrak{b}^c) \subseteq \mathfrak{p}$ where the last "iff" follows from the equivalence: $\mathfrak{q} \in \phi^*(V(\mathfrak{b}))$ iff $\mathfrak{q} = \phi^{-1}(\mathfrak{r})$ with $\mathfrak{b} \subseteq \mathfrak{r} \in X$ iff $\phi^{-1}(\mathfrak{b}) \subseteq \mathfrak{q}$ iff $\mathfrak{b}^c \subseteq \mathfrak{q}$ iff $\mathfrak{q} \in V(\mathfrak{b}^c)$.

iv) If ϕ is surjective, we know $A/\ker(\phi) \cong B$ and therefore, every prime ideal of B is in one-to-one correspondence to a prime ideal of A that contains $\ker(\phi)$. That is, $\phi^{-1}(\mathfrak{q}) \supseteq \ker(\phi)$ for every $\mathfrak{q} \in \mathrm{Spec}(B)$. Therefore, $\phi^* : \mathrm{Spec}(B) \to V(\ker(\phi))$ is a bijection. Next, i) established ϕ^* is a continuous map onto its image. All that remains is to show $(\phi^*)^{-1}$ is also continuous.

The map $(\phi^*)^{-1} : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is defined by $\mathfrak{p} \mapsto \mathfrak{q}$ where $\mathfrak{q} = \mathfrak{p}/\ker(\phi)$ is the corresponding prime ideal of $A/\ker(\phi)$. This map is continuous because for all closed subset of $\mathrm{Spec}(B)$, identifying the closed subset with $V(E/\ker\phi)$ for some E and $\mathrm{Spec}(B)$ with $\mathrm{Spec}(A/\ker\phi)$, the preimage is just the set $V(E)$ which is also a closed subset of $\mathrm{Spec}(A)$.

Now consider the ring homomorphism $\phi : A \to A/\mathfrak{N}$ which is surjective and the kernel is just $\mathfrak{N}$. From the first part, since ϕ is surjective, ϕ^* is a homeomorphism of $\mathrm{Spec}(A/\mathfrak{N})$ and $V(\ker(\phi)) = V(\mathfrak{N}) = \mathrm{Spec}(A)$ since every prime ideal contains the nilradical.

v) First Statement: To show $\phi^*(Y)$ is dense in X, we show $V(\phi^*(Y)) = X$. We know from iii) that $\overline{\phi^*(Y)} = \overline{\phi^*(V(\mathfrak{B}))} = V(\mathfrak{B}^c)$ where $\mathfrak{B}$ is the nilradical of B. The contraction of the nilradical of B contains the kernel, since $0 \in \mathfrak{B}$, so that means $\ker(\phi) \in \mathfrak{B}^c$. But we assumed $\ker(\phi) = 0$ so that means $V(\mathfrak{B}^c) = X$ since prime ideal contains 0. Hence, $\overline{\phi^*(Y)} = X$.

Second Statement: ($\Longrightarrow$): Suppose $\phi^*(Y)$ is dense in X. So, $V(\phi^*(\mathrm{Spec}(B))) = \mathrm{Spec}(A)$ and therefore, every $\mathfrak{p} \in \mathrm{Spec}(A)$ has $\phi^*(\mathrm{Spec}(B)) \subseteq \mathfrak{p}$. That means $\phi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$ for all $\mathfrak{q} \in \mathrm{Spec}(B)$ and every $\phi^{-1}(\mathfrak{q})$ contains the kernel of ϕ so that means $\ker(\phi) \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ was arbitrary, this holds for all $\mathfrak{p} \in \mathrm{Spec}(A)$. So we get $\ker(\phi) \subseteq \mathfrak{N} =: \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}$.

($\Longleftarrow$): To show density, we show $V(\phi^*(Y)) = X$ and only the containment $V(\phi^*(Y)) \supseteq X$ is nontrivial. We know $\overline{\phi^*(Y)} = \overline{\phi^*(V(\mathfrak{B}))} = V(\mathfrak{B}^c)$ where $\mathfrak{B}$ is the nilradical of B and we used iii). But $\mathfrak{B} = \sqrt{(0)}$, i.e. the nilradical is the radical of the zero ideal, and Exercise 1.18 [10] of AM tells us that $\mathfrak{B}^c = \sqrt{(0)^c} = \sqrt{\ker(\phi)}$. Then $\ker(\phi) = \mathfrak{N}$ because $\sqrt{\ker(\phi)}$ is

[8]We will freely use the fact that the Zariski closure of a set $E \subseteq \mathrm{Spec}(A)$ is just $V(E)$. This is just a direct generalization of what we did in Exercise 1.18ii) since nothing we did there actually required us to have a single point $x \in \mathrm{Spec}(A)$.

[9]The fact that $\phi(\mathfrak{a}) \subseteq \mathfrak{p}$ iff $\mathfrak{a}^e \subseteq \mathfrak{p}$ is immediate because $\mathfrak{a}^e$ is the ideal generated by $\phi(\mathfrak{a})$ and an ideal contains the generating set of another ideal iff it contains the ideal itself.

[10]In particular, we use the fact that $\sqrt{\mathfrak{b}}^c = \sqrt{(\mathfrak{b}^c)}$ which is on the RHS of the last line of the equations shown. Essentially, contraction commutes with taking radicals.

the intersection of all prime ideals containing $\ker(\phi)$, $\ker(\phi) \subseteq \mathfrak{N}$, and $\mathfrak{N} = \bigcap_{\mathfrak{p}\in X} \mathfrak{p}$. This shows $\sqrt{\ker(\phi)} = \mathfrak{N}$ and so, $V(\mathfrak{B}^c) = V(\sqrt{\ker(\phi)}) = V(\mathfrak{N}) = X$ as desired.

vi) To show that the two maps are equivalent, we check that they map elements of $\mathrm{Spec}(C)$ to the same elements of X. For $\mathfrak{p} \in \mathrm{Spec}(C)$,

$$(\psi \circ \phi)^*(\mathfrak{p}) = (\psi \circ \phi)^{-1}(\mathfrak{p}) = \psi^{-1}(\phi^{-1}(\mathfrak{p})) = \psi^*(\phi^{-1}(\mathfrak{p})) = \psi^* \circ \phi^*(\mathfrak{p}).$$

vii) By definition $\phi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. The prime ideals of B are precisely the ideals $((0,1))$ and $((1,0))$ i.e. the ideals generated by elements $(0,1)$ and $(1,0)$. Note that the the zero ideal is not prime in B since $(0,1)\cdot(1,0) = (0,0)$ but neither are in the the ideal.

Now we check what ϕ^{-1} does to the two ideals. We see[11] that $\phi^{-1}(((0,1)))$ consists of elements in A that lie in $\mathfrak{p}$ i.e. $\phi^*((0,1)) = \phi^{-1}(((0,1))) = \mathfrak{p}$. On the other hand, $\phi^{-1}(((1,0)))$ consists of elements that are zero but do not lie in $\mathfrak{p}$ if they are nonzero. Therefore, $\phi^*((1,0)) = \phi^{-1}(((1,0))) = 0$ is the zero ideal. Since $\mathrm{Spec}(B)$ and $\mathrm{Spec}(A)$ are both two element sets, our work shows that ϕ^* gives a bijection between $\mathrm{Spec}(B)$ and $\mathrm{Spec}(A)$.

To show that ϕ^* is not a homeomorphism, we exhibit an open subset of $\mathrm{Spec}(B)$ that does not map to an open subset of $\mathrm{Spec}(A)$ i.e. ϕ^* is not an open map. Consider $\mathrm{Spec}(B)_{(0,1)} := \{\text{prime ideals of } B \text{ not containing } (1,0)\}$ and the prime ideal $((0,1))$ is the only element of the set. The set $\mathrm{Spec}(B)_{(0,1)}$ is a basic open subset of $\mathrm{Spec}(B)$ so it is open. Then $\phi^*(\mathrm{Spec}(B)_{(0,1)}) = \phi^*(\{(0,1)\}) = \{\mathfrak{p}\}$. However, $\{\mathfrak{p}\}$ is not an open subset of $\mathrm{Spec}(A)$. □

Exercise 1.22

Proof. See Exercise II.2.19 of [3] and your solution of it. □

Exercise 1.23

Proof. □

Exercise 1.24
Exercise 1.25
Exercise 1.26
Exercise 1.27
Exercise 1.28

2. Modules

Exercise 2.1

Proof. It suffices to show that every element on the LHS is zero. Since m, n are coprime, choose x, y s.t. $xm + yn = 1$. Let $\overline{a} \otimes \overline{b} \in \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. Then,

$$\overline{a} \otimes \overline{b} = ((xm + yn)\overline{a} \otimes \overline{b}) = (xm\overline{a} + yn\overline{a} \otimes b) = (xm\overline{a} \otimes \overline{b}) + (yn\overline{a} \otimes \overline{b}) = (0 \otimes \overline{b}) + (\overline{a} \otimes yn\overline{b}) = 0.$$

□

Exercise 2.2

[11] In case it is not clear, this is an overburdening of that notation. We write (E) to mean the ideal generated by E. But we write $(0,1)$ for the element of the ring so $((0,1))$ is the monstrosity that arises.

PROOF. Consider the exact sequence of A-modules
$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0.$$
Tensoring by M gives an exact sequence
$$\mathfrak{a} \otimes M \to A \otimes M \to A/\mathfrak{a} \otimes M \to 0.$$
But $A \otimes M \cong M$ and $\mathfrak{a} \otimes M \cong \mathfrak{a}M$. Hence, $A/\mathfrak{a} \otimes M \cong M/\mathfrak{a}M$ by exactness of the sequence. □

Exercise 2.3

PROOF. Follow hint – straightforward once you know you have vector spaces. □

Exercise 2.4

PROOF. This is immediate if one considers how tensor products distribute over direct sums. □

Exercise 2.5

PROOF. It is clear that $A[x]$ is an A-algebra. Furthermore, we have an isomorphism of A-algebras:
$$A[x] \cong A \oplus Ax \oplus Ax^2 \oplus Ax^3 \oplus \dots$$
and Exercise 2.4 shows that $A[x]$ is flat. □

Exercise 2.6

PROOF. This is a consequence of an exercise you did for Math 200B. The idea is to define the isomorphism explicitly and use the universal property of tensor products. □

Exercise 2.7

PROOF. In general, if $\mathfrak{a} \subseteq A$ is an ideal,
$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0 \qquad \overrightarrow{\text{tensor } A[x] \otimes_A -} \qquad \mathfrak{a}[x] \to A[x] \to A/\mathfrak{a}[x] \to 0$$
which shows $A[x]/\mathfrak{a}[x] \cong A/\mathfrak{a}[x]$ for any ideal $\mathfrak{a}$. Since $A/\mathfrak{p}$ is an integral domain, $A/\mathfrak{p}[x]$ is an integral domain. Hence, $\mathfrak{p}[x]$ is a prime ideal of $A[x]$.

The statement for $\mathfrak{m} \subseteq A$ a maximal ideal is false. Consider $A = \mathbb{R}$ and $\mathfrak{m} = (0)$. Then $A[x]/\mathfrak{m}[x] \cong A/\mathfrak{m}[x] \cong A[x]$ which is certainly not a field. So, $\mathfrak{m}[x]$ is not maximal in $A[x]$. □

Exercise 2.8

PROOF.

(i) Let $L \hookrightarrow L'$ be an injective A-mdoule homomorphism. The $(M \otimes_A N) \otimes L \hookrightarrow (M \otimes_A N) \otimes L'$ because $(M \otimes_A N) \otimes L \cong M \otimes_A (N \otimes_L)$ and $N \otimes L \hookrightarrow N \otimes L'$ is an injective A-module homomorphims by flatness of N.

(ii) If $L \hookrightarrow L'$ is an injective A-module homomorphism, then
$$B \otimes_A L \hookrightarrow B \otimes_A L'$$
is an injective homomorphism of B-modules (and A-modules). Then, $N \otimes_B B \otimes_A L \hookrightarrow N \otimes_B B \otimes_A L'$ is an injective homomorphism of B-modules (and A-modules). By of Proposition 2.14, $N \otimes_A L \hookrightarrow N \otimes_A L'$ is an injective homomorphism of A-modules. This means N is a flat A-module.

□

Exercise 2.9

Proof. This is a standard exercise and is omitted intentionally. Just view M' as a submodule of M and $M'' \cong M/M'$. Since they are finitely generated, M has to be finitely generated. $\square$

Exercise 2.10

Proof. Let $\overline{u}$ denote the homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$. We know $\overline{u}(M/\mathfrak{a}M) = N/\mathfrak{a}N = N + \mathfrak{a}N$. But by definition, $\overline{u}(M/\mathfrak{a}M) = u(M) + \mathfrak{a}N$. Therefore, $u(M) + \mathfrak{a}N = N + \mathfrak{a}N$ which means $u(M) + \mathfrak{a}N = N$. Since N is a f.g. A-module and $u(M) \subseteq N$ is a submodule, Corollary 2.7 implies $u(M) = N$. Thus, $u : M \to N$ is surjective. $\square$

Exercise 2.11 Let A be a ring $\neq 0$. Show that $A^m \cong A^n \Rightarrow m = n$ [Let $\mathfrak{m}$ be a maximal ideal of A and let $\phi : A^m \to A^n$ be an isomorphism. Then $1 \otimes \phi : (A/\mathfrak{m}) \otimes A^m \to (A/\mathfrak{m}) \otimes A^n$ is an isomorphism between vector spaces of dimensions m and n over the field $k = A/m$. Hence $m = n$.] (Cf. Chapter 3, Exercise 15.)

If $\phi : A^m \to A^n$ is surjective, then $m \geq n$.

If $\phi : A^m \to A^n$ is injective, is it always the case that $m \leq n$?

Proof. **First statement:** Let $\mathfrak{m} \subseteq A$ and $\phi : A^m \to A^n$ be a maximal ideal of A and an isomorphism respectively. The existence of $\mathfrak{m}$ by choosing a maximal ideal containing the zero ideal. The map $1 \otimes \phi : (A/\mathfrak{m}) \otimes A^m \to (A/\mathfrak{m}) \otimes A^n$ is an isomorphism of vector spaces because $(A/\mathfrak{m}) \otimes -$ is just an extension of scalars so that $(A/\mathfrak{m}) \otimes A^m$ and $A/\mathfrak{m} \otimes A^n$ are $A/\mathfrak{m}$-vector spaces, ϕ is surjective and right exactness of the tensor product means $1 \otimes \phi$ is surjective, and Proposition 2.19iv) of AM shows that $1 \otimes \phi$ is injective since ϕ is injective and A^m, A^n are f.g. A-modules. It is a standard fact of linear algebra that an isomorphism of finite dimensional vector spaces implies the dimension of the vector spaces are equal. We know that $(A/\mathfrak{m}) \otimes A^m \cong (A/\mathfrak{m})^m$ and $(A/\mathfrak{m}) \otimes A^n \cong (A/\mathfrak{m})^n$ by commutativity of tensor products with direct sums so we deduce that $m = n$.
Second statement: Choose a maximal ideal $\mathfrak{m} \subseteq A$ and tensoring by $k := A/\mathfrak{m}$ gives a surjective linear transformation $1 \otimes \phi : k^m \to k^n$ because the tensor product is right exact. By the Rank-Nullity Theorem, we deduce that $m \geq n$.
Third statement:[12] Suppose not and $m < n$. By choosing a basis and reordering elements $e_i := (0, \ldots, 1, \ldots, 0)$ (1 is in the ith coordinate) of A^m, we may assume $\phi(e_j) = 0$ for all $j > m$. By Proposition 2.4, since we can view $A^m \subseteq A^n$ as a submodule and taking $\mathfrak{a} = (1)$, we know ϕ satisfies an equation of form

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0$$

where $a_i \in \mathfrak{a}$. Assume n is of minimal degree and by injectivity, we must have $a_n \neq 0$ (otherwise factor out ϕ which is nonzero unless mapping elements of its kernel). Then consider the element $e_m = (0, 0, \ldots, 1) \in A^m$ and observe that

$$(\phi^n + a_1\phi^{n-1} + \cdots + a_n)(e_m) = a_n \neq 0$$

which is a contradiction. $\square$

Exercise 2.12

[12]This is proof is inspired from a post here which I saw when I first did the problem. In particular, the proof via the Cayley-Hamilton Theorem which is quite beautiful.

Proof. The hint is essentially telling us to use the fact that a surjective module homomorphism onto a free module splits.

Since A^n is a free A-module of finite rank and $\phi : M \to A^n$ is a surjection, the morphism splits and there is an isomorphism $M \cong A^n \oplus \ker(\phi)$. Since M is finitely generated, this means $\ker(\phi)$ is finitely generated. □

Exercise 2.13

Proof. The fact that $g : N \hookrightarrow N_B$ is immediate since $1 \otimes y = 0$ iff $y = 0$. For the second statement, define $p : N_B \to N$ by $(b \otimes y) = by$. This is certainly a surjective map and $p \circ g = \mathrm{id}_N$ since $n \mapsto 1 \otimes n \mapsto n$ which means this is a split morphism. So, $N_B \cong N \oplus \ker p$ but since $g : N \hookrightarrow N_B$ was injective, $N \cong \operatorname{im} g$ and so, $N_B \cong \operatorname{im} g \oplus \ker p$. □

Exercise 2.14 "A *partially ordered* set I is said to be a directed set if for each pair i, j in I there exists $k \in I$ such that $i \leqslant k$ and $j \leqslant k$

Let A be a ring, let I be a directed set and let $(M_i)_{i \in I}$ be a family of A -modules indexed by I. For each pair i, j in I such that $i \leqslant j$, let $\mu_{ij} : M_i \to M_{9j}$ be an A-homomorphism, and suppose that the following axioms are satisfied:

(1) μ_{ii} is the identity mapping of M_i, for all $i \in I$;

(2) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$. Then the modules M_i and homomorphisms μ_{1j} are said to form a direct system $\mathbf{M} = (M_i, \mu_{ij})$ over the directed set $\boldsymbol{I}$.

We shall construct an A-module M called the direct limit of the direct system $\mathbf{M}$. Let C be the direct sum of the M_i, and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_1 - \mu_{ts}(x_i)$ where $i \leqslant j$ and $x_1 \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i

The module M, or more correctly the pair consisting of M and the family of homomorphisms $\mu_t : M_6 \to M$, is called the direct limit of the direct system M, and is written $\mathrm{lm} M_{\mathrm{i}}$. From the construction it is clear that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$."

Proof. There is nothing to prove besides checking that $\mu_i = \mu_j \circ \mu_{ij}$ when $i \leq j$.

Let $x_i \in M_i$ and $\mu_i(x_i) \in M$ be the image. Then, $\mu_j \circ \mu_{ij}(x_i) = \mu_j(x_i)$ and we have $\mu_j(x_i) = \mu_i(x_i)$ because μ_i and μ_j are just the restrictions of μ to M_i and M_j. □

Exercise 2.15

Proof. □

Exercise 2.17

Exercise 2.19 A sequence of direct systems and homomorphisms

$$\mathbf{M} \to \mathbf{N} \to \mathbf{P}$$

is *exact* if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$. Show that the sequence $M \to N \to P$ of direct limits is then exact. [Use Exercise 15.]

Proof. Suppose $\mathbf{M} := (M_i, \mu_{ij})$, $\mathbf{N} := (N_i, \nu_{ij})$, and $\mathbf{P} := (P_i, \gamma_{ij})$ are the direct systems. Let $\Phi : \mathbf{M} \to \mathbf{N}$ and $\Psi : \mathbf{N} \to \mathbf{P}$ be the homomorphisms of direct systems given as stated in Exercise 2.18[13]. For each $i \in I$, there is an exact sequence $M_i \overset{\phi_i}{\to} N_i \overset{\psi_i}{\to} P_i$. To show exactness of $M \overset{\phi}{\to} N \overset{\psi}{\to} P$, use Exercise 2.15[14].

First, show $\operatorname{im}(\phi) \subseteq \ker(\psi)$. Let $\phi(\mu_i(x_i)) \in \operatorname{im}(\phi)$ using Exercise 2.15 to represent the element of M as $\mu_i(x_i)$. Then, $\phi(\mu_i(x_i)) = \nu_i(\phi_i(x_i))$ by Exercise 2.18. Since $\phi_i(x_i) \in \ker(\psi_i)$ and $\psi_i \circ \phi_i = 0$,

$$\psi(\nu_i(\phi_i(x_i)) = \gamma_i(\psi_i(\phi_i(x_i))) = \gamma_i(0) = 0$$

where the first equality is from Exercise 2.18 and so $\nu_i(\phi_i(x_i)) \in \ker(\psi)$.

Now, show $\ker(\psi) \subseteq \operatorname{im}(\phi)$. Suppose we had $\nu_i(y_i) \in \ker(\psi)$ where $y_i \in N_i$ (using Exercise 2.15). Then by Exercise 2.18,

$$0 = \psi(\nu_i(y_i)) = \gamma_i(\psi_i(y_i)) \in P$$

and since $\psi_i(y_i) \in P_i$, Exercise 2.15 tells us that there exists a $j \geq i$ s.t. $\gamma_{ij}(\psi_i(y_i)) = 0$ and Exercise 2.14 then tells us that $\gamma_{ij}(\psi_i(y_i)) = \psi_i(y_i)$. Hence, $\psi_i(y_i) = 0$. By exactness, since this tells us $y_i \in \ker(\psi_i)$, we can find an $x_i \in \operatorname{im}(\phi_i)$ s.t. $\phi_i(x_i) = y_i$. So, in tandem with Exercise 2.18

$$\phi(\mu_i(x_i)) = \nu_i(\phi_i(x_i)) = \nu_i(y_i)$$

which shows that $\nu_i(y_i) \in \operatorname{im}(\phi)$.

Having shown the two containments, we deduce that $\operatorname{im}(\phi) = \ker(\psi)$ which means the sequence of direct limits was exact. $\square$

Exercise 2.20 Keeping the same notation as in Exercise 14, let N be any A-module. Then $(M_i \otimes N, \mu_{ij} \otimes 1)$ is a direct system; let $P = \varinjlim (M_i \otimes N)$ be its direct limit. For each $i \in I$ we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$, hence by Exercise 16 a homomorphism $\psi : P \to M \otimes N$. Show that ψ is an isomorphism, so that

$$\varinjlim (M_i \otimes N) \cong \left(\varinjlim M_i\right) \otimes N.$$

[For each $i \in I$, let $g_i : M_i \times N \to M_i \otimes N$ be the canonical bilinear mapping. Passing to the limit we obtain a mapping $g : M \times N \to P$. Show that g is A-bilinear and hence define a homomorphism $\phi : M \otimes N \to P$. Verify that $\phi \circ \psi$ and $\psi \circ \phi$ are identity mappings.]

Proof. First off, let us check that $(M_i \otimes N, \mu_{ij} \otimes 1)$ is a direct system. Clearly, the indexing set is a directed set because it is obtained from the direct system (M_i, μ_{ij}). Furthermore,

(1) $\mu_{ii} \otimes 1 : M_i \otimes N \to M_i \otimes N$ is the identity map since μ_{ii} is the identity map for all $i \in I$;
(2) $\mu_{ik} \otimes 1 = (\mu_{jk} \circ \mu_{ij} \otimes 1) = (\mu_{jk} \otimes 1) \circ (\mu_{ij} \circ 1)$ is clear.

Since we have a direct system, we can form a direct limit $P = \varinjlim(M_i \otimes N)$ with natural maps $\delta_i : M_i \otimes N \to P$ as in Exercise 2.14. For each $i \in I$, there are natural homomorphisms $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$ so the universal property of Exercise 2.16 tells us there exists a unique homomorphism $\psi : P \to M \otimes N$ and in particular, $\psi \circ \delta_i = \mu_i \otimes 1$.

For each $i \in I$, let $g_i : M_i \times N \to M_i \otimes N$ be the canonical bilinear map given by $(m, n) \mapsto m \otimes n$ which is just $\mu_{ii} \times \mathrm{Id}_N$. The maps $\{g_i\}$ form a family by which we can define

[13] "A **homomorphism** $\phi : \mathbf{M} \to \mathbf{N}$ is by definition a family of A-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_g \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leqslant j$."

[14] In the situation of Exercise 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.
Show also that if $\mu_i(x_i) = 0$ then there exists $j \geqslant i$ such that $\mu_i(x_i) = 0$ in M_s.

a map of direct limits $g : M \times N \to P$. First off, let $\mu_i(x_i), \mu_j(y_j) \in M$ for $x_i \in M_i$ and $y_j \in M_j$ by Exercise 2.15[15]. Then for any $a, b \in A$ and $n \in N$,

$$g(a\mu_i(x_i) + b\mu_j(y_j), n) = (a\mu_i(x_i) + b\mu_j(y_j)) \otimes n = a(\mu_i(x_i) \otimes n) + b(\mu_j(x_j) \otimes n)$$

because g is defined by using Exercise 2.18. From the universal property of tensor products, we can extend g to a homomorphism $\phi : M \otimes N \to P$. Explicitly if $\mu_i(m_i) \otimes n \in M \otimes N$, then $\phi((\mu_i(m_i) \otimes n)) = \delta_i(m_i \otimes n)$.

Let $(\delta_i)(m_i \otimes n) \in P$ and $\mu_i(m_i) \otimes n \in M \otimes N$ Then,

$$\phi(\psi((\delta_i)(m_i \otimes n))) = \phi((\mu_i \otimes 1)(m \otimes n)) = \phi(\mu_i(m_i) \otimes n) = \delta_i(m_i \otimes n)$$
$$\psi(\phi(\mu_i(m_i) \otimes n)) = \psi(\delta_i((m_i \otimes n))) = (\mu_i \otimes 1)(m_i \otimes n) = \mu_i(m_i) \otimes n$$

which shows the composite maps are both identity maps. Therefore, $\phi : M \otimes N \to P$ is an isomorphism and we have from definition,

$$(\varinjlim M_i) \otimes N \cong \varinjlim(M_i \otimes N).$$

□

Exercise 2.24 If M is an A-module, the following are equivalent:

i) M is flat;

ii) $\operatorname{Tor}_n^A(M, N) = 0$ for all $n > 0$ and all A -modules N;

iii) $\operatorname{Tor}_1^A(M, N) = 0$ for all A-modules N.

[To show that (i) ⇒ (ii), take a free resolution of N and tensor it with M. Since M is flat, the resulting sequence is exact and therefore its homology groups, which are the $\operatorname{Tor}_n^A(M, N)$, are zero for $n > 0$. To show that (iii) ⇒ (i), let $0 \to N' \to N \to N'' \to 0$ be an exact sequence. Then, from the Tor exact sequence,

$$\operatorname{Tor}_1(M, N) \to M \otimes N' \to M \otimes N \to M \otimes \mathrm{N}'' \to 0$$

is exact. Since $\operatorname{Tor}_1(M, N^n) = 0$ it follows that M is flat.]

Proof. The solution to the exercise is essentially spoiled by the authors so what we say is simply clarification of the details of the argument.

(i) $\Longrightarrow$ (ii): Consider a free resolution[16] of N,

$$\ldots \to F_2 \to F_1 \to F_0 \to N \to 0$$

and applying the functor $M \otimes -$, we get an complex,

$$\ldots \to M \otimes F_2 \to M \otimes F_1 \to M \otimes F_0 \to 0.$$

We do not necessarily have exactness at $M \otimes F_0$, but flatness of M guarantees we have exactness at $M \otimes F_n$ for all $n > 0$. But if we recall that $\operatorname{Tor}_n^A(M, N)$ is the homology at $M \otimes F_n$, exactness of implies $\operatorname{Tor}_n^A(M, N) = 0$ for all $n > 0$. In particular, if we denote $d_n : M \otimes F_{n+1} \to M \otimes F_n$, then we have

$$\operatorname{Tor}_n^A(M, N) = \ker(d_{n-1}) / \operatorname{im}(d_n) = 0$$

since $\ker(d_{n-1}) = \operatorname{im}(d_n)$ when $n \geq 1$. Note that we do not necessarily have $\operatorname{Tor}_0^A(M, N) = 0$ when $n = 0$ because $\operatorname{Tor}_0^A(M, N) \cong M \otimes N$ by observing the above sequence.

(ii) $\Longrightarrow$ (iii): This is obvious since we can apply (ii) with $n = 1$.

[15] We freely use this exercise from here on and will not quote it for ease of reading.

[16] Which exists by our work in Problem 1.

(iii) $\Longrightarrow$ (i): Let $0 \to N' \to N \to N'' \to 0$ be a SES and the LES of Tor[17] yields

$$\mathrm{Tor}_1(M, N'') \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$$

and by hypothesis, $\mathrm{Tor}_1(M, N'') = 0$ so we just have a SES

$$0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$$

i.e. M is a flat module.

□

Exercise 2.25

Proof. Consider the Tor exact sequence:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{Tor}_1(N', M) \arrow[r] & \mathrm{Tor}_1(N, M) \arrow[r] & \mathrm{Tor}_1(N'', M) \arrow[dlll] \\
N' \otimes M \arrow[r] & N \otimes M \arrow[r] & N'' \otimes M \arrow[r] & 0
\end{tikzcd}$$

By hypothesis, $\mathrm{Tor}_1(N, M) = 0$. If N' is flat, then $\mathrm{Tor}_1(N', M) = 0$ which implies $\mathrm{Tor}_1(N, M) = 0$ and by the preceding exercise, N is flat. If N is flat, then $\mathrm{Tor}_1(N, M) = 0$ which implies $\mathrm{Tor}_1(N', M) = 0$ by injectivity of the map $\mathrm{Tor}_1(N', M) \to \mathrm{Tor}_1(N', M)$. Therefore, $N' = 0$ by the preceding exercise. □

Exercise 2.27 A ring A is absolutely flat if every A-module is flat. Prove that the following are equivalent:

i) A is absolutely flat.
ii) Every principal ideal is idempotent.
iii) Every finitely generated ideal is a direct summand of A.
[i) $\Rightarrow$ ii). Let $x \in A$. Then $A/(x)$ is a flat A-module, hence in the diagram

$$\begin{tikzcd}
(x) \otimes A \arrow[r, "\beta"] \arrow[d] & (x) \otimes A/(x) \arrow[d, "\alpha"] \\
A \arrow[r] & A/(x)
\end{tikzcd}$$

the mapping α is injective. Hence $\mathrm{Im}(\beta) = 0$, hence $(x) = (x^2)$. ii) $\Longrightarrow$ iii). Let $x \in A$. Then $x = ax^2$ for some $a \in A$, hence $e = ax$ is idempotent and we have $(e) = (x)$. Now if e, f are idempotents, then $(e, f) = (e + f - ef)$. Hence every finitely generated ideal is principal, and generated by an idempotent e, hence is a direct summand because $A = (e) \oplus (1 - e)$. iii) $\Longrightarrow$ i). Use the criterion of Exercise 26.]

Proof. (i) $\Longrightarrow$ (ii) The map $\alpha : (x) \otimes A/(x) \to A/(x)$ is injective because $A?(x)$ is flat as an A-module and we tensor $0 \to (x) \to A \to A/(x) \to 0$. Now if we chase an element $ax \otimes b \in (x) \otimes A$ down and then to the right, we get $ax \otimes b \mapsto axb \mapsto \overline{0}$. Injectivity of α then implies $\mathrm{im}(\beta) = 0$. However, the map $A \to A/(x)$ is surjective and tensoring $(x) \otimes A \to (x) \otimes A/(x)$ means that $\mathrm{im}(\beta) = (x) \otimes A/(x) \cong (x)A/(x^2)A$ i.e. $(x) = (x^2)$. This proves (ii).

(ii) $\Longrightarrow$ (iii) If $(x) = (x^2)$, I know that $x = ax^2$. Therefore, $e = ax$ is an idempotent element and $(e) = (x)$. Now if e, f are idempotents, then $(e, f) = (e + f - ef)$ (multiply by

[17] Which is established in Problem 8.

the RHS by e or by f). Hence, every finitely generated ideal is principal and generated by an idempotent. Hence, they are direct summands because $A = (e) \oplus (1-e)$ whenever e is idempotnent.

(iii) $\implies$ (i) Every f.g. ideal is a summand of A which means $A \cong \mathfrak{a} \oplus B$ for $\mathfrak{a}$ a f.g. ideal. So, $A/\mathfrak{a} \cong B$ is a projective A-module. It follows that $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ because projective modules are flat. □

Exercise 2.28 A Boolean ring is absolutely flat. The ring of Chapter 1, Exercise 7 is absolutely flat. Every homomorphic image of an absolutely flat ring is absolutely fiat. If a local ring is absolutely flat, then it is a field.

If A is absolutely flat, every non-unit in A is a zero-divisor.

Proof. We use the previous exercise without any comments. A Boolean ring is absolutely flat since all elements are idempotent. The ring Chapter 1, Exercise 7 is absolute flat since every principal ideal is idempotent. The homomorphic image of an absolutely flat ring is absolutely flat by (ii) and (iii) of the last problem.

Let A be a local absolutely flat ring. Choose a nonunit $(x) \subseteq \mathfrak{m} \subseteq A$ contained in the unique maximal ideal. Then $A = (x) \oplus B$. But $A \cong (x) \oplus (1-x)$ by x idempotent. Then,

$$0 \to (x) \hookrightarrow A \to (1-x) \to 0$$

is a SES. The maximal ideal in $A = (e) \oplus (1-e)$ is the ideal generated (e). So $(e) = \mathfrak{m}$. But e was arbitrary so $(0) = \mathfrak{m}$.

Suppose A is absolutely flat. Let $a \in A$ be a non-unit and so, $(a) \oplus (1-a) \cong A$. But this means $(a, 0) \cdot (0, 1-a) = (0, 0)$ in A. So a was a zero-divisor. □

3. Rings and Modules of Fractions

Exercise 3.1

Proof. Let M be a f.g. R-module and $S^{-1}M$ be a localization.

$(\implies)$: If $S^{-1}M = 0$, then all fractions $\frac{m_i}{1} = 0$ for all m_i's generating M and there are finitely many. Choose s_i s.t. $s_i m_i = 0$ by the equivalence relation. Then $s := s_1 \ldots s_n$.

$(\impliedby)$: If there exists $s \in S$ s.t. $sM = 0$, then every fraction in $S^{-1}M$ has $\frac{m}{s'} = \frac{ms}{s's} = \frac{0}{ss'} = 0$. □

Exercise 3.2

Proof. Let $\frac{a}{s} \in S^{-1}\mathfrak{a}$. We use Proposition 1.9 to show $\frac{a}{s} \in \mathfrak{R}(S^{-1}A)$. Let $\frac{a'}{s'} \in S^{-1}A$. Then,

$$1 - \frac{a}{s}\frac{a'}{s'} = \frac{ss' - aa'}{ss'}$$

and the RHS is a unit iff $ss' - aa' \in 1 + \mathfrak{a}$. Suppose $s = 1 + b$ and $s' = 1 + b'$ for $b, b' \in \mathfrak{a}$¿ Then,

$$ss' - aa' = (1+b)(1+b') - aa' = 1 + b + b' + bb' - aa' \in 1 + \mathfrak{a}.$$

This proves the first statement.

Suppose $\mathfrak{a} = \mathfrak{a}M$. Then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$ because localization commutes with products by Proposition 3.11. Because $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$, we apply Nakayama's Lemma to deduce that $S^{-1}M = 0$. Then, Exercise 3.1 shows that $sM = 0$ for some $s \in S = 1 + \mathfrak{a}$. But $s \equiv 1 (\text{mod } \mathfrak{a})$ which is what we wanted in Corollary 2.5. □

Exercise 3.3

PROOF. This result follows immediately from the universal property of localization. Note that ST denotes the set of products of elements in S and T. It suffices to show $(ST)^{-1}A$ satisfies the universal property of $U^{-1}(S^{-1}A)$. Consider the diagram

$$\begin{tikzcd} S^{-1}A \arrow[r] \arrow[dr, "g"'] & (ST)^{-1}A \arrow[d, dotted] \\ & B \end{tikzcd}$$

where $g(u)$ is a unit in B for all $u \in U$. It follows that $g(u)$ is a unit in B for all $u \in T$ and since $S \subseteq (S^{-1}A)^\times$, the collection $g(ST) \subseteq B^\times$. By the universal property, there is a map $(ST)^{-1}A \to B$ that makes the diagram commute. So, there exists a morphism $(ST)^{-1}A \to U^{-1}(S^{-1}A)$ making the diagram commute. A map going in the other direction follows by the same set of reasoning. So, $(ST)^{-1}A \xrightarrow{\cong} U^{-1}(S^{-1}A)$ is an isomorphism. $\square$

Exercise 3.4 Let $f : A \to B$ be a homomorphism of rings and S a multiplicatively closed subset of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

PROOF. There is really only one natural way to define the isomorphism. Define $F : S^{-1}B \to T^{-1}B$ by $F(\frac{b}{s}) = \frac{b}{f(s)}$. Clearly, this map is surjective and is a well-defined morphism of $S^{-1}A$-modules if we define

$$\frac{a}{s}F\left(\frac{b}{s'}\right) = \frac{a}{s} \cdot \frac{b}{f(s')} = \frac{ab}{f(ss')} = F\left(\frac{a}{s} \cdot \frac{b}{s'}\right).$$

Now we check injectivity. Suppose $F(\frac{b}{s}) = 0$. Then $\frac{b}{f(s)} = 0$ which occurs iff there exists an $f(t) \in T$ s.t. $f(t)b = 0$. But that means $\frac{b}{s} = 0$ since $t \cdot b = f(t) \cdot b = 0$ (viewing $S^{-1}B$ as an $S^{-1}A$-module in the usual sense. $\square$

Exercise 3.5 Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain?

PROOF. **Statement One:** Corollary 3.12, Proposition 3.8, and abusive notation shows $\mathfrak{N}_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$ iff $\mathfrak{N} = 0$. In particular, we recall that Proposition 1.7 of AM shows $\mathfrak{N}$ is an ideal so it is an A-module.
Statement Two: *(Example 1)*[18] Consider $A := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then A has two prime ideals which are generated by $(0,1)$ and $(1,0)$ respectively. Let us show $\mathfrak{p} := ((0,1))$ gives an integral domain. When we localize at $\mathfrak{p}$, we invert $(1,0)$ and $(1,1)$. The elements of $A_\mathfrak{p}$ are therefore

$$\frac{(0,0)}{(1,1)}, \frac{(1,0)}{(1,1)}, \frac{(0,1)}{(1,1)}, \frac{(1,1)}{(1,1)}, \frac{(0,0)}{(1,0)}, \frac{(1,0)}{(1,0)}, \frac{(0,1)}{(1,0)}, \frac{(1,1)}{(1,0)}.$$

Notice that third element is equal to zero by the equivalence relation since $(1,0)(0,1) = (0,0)$. Similarly, $\frac{(0,1)}{(1,0)} = 0$. Also, the two elements with $(0,0)$ on top are equal to zero. The two

[18] I tend to write down a lot of examples since it strengthens my understanding of what goes wrong.

elements $\frac{(1,1)}{(1,1)}, \frac{(1,0)}{(1,0)}$ are equal. So, we are reduced to the elements

$$\frac{(0,0)}{(1,1)}, \frac{(1,0)}{(1,1)}, \frac{(1,1)}{(1,1)}, \frac{(1,1)}{(1,0)}.$$

Notice further that

$$\frac{(1,1)}{(1,0)} = \frac{(1,1)(1,0)}{(1,0)(1,0)} = \frac{(1,0)}{(1,0)} \qquad \& \qquad \frac{(1,0)}{(1,1)}\frac{(1,0)}{(1,0)} = \frac{(1,0)}{(1,0)}$$

which means we are now just left with two distinct elements

$$\frac{(0,0)}{(1,1)}, \frac{(1,1)}{(1,1)}.$$

A ring with two elements is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and hence, an integral domain. Since the proof that the localization at the prime ideal generated by $(0,1)$ is similar, we deduce that the localization of A at every prime ideal is an integral domain. However, A is clearly not and integral domain.

(Example 2) A more standard example might be to take $A := \mathbb{Z}/6\mathbb{Z}$. The only prime ideals of A are $2\mathbb{Z}/6\mathbb{Z}$ and $3\mathbb{Z}/6\mathbb{Z}$ since $2\mathbb{Z}, 3\mathbb{Z}$ are the only prime ideals that contain $6\mathbb{Z}$. If we localize $A_{2\mathbb{Z}}$, then the maximal ideal is $2\mathbb{Z}A_{2\mathbb{Z}}$ and every element in the maximal ideal has the form $\frac{r}{s}$ where $r \in \{0,2\}$ and $s \in \{1,3,5\}$. If $\frac{r}{s}$ is nonzero, we must have $r = 2$ but then, $3r = 0$ which means $\frac{r}{s} = 0$ by the equivalence relation on $A_{2\mathbb{Z}/6\mathbb{Z}}$ since $3 \in A \setminus 2\mathbb{Z}$. Therefore, every element of the maximal ideal of $A_{2\mathbb{Z}/6\mathbb{Z}}$ is zero so the zero ideal is prime. This means $A_{2\mathbb{Z}}$ is an integral domain. A similar method shows $A_{3\mathbb{Z}/6\mathbb{Z}}$ is also an integral domain. Therefore, the localization of A at any prime ideal is an integral domain, but A is clearly not.

(Nonexample) An easy first example one might consider is $\mathbb{Z}/4\mathbb{Z}$ which is a local ring with maximal ideal $\{0,2\}$ and it is clearly not an integral domain. Localizing at the prime ideal inverts $1,3$. However, $\frac{2}{1} \cdot \frac{2}{1} = \frac{0}{1}$ which means the localization at a prime ideal does not give an integral domain. So, this is not a counterexample.

(Crucial point) Being an integral domain is **not** a local property while being reduced is a local property. □

Exercise 3.6 Let A be a ring $\neq 0$ and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Σ has maximal elements and $S \in \Sigma$ is maximal if and only if $A - S$ is a minimal prime ideal of A.

Proof. Clearly, $\Sigma \neq \emptyset$ since we can take $S := \{1\} \in \Sigma$. To prove existence of maximal elements, we use Zorn's Lemma. Let order the elements of Σ by inclusion i.e. $S \leq T$ iff $S \subseteq T$.

Let $\{S_\alpha\}_{\alpha \in A}$ be a chain of elements in Σ. We claim $T := \bigcup_{\alpha \in A} S_\alpha$ is an upper bound for the chain. Clearly, $S_\alpha \leq T$ for all $\alpha \in A$. Meanwhile, $T \in \Sigma$ because if $h, g \in T$, we may choose a β s.t. $h, g \in S_\beta$ and so $hg \in S_\beta \subseteq T$. By Zorn's Lemma, there exists some maximal element in Σ.

Suppose $S \in \Sigma$ is a maximal element. We show $A - S$ is a minimal prime ideal in A. Certainly, $A - S$ is a prime ideal. Now suppose $\mathfrak{p} \subsetneq A - S$ were a proper prime ideal. Then $T := A - \mathfrak{p}$ is a multiplicatively closed subset satisfying $T \supsetneq S$ which contradicts maximality of S.

Now assume $A - S$ is a minimal prime ideal. If S is not maximal, then $S \subsetneq T$ which yields $A - T \subsetneq A - S$ which is a proper prime ideal contained in $A - S$. This contradicts minimality of $A - S$. $\square$

Exercise 3.7 A multiplicatively closed subset S of a ring A is said to be **saturated** if

$$xy \in S \iff x \in S \text{ and } y \in S.$$

Prove that
i) S is saturated $\iff$ $A - S$ is a union of prime ideals.
ii) If S is any multiplicatively closed subset of A, there is a unique smallest saturated multiplicatively closed subset $\overline{S}$ containing S, and that S is the complement in A of the union of the prime ideals which do not meet S. ($\overline{S}$ is called the saturation of S.)
If $S = 1 + \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of A, find $\overline{S}$.

Proof. **i)** Suppose $A - S$ is a union of prime ideals. Let $xy \in S$. Then $xy \notin \bigcup_i \mathfrak{p}_i = A - S$. So xy avoids the primes $\mathfrak{p}_i$. Now suppose not and that $x \notin S$ or $y \notin S$. So $x \in \mathfrak{p}_i$ for some i which means $xy \in \mathfrak{p}_i \subseteq \bigcup_i \mathfrak{p}_i$. That is a contradiction.

Suppose S is saturated. Then $xy \in A - S$ iff $x \in A - S$ or $y \in A - S$. This means $A - S \subseteq A$ is at best a union of prime ideals.

ii) Uniqueness is obvious so we show that $\overline{S} = \left(\bigcup_{\substack{\mathfrak{p} \in \mathrm{Spec}(A) \\ \mathfrak{p} \cap S = \emptyset}} \mathfrak{p}\right)^c$ is the saturation. Clearly, S is saturated. Observe further $A - \overline{S} = \bigcup_{\substack{\mathfrak{p} \in \mathrm{Spec}(A) \\ \mathfrak{p} \cap S = \emptyset}} \mathfrak{p}$.

Suppose T were another saturated set containing S. Then $A - T$ is a union of prime ideals $\bigcup_i \mathfrak{q}_i$. But then $\mathfrak{q}_i \cap S = \emptyset$ which means $\mathfrak{q}_i \subseteq A - \overline{S}$. It follows then that $A - T \subseteq A - \overline{S}$. Taking complements, $T \supseteq \overline{S}$.

iii) Let $S = 1 + \mathfrak{a}$. Then $\mathfrak{p} \cap S = \emptyset$ iff $\mathfrak{p} \cap (1 + \mathfrak{a}) = \emptyset$ iff there are no elements $1 + a \in \mathfrak{p} \cap (1 + \mathfrak{a})$ iff $a \notin \mathfrak{p}$ for all $a \in \mathfrak{a}$ nonzero.

It follows that $\mathfrak{p} \cap S = \emptyset$ iff $\mathfrak{p} \subseteq A - \mathfrak{a}$. So, $\overline{S} = \left(\bigcup_{\substack{\mathfrak{p} \in \mathrm{Spec}(A) \\ \mathfrak{p} \subseteq (A - \mathfrak{a}) \cup \{0\}}} \mathfrak{p}\right)^c$ $\square$

Exercise 3.8

Proof. Easy, use universal or follow nose. $\square$

Exercise 3.9

Proof. Follow nose. $\square$

Exercise 3.10 Let A be a ring.

i) If A is absolutely flat (Chapter 2, Exercise 27) and S is any multiplicatively closed subset of A, then $S^{-1}A$ is absolutely flat.

ii) A is absolutely flat $A_{\mathfrak{m}}$ is a field for each maximal ideal $\mathfrak{m}$.

Proof. (i) Let N be an $S^{-1}A$-module. By restriction of scalars, N is flat as an A-module. Apply $S^{-1}(-)$ which is an exact functor. Then this says $S^{-1}N = N$ is a flat $S^{-1}M$-module.
(ii) A being absolutely flat implies $A_{\mathfrak{m}}$ is absolutely flat by (i) and AM Exercise 2.28 says that the local ring $A_{\mathfrak{m}}$ is a field. This gives one direction.

Now assume $A_{\mathfrak{m}}$ are all fields. Let $(x)A$ be a principal ideal. Assume it is proper so that $x \in \mathfrak{m}A_{\mathfrak{m}}$. But that means $x = 0$ in this ring. SO x itself must have been idempotent. $\square$

Exercise 3.11

Exercise 3.12 Let A be an integral domain and M an A -module. An element $x \in M$ is a *torsion element* of M if $\operatorname{Ann}(x) \neq 0$, that is if x is killed by some non-zero element of A. Show that the torsion elements of M form a submodule of M. This submodule is called the *torsion submodule* of M and is denoted by $T(M)$. If $T(M)=0$ the module M is said to be torsion-free. Show that

i) If M is any A-module, then $M/T(M)$ is torsion-free.

ii) If $f: M \rightarrow N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

iii) If $0 \rightarrow M' \rightarrow M \rightarrow M''$ is an exact sequence, then the sequence $0 \rightarrow T(M') \rightarrow T(M) \rightarrow T(M')$ is exact.

iv) If M is any A -module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A.

[For iv), show that K may be regarded as the direct limit of its submodules $A\xi$ ($\xi \in K$); using Chapter 1, Exercise 15 and Exercise 20, show that if $1 \otimes x=0$ in $K \otimes M$ then $1 \otimes x=0$ in $A\xi \otimes M$ for some $\xi \neq 0$. Deduce that $\xi^{-1}x=0$.]

Proof. **i)** Let $\overline{m} \in M/T(M)$ be a nonzero element that was annihilated by some $a \in A \setminus \{0\}$ i.e. $a\overline{m}=0$. Then that means $am \in T(M)$ and we can choose a $a' \in A \setminus \{0\}$ s.t. $a'am=0$. Since A is an integral domain, $a'a \neq 0$ and this means $m \in T(M)$. Hence, $\overline{m}=0$ to being with. Therefore, $M/T(M)$ is torsionfree.

ii) If $m \in T(M)$, choose $a \in A \setminus \{0\}$ s.t. $am=0$. Then $af(m)=f(am)=f(0)=0$ implies $f(m) \in T(N)$ and $f(T(M)) \subseteq T(N)$. In particular, f induces a homomorphism $f': T(M) \rightarrow T(N)$.

iii) If $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact, then (ii) shows $\operatorname{Tor}(M') \rightarrow \operatorname{Tor}(M)$ is still injective so we need to show exactness at $T(M)$. Clearly, $\operatorname{im} f' \subseteq \operatorname{im} g'$ since $g(f(m))=0$ for any $m \in \operatorname{Tor}(M')$ by exactness of the original sequence. Conversely, let $m \in \ker g' \subseteq \operatorname{Tor}(M)$ and lift to get an $n \in M'$ s.t. $f'(n)=f(n)=m$. Since m is torsion, choose an $a \in A \setminus \{0\}$ s.t. $am=0$. Then, $am=f(an)=0$ and injectivity of f implies $an=0$ which means $n \in \operatorname{Tor}(M')$ as desired.

iv) [19] First, show that $K=\varinjlim_{\zeta \in K} A\zeta$. There is a natural inclusion $A\zeta \rightarrow A\eta$ when there exists an $a \in A$ s.t. $\zeta=a\eta$. Now if we observe that K has natural maps $A\zeta \rightarrow K$ which satisfies the universal property of the colimit, we get $K=\varinjlim_{\zeta \in K} A\zeta$.

By Exercise 2.20 of AM, $K \otimes M \cong \varinjlim_{\zeta \in K}(A\zeta \otimes M)$. Assume $1 \otimes x=0$. By Exercise 2.15 and Exercise 2.20 of AM, $1 \otimes x=(\mu_i \otimes 1)(1 \otimes x)$ for some $A\zeta_i \otimes M$ since $\mu_i: A\zeta_i \rightarrow K$ is a ring homomorphism. From how the direct system is defined, we can require $\zeta_j \neq 0$. By the second part of Exercise 2.15, we know $1 \otimes x=(\mu_{ij} \otimes 1)(1 \otimes x)=\mu_{ij}(1) \otimes x \in A\zeta_j \otimes M$. This means $1 \otimes x=0$ in $A\zeta_j \otimes M$ and $\zeta_j \neq 0$.

[19] It also looks like Atiyah-Macdonald intended to write *Chapter 2, Exercise 15 and Exercise 20* and not Chapter 1, Exercise 15 and Exercise 20. That might be worth adding to the errata list, but I do not use StackOverflow except for literature searching.

More "straightforward" approach that doesn't work: This was something that was discussed in an errata list for Atiyah-Macdonald here. Let $K=S^{-1}A$ where $S:=A \setminus \{0\}$. If $1 \otimes x=0$ in $K \otimes_A M$, then that means $\frac{x}{1}=0$ in $S^{-1}M \cong K \otimes_A M$. So, there is a nonzero $\zeta \in S$ s.t. $\zeta x=0$. But this means x is a torsion element or that x was zero to being with. Hence, kernel of the map is in $T(M)$ and the fact that $T(M)$ is in the kernel of the map is obvious writing out 1 as a ratio of $\frac{\zeta}{\zeta}$ of an element ζ that kills x.

This doesn't work because there may be extra relations that make $1 \otimes x=0$ in $K \otimes_A M$. The purpose of the direct limit is to avoid those extra relations.

With $1 \otimes x$ zero in $A\zeta \otimes M$, dropping the j, we deduce that

$$1 \otimes x = 0 \quad \Longrightarrow \quad \zeta\zeta^{-1} \otimes x = 0 \quad \Longrightarrow \quad \zeta \otimes \zeta^{-1}x = 0 \quad \Longrightarrow \quad \zeta^{-1}x = 0.$$

If we write $\zeta^{-1} = \frac{r}{s}$ with $r, s \neq 0$, then $\zeta^{-1}x = 0$ implies there is a nonzero $t \in A$ s.t. $trx = 0$. But $tr \neq 0$ since A is an integral domain which shows $x \in T(M)$.

□

Exercise 3.13 Let S be a multiplicatively closed subset of an integral domain A. In the notation of Exercise 12, show that $T\left(S^{-1}M\right) = S^{-1}(TM)$. Deduce that the following are equivalent:

i) M is torsion-free.
ii) $M_{\mathfrak{p}}$ is torsion-free for all prime ideals $\mathfrak{p}$.
iii) $M_{\mathfrak{m}}$ is torsion-free for all maximal ideals $\mathfrak{m}$.

PROOF. **Show $T(S^{-1}M) = S^{-1}(TM)$:** The result is obvious if $0 \in S$ since that means $S^{-1}M = 0$ so assume $0 \notin S$. The containment $S^{-1}(TM) \subseteq T(S^{-1}M)$ is clear because if $\frac{m}{s} \in S^{-1}(TM)$, we can choose an $a \in A\setminus\{0\}$ s.t. $am = 0$ and then, $\frac{a}{1} \neq 0$ is s.t. $\frac{a}{1}\frac{m}{s} = \frac{0}{s} = 0$. For the converse, suppose $\frac{m}{s} \in T(S^{-1}M)$ and choose $\frac{a}{r} \in S^{-1}A \setminus \{0\}$ s.t. $\frac{a}{r}\frac{m}{s} = 0$. Then there exists a $u \in S$ s.t. $uam = 0$. Since u, a are nonzero, $ua \in A \setminus \{0\}$ and this means $m \in T(M)$. Hence, $T(S^{-1}M) \subseteq S^{-1}(TM)$.

Equivalence of statements: The approach we take is to show (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii): If M is torsion-free, then $T(M) = 0$ and so $T(M_{\mathfrak{p}}) = T(M)_{\mathfrak{p}} = 0$ which shows $M_{\mathfrak{p}}$ is torsion-free. Since $\mathfrak{p} \in \operatorname{Spec}(A)$ was arbitrary, we are done.

(ii) $\Longrightarrow$ (iii): Obvious since all maximal ideals are prime.

(iii) $\Longrightarrow$ (i): By hypothesis, $0 = T(M_{\mathfrak{m}}) = T(M)_{\mathfrak{m}}$ for all maximal ideals and $T(M)$ is an A-module and so by Proposition 3.8 of AM, $T(M) = 0$.

More efficient: By Proposition 3.8, $T(M_{\mathfrak{m}}) = 0$ for all $\mathfrak{m} \in \operatorname{MaxSpec}(A)$ iff $T(M)_{\mathfrak{m}} = 0$ for all $\mathfrak{m} \in \operatorname{MaxSpec}(A)$ iff $T(M) = 0$ iff $T(M)_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$ iff $T(M_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$. □

Exercise 3.14 Let M be an A-module and $\mathfrak{a}$ an ideal of A. Suppose that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$. Prove that $M = \mathfrak{a}M$. [Pass to the $A/\mathfrak{a}$-module $M/\mathfrak{a}M$ and use (3.8).]

PROOF. It suffices to show $M/\mathfrak{a}M = 0$. By Proposition 3.8 of AM, it suffices to show that $(M/\mathfrak{a}M)_{\mathfrak{m}} = 0$ for all maximal ideals of $A/\mathfrak{a}$ (which correspond to maximal ideals of A containing $\mathfrak{m}$) and since localization commutes with quotients by Corollary 3.4 of AM,

$$(M/\mathfrak{a}M)_{\mathfrak{m}} \cong M_{\mathfrak{m}}/\mathfrak{a}M_{\mathfrak{m}} \cong 0/\mathfrak{a}M_{\mathfrak{m}} = 0.$$

□

Exercise 3.15 Let A be a ring, and let F be the A-module A^n. Show that every set of n generators of F is a basis of F. [Let $x_1, \ldots, x_n$ be a set of generators and $e_1, \ldots, e_n$ the canonical basis of F. Define $\phi : F \to F$ by $\phi(e_i) = x_i$. Then ϕ is surjective and we have to prove that it is an isomorphism. By (3.9) we may assume that A is a local ring. Let N be the kernel of ϕ and let $k = A/\mathfrak{m}$ be the residue field of A. Since F is a flat A -module, the exact sequence $0 \to N \to F \to F \to 0$ gives an exact sequence $0 \longrightarrow k \otimes N \longrightarrow k \otimes F \xrightarrow{1\otimes\phi} k \otimes F \longrightarrow 0$. Now $k \otimes F = k^n$ is an n -dimensional vector space over $k; 1 \otimes \phi$ is surjective, hence bijective, hence $k \otimes N = 0$.

Also N is finitely generated, by Chapter 2, Exercise 12, hence $N = 0$ by Nakayama's lemma. Hence ϕ is an isomorphism.] Deduce that every set of generators of F has at least n elements.

Proof. Choose generators $x_1, \dots, x_n$ for F and $e_1, \dots, e_n$ for the canonical basis of F. The map ϕ is surjective since it maps onto a set of generators. Our goal is to show injectivity and by localizing at a maximal ideal as needed, Proposition 3.9 ensures that we may assume WLOG that $(A, \mathfrak{m})$ is a local ring. Let $N = \ker\phi$ and $k = A/\mathfrak{m}$ the residue field. Since F is free, it is a flat A-module which means $\mathrm{Tor}_1(F, k) = 0$ since k is an A-module. The sequence

$$0 \to N \to F \xrightarrow{\phi} F \to 0$$

then induces[20] LES of Tor which has

$$\mathrm{Tor}_1(k, F) \to k \otimes N \to k \otimes F \xrightarrow{1\otimes\phi} k \otimes F \to 0$$

and since $\mathrm{Tor}_1(F, k) \cong \mathrm{Tor}_1(k, F)$, we deduce that we actually have an exact sequence

$$0 \to k \otimes N \to k \otimes F \xrightarrow{1\otimes\phi} k \otimes F \to 0.$$

Since tensor products commute with direct sums and k is an A-module, $k \otimes F \cong k^n$ so $1 \otimes \phi$ is a surjective linear transformation of vector spaces of the same degree. By the Rank-Nullity Theorem, the nullity of $1 \otimes \phi$ is zero which means $k \otimes N = 0$.

By Exercise 2.12 of AM, since $\phi : F \to F$ is surjective, $N = \ker\phi$, and F is a f.g. A-module, N must also be f.g. as an A-module. Since $k \otimes N \cong N/\mathfrak{m}N \cong 0$, the image of $0 \in N$ generates $N/\mathfrak{m}N$ and by Proposition 2.8, we deduce that 0 generates N. Hence, $N = 0$ as desired. □

Exercise 3.16 Let B be a flat A-algebra. Then the following conditions are equivalent:

i) $\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of A.

ii) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

iii) For every maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}^e \neq (1)$.

iv) If M is any non-zero A-module, then $M_B \neq 0$.

v) For every A-module M, the mapping $x \mapsto 1 \otimes x$ of M into M_B is injective.

[For i) $\Rightarrow$ ii), use (3.16). ii) $\Rightarrow$ iii) is clear.

iii) $\Rightarrow$ iv): Let x be a non-zero element of M and let $M' = Ax$. Since B is fat over A it is enough to show that $M'_B \neq 0$. We have $M' \cong A/a$ for some ideal $a \neq (1)$, hence $M'_B \cong B/\mathfrak{a}^e$. Now $\mathfrak{a} \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, hence $\mathfrak{a}^e \subseteq \mathfrak{m}^e \neq (1)$. Hence $M'_B \neq 0$.

iv) $\Rightarrow$ v) : Let M' be the kernel of $M \to M_B$. Since B is flat over A, the sequence $0 \to M'_B \to M_B \to (M_B)_B$ is exact. But (Chapter 2, Exercise 13, with $N = M_B$) the mapping $M_B \to (M_B)_B$ is injective, hence $M'_B = 0$ and therefore $M' = 0$.

v) $\Rightarrow$ i): Take $M = A/\mathfrak{a}$.]

B is said to be **faithfully flat** over A.

Proof. i)$\Rightarrow$ii) We know $\mathfrak{a}^{ec} = \mathfrak{a}$ for all $\mathfrak{a} \subseteq A$. So, $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is injective because if $\mathfrak{p} \in \mathrm{Spec}(A)$, then $\mathfrak{a} = \mathfrak{p}$ and $\mathfrak{a}^e \subseteq \mathrm{Spec}(B)$. But then $\mathfrak{a}^e \to \mathfrak{a}^{ec} = \mathfrak{a}$. This uses (3.16).

ii)$\Rightarrow$iii) is clear.

[20] I was quite confused on why this was possible. Initially, I thought that the residue field was somehow always flat, which felt too good to even be true. Discussing with Professor Bucur and the following discussion post helped clarify what I was missing. Importantly, Exercise 2.24 of AM must be used in tandum with the nontrivial fact that $\mathrm{Tor}_n(M, N) \cong \mathrm{Tor}_n(N, M)$.

iii)⇒iv)⇒v)⇒i) are all explained in the problem statement. Note that in the iii)⇒iv) step, $M'_B \neq 0$ is enough since B is flat and $M' = Ax$. If M is a nonzero B-module, then $M_B \neq 0$ iff multiplication by an element of M is nonzero. If $M'_B = (Ax)_B \neq 0$ for $x \in M$, then $(Ax)_B \neq 0$ and $\subseteq M_B$. □

Exercise 3.17 Let $A \xrightarrow{s} B \xrightarrow{g} C$ be ring homomorphisms. If $g \circ f$ is flat and g is faithfully flat, then f is flat.

Proof. Let $0 \to M' \to M \to M'' \to 0$ be a SES of A-modules. Tensor by $\otimes_A B$ gives a SESE. Now if I tensor by $\otimes_A C$, and $g \circ f$ is flat,

$$0 \to M' \otimes_A C \to M \otimes_A C \to M'' \otimes_A C \to 0.$$

Now, we look at the diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \otimes_A B \otimes_B C \arrow[r] & M \otimes_A B \otimes_B C \arrow[r] & M'' \otimes_A B \otimes_B C \arrow[r] & 0 \\
& M' \otimes_A B \arrow[u, hook] \arrow[r, "\phi"] & M \otimes_A B \arrow[u, hook, "\psi"] \arrow[r] & M'' \otimes_A B \arrow[u, hook] \arrow[r] & 0
\end{tikzcd}$$

We need to show that ϕ is injective. Observe that by commutativity of the diagram that $\psi \circ \phi$ is injective. So, ϕ is injective. This gives the desired result. Note that we used g faithfully flat to deduce the the map $M' \otimes_A B \otimes_B C \to M \otimes_A B \otimes_B C$ is injective (as $M' \otimes_A C \to M \otimes_A C$ is injective iff $M' \to M$ is injective). □

Exercise 3.18 Let $f : A \to B$ be a flat homomorphism of rings, let $\mathfrak{q}$ be a prime ideal of B and $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is surjective. [For $B_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ by (3.10), and $B_\mathfrak{q}$ is a local ring of $B_\mathfrak{p}$, hence is flat over $B_\mathfrak{p}$. Hence $B_\mathfrak{q}$ is flat over $A_\mathfrak{p}$ and satisfies condition (3) of Exercise 16.]

Proof. Flatness being local means $B_\mathfrak{p}$ is flat over $A_\mathfrak{p}$. Since $B_\mathfrak{q}$ is a local ring of $B_\mathfrak{p}$ (i.e. the localization of $B_\mathfrak{p}$), it is flat over $B_\mathfrak{p}$. So $B_\mathfrak{q}$ is flat over $A_\mathfrak{p}$ by transitivity of flatness. Now (iii) of AM Exercise 3.16 holds (since $\mathfrak{p}A_\mathfrak{p}$ is a maximal ideal and extends to $\mathfrak{q}B_\mathfrak{q} \neq (1)$). So (ii) of AM Exercise 3.16 says $f^* : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is surjective. □

Exercise 3.19 Let A be a ring, M an A-module. The support of M is defined to be the set $\mathrm{Supp}(M)$ of prime ideals $\mathfrak{p}$ of A such that $M_\mathfrak{p} \neq 0$. Prove the following results:

i) $M \neq 0 \Leftrightarrow \mathrm{Supp}(M) \neq \varnothing$.

ii) $V(\mathfrak{a}) = \mathrm{Supp}(A/\mathfrak{a})$.

iii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $\mathrm{Supp}\,(M) = \mathrm{Supp}\,(M') \cup \mathrm{Supp}\,(M')$.

iv) If $M = \sum M_i$, then $\mathrm{Supp}(M) = \bigcup \mathrm{Supp}\,(M_i)$.

v) If M is finitely generated, then $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$ (and is therefore a closed subset of $\mathrm{Spec}(A)$).

vi) If M, N are finitely generated, then $\mathrm{Supp}(M \otimes N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. [Use Chapter 2, Exercise 3.]

vii) If M is finitely generated and $\mathfrak{a}$ is an ideal of A, then $\mathrm{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \mathrm{Ann}(M))$.

viii) If $f : A \to B$ is a ring homomorphism and M is a finitely generated A module, then $\mathrm{Supp}\,(B \otimes_A M) = f^{*-1}(\mathrm{Supp}(M))$.

Proof. We remark that a lot of the facts proven can be found in an early chapter of Eisenbud's textbook [2].

(i) We know $M \neq 0$ iff $M_\mathfrak{p} \neq 0$ for some prime ideal $\mathfrak{p}$ iff $\mathrm{Supp}(M) \neq \emptyset$ (using Proposition 3.8).

(ii) We know $\mathfrak{p} \in V(\mathfrak{a})$ iff $\mathfrak{p} \supseteq \mathfrak{a}$ iff $(A/\mathfrak{a})_\mathfrak{p} \neq 0$.

(iii) We know that $\mathfrak{p} \in \mathrm{Supp}(M)$ iff $M_\mathfrak{p} \neq 0$ which then is equivalent to the SES

$$0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$$

having $M_\mathfrak{p} \neq 0$. But then it is equivalent to require $M'_\mathfrak{p}$ or $M''_\mathfrak{p}$ nonzero because if both were zero, we would have $M_\mathfrak{p} = 0$.

(iv) We know $\mathfrak{p} \in \mathrm{Supp}(M)$ iff $M_\mathfrak{p} \neq 0$ iff $\sum (M_i)_\mathfrak{p} \neq 0$ iff $(M_i)_\mathfrak{p} \neq 0$ for some i.

(v) Assume M is finitely generated and set $M = \sum Ax_i$ for x_i a finite set of generators. From the previous part, $\mathrm{Supp}(M) = \bigcup \mathrm{Supp}(Ax_i)$. We induct on the number of generators so assume $M = Ax$ (and note that the inductive step follows easily from Exercise 2.2. in the exposition of Chapter 2).

We are reduced to showing $\mathrm{Supp}(Ax) = V(\mathrm{Ann}(Ax))$. Let $(Ax)_\mathfrak{p} \neq 0$. Let $y \in \mathrm{Ann}(Ax)$ so that $yx = 0$. We show that $y \in \mathfrak{p}$. indeed, $y \in A$ so we can look at $\frac{y}{1} \in (Ax)_\mathfrak{p}$. If $y \notin \mathfrak{p}$, then $y \in (Ax)_\mathfrak{p}^\times$ which would mean that y is nonzero. Of course, this is absurd since $x \notin \mathfrak{p}$ (and use Exercise 3.1). Everything we did here goes the other direction so we get equality.

(vi) The result is trivial from Exercise 2.3 because $(M \otimes_A N)_\mathfrak{p}$ is nonzero iff $M_\mathfrak{p} \otimes N_\mathfrak{p}$ is nonzero iff $M_\mathfrak{p}$ or $N_\mathfrak{p}$ is nonzero.

(vii) Note that $M/\mathfrak{a}M \cong M \otimes A/\mathfrak{a}$. So, $\mathrm{Supp}(M/\mathfrak{a}M) = \mathrm{Supp}(M) \cap \mathrm{Supp}(A/\mathfrak{a})$ by vi). Then by ii) and v), as M is finitely generated,

$$\mathrm{Supp}(M/\mathfrak{a}M) = V(\mathrm{Ann}(M)) \cap V(\mathfrak{a}) = V(\mathfrak{a} + \mathrm{Ann}(M)).$$

(viii) Let $f : A \to B$ be a ring homomorphism and M be finitely generated. We get

$$\mathrm{Supp}(B \otimes_A M) = \mathrm{Supp}(B) \cap \mathrm{Supp}(M) = \mathrm{Supp}(B) \cap V(\mathrm{Ann}(M)) = \mathrm{Spec}\, B \cap V(\mathrm{Ann}(M))$$

On the other hand,

$$f^{*-1}(\mathrm{Supp}(M)) = f^{*-1}(V(\mathrm{Ann}(M))) = \mathrm{Spec}\, B \cap V(\mathrm{Ann}(M)).$$

□

Exercise 3.20 Let $f : A \to B$ be a ring homomorphism, $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the associated mapping. Show that

i) Every prime ideal of A is a contracted ideal $\iff$ f^* is surjective.

ii) Every prime ideal of B is an extended ideal $\Rightarrow f^*$ is injective.

Is the converse of ii) true?

PROOF. i) Both directions are immediate from Proposition 3.16.

ii) Suppose $f^*(\mathfrak{q}_1) = f^*(\mathfrak{q}_2)$. So that means $\mathfrak{q}_1^c = \mathfrak{q}_2^c$. But every prime ideal of B is an extended ideal. So assume $\mathfrak{q}_1 = \mathfrak{p}_1^e$ and $\mathfrak{q}_2 = \mathfrak{p}_2^e$. That means $\mathfrak{q}_1^{ce} = \mathfrak{q}_1$. But by Proposition 3.16,

$$\mathfrak{p}_1 = \mathfrak{p}_1^{ec} = \mathfrak{q}_1^c = \mathfrak{q}_2^c = \mathfrak{p}_2^{ec} = \mathfrak{p}_2.$$

It follows immediately that $\mathfrak{q}_1 = \mathfrak{q}_2$.

The converse of ii) is false. Consider the ring homomorphism $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ which maps $a \mapsto (a, a)$. This gives an injective $f^* : \mathrm{Spec}(\mathbb{Z} \times \mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z})$ but not every prime ideal of $\mathbb{Z} \times \mathbb{Z}$ is an extended prime ideal. For instnace, $p\mathbb{Z} \times q\mathbb{Z}$ is not an extended prime ideal for $p \neq q$ primes. □

Exercise 3.21

Proof. See Matsumura's Commutative Ring Theory for the details. ☐

Exercise 3.22

Proof. Clear. ☐

Exercise 3.23 23. Let A be a ring, let $X = \operatorname{Spec}(A)$ and let U be a basic open set in X (i.e., $U = X_f$ for some $f \in A$: Chapter 1, Exercise 17).

i) If $U = X_f$, show that the ring $A(U) = A_f$ depends only on U and not on f.

ii) Let $U' = X_g$ be another basic open set such that $U' \subseteq U$. Show that there is an equation of the form $g^n = u$ for some integer $n > 0$ and some $u \in A$, and use this to define a homomorphism $\rho : A(U) \to A(U')$ (i.e., $A_f \to A_g$) by mapping $a \mid f^m$ to au^m/g^{mn}. Show that ρ depends only on U and U'. This homomorphism is called the restriction homomorphism.

iii) If $U = U'$, then p is the identity map.

iv) If $U \supseteq U' \supseteq U''$ are basic open sets in X, show that the diagram

$$\begin{tikzcd} A(U) \arrow[rr] \arrow[dr] & & A(U'') \\ & A(U') \arrow[ur] & \end{tikzcd}$$

v) Let x $(= \mathfrak{p})$ be a point of X. Show that

$$\varinjlim_{U \ni x} A(U) \cong A_{\mathfrak{p}}.$$

The assignment of the ring $A(U)$ to each basic open set U of X, and the restriction homomorphisms p, satisfying the conditions iii) and iv) above, constitutes a presheaf of rings on the basis of open sets $(X_f)_{f \in A}$. v) says that the stalk of this presheaf at $x \in X$ is the corresponding local ring $A_{\mathfrak{p}}$.

Proof. This is just sheaf theory problem one also sees in Hartshorne. A full proof can actually be found in Hartshorne. ☐

Exercise 3.24 Show that the presheaf of Exercise 23 has the following property. Let $(U_i)_{i \in I}$ be a covering of X by basic open sets. For each $i \in I$ let $s_i \in A(U_i)$ be such that, for each pair of indices i, j, the images of s_i and s_j in $A(U_i \cap U_j)$ are equal. Then there exists a unique $s \in A(= A(X))$ whose image in $A(U_i)$ is s_i, for all $i \in I$. (This essentially implies that the presheaf is a sheaf.)

Proof. Again, see Hartshorne for full details. ☐

Exercise 3.25

Proof. ☐

Exercise 3.26

Proof. ☐

Exercise 3.27

Proof. ☐

Exercise 3.28

PROOF. $\square$

Exercise 3.29

PROOF. $\square$

Exercise 3.30

PROOF. $\square$

4. Primary Decomposition

Exercise 4.1

PROOF. Suppose $\mathfrak{a}$ has a primary decomposition into $\bigcap_{i=1}^{n} \mathfrak{q}_i$ for $\mathfrak{q}_i$ primary ideals. WLOG, assume the the decomposition is a minimal primary decomposition.

To show that $\operatorname{Spec}(A/\mathfrak{a})$ has only finitely many irreducible components, it suffices to show that the irreducible components correspond to the minimal prime ideals containing $\mathfrak{a}$. Indeed, the irreducible subsets of $\operatorname{Spec}(A/\mathfrak{a})$ are sets of the form $\operatorname{Spec}(A/\mathfrak{p})$ for a prime $\mathfrak{p}$ containing $\mathfrak{a}$. To get an irreducible component, we would need to get a maximal irreducible subset which then corresponds to a minimal prime ideal $\mathfrak{p}$. But the minimal prime ideals $\mathfrak{p}$ containing $\mathfrak{a}$ are precisely the prime ideals $r(\mathfrak{q}_i)$ and by hypothesis, there are only finitely many of them. $\square$

Exercise 4.2

PROOF. $\square$

Exercise 4.3

5. Integral Dependence and Valuations

Exercise 5.1

PROOF. Suppose $f : A \to B$ is an integral homomorphism of rings. Let $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be the induced mapping on the spectrum. Let $V(\mathfrak{a}) \subseteq \operatorname{Spec}(B)$ be a closed subset of B. By Theorem 5.10, if $\mathfrak{p} \in A$, there is a prime ideal $\mathfrak{q}$ for which $\mathfrak{q} \cap A = \mathfrak{p}$. By definition, we have $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Therefore, every prime ideal of $V(\mathfrak{a})$ gets mapped to a prime ideal in $V(f^{-1}(\mathfrak{a})) \subseteq \operatorname{Spec}(A)$. That is, $f^*(V(\mathfrak{a})) = V(f^{-1}(\mathfrak{a}))$ which is a closed set of $\operatorname{Spec}(A)$. This shows $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a closed mapping. $\square$

Exercise 5.2

PROOF. Easy exercise using Zorn's Lemma. $\square$

Exercise 5.3 Let $f : B \to B'$ be a homomorphism of A-algebras, and let C be an A-algebra. If f is integral, prove that $f \otimes 1 : B \otimes_A C \to B' \otimes_A C$ is integral. (This includes (5.6)ii) as a special case.)

Proof. If $f : B \to B'$ is integral, then B' is an integral B-algebra. To show $f \otimes 1 : B \otimes_A C \to B' \otimes_A C$ is integral, let $b' \otimes c \in B' \otimes_A C$. Assume the monic equation that b' satisfies is $\sum_{i=0}^n b_i x^i$ where $b_i \in f(B)$. Then the desired equation for integral dependence is $\sum_{i=0}^n (b_i \otimes c^{n-i}) x^i$. Indeed,

$$\sum_{i=0}^n (b_i \otimes c^{n-i})(b' \otimes c)^i = \sum_{i=0}^n (b_i b'^i \otimes c^n) = \left(\sum_{i=0}^n b_i b'^i\right) \otimes c^n = 0 \otimes c^n = 0.$$

Now that we have the result for pure tensors, we observe that the sum of two integral elements is till integral and linear multiples of integral elements are still integral. This follows from the fact that the set of integral elements forms a ring. $\square$

Exercise 5.4 Let A be a subring of a ring B such that B is integral over A. Let $\mathfrak{n}$ be a maximal ideal of B and let $\mathfrak{m} = \mathfrak{n} \cap A$ be the corresponding maximal ideal of A. Is $B_\mathfrak{n}$ necessarily integral over $A_\mathfrak{m}$?

[Consider the subring $k[x^2-1]$ of $k[x]$, where k is a field, and let $\mathfrak{n} = (x-1)$. Can the element $1/(x+1)$ be integral?]

Proof. No and a counterexample is provided for in the hint. Consider the subring $A := k[x^2-1]$ of $B := k[x]$. Clearly $\mathfrak{n}$ is a maximal ideal in A since $x-1$ is irreducible over k. We show that $k[x]_{(x-1)}$ is not integral over $k[x^2-1]_{(x-1)\cap k[x^2-1]}$ by showing that $\frac{1}{x+1}$ is not integral. Observe that $\mathfrak{m} := (x-1) \cap k[x^2-1] = (x^2-1)$ inside $k[x^2-1]$.

For the element $\frac{1}{(x+1)}$ to be integral, it would need to satisfy a monic equation in $A_\mathfrak{m}$. So suppose

$$\frac{1}{(x+1)^n} + \frac{a_1}{b_1}\frac{1}{(x+1)^{n-1}} + \cdots + \frac{a_n}{b_n} = 0$$

where $\frac{a_i}{b_i} \in A_\mathfrak{m}$. This gives

$$1 + \frac{a_1}{b_1}(x+1) + \cdots + \frac{a_n}{b_n}(x+1)^n = 0.$$

Clearing denominators gives

$$c_0 + a_1 c_1 (x+1) + \cdots + a_n c_n (x+1)^n = 0$$

where $c_i \in A \setminus \mathfrak{m}$. Observe that this implies $x+1$ divides $c_0 \in A \setminus \mathfrak{m}$ as elements of $k[x]$. That means $c_0 \in (x+1)$, the ideal generated by $x+1$ in A. Thus, $c_0 \in \mathfrak{n}$ which implies $c_0 \in \mathfrak{m}$ whcih is a contradiction.

Alternatively, by adding by -1

$$\frac{a_1}{b_1}(x+1) + \cdots + \frac{a_n}{b_n}(x+1)^n = -1$$

and we deduce that $(x+1)$ divides -1 in $A_\mathfrak{m}$. That means $\frac{-1}{(x+1)} = \frac{a}{b} \in A_\mathfrak{m}$. But observe that this is impossible because the denominator and numerator of elements of $A_\mathfrak{m}$ consist of polynomials with even degree while the LHS $\frac{-1}{(x+1)}$ has total degree -1.

$\square$

Exercise 5.5

Proof. Follow nose. Did it before. Second statement you use a the result in 1.

(i) Assume $x \in A$ is a unit in B. Let $x^{-1} \in B$ be its inverse. We know x^{-1} is integral over A so we may write

$$x^{-n} + a_1 x^{-n+1} + \cdots + a_{n-1}x^{-1} + a_n = 0.$$

Multiply across by x^n to get

$$1 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n = 0.$$

After some rewriting, we deduce that

$$1 = -x(a_1 + \cdots + a_{n-1}x^{n-2} + a_n x^{n-1}) = 0.$$

However, $-(a_1 + \cdots + a_{n-1}x^{n-2} + a_n x^{n-1})$ lays entirely in A so that means $x^{-1} \in A$.
(ii) Recall from AM Proposition 1.9 that x is in the Jacobson radical of a ring iff $1 - xy$ is a unit for all elements y in the ring. Let $\mathfrak{J}$ be the Jacobson radical of B and $\mathfrak{K}$ be the Jacobson radical of A. It is clear that $\mathfrak{K} \subseteq \mathfrak{J}^c$ so we prove $\mathfrak{K} \supseteq \mathfrak{J}^c$. Let $x \in \mathfrak{J}^c$ i.e. if $i : A \hookrightarrow B$, then $i(x) \in \mathfrak{J}$. That means $1 - i(x)y = 1 - xy$ is a unit in B for all $y \in A$. But part (i) then says $1 - xy$ is a unit in A for all y. Proposition 1.9 says $x \in \mathfrak{K}$. □

Exercise 5.6

Proof. Clear; by induction, consider the case of $n = 2$. Then $(b_1, b_2) \in B_1 \times B_2$ satisfies $f_1 \times f_2$ where f_1 and f_2 are equations of integral dependence for b_1 and b_2 over A respectively. □

Exercise 5.7

Proof. Let $A \subseteq B$ be a subring and assume $B - A$ is closed under multiplication. Let $x \in B$ be an element integral over A. We show $x \in A$. We know

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

for $a_i \in A$. □

Exercise 5.9 Let A be a subring of a ring B and let C be the integral closure of A in B. Prove that $C[x]$ is the integral closure of $A[x]$ in $B[x]$.

[If $f \in B[x]$ is integral over $A[x]$, then

$$f^m + g_2 f^{m-1} + \cdots + g_m = 0 \quad (g_i \in A[x])$$

Let r be an integer larger than m and the degrees of $g_1, \ldots, g_m$, and let $f_1 = f - x^r$, so that

$$(f_1 + x^r)^m + g_1 (f + x^r)^{m-1} + \cdots + g_m = 0$$

or say

$$f_1^m + h_1 f_1^{m-1} + \cdots + h_m = 0$$

where $h_m = (x^r)^m + g_1 (x^r)^{m-1} + \cdots + g_m \in A[x]$. Now apply Exercise 8 to the polynomials $-f_1$ and $f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}$.]

Proof. We must show that if $f \in B[x]$ is integral over $A[x]$, then $f \in C[x]$ and since most of the hint is written out for us, we shall focus on explicating the nontrivial details. The relation $f_1^m + h_1 f^{m-1} + \ldots h_m = 0$ follows from the preceding relation by expanding each term in the preceding equation via the binomial formula and then setting $h_i := (x^r)^i + g_1(x^r)^{i-1} + \cdots + g_i$. Next,

$$-f_1(f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}) = h_m \in A[x] \subseteq C[x]$$

and each factor on the LHS lie inside $B[x]$. Indeed, $f_1 = f - x^r \in B[x]$ while the other factor has terms $f_1 \in B[x]$ and $h_i \in A[x] \subseteq B[x]$ so it must also lie in $B[x]$. Then Exercise

5.8ii, which has no requirement on the rings, tells us that $-f_1 \in B[x]$. Of course, this means $f = x^r + f_1 \in B[x]$. □

Exercise 5.14

PROOF. **[Incomplete]** □

Exercise 5.15

PROOF. **[Incomplete]** □

Exercise 5.28 Let A be an integral domain, K its field of fractions. Show that the following are equivalent:

(1) A is a valuation ring of K;

(2) If $\mathfrak{a}, \mathfrak{b}$ are any two ideals of A, then either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$.

Deduce that if A is a valuation ring and $\mathfrak{p}$ is a prime ideal of A, then $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are valuation rings of their fields of fractions.

PROOF. [21]

(1) $\Longrightarrow$ (2) Assume A is a valuation ring of K and $\mathfrak{a}, \mathfrak{b} \subseteq A$ are ideals. Suppose $\mathfrak{b} \not\subseteq \mathfrak{a}$. Then choose $x \in \mathfrak{b} \setminus \mathfrak{a}$. Suppose $y \in \mathfrak{a}$. We show $y \in \mathfrak{b}$.

We know $\frac{x}{y} \in A$ or $\frac{y}{x} \in A$. If $\frac{y}{x} \in A$, we know $y = x \cdot \frac{y}{x} \in \mathfrak{b}$ since $x \in \mathfrak{b}$. Meanwhile, if $\frac{x}{y} \in A$, we get $x = y \cdot \frac{x}{y} \in \mathfrak{a}$ since $y \in \mathfrak{a}$ which contradicts $x \notin \mathfrak{a}$.

(2) $\Longrightarrow$ (1) Let $\frac{x}{y} \in K$ be a nonzero element and we must show $\frac{x}{y} \in B$ or $\frac{y}{x} \in B$. But the ideals of B are totally ordered by inclusion so $(x) \subseteq (y)$ or $(y) \subseteq (x)$. WLOG assume $(x) \subseteq (y)$ which means $x = my$ for some $m \in B$ and so, $\frac{x}{y} = \frac{my}{y} = m \in B$.

Remaining Statements:

(a): Assume A is a valuation ring and $\mathfrak{p}$ is a prime ideal of A. To show $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are valuation rings in their field of fractions, we show they both satisfy (2).

The maximal ideal of $A_{\mathfrak{p}}$ is $\mathfrak{p}A_{\mathfrak{p}}$. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of $A_{\mathfrak{p}}$ and they are contained in $\mathfrak{p}A_{\mathfrak{p}}$. Suppose $\mathfrak{a} \not\subseteq \mathfrak{b}$ and then, there exists $x \in \mathfrak{a} \setminus \mathfrak{b}$. Let $y \in \mathfrak{b}$ and we show $y \in \mathfrak{a}$. Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$ where we assume $y \neq 0$ since we are done if so. It suffices to show $c \in \mathfrak{a}$. Consider the principal ideals (a) and (c) generated in A. Then $(c) \subseteq (a)$ because if not, $(a) \subseteq (c)$ which means $a = mc$ for some m and then $x = \frac{mc}{b} \in \mathfrak{b}$ since it is multiple of $y \in \mathfrak{b}$. But if $(c) \subseteq (a)$, then $c = na$ for some n and then $y = \frac{na}{d} \in \mathfrak{a}$ because $y = \frac{bn}{1}\frac{a}{b} = \frac{bn}{1}x \in \mathfrak{a}$. This shows that $\mathfrak{b} \subseteq \mathfrak{a}$ as desired.

(b): First, $A/\mathfrak{p}$ is an integral domain so we can use the equivalence from before. The ideals of $A/\mathfrak{p}$ are in correspondence with those of A containing $\mathfrak{p}$. So if $\overline{\mathfrak{a}}, \overline{\mathfrak{b}}$ are ideals of $A/\mathfrak{p}$, then they can be lifted to ideals $\mathfrak{a}, \mathfrak{b}$ of A and then $\mathfrak{a} \subseteq \mathfrak{b}$ of $\mathfrak{b} \subseteq \mathfrak{a}$ which implies (2) for $A/\mathfrak{p}$. □

Exercise 5.31
Exercise 5.32
Exercise 5.33

[21]Valuation rings are relatively new concepts for me and I was curious on what they are useful for (I know number theorists love these things). So, here's a nice reference from Ash's Commutative Algebra here which talks about uniformizers a bit more than AM.

6. Chain Conditions

Exercise 6.1 i) Let M be a Noetherian A -module and $u : M \to M$ a module homomorphism. If u is surjective, then u is an isomorphism.

ii) If M is Artinian and u is injective, then again u is an isomorphism.

[For (i), consider the submodules $\operatorname{Ker}(u^n)$; for (ii), the quotient modules Coker (u^n) .]

PROOF. **i)** Let M be a Noetherian A-module and $u : M \to M$ be a surjective module homomorphism. Consider the submodules $\ker(u^n)$ for $n \in \mathbb{N}$. We have an ascending chain of submodules

$$\ker(u) \subseteq \ker(u^2) \subseteq \ker(u^3) \subseteq \dots \tag{2}$$

and by M Noetherian, this chain terminates. So choose an i s.t. $\ker(u^i) = \ker(u^{i+1})$. Suppose not and $\ker(u) \neq 0$ and choose an $m \in M \setminus 0$ s.t. $u(m) = 0$. By surjectivity, we can find an l s.t. $u^i(l) = m$. Then, $u^{i+1}(l) = 0$ but $l \notin \ker(u^i)$ which is a contradiction of $\ker(u^i) = \ker(u^{i+1})$.

ii)[22] Let M be Artinian and u injective. The descending chain of submodules

$$\operatorname{im}(u) \supseteq \operatorname{im}(u^2) \supseteq \dots$$

terminates so $\operatorname{im}(u^i) = \operatorname{im}(u^{i+1})$ for all large i. For any $n \in M$, there exists $l \in M$ s.t. $u^i(n) = u^{i+1}(l)$. Then,

$$u^i(n) - u^{i+1}(l) = 0 \qquad \Longrightarrow \qquad u^i(n - u(l)) = 0.$$

Injectivity says $n = u(l)$. This means $u : M \to M$ is surjective and therefore, an isomorphism. □

Exercise 6.2

PROOF. Let M be an A-module. Assume every non-empty set of f.g. submodule has a maximal element. To show that M is Noetherian, it suffices, by Proposition 6.2, to every submodule of M is finitely generated. Let N be a submodule of M. Let Σ be the set of submodules of N that are finitely generated. Then there is a maximal element $N^* \in \Sigma$. It suffices to show $N^* = M$.

Suppose $N^* \subsetneq M$. Then, there is an element of $m \in M$ for which $m \notin N^*$. But since $N^* = (m_1, \dots, m_n)$ is finitely generated, we can form a new module $N^{**} = (m_1, \dots, m_n, m)$ which contradicts the maximality of N^*.

Therefore, we must have $N^* = M$ and hence, M is a f.g. A-module. □

Exercise 6.3

PROOF. Consider the exact sequence

$$0 \to M/N_1 \to M/(N_1 \cap N_2) \to M/N_2 \to 0. \tag{3}$$

Because M/N_1 and M/N_2 are Noetherian (resp. Artinian), we conclude by Proposition 6.3 that $M/(N_1 \cap N_2)$ is Noetherian (resp. Artinian). □

[22] I am unsure what the cokernel can do for us here. The quotient modules are not necessarily submodules of M and observing that $\operatorname{coker}(u^i) = \operatorname{coker}(u^{i+1})$ doesn't seem to give any more insight than observing the image.

Exercise 6.4 Let M be a Noetherian A-module and let $\mathfrak{a}$ be the annihilator of M in A. Prove that $A/\mathfrak{a}$ is a Noetherian ring. If we replace "Noetherian" by "Artinian" in this result, is it still true?

Proof. Since M is a Noetherian A-module, it is a finitely generated A-module so take $M = \sum_{i=1}^{n} Am_i$. Define an A-module homomorphism $\varphi : A \to M^n$ via $1 \mapsto (m_1, \dots, m_n)$ and extend A-linearly. Then the kernel of this map is precisely $\mathfrak{a}$ since $a \mapsto (am_1, \dots, am_n) = 0$ for $a \in \mathfrak{a}$. So we have an A-module isomorphism $A/\mathfrak{a} \cong \varphi(A)$. But M^n is a direct sum of Noetherian A-modules so $\varphi(A)$ is also[23] Noetherian as an A-module. Passing to the quotient implies $A/\mathfrak{a}$ is Noetherian as an $A/\mathfrak{a}$-module and therefore, as a ring.

The result is not true with an Artinian hypothesis despite the fact that we used Proposition 6.3, Corollary 6.4, and Proposition 6.6 which all hold with the Artinian hypothesis. The issue is that we claimed that M is a f.g. A-module which is not necessarily true if M were Artinian. $\square$

Exercise 6.5

Proof. This is an exercise in Hartshorne I.1. In particular, it is exercise I.1.7(b) and I.1.7(c). $\square$

Exercise 6.6

Proof. The implications (i) $\implies$ (iii) $\implies$ (ii) $\implies$ (i) will be our approach to the problem. The fact that (i) $\implies$ (iii) follows immediately by the preceding exercise. The statement (iii) $\implies$ (ii) is trivial. So the difficulty is in showing (ii) $\implies$ (i).

Assume every open subspace of X is quasi-compact. Let

$$U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \dots$$

be an ascending chain of open subsets of X. Now we consider the open subset $V := \bigcup_{i=0}^{\infty} U_i$ and by the proper containments, the set $\{U_i\}_{i \in \mathbb{N}_0}$ forms an open cover of V. By quasi-compactness of V, there is a finite subcover, say $U_{i_1}, \dots, U_{i_k}$. Now assuming WLOG U_{i_k} is the open subset with $i_k \in \mathbb{N}_0$ the largest value, we deduce that $U_{i_k} = V$. So the ascending chain actually terminated. $\square$

Exercise 6.7

Proof. See Proposition I.1.5 of [3]. $\square$

Exercise 6.8

Proof. Suppose A is Noetherian. To show Spec(A) is a Noetherian topological space, consider a descending chain of closed subsets

$$Z_1 \supseteq Z_2 \supseteq Z_3 \supseteq \dots . \tag{4}$$

By definition, such closed subsets are of the form $Z_i = V(I_i)$ for some ideal I_i[24]. Therefore, we have a descending chain of closed subsets

$$V(I_1) \supseteq V(I_2) \supseteq V(I_3) \supseteq \dots \tag{5}$$

[23]This uses Proposition 6.3. Indeed, we consider the exact sequence $0 \to \ker \psi \to M^n \to^{\psi} \varphi(M) \to 0$.

[24]Exercise 1.15 says that $Z_i = V(E_i)$ for some *arbitrary subset* $E_i \subseteq A$. However, $V(E) = V(\langle E \rangle)$ where $\langle E \rangle$ is the ideal generated by E¿

which is associated to an ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots \tag{6}$$

and because A is Noetherian, the chain of ideals terminates. Therefore, the chain (5) terminates which shows that $\operatorname{Spec}(A)$ is a Noetherian topological space.

The converse is false in general. Consider the polynomial ring in infinitely many variables

$$A := k[x_1, x_2, x_3, \dots]/(x_1^3, x_2^3, x_3^3, \dots). \tag{7}$$

This is clearly a nonnoetherian ring because the chain of ideals

$$(x_1) \subseteq (x_1, x_2) \subseteq (x_1, x_2, x_3) \subseteq \dots \tag{8}$$

does not terminate. On the other hand, $\operatorname{Spec}(A)$ is Noetherian because there is only one closed subset. Namely, the set containing the prime ideal $\mathfrak{p} = (x_1, x_2, x_3, \dots)$. Indeed, $\mathfrak{p}$ is the only prime ideal that contains $(x_1^3, x_2^3, x_3^3, \dots)$. □

Exercise 6.9

Proof. Suppose $\mathfrak{p}$ is a minimal prime ideal in a Noetherian ring A. Then, $V(\mathfrak{p})$ is a maximal, irreducible closed subset of $\operatorname{Spec}(A)$. Let $\mathscr{S}$ be the set of all minimal prime ideals of A. We form a chain from ideals $\mathfrak{p}_i \in \mathscr{S}$:

$$V(\mathfrak{p}_1) \supseteq V(\mathfrak{p}_1, \mathfrak{p}_2) \supseteq V(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3) \supseteq \dots. \tag{9}$$

By Exercise 6.8, we know that $\operatorname{Spec}(A)$ is Noetherian and hence, the above chain terminates. Therefore, there is some i for which

$$V(\mathfrak{p}_1, \dots, \mathfrak{p}_i) = V(\mathfrak{p}_1, \dots, \mathfrak{p}_{i+1}).$$

By definition, $\mathfrak{p}_{i+1} \in V(\mathfrak{p}_1, \dots, \mathfrak{p}_i)$ and hence, $\mathfrak{p}_{i+1}$ is not minimal. Thus, $\mathscr{S}$ is a finite set. □

Exercise 6.10 Let A be a ring such that $\operatorname{Spec}(A)$ is a Noetherian space. Show that the set of prime ideals of A satisfies the ascending chain condition. Is the converse true?

Proof. Let $\operatorname{Spec}(A)$ be a Noetherian topological space. That means it satisfies the ascending chain condition on open subsets and equivalently, the descending chain condition on closed subsets.

Now consider sets of prime ideals of A. Assume it is ascending i.e. if

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \dots$$

is an ascending chain of prime ideals, then it terminates. Of course, that means we have a descending chain of closed subsets

$$V(\mathfrak{p}_0) \supsetneq V(\mathfrak{p}_1) \supsetneq V(\mathfrak{p}_2) \dots$$

and so it stabilizes $V(\mathfrak{p}_n) = V(\mathfrak{p}_{n+1})$ for all $n \geq N$ and some N. Taking $I(-)$, we get $I(V(\mathfrak{p}_n)) = I(V(\mathfrak{p}_{n+1}))$ for all $n \geq N$. By Hilbert's Nullstellensatz, $I(V(\mathfrak{p})) = \mathfrak{p}$ when $\mathfrak{p}$ is prime which means $\mathfrak{p}_n = \mathfrak{p}_{n+1}$ for $n \geq N$.

As a minor remark, $\operatorname{Spec}(A)$ is a Noetherian topological space iff A satisfies the ascending chain condition on *radical ideals*. See this post. □

Exercise 6.11

PROOF. Assume M is a Noetherian module. Then every submodule $N \subseteq M$ is finitely generated. But that means $\mathrm{Supp}(N) = V(\mathrm{Ann}(N))$ by Exercise 3.19.

We start by showing $\mathrm{Supp}(M)$ is a closed subspace. It is clear that $M_{\mathfrak{p}} \neq 0$ iff $N_{\mathfrak{p}} \neq 0$ for every submodule $N \subseteq M$. $\square$

Exercise 6.12

PROOF. Let A be a ring s.t. $\mathrm{Spec}(A)$ is a Noetherian space. Given an ascending chain of prime ideals

$$\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \mathfrak{p}_3 \dots \tag{10}$$

and take $V(\cdot)$ and consider the descending chain of irreducible closed subsets

$$V(\mathfrak{p}_1) \supseteq V(\mathfrak{p}_2) \supseteq V(\mathfrak{p}_3) \supseteq \dots . \tag{11}$$

Because $\mathrm{Spec}(A)$ is Noetherian, the above chain terminates and $V(\mathfrak{p}_i) = V(\mathfrak{p}_{i+1})$ for some $i \in \mathbb{N}$. But $V(\mathfrak{p}_i)$ is the set of prime ideals containing $\mathfrak{p}_i$ and $\mathfrak{p}_i$ is prime and so, $\mathfrak{p}_i = \mathfrak{p}_{i+1}$. Therefore, A has the ACC on the set of prime ideals.

The converse is true: see Exercise 2.22 of [2].

$\square$

7. Noetherian rings

Exercise 7.1

PROOF. Let Σ be the set of ideals in A which are not f.g. and assume A is non-Noetherian. Because a ring is Noetherian iff every ideal is f.g., we know there is an ideal that is not finitely generated in A. Then, Σ is nonempty. Certainly, Σ is nonempty because A.To see that Σ has a maximal element, order the set by inclusion. Now, if we for any chain $\{\mathfrak{a}_\alpha\}_{\alpha \in A}$, we can get a maximal element by taking $(\cup_{\alpha \in A} \mathfrak{a}_\alpha)$, the ideal generated by the union of the ideals. By Zorn's Lemma, there is a maximal element for the set Σ.

Let $\mathfrak{a} \in \Sigma$ be a maximal element. Suppose $\mathfrak{a}$ were not prime. Then there are $x, y \notin \mathfrak{a}$ for which $xy \in \mathfrak{a}$. Therefore, there is a f.g. ideal $\mathfrak{a}_0$ containing y for which $\mathfrak{a}_0 + (x) = \mathfrak{a} + (x)$. Multiplying through by $(\mathfrak{a} : x)$, we get

$$\mathfrak{a} = \mathfrak{a}_0 + x \cdot (\mathfrak{a} : x). \tag{12}$$

Because $(\mathfrak{a} : x)$ strictly contains $\mathfrak{a}$ and $\mathfrak{a}$ is maximal in Σ, we conclude that $(\mathfrak{a} : x)$ is f.g. and $\mathfrak{a}$ is as well. Contradiction.

We have shown that if A is not Noetherian, then Σ has maximal not f.g. ideals which are prime ideals of A. Conversely, if does Σ not have prime ideals, then A is Noetherian. Equivalently, if every prime ideal of A is f.g., then A is Noetherian. $\square$

Exercise 7.2

PROOF. Suppose A is Noetherian and $f = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$.

Suppose f is nilpotent. Therefore, $f^m = 0$ for some m. Use the Cauchy product for infinitely series and note that each coefficient of terms of f^n has to be zero. But this implies that each a_n is nilpotent.

Conversely, assume each a_n is nilpotent. Then by the Noetherian hypothesis, we can choose an m sufficiently large for which $f^m = 0$. $\square$

Exercise 7.3

Proof. □

Exercise 7.4 Which of the following rings are Noetherian?

i) The ring of rational functions of z having no pole on the circle $|z| = 1$.

ii) The ring of power series in z with a positive radius of convergence.

iii) The ring of power series in z with an infinite radius of convergence.

iv) The ring of polynomials in z whose first k derivatives vanish at the origin (k being a fixed integer).

v) The ring of polynomials in z, w all of whose partial derivatives with respect to w vanish for $z = 0$.

In all cases the coefficients are complex numbers.

Proof. **i)** Yes the ring R is Noetherian. We construct the ring R as the localization of $k[z]$. Let S be the multiplicatively closed subset consisting of $f(z) \in k[z]$ with no zeros on $|z| = 1$. This is multiplicatively closed because if $f(z), g(z) \in S$, then $f(z) \cdot g(z) \in S$ since the product has the set of zero equal to the union of the set of zeros of $f(z)$ and $g(z)$. Meanwhile, $1 \in S$ because it is nonzero on all of $\mathbb{C}$. Then, $R = S^{-1}k[z]$ and since $k[z]$ is Noetherian, R is as well.

ii)[25] Yes, this is Noetherian. If R is given ring, then we may identify the ring R with the ring of power series which are centered at the origin. Then, observe that every $f(z) \in R$ has a factorization $f(z) = z^k h(z)$ for some $h(z) \in R$ and $k \geq 0$. Observe that $(z^m) \subseteq (z^n)$ whenever $n \leq m$. Now, if we have an ascending chain of ideals $(z^{n_1}) \subseteq (z^{n_2}) \subseteq \dots$ then the chain must terminate because $n_1 \geq n_2 \geq \dots \geq 0$. Now, if we consider an arbitrary chain of ideals $I_1 \subseteq I_2 \subseteq \dots$ inside R, then we can identify the ideals I_n with (z^{m_n}) for some m_n since each $f(z) \in I_n$ can be written as $z^k h(z)$ where $h(z)$ is nonzero at $z = 0$ with positive radius of convergence. Then take m_n to be the minimum k which appears and so $I_n \subseteq (z^{m_n})$ while $(z^{m_n}) \subseteq I_n$ because $z^{m_n} = z^{m_n} h(z) \cdot \frac{1}{h(z)} \in I_n$. Indeed, if $h(z)$ has positive radius of convergence and is nonzero at $z = 0$, then $\frac{1}{h(z)}$ also has positive radius of convergence about $z = 0$. The result we used here is a standard complex analytic result[26]. This implies $I_n = (z^{m_n})$ and then the termination of the chain follows by the same argument as above.

iii) No, this is not Noetherian. Let R be the given ring. Define ideals $I_N := \{f \in R : \forall |n| \geq N, f(2\pi n) = 0\}$. Clearly, if $f, g \in I_n$, then $f + g \in I_N$ since the set of zeros is the intersection of the set of zeros of f and g respectively. Meanwhile, if $f \in I_N$ and $h \in R$, then $f \cdot h \in I_N$ since $f(2\pi n)h(2\pi n) = 0$ for all $|n| \geq N$. We claim that $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$ is a strictly increasing sequence of ideals. The containments are clear so we just show equality doesn't hold in general. Define $f_N(z) := \frac{\cos(z)}{\prod_{k=-N}^{N}(z - 2\pi k)}$ but observe that we have removable singularities[27] at $z = 2\pi k$, so we can extend $f_N(z)$ is an analytic function at $z = 2\pi k$. Hence, $f_N(z)$ is an entire function and so $f_N(z) \in R$. Furthermore, we observe that $f_N(z) \in I_{N+1}$ but $f_N(z) \notin I_N$ because $f_N(2\pi N) \neq 0$ and $f_N(-2\pi N) \neq 0$.

iv) Yes. The only way for the first k derivatives to vanish at the origin is for there to be no terms in $z, z^2, \dots, z^{k-1}, z^k$ appearing. So the ring in question is $R := \mathbb{C} + z^{k+1}\mathbb{C}[z]$. It suffices

[25] I was honestly stumped on this problem for a while. Failing to come up with an example, I looked around online for a hint and all of the proofs used sheaf theory. Personally, I didn't want to use sheaves or germs because I doubt that was how Atiyah-Macdonald intended for this to be done. The sheaf idea did bring up the idea to use the result from Churchill and Brown.

[26] See Churchill and Brown's "Complex Variables and Applications" 9th Edition p. 248 or Section 82.

[27] See Churchill and Brown p. 239 (couldn't find in Conway).

then to show that $z^{k+1}\mathbb{C}[z]$ is Noetherian. Indeed, $\mathbb{C}+z^{k+1}\mathbb{C}[z]=\mathbb{C}[z^{k+1},z^{k+2},\ldots,z^{2k+1}]\cong\mathbb{C}[x_1,\ldots,x_m]$ and observe that the $z^{k+1},\ldots,z^{2k+1}$ are linearly independent so the last isomorphism with $m\in\mathbb{N}$ is valid[28]. Clearly, the polynomial ring is Noetherian so R itself is Noetherian.

v) No. Let S be the ring stated and we observe that $S\subseteq\mathbb{C}[z,w]$. For the partial derivatives of $f\in\mathbb{C}[z,w]$ w.r.t to w to vanish when $z=0$ there must be a multiple of z in every nonconstant term of f. Thus, $S:=\mathbb{C}[z,zw,zw^2,zw^3,\ldots]$. One can then consider the chain of ideals $(z)\subseteq(z,zw^2)\subseteq(z,zw^2,zw^3)\subseteq\ldots$ which never terminates. Thus, the ascending chain condition is not satisfied. □

Exercise 7.5

Proof. Trivial. □

Exercise 7.6 If a finitely generated ring K is a field, it is a finite field.

[If K has characteristic 0, we have $\mathbb{Z}\subset\mathbb{Q}\subseteq K$. Since K is finitely generated over $\mathbb{Z}$ it is finitely generated over $\mathbb{Q}$, hence by (7.9) is a finitely generated $\mathbb{Q}$-module. Now apply (7.8) to obtain a contradiction. Hence K is of characteristic $p>0$, hence is finitely generated as a $\mathbb{Z}/(p)$ -algebra. Use (7.9) to complete the proof.]

Proof. If K is characteristic zero, then $\mathbb{Z}\subseteq\mathbb{Q}\subseteq K$. Because K is a f.g. ring, it is a f.g. $\mathbb{Z}$-algebra ($\mathbb{Z}$ is an initial object in the category of rings). Since K is a f.g. $\mathbb{Z}$-algebra, the same generators turn K into a f.g. $\mathbb{Q}$-algebra. By Proposition 7.9, K is a finite algebraic extension of $\mathbb{Q}$. Proposition 7.8 then says that $\mathbb{Q}$ is a f.g. $\mathbb{Z}$-module which is absurd. Indeed, $\mathbb{Z}$ is Noetherian, C is a f.g. $\mathbb{Z}$-algebra, ad C is a f.g. $\mathbb{Q}$-vector space so Proposition 7.8 applies.

Since $\operatorname{char}K:=p>0$ must occur, we know K is a f.g. $\mathbb{Z}/(p)$-algebra. By Proposition 7.9, K is a finite algebraic extension of $\mathbb{Z}/(p)$ and therefore $K:=\mathbb{F}_{p^n}$ for some n. So K is a finite field. □

Exercise 7.7

Proof. □

Exercise 7.8 If $A[x]$ is Noetherian, is A necessarily Noetherian?

Proof. Yes.

Proof 1: Consider the quotient $A[x]/(x)\cong A$. This is clear since $\varphi:A[x]\to A$ define by $f\mapsto f(0)$ is surjective map which has kernel (x). Since A is then a homomorphic image of a Noetherian ring, $A[x]$ is Noetherian.

Proof 2:[29] Suppose $I_1\subseteq I_2\subseteq\ldots$ is an ascending chain of ideals in A. Let $\phi:A\hookrightarrow A[x]$ be the natural inclusion. Then $\{I_n^e\}_{n\in\mathbb{N}}$ is also an ascending chain of ideals in $A[x]$ and therefore, there is an $M\in\mathbb{N}$ for which $I_m^e=I_{m+1}^e$ holds for all $m\geq M$. Taking the contraction and observing $I_n^{ec}\supseteq I_n$, for all $m\geq M$, $I_m\subseteq I_m^{ec}=I_{m+1}^{ec}$. To finish the proof, it suffices to show $I_n^{ec}=I_n$. Indeed, the contraction of an ideal J of $A[x]$ is the ideal consisting of the coefficients of polynomials in J but the extension I^e does not introduce any polynomials

[28]To see this, one observes the degree of the elements. If we have z^{2k+2}, then we could just write $z^{2k+2}=z^kz^{k+2}$. Meanwhile, there is a no way to write z^{2k+1} as a linear combination using elements with degree between $k+1$ and $2k$.

[29]For some reason I was convinced this second proof would be quicker but discussing with Scotty changed my mind.

whose coefficients are not already in the ideal I. So, $I_m = I_m^{ec} = I_{m+1}^{ec} = I_{m+1}$ which means the chain $\{I_n\}_{n\in\mathbb{N}}$ in A terminates. □

Exercise 7.9

Proof. □

Exercise 7.10 Let M be a Noetherian A-module. Show that $M[x]$ (Chapter 2, Exercise 6) is a Noetherian $A[x]$-module.

Proof. The proof is essentially a repeat[30] the of Hilbert Basis Theorem except we show that a submodule is necessarily finitely generated.

Let $M' \subseteq M[x]$ be a submodule and the leading coefficients of elements of M' clearly form a submodule $M'' \subseteq M$. Since M is Noetherian, choose generators $M'' = \sum_{i=0}^n Am_i$ and for each i, there is an $f_i \in M[x]$ of the form $f_i = m_i x^{r_i} +$ (lower order terms). Set $r := \max_{i=1}^n r_i$ and then the f_i generate a submodule $N \subseteq M'$ in $M[x]$.

If $f = ax^m +$ (lower order terms) $\in M'$, we know $a \in M''$. Set $a := \sum_{i=1}^n u_i a_i$ with $u_i \in M$ and then $f - \sum u_i f_i x^{m-r_i}$ is in M and has degree strictly less than m. Repeated application and finiteness allows us to deduce that $f = g + h$ where $h \in N$ and $\deg g < r$.

Let L be the submodule of $M[x]$ generated by $1, x, \dots, x^{r-1}$ and the decomposition shows that $M' = (M' \cap L) + N$. Since L is f.g. as an $A[x]$-module, it is Noetherian and hence, $M' \cap L$ is f.g. as an $A[x]$-module. Furthermore, N is f.g. as an $A[x]$-module so that means M' itself is f.g. as an $A[x]$-module. Thus, M is a Noetherian $A[x]$-module. □

Exercise 7.11

Proof. No, it is not necessarily the case that A is Noetherian. Consider the ring

$$A := k[x_1, x_2, x_3, \dots]/(x_1^2, x_2^2, x_3^2, \dots). \tag{13}$$

Certainly, A is nonnoetherian because the ideal $(x_1, x_2, x_3, \dots)$ is not finitely generated. On the other hand, the localization at a prime ideal is Noetherian. All prime ideals are generated by elements x_i's. Consider $\mathfrak{p} = (x_1)$. Then,

$$A_{\mathfrak{p}} := k[x_1]/(x_1^2) \tag{14}$$

because by localizing at $\mathfrak{p}$, we have just inverted zero divisors $x_2, x_3, \dots$ and therefore, $A_{\mathfrak{p}}$ is isomorphic to $k[x_1]/(x_1^2)$. □

Exercise 7.12

Proof. □

Exercise 7.13

Proof. □

Exercise 7.14 Let k be an algebraically closed field, let A denote the polynomial ring $k[t_1, \dots, t_n]$ and let $\mathfrak{a}$ be an ideal in A. Let V be the variety in k^n defined by the ideal $\mathfrak{a}$, so that V is the set of all $x = (x_1, \dots, x_n) \in k^n$ such that $f(x) = 0$ for all $f \in \mathfrak{a}$. Let $I(V)$ be the ideal of V, i.e. the ideal of all polynomials $g \in A$ such that $g(x) = 0$ for all $x \in V$. Then $I(V) = r(\mathfrak{a})$.

[30]Unfortunately, I could not deduce a way to prove this result without repeating the argument. It isn't clear how to use the tensor product to somehow import the Hilbert Basis Theorem for free.

[It is clear that $r(\mathfrak{a}) \subseteq I(V)$. Conversely, let $f \notin r(\mathfrak{a})$, then there is a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ such that $f \notin \mathfrak{p}$. Let $\overline{f}$ be the image of f in $B = A/\mathfrak{p}$, let $C = B_f = B[1/\overline{f}]$, and let $\mathfrak{m}$ be a maximal ideal of C. Since C is a finitely generated k-algebra we have $C/\mathfrak{m} \cong k$, by (7.9). The images x_i in $C/\mathfrak{m}$ of the generators t_i of A thus define a point $x = (x_1, \ldots, x_n) \in k^n$, and the construction shows that $x \in V$ and $f(x) \neq 0$.]

Proof. [31] Clearly $r(\mathfrak{a}) \subseteq I(V)$ since $\mathfrak{a} \subseteq I(V)$ and $f^n(x) = 0$ implies $f(x) = 0$ for all $x \in V$.

For the converse, we show the contrapositive so suppose $f \notin r(\mathfrak{a})$. Since $r(\mathfrak{a})$ is an intersection of prime ideals containing $\mathfrak{a}$, choose a prime ideal $\mathfrak{p}$ s.t. $\mathfrak{a} \subseteq \mathfrak{p}$ and $f \notin \mathfrak{a}$. Then set $C := B_f$ as stated in the problem and $\mathfrak{m}$ a maximal ideal of C. Since C is a quotient of a polynomial ring in k, it is a f.g. k-algebra. By Corollary 7.10, $C/\mathfrak{m} \cong k$. Let x_i be the image of t_i under the composition of maps $A \to B \to B_f =: C$. Consider the point in affine space $x := (x_1, \ldots, x_n) \in k^n$. Clearly $x \in V$ because for all $g \in \mathfrak{a}$, we get $g(x) \in \mathfrak{p}$ which is zero in B and hence, in C. However, $f(x) \neq 0$ in C because $f \notin \mathfrak{m}$ and so $f(x) \neq 0$ in $C/\mathfrak{m}$ when evaluated at x. □

Exercise 7.15 Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal and k its residue field, and let M be a finitely generated A -module. Then the following are equivalent:

i) M is free;
ii) M is flat;
iii) the mapping of $\mathfrak{m} \otimes M$ into $A \otimes M$ is injective;
iv) $\operatorname{Tor}_1^A(k, M) = 0$.

[To show that iv) $\implies$ i), let $x_1, \ldots, x_n$ be elements of M whose images in $M/\mathfrak{m}M$ form a k-basis of this vector space. By (2.8), the x_i generate M. Let F be a free A-module with basis $e_1, \ldots, e_n$ and define $\phi : F \to M$ by $\phi(e_i) = x_i$. Let $E = \ker(\phi)$. Then the exact sequence $0 \to E \to F \to M \to 0$ gives us an exact sequence

$$0 \longrightarrow k \otimes_A E \longrightarrow k \otimes_A F \xrightarrow{1\otimes\phi} k \otimes_A M \longrightarrow 0$$

Since $k \otimes F$ and $k \otimes M$ are vector spaces of the same dimension over k, it follows that $1 \otimes \phi$ is an isomorphism, hence $k \otimes E = 0$, hence $E = 0$ by Nakayama's Lemma (E is finitely generated because it is a submodule of F, and A is Noetherian).]

Proof. **i) $\Longrightarrow$ ii):** This is clear since free modules are flat.
ii) $\Longrightarrow$ iii): Since there is an injection $\mathfrak{m} \hookrightarrow A$, injectivity of $\mathfrak{m} \otimes M \to A \otimes M$ follows from definition of flatness which tells us that $\otimes M$ is a left exact functor.
iii) $\Longrightarrow$ iv): Consider the SES

$$0 \to \mathfrak{m} \to A \to k \to 0.$$

Then we have an induced LES of Tor modules

$$\ldots \to \operatorname{Tor}_1^A(k, M) \to \mathfrak{m} \otimes M \to A \otimes M \to k \otimes M \to 0.$$

and since $\mathfrak{m} \otimes M \to A \otimes M$ is injective, $\operatorname{Tor}_1^A(k, M) = 0$.
iv) $\Longrightarrow$ i): It follows by M f.g. as an A-module that we can find $x_1, \ldots, x_n$ whose images generate $M/\mathfrak{m}M$ and then Proposition 2.8 tells us that these x_i must generate M. Let

[31] The hint should say to use Corollary 7.10 rather than Proposition 7.9, but this is a minor nitpick. They're the same result.

$F := \bigoplus_{i=1}^{n} Ae_i$ and define $\phi : F \to M$ by $\phi(e_i) = x_i$. Set $E := \ker(\phi)$ and we have an exact sequence

$$0 \to E \to F \to M \to 0.$$

From the LES of Tor modules and the fact that $\mathrm{Tor}_1^A(k, M) = 0$, we have an exact sequence

$$0 \to k \otimes_A E \to k \otimes_A F \stackrel{1\otimes\phi}{\to} k \otimes_A M \to 0.$$

Since $k \otimes F$ and $k \otimes M \cong M/\mathfrak{m}M$ are vector spaces of the same dimension, it follows from the Rank-Nullity Theorem that the surjection $1 \otimes \phi$ is an isomorphism. Thus, $E/\mathfrak{m}E \cong k \otimes E = 0$ and so $E = \mathfrak{m}E$. Recall that E was a submodule of F which is f.g. so E is also f.g. and then applying Nakayama's Lemma tells us that $E = 0$. Since $\phi : F \to M$ was surjective, $E = 0$ implies that ϕ is an isomorphism. Hence, M is a free A-module. □

Exercise 7.16 Let A be a Noetherian ring, M a finitely generated A -module. Then the following are equivalent:

i) M is a flat A-module;

ii) $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module, for all prime ideals $\mathfrak{p}$;

iii) $M_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module, for all maximal ideals $\mathfrak{m}$. In other words, flat = locally free. [Use Exercise 15.]

Proof. **i) $\implies$ ii):** Let M be a flat A-module. The localization $M_\mathfrak{p}$ is a f.g. $A_\mathfrak{p}$-module and $A_\mathfrak{p}$ is a Noetherian local ring with residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} =: k$. Since M was flat as an A-module, $M_\mathfrak{p}$ is flat as an $A_\mathfrak{p}$-module since flatness is a local property (Proposition 3.10). The preceding exercise (in particular the equivalence of Exercise 7.15 (i) and (ii)) then tells us that $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module.

ii) $\implies$ iii): This is clear since maximal ideals are prime.

iii) $\implies$ i): Since $M_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ and free modules are flat, we know $M_\mathfrak{m}$ is a flat A-module for all maximal ideals of A. Since flatness is a local property that can be checked on maximal ideals (again by Proposition 3.10), we deduce that M is flat as an A-module. □

Exercise 7.17

Proof. □

Exercise 7.18

Proof. □

Exercise 7.19

Proof. □

Exercise 7.20

Proof. □

Exercise 7.21

Proof. □

Exercise 7.22

Proof. □

Exercise 7.23

Proof. $\square$

Exercise 7.24

Proof. $\square$

Exercise 7.25

Proof. $\square$

Exercise 7.26 Let A be a Noetherian ring and let $F(A)$ denote the šet of all isomorphism classes of finitely generated A-modules. Let C be the free abelian group generated by $F(A)$. With each short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated A-modules we associate the element $(M')-(M)+(M'')$ of C, where (M) is the isomorphism class of M, etc. Let D be the subgroup of C generated by these elements, for all short exact sequences. The quotient group C/D is called the Grothendieck group of A, and is denoted by $K(A)$. If M is a finitely generated A-module, let $\gamma(M)$, or $\gamma_A(M)$, denote the image of (M) in $K(A)$

i) Show that $K(A)$ has the following universal property: for each additive function λ on the class of finitely generated A-modules, with values in an abelian group G, there exists a unique homomorphism $\lambda_0 : K(A) \to G$ such that $\lambda(M) = \lambda_0(\gamma(M))$ for all M.

ii) Show that $K(A)$ is generated by the elements $\gamma(A/\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of A. [Use Exercise 18.]

iii) If A is a field, or more generally if A is a principal ideal domain, then $K(A) \cong \mathbb{Z}$

iv) Let $f : A \to B$ be a finite ring homomorphism. Show that restriction of scalars gives rise to a homomorphism $f_! : K(B) \to K(A)$ such that $f\,(\gamma_B(N)) = \gamma_A(N)$ for a B-module N. If $g : B \to C$ is another finite ring homomorphism, show that $(g \circ f)_! = f_! \circ g_!$.

Proof. **i)** This is straightforward. The additive function λ satisfies $\lambda(M'-M+M'') = 0$ for all $(M') - (M) + (M'')$ elements of D. So the factorization exists by the universal property of quotients for groups.
ii) Let (M) be an isomorphism class in $K(A)$. Then from Exercise 18, there is a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$$

s.t. $M_i/M_{i-1} \cong A/\mathfrak{p}_i$. The SESs $0 \to M_i \to M_{i+1} \to A/\mathfrak{p}_i \to 0$ show that $(M_i) - (M_{i+1}) + (A/\mathfrak{p}_i)$ is zero in $K(A)$. Then,

$$(M_r) = (M_{r-1}) + (A/\mathfrak{p}_{r-1}) = \cdots = \sum_{i=0}^{r-1} (A/\mathfrak{p}_{r-1}).$$

iii) We will just prove the result for when A is a PID. From the previous part, $K(A)$ is generated by $\gamma(A/(p))$ for p an irreducible element. It follows from $0 \to (p) \to A \to A/(p) \to 0$ that $(A/(p)) = (A) - (p)$ inside $K(A)$. Then, a SES

$$0 \to (p) \overset{\text{division by } p}{\to} A \to 0 \to 0$$

shows that $(A/(p)) = (A)$. So, $K(A)$ is the free abelian group generated by (A). So it is isomorphic to $\mathbb{Z}$.
iv) The condition that $f : A \to B$ be a finite ring homomorphism ensures that restriction of scalars of a finitely generated B-module M gives a finitely generated A-module M. So the statement $(g \circ f)_! = f_! \circ g_!$ is clear. $\square$

Exercise 7.27 Let A be a Noetherian ring and let $F_1(A)$ be the set of all isomorphism classes of finitely generated flat A-modules. Repeating the construction of Exercise 26 we obtain a group $K_1(A)$. Let $\gamma_1(M)$ denote the image of (M) in $K_1(A)$.

i) Show that tensor product of modules over A induces a commutative ring structure on $K_1(A)$, such that $\gamma_1(M) \cdot \gamma_1(N) = \gamma_1(M \otimes N)$. The identity element of this ring is $\gamma_1(A)$.

ii) Show that tensor product induces a $K_1(A)$-module structure on the group $K(A)$, such that $\gamma_1(M) \cdot \gamma(N) = \gamma(M \otimes N)$.

iii) If A is a (Noetherian) local ring, then $K_1(A) \cong \mathbb{Z}$.

iv) Let $f : A \to B$ be a ring homomorphism, B being Noetherian. Show that extension of scalars gives rise to a ring homomorphism $f^! : K_1(A) \to K_1(B)$ such that $f^!(\gamma_1(M)) = \gamma_1(B \otimes_A M)$. [If M is flat and finitely generated over A, then $B \otimes_A M$ is flat and finitely generated over B.] If $g : B \to C$ is another ring homomorphism (with C Noetherian), then $(f \circ g)^! = f^! \circ g^!$.

v) If $f : A \to B$ is a finite ring homomorphism then

$$f_!\left(f^!(x)y\right) = xf_!(y)$$

for $x \in K_1(A), y \in K(B)$. In other words, regarding $K(B)$ as a $K_1(A)$-module by restriction of scalars, the homomorphism $f^!$ is a $K_1(A)$-module homomorphism.

Remark. Since $F_2(A)$ is a subset of $F(A)$ we have a group homomorphism $\epsilon : K_1(A) \to K(A)$, given by $\epsilon(\gamma_1(M)) = \gamma(M)$. If the ring A is finite-dimensional and regular, i.e., if all its local rings $A_\mathfrak{p}$ are regular (Chapter 11) it can be shown that ϵ is an isomorphism.

PROOF. **i)** As we are working in $F_1(A)$, if we have a SES $0 \to M' \to M \to M'' \to 0$ of f.g. flat A-modules, then tensor by a f.g. flat A-module N gives a SES $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$. So the multiplicative structure is well-defined. The fact that $\gamma_1(A)$ is the identity follows from how $M \otimes_A A \cong M$ for all $(M) \in K_1(A)$.
ii) Checking that we have a $K_1(A)$-module structure is rather easy. Clearly, $K(A)$ is already an abelian groups o we just check properties the multiplication of $K_1(A)$ on $K(A)$. Clearly, $\gamma_1(M) \cdot \gamma(N) = \gamma(M \otimes N)$ gives a module $M \otimes N$ which is an element of $K(A)$. Furthermore, the tensor product is distributive over the addition operation in the group.
iii) If A is a Noetherian local ring, we can repeat ii) of the previous exercise to deduce that $K_1(A)$ is generated by $\gamma_1(A/\mathfrak{p})$ for $\mathfrak{p}$ a prime ideal of A. But we know from Exercise 15 that $A/\mathfrak{p}$ is a flat A-module that is finitely generated iff it is free. So the only possibility is for $A/\mathfrak{p} \cong A$ meaning $\mathfrak{p} = 0$. So $K_1(A)$ is generated by $\gamma_1(A)$. Therefore, $K_1(A) \cong \mathbb{Z}$.
iv) This is obvious from definition of $f^!$.
v) We have $x \in K_1(A)$. It follows that $f^!(x)y$ is the isomorphism class $(B \otimes_A x) \otimes_A y$. Apply a restriction of scalars to get $f_!(f^!(x)y) = x \otimes_A y$. This is the same as $xf_!(y)$. □

8. Artin Rings

Exercise 8.1 Let $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = 0$ be a minimal primary decomposition of the zero ideal in a Noetherian ring, and let $\mathfrak{q}_i$ be $\mathfrak{p}_i$-primary. Let $\mathfrak{p}_i^{(r)}$ be the rth symbolic power of $\mathfrak{p}_i$ (Chapter 4, Exercise 13). Show that for each $i = 1, \ldots, n$ there exists an integer r_i such that $\mathfrak{p}_i^{(r_i)} \subseteq q_i$

Suppose $\mathfrak{q}_1$ is an isolated primary component. Then $A_{\mathfrak{p}_1}$ is an Artin local ring, hence if $\mathfrak{m}_i$ is its maximal ideal we have $\mathfrak{m}_i^r = 0$ for all sufficiently large r, hence $\mathfrak{q}_i = \mathfrak{p}_i^{(r)}$ for all large r.

Proof. □

Exercise 8.2 Let A be a Noetherian ring. Prove that the following are equivalent:

i) A is Artinian;

ii) $\mathrm{Spec}(A)$ is discrete and finite;

iii) $\mathrm{Spec}(A)$ is discrete.

Proof. **i) $\implies$ ii):** Since A is Artinian, we know $\dim(A) = 0$ and there are only finitely many prime ideals by Proposition 8.1 in conjunction with Proposition 8.3. Hence, $\mathrm{Spec}(A)$ is finite.

If $\mathfrak{p} \in \mathrm{Spec}(A)$, then $V(\mathfrak{p})$ is a closed set and $V(\mathfrak{p}) = \{\mathfrak{p}\}$ by maximality of $\mathfrak{p}$. Since the space is finite and every point is closed, the topology is discerete.

ii) $\implies$ iii): This is clear.

iii) $\implies$ i): It suffices to show that $\dim(A) = 0$ by Theorem 8.5. This means we must show every prime ideal is maximal. Since $\mathrm{Spec}(A)$ is discrete, $\{\mathfrak{p}\}$ is a closed set for every $\mathfrak{p} \in \mathrm{Spec}\, A$. Thus, $\{\mathfrak{p}\} = V(I)$ for some ideal $I \subseteq A$. Since $I \supseteq \mathfrak{p}$, we get $V(I) \supseteq V(\mathfrak{p})$ and so $\{\mathfrak{p}\} \subseteq V(\mathfrak{p}) = V(I) = \{\mathfrak{p}\}$, we deduce that $V(\mathfrak{p}) = \{\mathfrak{p}\}$. Hence, the longest chain of prime ideals in A is only ever going to consist of one ideal (if $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \mathfrak{p}_2 \cdots \subseteq \mathfrak{p}_n$ is as strictly ascending chain with $n > 0$, then $V(\mathfrak{p}_0) = \{\mathfrak{p}_0, \ldots, \mathfrak{p}_n\} = \{\mathfrak{p}_0\}$ contradiction). Therefore, $\dim A = 0$. □

Exercise 8.3 Let k be a field and A a finitely generated k -algebra. Prove that the following are equivalent:

i) A is Artinian;

ii) A is a finite k -algebra.

[To prove that i) $\Rightarrow$ ii), use (8.7) to reduce to the case where A is an Artin local ring. By the Nullstellensatz, the residue field of A is a finite extension of k. Now use the fact that A is of finite length as an A-module. To prove ii) $\Rightarrow$ i), observe that the ideals of A are k-vector subspaces and therefore satisfy d.c.c.]

Proof. **i) $\implies$ ii):** Assume A is Artinian. By Theorem 8.7, $A \cong \prod_{i=1}^{n} A_i$ where the A_i are Artin local rings. If we show that the A_i are finite k-algebras and since A is a finite product of finite k-algebras, A is therefore a finite k-algebras.

Assume that $(A, \mathfrak{m})$ is a f.g. k-algebra which is an Artin local ring. Then $A/\mathfrak{m} =: K$ is a f.g. k-algebra and since K is a field, K is a finite algebraic extension of k by Proposition 7.9.

Since A is a f.g. k-algebra it is a f.g. k-module and in particular a finite dimensional k-vector space. Then Proposition 6.10 tells us that A must have finite length.

This argument is wrong. f.g. k-algebra does not imply f.g. k-module in general. Patrick: You need to use that A has finite length, one argument would be to then write down a composition series and use the fact that a simple A-module must be isomorphic to A/m (see here: https://math.stackexchange.com/questions/96052/finite-length-modules-over-local-rings).

ii) $\implies$ i): Assume A is a finite k-algebra which implies it is a finite dimensional k-vector spaces. An ideal $\mathfrak{a} \subseteq A$ is an A-submodule and by restriction of scalars, $\mathfrak{a}$ is a k-vector subspace of A. The vector subspaces satisfy the d.c.c. and therefore, the ring A satisfies the d.c.c. on ideals. Proposition 6.10 then says A is Artinian. □

Exercise 8.4

Proof. See Hartshorne's text i.e. the exercises in II.3. □

Exercise 8.5 In Chapter 5, Exercise 16, show that X is a finite covering of L (i.e., the number of points of X lying over a given point of L is finite and bounded).

PROOF. □

Exercise 8.6 Let A be a Noetherian ring and q a p-primary ideal in A. Consider chains of primary ideals from q to p. Show that all such chains are of finite bounded length, and that all maximal chains have the same length.

PROOF. □

9. Discrete Valuation Rings and Dedekind Domains

Exercise 9.1 Let A be a Dedekind domain, S a multiplicatively closed subset of A. Show that $S^{-1}A$ is either a Dedekind domain or the field of fractions of A.

Suppose that $S \neq A - \{0\}$, and let H, H' be the ideal class groups of A and $S^{-1}A$ respectively. Show that extension of ideals induces a surjective homomorphism $H \to H'$.

PROOF. **First statement:** Suppose $S^{-1}A$ is the field of fractions of A. Then $S^{-1}A$ is not a Dedekind domain since it is a field which means it has dimension zero.

Suppose $S^{-1}A$ is not a field. The prime ideals of $S^{-1}A$ are generated by those $\mathfrak{p} \in \operatorname{Spec}(A)$ s.t. $\mathfrak{p} \cap S = \emptyset$. Since A is a Dedekind domain, Theorem 9.3 tells us that A is a Noetherian domain of dimension one which is integrally closed. Since it is a domain and of dimension 1 and $S \neq A \setminus \{0\}$, we deduce that $S^{-1}A$ also has dimension 1. Indeed, we can consider the chain of prime ideals $(0) \subseteq S^{-1}S^c$ where $S^{-1}S^c$ is the extension of the prime ideal $S^c \subseteq A$ (primality follows by definition of a multiplicatively closed set) and this is the longest chain of prime ideals we can obtain because of the correspondence between $\operatorname{Spec}(S^{-1}A)$ and $\operatorname{Spec} A$. It is clearly still a domain since A has no zero divisors and it is Noetherian since localization preserves Noetherianness. It now suffices to show that $S^{-1}A$ is integrally closed.

First, the integral closure of A in its field of fractions $K := T^{-1}A$ is just A where $T := A \setminus 0$. The field of fractions of $S^{-1}A$ is isomorphic to K. By Proposition 5.12, then integral closure of $S^{-1}A$ in $S^{-1}K = K$ is going to be $S^{-1}A$. Hence, $S^{-1}A$ is integrally closed.
Second statement: Since $S \neq A - \{0\}$, we may consider the ideal class groups as claimed. Recall that the ideal class group H is defined as the quotient I/P where I is the group of non-zero fractional ideals and P is the group of principal fractional ideals.

First, we show that the map e exists. Naturally, we have a map $e : I \to H'$ because we can take $e : I \to I'$ and then compose with the quotient. Obtaining a map from H then entails using the universal property of quotients. Indeed, if $\mathfrak{a} \in I$ is a principal ideal, then $e(\mathfrak{a})$ is still principal and generated by the same generator inside $S^{-1}A$. Thus, we obtain a map $e : H \to H'$ from the quotient. Now, $e : H \to H'$ is an induced map which is defined by mapping a representative $\mathfrak{a}$ of H to $\mathfrak{a}^e$ which represents its coset in H'. Doing it in this manner, we would get that e is a well-defined group homomorphism, but we give a proof from more elementary means.

First, we verify e is well-defined. Let $\mathfrak{a}, \mathfrak{b}$ represent the same coset in H. Then $\mathfrak{a}\mathfrak{b}^{-1} = (u)$ for some $u \in K^*$. We show that $e(\mathfrak{a}) = e(\mathfrak{b})$. Indeed, $e(\mathfrak{a})e(\mathfrak{b})^{-1} = e(u)$ because extension commutes with products;

$$\begin{aligned} e(\mathfrak{a})e(\mathfrak{b})^{-1} &= e(\mathfrak{a})(S^{-1}A : S^{-1}\mathfrak{b}) = e(\mathfrak{a})(S^{-1}A : e(\mathfrak{b})) \\ &= e(\mathfrak{a})S^{-1}(A : \mathfrak{b}) \\ &= e(\mathfrak{a})e(A : \mathfrak{b}) = e(\mathfrak{a}(A : \mathfrak{b})) \\ &= e(u). \end{aligned}$$

Note that A is Noetherian so $\mathfrak{b}$ is f.g. which gives the third equality. This means we have a well-defined map $e : H \to H'$. Next, this is a homomorphism because given two non-zero fractional ideals of A and applying Exercise 1.18 (the in-chapter one), $e(\mathfrak{a})e(\mathfrak{b}) = e(\mathfrak{a}\mathfrak{b})$.

Recall from Proposition 3.11 that every $S^{-1}\mathfrak{a}$ ideal in $S^{-1}A$ is an extended ideal so surjectivity follows since the inverse of ideals of $S^{-1}A$ in H' are going to be the extension of the inverse of the ideal $\mathfrak{a} \subseteq A$. □

Exercise 9.2 Let A be a Dedekind domain. If $f = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial with coefficients in A, the content of f is the ideal $c(f) = (a_0, \ldots, a_n)$ in A. Prove Gauss's lemma that $c(fg) = c(f)c(g)$.

[Localize at each maximal ideal.]

Proof. [32] We observe that $c(f)$ is an ideal as opposed to a number in the usual Gauss's Lemma. It also suffices to assume $f, g \neq 0$ because if either of them were zero, then $c(fg) = 0$ and $c(f)(g) = 0$.

Following the hint, let us localize at a maximal ideal $\mathfrak{p}$ and prove the result in that situation i.e. we are assuming $f \in A_\mathfrak{p}[x]$ and $g \in A_\mathfrak{p}[x]$ so $f = \frac{a_0}{1} + \cdots + \frac{a_n}{1}x^n$ and similarly for g. To show that $c(fg)$ is not divisible by any maximal ideal $\mathfrak{p}$.

We know $A_\mathfrak{p}$ is a DVR so every ideal has the form (w^k) for some $k \geq 0$ and $v(w) = 1$ where v is the discrete valuation on $A_\mathfrak{p}$ and $\mathfrak{p}A_\mathfrak{p} = (w)$. Then we can define $v(f) = v(w^k) = k$ where $w^k | f(t)$ and k is the largest integer to do so. Clearly, $v(fg) = v(f) + v(g)$ since we can just factor powers of w. Now let $c_\mathfrak{p}(f)$ be the ideal generated by the coefficients in $A_\mathfrak{p}$.

We show that $c_\mathfrak{p}(f) = c_\mathfrak{p}(g) = 1$ implies $c_\mathfrak{p}(fg) = 1$. Indeed, $c_\mathfrak{p}(f) = 1$, iff w^k does not divide f for any $k > 0$ iff $v(f) = 0$. Thus, $v(fg) = v(f)+v(g) = 0$ and so $c_\mathfrak{p}(fg) = c_\mathfrak{p}(f)c_\mathfrak{p}(f)$.

For the general case, we write $f = w^{v(f)}\widetilde{f}$ and $g = w^{v(g)}\widetilde{g}$. Then because, $c_\mathfrak{p}(f) = (w^{v(f)})$ which is due to the fact that the ideal $c_\mathfrak{p}(f)$ is of the form (w^l) for some l,

$$c_\mathfrak{p}(f)c_\mathfrak{p}(g) = (w^{v(f)})c(\widetilde{f})(w^{v(g)})c_\mathfrak{p}(\widetilde{g}) = (w^{v(f)+v(g)})c(\widetilde{f}\widetilde{g}) = (w^{v(f)+v(g)}) = (w^{v(fg)}) = c_\mathfrak{p}(fg).$$

Now we return to the case of A. We know that $c_\mathfrak{p}(fg) = c_\mathfrak{p}(f)c_\mathfrak{p}(g)$ for every $\mathfrak{p} \in \operatorname{Spec} A$. But we observe that $c(f) = \prod_{\mathfrak{p}\in\operatorname{Spec} A} \mathfrak{p}^{v_\mathfrak{p}(f)} = \prod_{\mathfrak{p}\in\operatorname{Spec} A} c_\mathfrak{p}(f)$ where $v_\mathfrak{p}$ denotes the discrete valuation on $A_\mathfrak{p}$. indeed, this is because $c_\mathfrak{p}(f)$ is generated by some (w^m) and if $w = \frac{a}{b}$, then b is just a unit so $(w^m) = (a^m)$. But then,

$$c(f)c(g) = \prod_{\mathfrak{p}\in\operatorname{Spec} A} \mathfrak{p}^{v_\mathfrak{p}(f)}\mathfrak{p}^{v_\mathfrak{p}(g)} = \prod_{\mathfrak{p}\in\operatorname{Spec} A} \mathfrak{p}^{v_\mathfrak{p}(fg)} = c(fg).$$

□

[32] I was stuck on this problem for more or less quite some time. I did some reading around online and found most references to be either too terse to follow, or largely unmotivated and not really providing an idea on what DVRs really do for us. Eventually, I found this essay on Dedekind domain which was helpful and I saved it in my "good exposition" folder.

Exercise 9.3 A valuation ring (other than a field) is Noetherian if and only if it is a discrete valuation ring.

PROOF. ($\implies$): Let A be a Noetherian valuation ring other than a field. Proposition 5.18i) tells us that A is a local ring and integrally closed in its field of fractions K.

Since A is a not a field, we know $\dim A > 0$ (an integral domain has dimension zero iff it is a field). Since A is Noetherian, the maximal ideal is finitely generated so let $\mathfrak{m} = (x_1, \ldots, x_n)$. We know from Exercise 5.28 that either $x_i | x_j$ or $x_j | x_i$ for all $i \neq j$ and so picking the element x_i which divides all of the others, we deduce that $\mathfrak{m} = (x)$ where $x := x_i$. By the same line of reasoning, we know that all the ideals of A are principal. Hence, A is a PID which is not a field.

Now we show[33] the more general statement that any PID which is not a field has dimension 1. Let B be such a PID. Again, $\dim B > 0$ and we can choose a maximal chain of prime ideals

$$(0) \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \cdots \subsetneq \mathfrak{p}_n$$

and assume $\mathfrak{p}_i = (x_i)$. First, we observe that if $\mathfrak{p} = (x_i)$ is a prime ideal, then x_i is an irreducible element. Indeed, if $ab = x_1$, then WLOG $a \in (x_1)$ which means $a = x_1 m$ for some m which then means $x_1 mb = x_1$ and hence, $mb = 1$ which means b was a unit. Now if $(x_1) \subseteq (x_2)$, we can find an m s.t. $x_1 = mx_2$. But x_1 is irreducible so either m or x_2 is a unit. Since $\mathfrak{p}_2$ is a prime ideal, x_2 cannot be a unit so that means m is a unit. But then $(x_1) \subseteq (x_2) = (mx_2) = (x_1)$ implies $(x_1) = (x_2)$. Hence, $\mathfrak{p}_1 = \mathfrak{p}_2$. By the same argument for each nonzero prime ideal in the maximal chain, $n = 1$. Hence, $\dim B = 1$.

Our work above has shown that A is a Noetherian local ring which is integrally closed and of dimension 1. Apply Proposition 9.2.

($\impliedby$): By the remarks preceding Proposition 9.2, if A is a DVR, then it is a Noetherian local domain of dimension 1. Hence, it is Noetherian and the discrete valuation on A makes A a valuation ring. Being dimension 1 implies that A cannot be a field. $\square$

Exercise 9.4 Let A be a local domain which is not a field and in which the maximal ideal m is principal and $\cap_{n=1}^{\infty} m^n = 0$. Prove that A is a discrete valuation ring.

PROOF. $\square$

Exercise 9.5 Let M be a finitely-generated module over a Dedekind domain. Prove that M is flat $\Leftrightarrow$ M is torsion-free.

[Use Chapter 3, Exercise 13 and Chapter 7, Exercise 16.]

PROOF. $\square$

Exercise 9.6 Let M be a finitely-generated torsion module ($T(M) = M$) over a Dedekind domain A. Prove that M is uniquely representable as a finite direct sum of modules $A/\mathfrak{p}_i^{n_i}$, where $\mathfrak{p}_i$ are non-zero prime ideals of A. [For each $\mathfrak{p} \neq 0, M_{\mathfrak{p}}$ is a torsion $A_{\mathfrak{p}}$-module; use the structure theorem for modules over a principal ideal domain.]

PROOF. $\square$

Exercise 9.7 Let A be a Dedekind domain and $\mathfrak{a} \neq 0$ an ideal in A. Show that every ideal in $A/\mathfrak{a}$ is principal. Deduce that every ideal in A can be generated by at most 2 elements.

[33] This was proven in Math 200B, but we prove it here for completeness.

PROOF. **First Statement:** Since $\mathfrak{a}$ is a nonzero ideal, $\mathfrak{a} = \mathfrak{p}_1^{v_1} \dots \mathfrak{p}_n^{v_n}$ for some distinct prime ideals $\mathfrak{p}_i$ which are uniquely determined up to rearrangement. Then, the Chinese Remainder Theorem tells us that

$$A/\mathfrak{a} \cong A/\mathfrak{p}_1^{v_1} \oplus \cdots \oplus A/\mathfrak{p}_n^{v_n},$$

but we must show that if $\mathfrak{p}$ and $\mathfrak{q}$ are distinct primes, then $\mathfrak{p}^n$ and $\mathfrak{q}^m$ are coprime ideals in A. Indeed, $\mathfrak{p}^n + \mathfrak{q}^m$ is the smallest ideal containing both $\mathfrak{p}^n$ and $\mathfrak{q}^m$ and the only ideals that contain $\mathfrak{p}^n$ are $\mathfrak{p}^{n-1}, \dots, \mathfrak{p}$ and similar for $\mathfrak{q}^m$. This means I must have form $\mathfrak{p}^i$ and $\mathfrak{q}^j$ for some i and j which contradicts unique factorization into prime ideals.

If we show each $A/\mathfrak{p}_i^{v_i}$ is a principal ring, then the fact that a direct sum of principapl rings is still a principal ring finishes the proof. So we show WLOG that $A/\mathfrak{p}^v$ is a princial ring. Consider the projection map $\phi : A \to A/\mathfrak{p}^v$ and the ideals of $A/\mathfrak{p}^v$ correspond to ideals of A containing $\mathfrak{p}^v$. The only such ideals are $\mathfrak{p}, \mathfrak{p}^2, \dots, \mathfrak{p}^v$ and so $\mathfrak{p}$ is the maximal ideal of $A/\mathfrak{p}^v$. So because A is Noetherian, $A/\mathfrak{p}^v$ is a Noetherian local ring.

Next, we observe that[34]

$$A/\mathfrak{p}^v \cong (A/\mathfrak{p}^v)_{\mathfrak{p}} \cong A_{\mathfrak{p}}/\mathfrak{p}^v A_{\mathfrak{p}}$$

and the first isomorphism follows from the fact that $\mathfrak{p}/\mathfrak{p}^v$ is the unique maximal ideal of $A/\mathfrak{p}^v$ so localizing at $\mathfrak{p}$ means inverting the elements which were already units. The second isomorphism is due to the fact that quotients commute with localizations.

Now the quotient of a PID by an ideal is still a principal ring (as it is not generally a domain, consider $\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$). This is simply a consequence of the correspondence theorem. Since $A_{\mathfrak{p}}$ is a DVR, it is certainly a PID so the above isomorphism shows $A/\mathfrak{p}^v$ is a principal ring which is what we wanted to show.

Second Statement: Let $\mathfrak{b}$ be a proper ideal of A (if it were just A, then $\mathfrak{b} = (1)$). Choose a nonzero element $b \in \mathfrak{b}$ and consider $\mathfrak{b}/(b) \subseteq A/(b)$. Then $\mathfrak{b}/(b)$ is principal and suppose it is generated by $a + (b)$. But that means $(a) + (b) = \mathfrak{b} + (b)$ which implies $\mathfrak{b} = (a) + (b) - (b) = (a) + (b)$ so it is generated by possibly two elements. Hence, every ideal in A can be generated by at most 2 elements. □

Exercise 9.8 Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be three ideals in a Dedekind domain. Prove that

$$\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{c})$$
$$\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c}).$$

[Localize.]

PROOF. The hint to localize indicates that we should check that both the LHS and RHS have the same powers n_i for prime ideals $\mathfrak{p}_i^{n_i}$ appearing int he unique factorization of ideals. Note that $\mathfrak{p}^e \cap \mathfrak{p}^f = \mathfrak{p}^{\max\{e,f\}}$ and $\mathfrak{p}^e + \mathfrak{p}^f = \mathfrak{p}^{\min\{e,f\}}$. Now if we localize at $\mathfrak{p}$, the LHS and RHS essentially of the first equation will be $\mathfrak{p}^{\max\{e,\min\{f,g\}\}}$ while the RHS is $\mathfrak{p}^{\min\{\max\{e,f\},\max\{e,g\}\}}$. These are clearly equal. For the second equation, the LHS is $\mathfrak{p}^{\min\{e,\max\{f,g\}\}}$ while the RHS is $\mathfrak{p}^{\max\{\min\{e,f\},\min\{e,g\}\}}$. These are clearly equal. □

Exercise 9.9 (Chinese Remainder Theorem). Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals and let $x_1, \dots, x_n$ be elements in a Dedekind domain A. Then the system of congruences $x = x_1 \,(\mathrm{mod}\, \mathfrak{a}_1)\, (1 \leqslant i \leqslant n)$ has a solution x in $A \Leftrightarrow x_1 \equiv x_1 \,(\mathrm{mod}\, \mathfrak{a}_i + \mathfrak{a}_j)$ whenever $i \neq j$.

[34]This approach to this part of the problem isn't from my own creativity. I recall doing a similar problem when auditing Math 204A and had read about this method. It was either this argument or to consider $x \in \mathfrak{p}/\mathfrak{p}^2$ and prove it directly.

[This is equivalent to saying that the sequence of A-modules

$$A \xrightarrow{\phi} \bigoplus_{i=1}^{n} A/a_i \xrightarrow{\psi} \bigoplus_{i<j} A/\left(\mathfrak{a}_i+\mathfrak{a}_j\right)$$

is exact, where ϕ and ϕ are defined as follows: $\phi(x)=\left(x+\mathfrak{a}_1, \ldots, x+\mathfrak{a}_n\right) ; \psi\left(x_1+\mathfrak{a}_1, \ldots, x_n+\mathfrak{a}_n\right)$ has (i, j)-component $x_i-x_j+\mathfrak{a}_i+\mathfrak{a}_j$. To show that this sequence is exact it is enough to show that it is exact when localized at any $p \neq 0$: in other words we may assume that A is a discrete valuation ring, and then it is easy.]

Proof. □

10. Completions

Exercise 10.1 Let $\alpha_n: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ be the injection of abelian groups given by $\alpha_n(1)=p^{n-1}$, and let $\alpha: A \rightarrow B$ be the direct sum of all the α_n (where A is a countable direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$, and B is the direct sum of the $\mathbb{Z}/p^n\mathbb{Z}$). Show that the p-adic completion of A is just A but that the completion of A for the topology induced from the p-adic topology on B is the direct product of the $\mathbb{Z}/p\mathbb{Z}$. Deduce that p-adic completion is not a right-exact functor on the category of all $\mathbb{Z}$-modules.

Proof. □

Exercise 10.2 In Exercise 1, let $A_n=\alpha^{-1}\left(p^n B\right)$, and consider the exact sequence

$$0 \rightarrow A_n \rightarrow A \rightarrow A \mid A_n \rightarrow 0.$$

Show that $\varprojlim$ is not right exact, and compute $\varprojlim^1 A_n$.

Proof. □

Exercise 10.3 Let A be a Noetherian ring, a an ideal and M a finitely-generated A-module. Using Krull's Theorem and Exercise 14 of Chapter 3, prove that

$$\bigcap_{n=1}^{\infty} \mathfrak{a}^n M=\bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \operatorname{Ker}\left(M \rightarrow M_{\mathfrak{m}}\right)$$

where $\mathfrak{m}$ runs over all maximal ideals containing $\mathfrak{a}$.

Deduce that

$$\widehat{M}=0 \Rightarrow \operatorname{Supp}(M) \cap V(\mathfrak{a})=\emptyset \qquad (\text{ in } \operatorname{Spec}(A))$$

[The reader should think of $\widehat{M}$ as the "Taylor expansion" of M transversal to the subscheme $V(\mathfrak{a})$: the above result then shows that M is determined in a neighborhood of $V(\mathfrak{a})$ by its Taylor expansion.]

Proof. □

Exercise 10.4 Let A be a Noetherian ring, $\mathfrak{a}$ an ideal in A, and A the $\mathfrak{a}$-adic completion. For any $x \in A$, let $\hat{x}$ be the image of x in $\hat{A}$. Show that

$$x \text{ not a zero-divisor in } A \implies \hat{x} \text{ not a zero-divisor in } A.$$

Does this imply that

$$A \text{ is an integral domain } \implies A \text{ is an integral domain?}$$

[Apply the exactness of completion to the sequence $0 \rightarrow A \xrightarrow{x} A$.]

PROOF. If x is not a zero-divisor in A, then $0 \to A \xrightarrow{x} A$ is exact where $A \to A$ is given by $a \mapsto ax$. Now taking the completion, we get $0 \to \widehat{A} \xrightarrow{\widehat{x}} \widehat{A}$ which is also exact by Proposition 10.12 and the second map is given by $\widehat{a} \mapsto \widehat{a}\widehat{x}$. Hence, $\widehat{x}$ is not a zero-divisor of A.

No[35], it does not. Note that $\mathbb{Z}$ with the p-adic completion is not a counterexample since $\mathbb{Z}_p$ is a principal ideal domain. What we need to observe is that $\widehat{A}$ may introduce new elements which may potentially be zero-divisors. For a counterexample, take $k[x]$ with the completion along $\mathfrak{a} = (x(x+1))$. Then,

$$\widehat{k[x]} \cong \varprojlim k[x]/(x(x-1))^n \cong \varprojlim k[x]/(x)^n \oplus k[x]/(x+1)^n \cong \varprojlim k[x]/(x)^n \oplus \varprojlim k[x]/(x-1)^n$$

where the second isomorphism is by the Chinese Remainder Theorem and the third is by the fact that the inverse limit distributes over finite direct sums. Another way to do this is as follows. We have

$$k[x]/(x(x+1))^n \cong k[x]/(x)^n \oplus k[x]/(x+1)^n$$

by the Chinese Remainder Theorem. Then, taking the inverse limit and that implies $\widehat{k[x]}_{x(x+1)} \cong \widehat{k[x]}_{(x)} \oplus \widehat{k[x]}_{(x+1)}$.

$\square$

Exercise 10.5 Let A be a Noetherian ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals in A. If M is any A-module, let $M^{\mathfrak{a}}, M^{\mathfrak{b}}$ denote its $\mathfrak{a}$-adic and $\mathfrak{b}$-adic completions respectively. If M is finitely generated, prove that $(M^{\mathfrak{a}})^{\mathfrak{b}} \cong M^{\mathfrak{a}+\mathfrak{b}}$.

[Take the $\mathfrak{a}$-adic completion of the exact sequence

$$0 \to \mathfrak{b}^m M \to M/\mathfrak{b}^m M \to 0$$

and apply (10.13). Then use the isomorphism

$$\varprojlim_m \left(\varprojlim_n M/\left(\mathfrak{a}^n M + \mathfrak{b}^m M\right) \right) \cong \varprojlim_n M/\left(\mathfrak{a}^n M + \mathfrak{b}^n M\right)$$

and the inclusions $(\mathfrak{a}+\mathfrak{b})^{2n} \subseteq \mathfrak{a}^n + \mathfrak{b}^n \subseteq (\mathfrak{a}+\mathfrak{b})^n$.]

PROOF. $\square$

Exercise 10.6
Exercise 10.7
Exercise 10.8
Exercise 10.9 Let A be a local ring, $\mathfrak{m}$ its maximal ideal. Assume that A is $\mathfrak{m}$-adically complete. For any polynomial $f(x) \in A[x]$, let $f(x) \in (A/\mathfrak{m})[x]$ denote its reduction mod. $\mathfrak{m}$. Prove *Hensel's lemma*: if $f(x)$ is monic of degree n and if there exist coprime monic polynomials $\bar{g}(x), \bar{h}(x) \in (A/\mathfrak{m})[x]$ of degrees $r, n-r$ with $f(x) = \bar{g}(x)F(x)$, then we can lift $\bar{g}(x), \bar{h}(x)$ back to monic polynomials $g(x), h(x) \in A[x]$ such that $f(x) = g(x)h(x)$.

[Assume inductively that we have constructed $g_k(x), h_k(x) \in A[x]$ such that $g_k(x)h_k(x) - f(x) \in \mathfrak{m}^k A[x]$. Then use the fact that since $\bar{g}(x)$ and $\bar{h}(x)$ are coprime we can find

[35]The following is an incorrect proof claiming the opposite:
Yes, it does. Recall that the $\mathfrak{a}$-adic completion is just $\varprojlim A/\mathfrak{a}^n$. This is given the ring of coherent sequences $(a_0, a_1, \dots,)$ where $\theta_{n+1}(a_{n+1}) = a_n$ where $\theta_{n+1} : \mathfrak{a}^{n+1} \to \mathfrak{a}^n$. So if $\widehat{A}$ is not an integral domain, we would have nonzero elements s.t. $(a_0, a_1, \dots)(b_0, b_1, \dots) = (0, 0, \dots)$ which is impossible since we would need $a_0 b_0 = 0$ where $a_0, b_0 \in A$ which means WLOG that $a_0 = 0$. Indeed, by coherence, this implies $a_n = 0$ for all n so that $(a_0, a_1, \dots) = (0, 0, \dots)$.

$\bar{a}_p(x), b_p(x)$, of degrees $\leqslant n-r, r$ respectively, such that $x^p = \bar{a}_p(x)\bar{g}_k(x) + b_p(x)h_k(x)$, where p is any integer such that $1 \leqslant p \leqslant n$ Finally, use the completeness of A to show that the sequences $g_k(x), h_k(x)$ converge to the required $g(x), h(x)$.]

Proof. See Neukirch's "Algebraic Number Theory" Chapter II. The proof of Hensel's lemma is highly nontrivial and requires a significant amount of effort from the reader. For my number theory course, the proof of the lemma took at least half an hour to complete. □

Exercise 10.10
Exercise 10.11
Exercise 10.12

11. Dimension Theory

Exercise 11.1 Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial over an algebraically closed field k. A point P on the variety $f(x) = 0$ is non-singular $\Leftrightarrow$ not all the partial derivatives $\partial f / \partial x_i$ vanish at P. Let $A = k[x_1, \ldots, x_n]/(f)$, and let $\mathfrak{m}$ be the maximal ideal of A corresponding to the point P. Prove that P is non-singular $\Leftrightarrow$ $A_{\mathfrak{m}}$ is a regular local ring.

[By (11.18) we have $\dim A = n-1$. Now

$$\mathfrak{m}/\mathfrak{m}^2 \cong (x_1, \ldots, x_n) / (x_1, \ldots, x_n)^2 + (f)$$

and has dimension $n-1$ if and only if $f \notin (x_1, \ldots, x_n)^2$.]

Proof. □

Exercise 11.2 In (11.21) assume that A is complete. Prove that the homomorphism $k[[t_1, \ldots, t_d]] \to A$ given by $t_i \mapsto x_i$ $(1 \leqslant i \leqslant d)$ is injective and that A is a finitely-generated module over $k[[t_1, \ldots, t_d]]$. [Use (10.24).].

Proof. □

Exercise 11.3 Extend (11.25) to non-algebraically-closed fields. [If $\overline{k}$ is the algebraic closure of k, then $\overline{k}[x_1, \ldots, x_n]$ is integral over $k[x_1, \ldots, x_n]$.]

Proof. □

Exercise 11.4 An example of a Noetherian domain of infinite dimension (Nagata). Let k be a field and let $A = k[x_1, x_2, \ldots, x_n, \ldots]$ be a polynomial ring over k in a countably infinite set of indeterminates. Let $m_1, m_2, \ldots$ be an increasing sequence of positive integers such that $m_{1+1} - m_1 > m_1 - m_{1-1}$ for all $i > 1$. Let $\mathfrak{p}_i = (x_{m_1+1}, \ldots, x_{m_1+1})$ and let S be the complement in A of the union of the ideals $\mathfrak{p}_i$. Each $\mathfrak{p}_i$ is a prime ideal and therefore the set S is multiplicatively closed. The ring $S^{-1}A$ is Noetherian by Chapter 7, Exercise 9. Each $S^{-1}\mathfrak{p}_i$ has height equal to $m_{i+1} - m_1$, hence $\dim S^{-1}A = \infty$.

Proof. □

Exercise 11.5 Reformulate (11.1) in terms of the Grothendieck group $K(A_0)$ (Chapter 7 , Exercise 25).

Proof. □

Exercise 11.6 Let A be a ring (not necessarily Noetherian). Prove that

$$1 + \dim A \leqslant \dim A[x] \leqslant 1 + 2 \dim A.$$

[Let $f : A \to A[x]$ be the embedding and consider the fiber of $f^* : \mathrm{Spec}(A[x]) \to \mathrm{Spec}\ (A)$ over a prime ideal p of A. This fiber can be identified with the spectrum of $k \otimes_A A[x] \cong k[x]$, where k is the residue field at $\mathfrak{p}$ (Chapter 3, Exercise 21), and $\dim k[x] = 1$. Now use Exercise 7(ii) of Chapter 4.]

Proof. □

Exercise 11.7 Let A be a Noetherian ring. Then $\dim A[x] = 1 + \dim A$ and hence, by induction on n, $\dim A\,[x_1, \ldots, x_n] = n + \dim A$.

[Let $\mathfrak{p}$ be a prime ideal of height m in A. Then there exist $a_1, \ldots, a_m \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime ideal belonging to the ideal $\mathfrak{a} = (a_1, \ldots, a_m)$. By Exercise 7 of Chapter 4, $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$ and therefore height $\mathfrak{p}[x] \leqslant m$. On the other hand, a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m = \mathfrak{p}$ gives rise to a chain $\mathfrak{p}_0[x] \subset \ldots \subset p_m[x] = p[x]$, hence height $\mathfrak{p}[x] \geqslant m$. Hence height $\mathfrak{p}[x] =$ height $\mathfrak{p}$. Now use the argument of Exercise 6.]

Proof. □

CHAPTER 2

Commmutative Algebra: With a View Toward Algebraic Geometry by Eisenbud

CHAPTER 3

Cohen-Macaulay Rings by Bruns and Herzog

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